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RICCI FLOW OF CONFORMALLY COMPACT METRICS

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Abstract. In this paper we prove that given a smoothly conformally compact metric there is a short-time solution to the Ricci flow that remains smoothly conformally compact. We apply recent results of Schnürer, Schulze and Simon to prove a stability result for conformally compact metrics sufficiently close to the hyperbolic metric.

1. Introduction

In 1989, W. X. Shi initiated the study of the Ricci flow on a noncompact manifold by proving that there is a short-time solution to the flow starting at a complete metric of bounded curvature, and moreover the flow remains in this class. Recently there has been intense activity to understand to what extent the Ricci flow preserves other geometric conditions on noncompact manifolds, see [2, 5, 13, 11, 12, 18] for examples. In this paper we prove that the Ricci flow preserves the set of smoothly conformally compact metrics in general dimension for a short time. We begin by introducing these metrics.

Let M^{n+1} be the interior of a compact manifold with boundary $\overline{M}$. Suppose that x is a boundary defining function for ∂M. This is to say that x is a smooth non-negative function on $\overline{M}$ that vanishes to first order precisely at ∂M. We say that a metric h is smoothly *conformally compact* if $\overline{h} := x^2 h$ extends to be smooth metric on $\overline{M}$. The Poincaré ball model of hyperbolic space provides an easy example.

We may use x to identify a collar neighbourhood of ∂M in $\overline{M}$ with $[0, \epsilon) \times \partial M$. We then write h as

$$h = \frac{dx^2 + \hat{h}(x)}{x^2},$$

for a smooth family of metrics $\hat{h}$ on ∂M.

If h is smoothly conformally compact, with $|dx|^2_{\overline{h}} = 1$ on ∂M, and K denotes the curvature 4-tensor of constant sectional curvature $+1$, then the curvature 4-tensor R of h satisfies

$$|R + K|_h = O(x), \text{ and}$$
$$|\nabla^{(j)}_h R|_h = O(x), \text{ for all } j.$$

For this reason conformally compact manifolds with $|dx|^2_{\overline{h}} = 1$ on ∂M are asymptotically hyperbolic. It is well known that h is complete and of bounded geometry.

Recall the Ricci flow is the system of equations

$$\begin{cases} \partial_t g &= -2Rc\, g(t), \\ g(0) &= h. \end{cases} \tag{1.1}$$

As previously mentioned, it follows from [24] that there is a solution to the Ricci flow, $g(t)$, with initial metric h for a short time.

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It will be more convenient to study a normalized Ricci flow. Set $\tau = \frac{1}{2n}(e^{2nt} - 1)$, and set $g^N(x,t) = \frac{1}{1+2n\tau} g(x,t)$. It is a simple matter to verify that g_N solves the normalized Ricci flow:

$$\begin{cases} \partial_\tau g^N_{ij} & = -2n g^N_{ij} - 2Rc\, g^N_{ij}, \\ g^N(0) & = h. \end{cases} \tag{1.2}$$

As solutions to the Ricci flow and normalized Ricci flow differ by a rescaling in the time variable, we see that spatial regularity is preserved. Thus a conformally compact solution to the normalized Ricci flow yields a conformally compact solution to the Ricci flow and vice versa.

The first main result of this paper is the following

Theorem A. *If h is smoothly conformally compact then there exists a unique smoothly conformally compact solution $g(t)$ to (1.2) (and hence (1.1)) for a short time.*

The proof of this Theorem proceeds in three steps. First we apply the DeTurck trick to obtain a system that may be solved by parabolic PDE techniques. Then conditioning the equation appropriately we are able to apply a contraction mapping argument to reprove the existence of a short-time solution to the flow in 0-Hölder spaces, which are Hölder spaces associated to conformally compact metrics that respect the interior geometry. Finally we prove that the solution is smoothly conformally compact by applying regularity techniques modeled on [20]. By chasing regularity at each step of the argument it would be possible to give a finite regularity version of this theorem.

Given short-time existence for the Ricci flow it is natural to study the stability of the flow about fixed points. Recently, Schnürer, Schulze and Simon studied the stability of hyperbolic space under the Ricci flow [23]. Using their work and the regularity result underlying Theorem A, we are able to quickly obtain the second main result of this paper.

Theorem B. *Let $n+1 \geq 4$. For all $K > 0$ there exists $\epsilon_1 = \epsilon_1(n, K) > 0$ such that the following holds. Let g_0 be a smoothly conformally compact metric close to the hyperbolic metric h on the unit ball in the sense that*

$$\int_{\mathbb{H}^{n+1}} |g_0 - h|_h^2 dvol_h \leq K,$$

and

$$\sup_{\mathbb{H}^{n+1}} |g_0 - h|_h \leq \epsilon_1.$$

Then there exists a long-time solution $g(t)$ to the Ricci-DeTurck flow (with initial metric g_0) such that

$$\sup_{\mathbb{H}^{n+1}} |g(t) - h|_h \leq C(n,K) e^{-\frac{1}{4(n+3)}t}.$$

Moreover, $g(t) \longrightarrow h$ exponentially and $g(t)$ remains conformally compact for all time.

Note in the above theorem, unlike in the original source, we have transcribed the dimension to $n+1$ to match the convention of the rest of the paper.

It follows from the work of Fefferman and Graham [9] that there is an obstruction to finding a smooth conformally compact Einstein metric on generic manifolds when the boundary dimension (in our convention n) is even. In view of this fact, one expects there to be considerable challenges to using the Ricci flow to produce conformally compact Einstein metrics more generally.

We take this opportunity to mention two related papers. First, recent work by Hu, Qing and Shi [11] proves the Ricci flow preserves a certain class of asymptotically hyperbolic metrics for a short-time. These metrics are defined by curvature decay conditions and, as shown in [3] and [11], are conformally compact of only a limited regularity. Hu, Qing and Shi subsequently prove an interesting rigidity result. On the other hand, in view of the applications of smoothly conformally compact metrics to geometry and physics (see for example [6] and references therein), it is natural to study the Ricci flow in the smooth conformally compact setting. Second, the author and Helliwell have recently proved short-time existence results for higher-order geometric flows on compact manifolds [4]. We observed that many short-time existence results depend only on the special algebraic structure of the flow. Both [4] and the present paper were developed in parallel, and were inspired by recent work of Koch and Lamm [14]. The short-time existence of the Ricci flow we give here, while in the setting of conformally compact metrics, may be regarded as a concrete application of the ideas in [4].

This paper is structured as follows. In Section 2, we outline the DeTurck trick and reduction of the flow to a parabolic system. In Section 3, we define function spaces and outline the main results from linear parabolic theory on conformally compact manifolds. This theory is based on the edge and heat calculus for 0-operators that appears in [19] and [1]. In order to not distract from the main Ricci flow argument, we have kept this section short and instead sketched several of the proofs of the analytic results in the Appendix. In Section 4, we condition the flow equations and provide the contraction mapping argument. We discuss the regularity argument in Section 5, and the stability argument in Section 6. Finally, in the Appendix we provide sketches for the various analytic facts quoted in Section 3.

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2. Preliminaries

As is well known, the Ricci flow is not a parabolic system due to the diffeomorphism invariance of the Ricci tensor. We will break this invariance using the standard DeTurck trick. Choosing h as the background metric, and writing all Christoffel symbols and curvature quantities with respect to h with tildes, we define a time dependent vector field

$$W^k = g^{pq}\left(\Gamma^k_{pq} - \widetilde{\Gamma^k_{pq}}\right).$$

The normalized Ricci-DeTurck flow is given by

$$\begin{cases} \partial_t g_{ij} &= -2ng_{ij} - 2Rc\; g_{ij} + \nabla_i W_j + \nabla_j W_i, \\ g(0)_{ij} &= h_{ij}. \end{cases} \tag{2.1}$$

Standard computations, for example given in [24], show that this flow may be written

$$\begin{cases} 0 &= \partial_t g_{ij} - g^{ab}\widetilde{\nabla}_a\widetilde{\nabla}_b g_{ij} + 2ng_{ij} + g^{ab}g_{ip}h^{pq}\widetilde{R}_{jaqb} + g^{ab}g_{jp}h^{pq}\widetilde{R}_{iaqb} \\ & \quad -\frac{1}{2}g^{ab}g^{pq}\Big(\widetilde{\nabla}_i g_{pa}\widetilde{\nabla}_j g_{qb} + 2\widetilde{\nabla}_a g_{jp}\widetilde{\nabla}_q g_{ib} - 2\widetilde{\nabla}_a g_{jp}\widetilde{\nabla}_b g_{iq} \\ & \quad - 2\widetilde{\nabla}_j g_{pa}\widetilde{\nabla}_b g_{iq} - 2\widetilde{\nabla}_i g_{pa}\widetilde{\nabla}_b g_{jq}\Big), \\ g(0) &= h. \end{cases} \tag{2.2}$$

From this equation we see the Ricci-DeTurck flow is a quasilinear parabolic system for the metric.

Once we prove short-time existence of a smoothly conformally compact solution g to the Ricci-DeTurck flow, the time-dependent vector field W^k will have coefficients smooth up to the boundary of $\overline{M}$ and vanishing to first order there. If ϕ_t denotes the flow generated by W^k, then $\hat{g} = \phi_t^* g$ is a solution to the normalized Ricci flow [8]. It is straightforward to see that $\hat{g}$ is smoothly conformally compact.

Finally, we only prove the existence of a short-time solution to the Ricci flow. The uniqueness assertion in Theorem A follows from the work of Chen and Zhu [7].

3. Parabolic theory on conformally compact spaces

In this section we outline linear parabolic theory for uniformly degenerate operators on conformally compact manifolds. We just state the results we need here; sketches of proofs are deferred to the appendix. The primary references for the material in this section are [19] and [1].

Let (M, h) be a smoothly conformally compact manifold as defined in the introduction. Suppose that x is a boundary defining function and that $\{y^1, \cdots, y^n\}$ are coordinates on the boundary, extended to be constant in x. We will refer to these coordinates as background coordinates. The metric h decomposes as

$$h = \frac{dx^2 + \hat{h}_{ab}(x,y)dy^a dy^b}{x^2},$$

where the components of $\hat{h}$ are smooth up to the boundary.

The 0-*vector fields* are generated by

$$\left\{ x\partial_x, x\partial_{y^1}, \cdots, x\partial_{y^n} \right\},$$

and form the basis of a vector bundle, the 0-tangent bundle ${}^0TM$. We will also have occasion to discuss *b-vector fields*, which are generated by

$$\left\{ x\partial_x, \partial_{y^1}, \cdots, \partial_{y^n} \right\}.$$

We will denote the space of smooth functions on M by $C^\infty(M)$ and functions smooth up to the boundary by $C^\infty(\overline{M})$. The vector bundle of symmetric 2-tensors on M will be denoted $\Sigma^2(M)$. We will use $\frac{dx}{x}$ and $\frac{dy^b}{x}$ as the preferred basis for this bundle.

An operator L on functions is *uniformly degenerate* of order m if in local coordinates it is given by:

$$L = \sum_{j+|\beta|\leq m} a_{j,\beta}(x,y,t)(x\partial_x)^j (x\partial_y)^\beta.$$

where the coefficients $a_{j,\beta}$ are at least continuous up to the boundary. In order to use Albin's heat calculus, we require that $a_{j,\beta}$ be smooth up to the boundary and independent of time.

The principal symbol of a uniformly degenerate operator L is a homogeneous polynomial on ${}^0T^*M$ given by

$${}^0\sigma(L)(\xi,\eta) = \sum_{j+|\beta|=m} a_{j,\beta}\xi^j\eta^\beta.$$

We say that L is *elliptic* if ${}^0\sigma(L)$ is invertible away from $(\xi,\eta) = 0$.

For the Ricci flow analysis, we will have to deal with systems of equations as our operators will act on the vector bundle of symmetric two tensors. An operator between tensor bundles E and F is uniformly degenerate if in local coordinates it may be written as a system:

$$(Lu)_i = \sum_{j+|\beta|\leq m} (a_{j,\beta})_i^k (x\partial_x)^j (x\partial_y)^\beta u_k.$$

where the coefficients $a_{j,\beta}$ are now entries of a $\dim F \times \dim E$ matrix that is at least continuous up to the boundary. The principal symbol is defined as before. We will not need to consider the most general notions of ellipticity for systems as the Ricci flow system (2.2) is 'diagonal' at top order, i.e. $(a_{j,\beta})^i_k = (a_{j,\beta}) \cdot \delta^i_k$. From this we can see that all coupling occurs at lower order. We now say that L is elliptic if $\dim F = \dim E$ and the symbol is invertible away from $(\xi, \eta) = 0$.

For the remainder of this section we suppose that L is a second order uniformly degenerate elliptic operator with diagonal principal symbol.

3.1. **Function spaces.** We work in the 0-Hölder spaces defined for example in [17, 19, 21]. We describe the anisotropic version of these Hölder function spaces, and refer the reader to the references for the purely spatial version. For any manifold M, the notation M_T will denote the cylinder $M \times [0, T)$. Fix a smoothly conformally compact metric h, which in the Ricci flow analysis, will be the initial metric. We assume a covering of $\overline{M}$ by background coordinates has been fixed.

Cover M by a Whitney decomposition of countably many uniformly locally finite coordinate balls B_i with centre (x_i, y_i) and radius $\frac{1}{2}x_i$. We will consider the product of each ball with a time interval $[0, T)$. For any $0 < a < 1$, consider the norm

$$||u||_{a,\frac{a}{2}} := ||u||_\infty + \sup_i \left\{ \sup_{(x,y,t)\neq(x',y',t)\in(B_i)_T} \frac{(x+x')^a\,|u(x,y,t)-u(x',y',t)|}{|x-x'|^a+|y-y'|^a} + \sup_{(x,y,t)\neq(x,y,t')\in(B_i)_T} \frac{|u(x,y,t)-u(x,y,t')|}{|t-t'|^{a/2}} \right\}.$$

The prefactor $x + x'$ comes from using the euclidean metric in background coordinates instead of the intrinsic g-distance, see [21]. Note that we may also use an affine map $f_i : B_T \to (B_i)_T$ from a fixed cylinder B_T to define these norms.

Let $C_e^{a,\frac{a}{2}}(M_T)$ be the closure of $C^\infty(\overline{M}_T)$ with respect to this norm. We define $C_e^{k+a,\frac{k+a}{2}}(M_T)$ to consist of all functions u such that $(\partial_t)^i(x\partial_x)^j(x\partial_y)^\beta u \in C_e^{a,\frac{a}{2}}(M_T)$ for all $2i + j + |\beta| \leq k$. Note that we also weight these spaces: $u \in x^\nu C_e^{k+a,\frac{k+a}{2}}(M_T)$ if and only if $u = x^\nu v$ for some $v \in C_e^{k+a,\frac{k+a}{2}}(M_T)$.

We will also need Hölder spaces of tensors. As previously stated, we use the vector fields $x\partial_x$ and $x\partial_{y^b}$ and covector fields dx/x and dy^b/x as a basis for bundles of tensors, and with this convention ∇^h involves only derivatives by the 0-vector fields. In this way a section of a tensor bundle is an element of a Hölder space if and only if its components are. Furthermore, for $j \leq k$

$$(\nabla^h)^j : x^\nu C_e^{k+a,\frac{k+a}{2}}(M_T; E) \longrightarrow x^\nu C_e^{a,\frac{a}{2}}(M_T; E \otimes {}^0T^*M).$$

Finally, in what follows since we always deal with the bundle of symmetric 2-tensors, we will not explicitly mention it in the notation.

We mention also that the norm on $C_e^k(M)$ is equivalent to the usual intrinsic C^k norm, $\sum_{i=0}^k ||(\nabla^h)^i u||_{L^\infty(M)}$.

As proved in [17, 19], elliptic estimates in 0-Hölder spaces are proved from scaling and classical interior elliptic estimates on the balls B_i, as the pullback of a uniformly degenerate elliptic operator under f_i becomes uniformly elliptic. Similarly we may obtain parabolic estimates from scaling and classical parabolic estimates. In particular we have the following regularity result, see [15, Theorem 8.11.1, Theorem 8.12.1] for the classical parabolic statements.

Proposition 3.1 (Parabolic regularity)**.** *Let L be a second order uniformly degenerate elliptic operator. Suppose that $D^\gamma a_{j,\beta} \in C_e^{a,\frac{a}{2}}(M_T)$ for $|\gamma| \leq k$, and $D^\gamma f \in C_e^{a,\frac{a}{2}}(M_T)$, $D^\gamma \phi \in C_e^a(M)$ for all $|\gamma| \leq k$. If $u \in C_e^{2+a,\frac{2+a}{2}}(M_T)$ is a solution to $(\partial_t - L)u(\zeta,t) = f(\zeta,t)$ then $D^\gamma u \in C_e^{2+a,\frac{2+a}{2}}(M_T)$ for all $|\gamma| \leq k$.*

3.2. **Parabolic Schauder estimates.** We now state the main facts from linear parabolic PDE theory that we need. We will be interested in the following problem

$$\begin{cases} (\partial_t - L)u(\zeta,t) &= f(\zeta,t) \\ u(\zeta,0) &= 0, \end{cases} \tag{3.1}$$

The basic result is

Theorem 3.2. *Suppose L is a second order uniformly degenerate elliptic operator with time-independent coefficients. For every $f \in x^\mu C_e^{a,\frac{a}{2}}(M_T)$ there is a solution u to (3.1) in $x^\mu C_e^{2+a,\frac{2+a}{2}}(M_T)$. Moreover, u satisfies the parabolic Schauder estimate*

$$||u||_{x^\mu C_e^{2+a,\frac{2+a}{2}}(M_T)} \leq K||f||_{x^\mu C_e^{a,\frac{a}{2}}(M_T)}. \tag{3.2}$$

The Schauder constant K that appears in the statement depends on T but remains bounded as $T \to 0$.

3.3. **Mapping properties of the heat operator.** Given the homogeneous Cauchy problem

$$\begin{cases} (\partial_t - L)u(\zeta,t) &= 0 \\ u(\zeta,0) &= \phi(\zeta), \end{cases} \tag{3.3}$$

let A denote the heat operator such that takes ϕ to the solution of this problem, i.e. $(A\phi)(\zeta,t) = u(\zeta,t)$. We also use the notation that $A = e^{tL}$. In the appendix we describe how A is given by an integration against a specific polyhomogeneous distribution on a certain manifold with corners that covers $M \times M \times \mathbb{R}^+$. The mapping properties of such operators follow from the asymptotics at each of the boundary hypersurfaces. A key result that we will need is that if V_b is a b-vector field and A is a heat operator, then the commutator $[A, V_b]$ has the same asymptotics as A, and will enjoy the same mapping properties. See Proposition A.6 for a precise formulation.

Let H denote the following time convolution of the heat operator

$$(Hf)(\zeta,t) = \int_0^t e^{(t-s)L} f(\cdot,s)ds.$$

This operator provides a solution to the inhomogeneous Cauchy problem with zero initial data. The precise mapping properties we need are given in the following

Proposition 3.3. *(see Corollary A.4) If $\phi \in x^\mu C^\infty(\overline{M})$ and $f \in x^\mu C^\infty(\overline{M}_T)$ then*

1. *$A\phi \in x^\mu C^\infty(\overline{M}_T)$.*
2. *$Hf \in x^\mu C^\infty(\overline{M}_T)$.*

4. Short-time existence

In this section we prove short-time existence of a solution to (2.2) in the 0-Hölder spaces. This is based on a contraction mapping argument.

We begin by making several observations that will be needed later. Let $E = R + K$ be the curvature 'error' tensor for the conformally compact metric h, where K denotes the

$+1$ constant curvature 4-tensor. By our convention for function spaces, if h is smoothly conformally compact then

$$h = \overline{h}_{ij}\frac{dx^i}{x}\frac{dx^j}{x} \in C_e^\infty(M).$$

We also have $E \in xC_e^\infty(M; T^4M)$.

We need an expansion for the inverse of the metric. Suppose that $v \in xC_e^{k+a,\frac{k+a}{2}}(M_T)$ with sufficiently small norm, then the symmetric 2-tensor $h+v$ will be invertible and $(h+v)^{-1} \in C_e^{k+a,\frac{k+a}{2}}(M_T)$. Furthermore, we document a useful expansion

$$(h+v)^{ab} = h^{ab} - h^{al}h^{bm}v_{ml} + (h+v)^{bl}h^{am}h^{pq}v_{lp}v_{mq}. \tag{4.1}$$

4.1. **Conditioning the Ricci-DeTurck system.** Here we pursue short-time existence of the normalized Ricci-DeTurck flow. We will look for a solution of the form

$$g_{ij}(x,y,t) = h_{ij}(x,y) + v_{ij}(x,y,t)$$

where $v_{ij} \in xC_e^{k+a,\frac{k+a}{2}}(M_T)$. The system (2.2) for v may be written in the following way, which will facilitate treating the quasilinear system with a contraction mapping argument. Here we handle the quasilinearity as a quadratic error.

$$\begin{cases} 0 & = \partial_t v_{ij} - h^{ab}\widetilde{\nabla}_a\widetilde{\nabla}_b v_{ij} - \left((h+v)^{ab} - h^{ab}\right)\widetilde{\nabla}_a\widetilde{\nabla}_b v_{ij} + 2n(h+v)_{ij} \\ & \quad -(h+v)^{ab}(h+v)_{ip}h^{pq}\widetilde{R}_{jaqb} - (h+v)^{ab}(h+v)_{jp}h^{pq}\widetilde{R}_{iaqb} \\ & \quad +[(h+v)^{-1} * (h+v)^{-1} * \widetilde{\nabla}v * \widetilde{\nabla}v]_{ij}, \\ v(0) & = 0. \end{cases} \tag{4.2}$$

Note that in this expression we have switched curvature sign conventions from [24]. Shi lowers an index in the curvature tensor to the third slot whereas I lower to the fourth slot. The asterisk denotes linear contractions whose precise formula is unimportant for what follows.

Let us introduce notation for some of the terms above. Define

$$(T_1v)_{ij} := \left((h+v)^{ab} - h^{ab}\right)\widetilde{\nabla}_a\widetilde{\nabla}_b v_{ij},$$
$$(T_2v)_{ij} := 2n(h_{ij}+v_{ij}) + \left(-(h+v)^{ab}(h+v)_{ip}h^{pq}\widetilde{R}_{jaqb} - (h+v)^{ab}(h+v)_{jp}h^{pq}\widetilde{R}_{iaqb}\right),$$
$$(T_3v)_{ij} := (h+v)^{-1} * (h+v)^{-1} * \widetilde{\nabla}v * \widetilde{\nabla}v.$$

We begin by studying the various mapping properties of the terms of this equation. Much of the argument depends on the special algebraic structure of these equations. We introduce the following notation. We will say various terms are $\mathcal{Q}(v)$ if they are linear combinations of contractions of bounded tensors with either v or its first two h-covariant derivatives. We will loosely refer to this dependence as being 'quadratic', and we will make precise the estimates we need at the end of this section. Note that indices on $\mathcal{Q}$ index the term of origin in the decomposition above.

Lemma 4.1.

$$T_1v = \mathcal{Q}_1(v),\ \textit{and}$$

$$\mathcal{Q}_1 : x^\nu C_e^{k+a;\frac{k+a}{2}}(M_T) \longrightarrow x^{2\nu} C_e^{k-2+a;\frac{k-2+a}{2}}(M_T).$$

Proof. We begin by applying the expansion for the inverse in equation 4.1

$$\begin{aligned}(T_1 v)_{ij} &:= \left((h+v)^{ab} - h^{ab}\right)\widetilde{\nabla}_a \widetilde{\nabla}_b v_{ij} \\ &= \left(h^{ab} - h^{al}h^{bm}v_{ml} + (h+v)^{bl}h^{am}h^{pq}v_{lp}v_{mq} - h^{ab}\right)\widetilde{\nabla}_a \widetilde{\nabla}_b v_{ij} \\ &= h^{-1} * h^{-1} * v * \widetilde{\nabla}^2 v + (h+v)^{-1} * h^{-1} * h^{-1} * v * v * \widetilde{\nabla}^2 v,\end{aligned}$$

which shows the expression is quadratic in v. Noting that $h^{-1} \in C_e^\infty(M)$ and $v \in x^\nu C_e^{k+a;\frac{k+a}{2}}(M_T)$, we see that while we lose two 0-derivatives we gain decay in x, i.e. $\mathcal{Q}_1 v \in x^{2\nu} C_e^{k-2+a;\frac{k-2+a}{2}}(M_T)$. □

The expression for T_2 simplifies considerably.

Lemma 4.2.

$$(T_2 v)_{ij} = 2E_{ij} + 2nv_{ij} + v_{ip}\widetilde{Rc}_j^{\ p} + v_{jp}\widetilde{Rc}_i^{\ p} + 2v_{ml}\widetilde{R}^m_{\ ij}{}^l + \mathcal{Q}_2(v)_{ij}.$$

$$\mathcal{Q}_2 : x^\nu C_e^{k+a;\frac{k+a}{2}}(M_T) \longrightarrow x^{2\nu} C_e^{k+a;\frac{k+a}{2}}(M_T).$$

Proof. By applying the expansion for the inverse to terms in T_2 we find the expression contains inhomogeneous terms as well as terms linear in v which we must separate from the main expression. In particular, considering one of the constituent terms in T_2 we find

$$\begin{aligned}-(h+v)^{ab}(h+v)_{ip}h^{pq}\widetilde{R}_{jaqb} &= -(h+v)^{ab}(h+v)_{ip}\widetilde{R}_{ja\ b}^{\ \ p} \\ &= -\left(h^{ab} - h^{al}h^{bm}v_{ml} + (h+v)^{bl}h^{am}h^{pq}v_{lp}v_{mq}\right)(h_{ip}+v_{ip})\widetilde{R}_{ja\ b}^{\ \ p} \\ &= \widetilde{Rc}_{ij} + v_{ip}\widetilde{Rc}_j^{\ p} - v_{ml}\widetilde{R}^m_{\ ij}{}^l + [h^{-1} * h^{-1} * v * v * \widetilde{R}]_{ij},\end{aligned}$$

where $\widetilde{R}$ in this calculation denotes the $(1,3)$ tensor. One may check that the final quadratic contraction terms map $x^\nu C_e^{k+a;\frac{k+a}{2}}(M_T) \longrightarrow x^{2\nu} C_e^{k+a;\frac{k+a}{2}}(M_T)$.

Consequently,

$$\begin{aligned}-(h+v)^{ab}(h+v)_{ip}h^{pq}\widetilde{R}_{jaqb} &- (h+v)^{ab}(h+v)_{jp}h^{pq}\widetilde{R}_{iaqb} \\ &= 2\widetilde{Rc}_{ij} + v_{ip}\widetilde{Rc}_j^{\ p} + v_{jp}\widetilde{Rc}_i^{\ p} - 2v_{ml}\widetilde{R}^m_{\ ij}{}^l + \mathcal{Q}_2(v)_{ij}\end{aligned}$$

Note that by the curvature asymptotics $\widetilde{Rc}_{ij} = -nh_{ij} + E_{ij}$ where $E_{ij} \in xC^\infty(\overline{M})$. Therefore, re-assembling T_2 we find

$$(T_2 v)_{ij} = 2E_{ij} + 2nv_{ij} + v_{ip}\widetilde{Rc}_j^{\ p} + v_{jp}\widetilde{Rc}_i^{\ p} - 2v_{ml}\widetilde{R}^m_{\ ij}{}^l + \mathcal{Q}_2(v)_{ij}$$

□

The third term requires no additional conditioning.

Lemma 4.3.

$$(T_3 v)_{ij} = \mathcal{Q}_3(v), \textit{ and}$$

$$\mathcal{Q}_3 : x^\nu C_e^{k+a;\frac{k+a}{2}}(M_T) \longrightarrow x^{2\nu} C_e^{k-1+a;\frac{k-1+a}{2}}(M_T).$$

The preceding lemmas allows us to condition the equation for v further. We now move the terms linear in v to the other side of the equation. We also have from [8] that the term $h^{ab}\widetilde{\nabla}_a\widetilde{\nabla}_b v_{ij}$ is the rough Laplacian on 2-tensors. In fact, we see the linear elliptic part of the equation is the Lichnerowicz Laplacian on 2-tensors,

$$L = \Delta_L^h v_{ij} + 2nv_{ij} = h^{ab}\widetilde{\nabla}_a\widetilde{\nabla}_b v_{ij} + v_{ip}\widetilde{Rc}_j^{\ p} + v_{jp}\widetilde{Rc}_i^{\ p} - 2v_{ml}\widetilde{R}^m_{\ ij}{}^{l} + 2nv_{ij}.$$

We may write:

$$\begin{cases} \partial_t v_{ij} - (Lv)_{ij} & = \mathcal{Q}v_{ij} + 2E_{ij}, \\ v_{ij}(0) & = 0. \end{cases} \tag{4.3}$$

For the remainder of the argument we drop indices.

To summarize the argument so far, we have conditioned the flow equations to recognize a strongly parabolic equation for the metric. As the quadratic terms $\mathcal{Q}$ depend on v and up to its first two covariant derivatives in a polynomial fashion, there is a constant $C > 0$ depending on the algebraic structure of $\mathcal{Q}$ such that for all $u, v \in x^\mu C^{2+a,\frac{a}{2}}(M_T)$,

(1)
$$||\mathcal{Q}(v)||_{x^\mu C_e^{a,\frac{a}{2}}(M_T)} \leq C||v||^2_{x^\mu C_e^{2+a,\frac{a}{2}}(M_T)},$$

(2)
$$\begin{aligned} &||\mathcal{Q}(u) - \mathcal{Q}(v)||_{x^\mu C_e^{a,\frac{a}{2}}(M_T)} \\ &\qquad \leq C \max\left\{||u||_{x^\mu C_e^{2+a,\frac{a}{2}}(M_T)}, ||v||_{x^\mu C_e^{2+a,\frac{a}{2}}(M_T)}\right\} ||u-v||^2_{x^\mu C_e^{2+a,\frac{a}{2}}(M_T)}. \end{aligned}$$

Note in these estimates that we are relaxing control of one time derivative. This will facilitate the contraction mapping argument given in the next section. Note also that this part of the argument will not explicitly use the gain of decay by $\mathcal{Q}$.

In the regularity argument of Section 5, we will need this additional decay. We conclude with the following lemma.

Lemma 4.4. *All of the quadratic mapping terms satisfy*

$$\mathcal{Q} : x^\nu C_e^{k+a;\frac{k+a}{2}}(M_T) \longrightarrow x^{2\nu} C_e^{k-2+a;\frac{k-2+a}{2}}(M_T).$$

Moreover, if $w = w' + w''$, *where* $w' \in x^\nu C_e^{k+a;\frac{k+a}{2}}(M_T)$ *and* $w'' \in x^\mu C_e^{k+a;\frac{k+a}{2}}(M_T)$, $(\nu < \mu)$ *then*

$$\mathcal{Q}(w) \in x^{2\nu} C_e^{k-2+a;\frac{k-2+a}{2}}(M_T) + x^{\mu+\nu} C_e^{k-2+a;\frac{k-2+a}{2}}(M_T).$$

Proof. The first mapping property stated follows from the previous lemmas. We need only check the final claim. The explicit contraction structure of each of the terms that form $\mathcal{Q}$ are:

$$\begin{aligned} \mathcal{Q}_1 v &= h^{-1} * h^{-1} * v * \widetilde{\nabla}^2 v \\ \mathcal{Q}_2 v &= h^{-1} * h^{-1} * v * v * \widetilde{R} \\ \mathcal{Q}_3 v &= (h+v)^{-1} * (h+v)^{-1} * \widetilde{\nabla} v * \widetilde{\nabla} v. \end{aligned}$$

Now insert $w = w' + w''$ into the expression and notice the cross terms have the decay expected of $w' * w''$. ∎

4.2. **The contraction mapping argument.** We now explain the contraction mapping argument that leads to short-time existence for equation (4.3). Write the heat operator for $\partial_t - L$ as e^{tL}. Apply Duhamel's principle to (4.3) to get an equivalent integral equation

$$v(t) = \underbrace{\int_0^t e^{(t-s)L} \left(E + \mathcal{Q}(v) \right) ds}_{:=\Psi v} . \tag{4.4}$$

Note the definition of the map Ψ in the displayed equation above.

For a parameter μ and T to be specified, define a subspace $\mathcal{Z}_{\mu,T}$ of $xC_e^{2+a,\frac{a}{2}}(M_T)$ by

$$\mathcal{Z}_{\mu,T} = \left\{ u \in xC_e^{2+a,\frac{a}{2}}(M_T) : u(x,0) = 0, ||u||_{xC_e^{2+a,\frac{a}{2}}(M_T)} \leq \mu. \right\}.$$

This is a closed subset of a Banach space.

Suppose that $u \in Z_{\mu,T}$, it follows that $v = \Psi u$ is a solution to

$$\begin{cases} (\partial_t - L)v & = \mathcal{Q}(u) + E, \\ v(0) & = 0. \end{cases}$$

As $\mathcal{Q}(u) + E \in xC^{a,\frac{a}{2}}(M_T)$, the Schauder estimate implies $v \in xC_e^{2+a,1+\frac{a}{2}}(M_T) \subset xC_e^{2+a,\frac{a}{2}}(M_T)$, and so

$$\Psi : \mathcal{Z}_{\mu,T} \longrightarrow xC_e^{2+a,\frac{a}{2}}(M_T).$$

We would like to prove that Ψ is in fact an automorphism of $\mathcal{Z}_{\mu,T}$ and a contraction for μ and T sufficiently small.

Lemma 4.5. *$\Psi : Z_{\mu,T} \longrightarrow Z_{\mu,T}$ for μ and T sufficiently small.*

Proof. To begin, let $u \in \mathcal{Z}_{\mu,T}$ and set

$$v_1 := \int_0^t e^{(t-s)L} \mathcal{Q}(u) ds$$

$$v_2 := \int_0^t e^{(t-s)L} E ds.$$

Consider v_1. This is a solution to

$$\begin{cases} (\partial_t - L)v_1 & = \mathcal{Q}(u), \\ v_1(0) & = 0. \end{cases}$$

The Schauder estimate, followed by the estimate for $\mathcal{Q}$ given on page 9 gives

$$\begin{aligned} ||v_1||_{xC_e^{2+a,\frac{a}{2}}(M_T)} &\leq ||v_1||_{xC_e^{2+a,1+\frac{a}{2}}(M_T)} \\ &\leq K||\mathcal{Q}u||_{xC_e^{a,\frac{a}{2}}(M_T)} \\ &\leq KC||u||^2_{xC_e^{2+a,\frac{a}{2}}(M_T)} \\ &\leq KC\mu||u||_{xC_e^{2+a,\frac{a}{2}}(M_T)}. \end{aligned}$$

Taking μ sufficiently small allows us to force $KC\mu < \frac{1}{2}$. So $||v_1||_{xC_e^{2+a,\frac{a}{2}}(M_T)} \leq \frac{\mu}{2}$. Note that this same μ works if we shrink T.

Regarding v_2, note that this is a solution to

$$\begin{cases} (\partial_t - L)v_2 & = E, \\ v_2(0) & = 0. \end{cases}$$

We recall that E and the coefficients of L are smooth, time-independent and have bounded 0-derivatives of all orders, and so by parabolic regularity any finite number of derivatives of v_2 are bounded. Fixing any ζ we may write

$$v_2(\zeta, t) = \int_0^t E(\zeta) + Lv_2(\zeta, s)ds, \;\; t \in [0, T).$$

We may now estimate the $xC_e^{2+a,1+\frac{a}{2}}(M_T)$ norm of v_2. The L^∞ norm of spatial derivatives may be controlled through the Schauder estimates by the norm of E, and can be made as small as we like by choosing T sufficiently small. Further, as the time derivative of v_2 is bounded and $v_2(x, 0) = 0$, the $C_e^{a,\frac{a}{2}}(M_T)$ norm of v_2 can be made arbitrarily small by choosing T sufficiently small. We conclude for T small enough

$$||v_2||_{xC^{2+a,\frac{a}{2}}(M_T)} \leq \frac{\mu}{2}.$$

Thus $\Psi : \mathcal{Z}_{\mu,T} \longrightarrow \mathcal{Z}_{\mu,T}$ for $t \in [0, T]$. □

Lemma 4.6. *For the μ and T specified in the previous lemma, $\Psi : \mathcal{Z}_{\mu,T} \longrightarrow \mathcal{Z}_{\mu,T}$ is a contraction.*

Proof. Schauder's estimate applied to $\Psi u - \Psi v$ implies

$$\begin{aligned} &||\Psi u - \Psi v||_{xC_e^{2+a,\frac{a}{2}}(M_T)} \\ &\quad \leq ||\Psi u - \Psi v||_{xC_e^{2+a,1+\frac{a}{2}}(M_T)} \\ &\quad \leq K||\mathcal{Q}u - \mathcal{Q}v||_{xC_e^{a,\frac{a}{2}}(M_T)} \\ &\quad \leq KC\max\{||u||_{xC_e^{2+a,\frac{a}{2}}(M_T)}, ||v||_{xC_e^{2+a,\frac{a}{2}}(M_T)}\}||u - v||_{xC_e^{2+a,\frac{a}{2}}(M_T)} \\ &\quad \leq KC\mu||u - v||_{xC_e^{2+a,\frac{a}{2}}(M_T)}. \end{aligned}$$

Where K and C are the same constants from the previous proof. Consequently $KC\mu < \frac{1}{2}$, and Ψ is a contraction. □

We are now ready to prove the existence of a solution to the Ricci-DeTurck flow with full 0-regularity.

Theorem 4.7. *If h is a smoothly conformally compact metric, then there exists $T > 0$ and a solution $g \in C_e^{\infty,\infty}(M_T)$ to (2.2).*

Proof. The existence of a solution to (4.3) in $\mathcal{Z}_{\mu,T}$ follows from the Banach fixed point theorem. The Schauder estimate applied to the fixed point equation shows that the solution lies in $C_e^{2+a,\frac{2+a}{2}}(M_T)$. This short-time solution yields a solution in the same space to the Ricci-DeTurck flow by taking $g = h + v$. We now improve the regularity by using a bootstrap procedure, applied to the system (2.2). We may write this abstractly as

$$\partial_t g + \sum_{|\beta|=0}^{2} a_\beta(h, g)D^\beta g,$$

where the coefficients a_β at worst satisfy $D^\gamma a_\beta \in C^{a,\frac{a}{2}}(M_T)$, for $|\gamma| = 1$. By parabolic regularity (c.f. Proposition 3.1) we conclude $D^\gamma g \in C_e^{2+a,\frac{2+a}{2}}(M_T)$ for all $|\gamma| = 1$, which allows us to improve the spatial regularity. By bootstrapping, and then using the equation to improve regularity in time, we find $g \in C_e^{\infty,\infty}(M_T)$. □

5. Beyond 0-regularity

In the previous section we proved short-time existence of the Ricci-DeTurck flow starting at a smoothly conformally compact metric. The solution was constructed in 0-Hölder spaces and is smooth in time and 0-derivatives. We can expect more as the inhomogeneous terms in equation (4.4) are smooth up to the boundary with respect to background derivatives. In this section we prove the solution remains smoothly conformally compact on the entire interval of existence. The arguments of this section are modeled on the arguments in [20].

We begin by writing (4.4) more compactly as

$$v = -HE - H\mathcal{Q}v, \tag{5.1}$$

where H is the time convolution of the heat operator appearing in (4.4).

We will need to introduce function spaces intermediate between the 0-Hölder spaces and functions smooth up to the boundary. Define $x^\mu C_b^k(M_T)$ to be the space of functions such that up to k b-derivatives of the function lie in $C_e^{a,\frac{a}{2}}(M_T)$. The following lemma gives the main mapping property of H on these spaces.

Lemma 5.1. *For $k \geq 1$,*

$$H : x^\mu C_b^k(M_T) \longrightarrow x^\mu C_b^k(M_T)$$

Proof. Suppose that $f \in x^\mu C_b^1(M_T)$. Consider taking an arbitrary b-derivative of Hf, we write

$$\partial_y(Hf) = H(\partial_y f) + [H, \partial_y]f.$$

Since $\partial_y f \in x^\mu C_e^{a,\frac{a}{2}}(M_T)$, and $H : x^\mu C_e^{a,\frac{a}{2}}(M_T) \longrightarrow x^\mu C_e^{2+a,\frac{2+a}{2}}(M_T)$, the first term lies again in $x^\mu C_e^{a,\frac{a}{2}}(M_T)$. For the other term, by Proposition A.6 and Corollary A.5, $[H, \partial_y]$ has the same mapping properties as H, and again maps $x^\mu C_b^1(M_T) \subset x^\mu C_e^{a,\frac{a}{2}}(M_T)$ to $x^\mu C_e^{a,\frac{a}{2}}(M_T)$. This implies that $Hf \in x^\mu C_b^1(M_T)$.

By iteration one shows $H : x^\mu C_b^k(M_T) \longrightarrow x^\mu C_b^k(M_T)$. □

We now state the main result of this section

Theorem 5.2. *Let g be a solution to (2.2) in $C_e^{\infty,\infty}(M_T)$, with h smoothly conformally compact. Then $g(t)$ is smoothly conformally compact for all $t \in [0, T)$.*

Proof. Consider the term HE. As h is smoothly conformally compact, $E \in xC^\infty(\overline{M}_T)$. Now H preserves polyhomogeneity via Proposition 3.3, and so $HE \in xC^\infty(\overline{M}_T)$. Thus we need only focus on the second term.

In order to handle the term $H\mathcal{Q}v$ we take advantage of the improved decay of $\mathcal{Q}v$. If $v \in xC_e^{\infty,\infty}(M_T)$, then $\mathcal{Q}v \in x^2C_e^{\infty,\infty}(M_T)$, which H maps to $x^2C_e^{\infty,\infty}(M_T)$. Consequently, taking any b-derivative of $H\mathcal{Q}v$ yields

$$\partial_y H\mathcal{Q}v = x^{-1}\left(x\partial_y(H\mathcal{Q}v)\right) \in xC_e^{\infty,\infty}(M_T),$$

which shows a gain of one tangential derivative. This argument iterates, as we now show.

Step 1: The tensor v is fully tangentially regular, i.e. $v \in xC_b^\infty(M_T)$.

Fix any finite k. Given control of k 0-derivatives, we iterate the fixed point equation k times to obtain an expression of the form

$$v = \sum_{j=0}^{k-1} (H\mathcal{Q})^j HE + (H\mathcal{Q})^k v,$$

The summation term causes no problems, already being smooth up to the boundary. The term $(H\mathcal{Q})^k v \in x^{2k} C_e^{\infty,\infty}(M_T)$, and upon taking $2k-1$ tangential derivatives we find $(\partial_y)^{2k-1}(H\mathcal{Q})^k v \in x C_e^{\infty,\infty}(M_T)$. We conclude $v \in x C_b^{2k-1}(M_T)$. As k is arbitrary, this step is complete.

Step 2: $v \in xC^\infty(\overline{M}_T)$.
We return to the equation

$$v = -HE - H\mathcal{Q}v,$$

Note that initially, $v \in xC_b^\infty(M_T)$. Since 0-vector fields are combinations of $x\partial_x$ and $x\partial_y$, we find that $\mathcal{Q}(v) \in x^2 C_b^\infty(M_T)$.

We will now use the structure of the heat kernel as a polyhomogeneous distribution to prove polyhomogeneity of v. Given that $v \in xC_b^\infty(M_T)$ satisfies

$$v = -HE - H\mathcal{Q}v,$$

We see that $\mathcal{Q}v \in x^2 C_b^\infty(M_T)$, and so we may decompose v as

$$v = v' + v'' \in xC^\infty(\overline{M}_T) + x^2 C_b^\infty(M_T).$$

We now insert this back into (5.1). Using Lemma 4.4, we find

$$\mathcal{Q}(v' + v'') = x^2 C^\infty(\overline{M}_T) + x^3 C_b^\infty(M_T).$$

Equation (5.1) now lets us conclude

$$v \in xC^\infty(\overline{M}_T) + x^3 C_b^\infty(M_T).$$

Iterating we conclude $v \in xC^\infty(\overline{M}_T)$. □

We have proved $v \in xC^\infty(\overline{M}_T)$, i.e. that

$$v = x\overline{v}_{ij} \frac{dx^i}{x} \frac{dx^j}{x},$$

where $\overline{v}_{ij}$ is smooth up to the boundary. So now $x^2 v = x\overline{v}_{ij} dx^i dx^j$ and consequently $g = h + v$ is smoothly conformally compact. This completes the proof of Theorem A.

6. Stability about hyperbolic space

In [23] the authors proved a stability result for hyperbolic space under the Ricci flow. We review their main theorem. In the following h denotes the standard hyperbolic metric on $\mathbb{H}^n$. C^k denotes the space of sections of a bundle with up to k continuous covariant derivatives with respect to h in $L^\infty(\mathbb{H}^n)$. $\mathcal{M}^0(\mathbb{H}^n)$ denotes the space of continuous metrics on $\mathbb{H}^n$. $\mathcal{M}_0^\infty(\mathbb{H}^n \times [0,\infty))$ denotes the space of continuous metrics which are smooth for positive times and, when restricted to time intervals of the form $[\delta,\infty)$ are uniformly bounded in C^k.

In particular the main result of [23] is

Theorem 6.1. *Let $n \geq 4$. For all $K > 0$ there exists $\epsilon_1 = \epsilon_1(n, K) > 0$ such that the following holds. Let $g_0 \in \mathcal{M}^0(\mathbb{H}^n)$ satisfy*

$$\int_{\mathbb{H}^n} |g_0 - h|_h^2 dvol_h \leq K,$$

and

$$\sup_{\mathbb{H}^n} |g_0 - h|_h \leq \epsilon_1.$$

Then there exists a long-time solution $g \in \mathcal{M}_0^\infty(\mathbb{H}^n, [0, \infty))$ to the Ricci-DeTurck flow (with initial metric g_0) such that

$$\sup_{\mathbb{H}^n} |g(t) - h|_h \leq C(n, K) e^{-\frac{1}{4(n+2)} t}.$$

Moreover, $g(t) \longrightarrow h$ exponentially in C^k as $t \to \infty$ for all $k \in \mathbb{N}$.

As the reader may check, C^k in this theorem corresponds to C_e^k: covariant derivatives with respect to a smoothly conformally compact metric (in this case h) are bounded if and only if the 0-derivatives are bounded. In view of this, starting with an initial metric sufficiently close to h in the C^0 sense, and of bounded distance from h in the L^2 sense, the solution to the Ricci-DeTurck flow converges in the 0-Hölder spaces to h.

In view of our work, if g_0 is initially smoothly conformally compact, then this shows that the Ricci-DeTurck flow $g(t)$ exists for all time and converges to h in C_e^∞. By Theorem 5.2 in the previous section, $g(t)$ remains smoothly conformally compact for all time. This proves Theorem B.

Appendix A. More on Linear Parabolic PDE theory

In this appendix we give more detail surrounding linear parabolic theory on conformally compact manifolds. Our approach to understanding these operators is based on the edge heat calculus developed in [1]. Note that in this appendix we deal exclusively with the 0-case but the arguments generalize in a straightforward manner to the full complete edge case.

The point of view we adopt is that for a second order uniformly degenerate elliptic operator with time-independent coefficients, we can explicitly construct the heat kernel as a polyhomogeneous distribution on an appropriate manifold with corners that covers $M \times M \times \mathbb{R}^+$. In this section we will first describe this blow up space. We then proceed to discuss the heat kernel as constructed in [1]. We then prove several mapping properties of these kernels. We conclude by proving Schauder type estimates.

We now introduce the appropriate blow up spaces for the construction of the heat kernel. First we define the 0-double space: M_e^2, originally introduced in [19] for the elliptic edge calculus. This is a manifold with corners that covers M^2, and is obtained by introducing polar coordinates around the submanifold

$$\partial M \times_B \partial M = \{(w, w') \in \partial M \times \partial M : w = w'\}.$$

So $M_e^2 = [M \times M; \partial M \times_B \partial M]$. This will introduce three new boundary hypersurfaces; following Albin we denote these by B_{11} (the front face), B_{01} (the right boundary) and B_{10} (the left boundary). We denote the blowdown map

$$\beta_e : M_e^2 \to M^2,$$

and the edge diagonal by

$$\text{diag}_e = \overline{\beta_e^{-1}(\text{diag} \setminus \partial M \times_B \partial M)}.$$

We describe the edge double space in terms of coordinate charts. In the interior of M_e^2 we may use the usual coordinates

$$\big((x,y),(x',y')\big) = \big(\zeta,\zeta'\big)\,,$$

where y will always denote coordinates along B and z will always denote coordinates along F. We will favour the following projective coordinates for M_e^2, defined away from B_{10} and that express the edge diagonal easily are given by

$$\left((x,y,z),\left(s := \frac{x'}{x}, v := \frac{y'-y}{x}\right)\right).$$

Note that in these coordinates, $s=0$ is a defining function for B_{01} and $x=0$ for the front face (away from B_{10}). By reversing the roles of x and x' in the obvious manner, one may obtain a second chart covering the remainder of M_e^2.

We now introduce the heat space HM_e^2. This is given by a parabolic blow up of the manifold $M_e^2 \times \mathbb{R}_+$ along the submanifold $\mathrm{diag}_e \times \{0\}$. This gives us a number of new boundary hypersurfaces. We keep Albin's notation for these, illustrated in Figure 1.

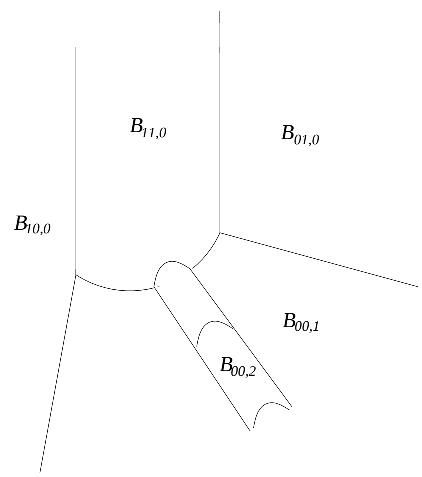

FIGURE 1. The blown up heat space, HM_e^2

We now discuss the coordinate systems we can use on HM_e^2. In what follows we work away from $B_{10,0}$ (i.e. away from $x=0$). Near $B_{11,0}$, and away from $B_{00,1}$ we can use

$$\left((x,y),\left(s' := \frac{x'}{x}, v' := \frac{y'-y}{x}\right), \tau := t^{1/2}\right).$$

Near $B_{11,0}$ and the 'top' of $B_{00,2}$ we may use

$$\big((S,U),\zeta',\tau\big) := \left(\left(\frac{x-x'}{x't^{1/2}}, \frac{y-y'}{x't^{1/2}}\right),(x',y'),t^{1/2}\right). \tag{A.1}$$

Finally, near $B_{11,0}$ and the 'bottom' of $B_{00,2}$, close to $B_{00,1}$ we appear to need to introduce another coordinate system. However, we observe that this region is reached using the above coordinates as $|(S,U)| \to +\infty$. We will soon see that our heat kernels vanish to infinite order along this boundary.

We will denote the full blow down map $\beta : HM_e^2 \to M^2 \times \mathbb{R}^+$.

Given a manifold with corners M, $C^\infty(\overline{M})$ denotes functions on M that are smooth in the interior and smooth up to all boundary hypersurfaces. The space $\dot{C}^\infty(M)$ will denote smooth functions vanishing to all orders at the boundary hypersurfaces. If $\mathcal{F}$ denotes a list of boundary hypersurfaces then $\dot{C}^\infty_{\mathcal{F}}(M)$ denotes smooth functions vanishing to all orders at all boundary hypersurfaces except those in $\mathcal{F}$; at the other hypersurfaces we demand the functions are smooth up to the boundary.

We will also need to define sets of functions that have asymptotic expansions at the boundary hypersurfaces. Let M be a manifold with corners with boundary defining functions x_i. A distribution u is polyhomogeneous conormal[1] if:

$$u \sim \sum_{Res_j \to \infty} \sum_{p=0}^{p_j} a_{j,p}(x,y) x^{s_j} (\log x)^p,$$

where $a_{j,p} \in C^\infty(\overline{M})$. We'll denote the set of such distributions $\mathcal{A}^*_{phg}$. We can also restrict the set of exponents that may occur above. Define an index set to be a discrete subset $E \subset \mathbb{C} \times \mathbb{N}_0$ such that

(1) if $(s_j, p_j) \in E$ and $|(s_j, p_j)| \longrightarrow \infty$, then $Re(s_j) \longrightarrow \infty$.

(2) if $(s,p) \in E$ then $(s+k, p-l) \in E$ for any $k, l \in \mathbb{N}, l \leq p$.

Given a set of index sets $\mathcal{E}$ for each boundary hypersurface, we denote by $\mathcal{A}^{\mathcal{E}}_{phg}$ the set of polyhomogeneous conormal functions with exponents ranging in $\mathcal{E}$. Note that we will use a few special notations for index sets. The empty set will denote the index set for a function vanishing to all orders along a hypersurface. A single number $n \in \mathbb{N}_0$ will denote the index set $\{(j,0) : j \in \mathbb{N}, j \geq n\}$ of functions vanishing to order n. Note that the index set $\{0\}$ represents functions smooth up to the hypersurface. For more details about operations on these sets, see the concise review in [19, Appendix A].

A.1. **The heat kernel of a uniformly degenerate elliptic operator.** Let L be a second order uniformly degenerate elliptic operator. We consider a heat type equation

$$\begin{cases} (\partial_t - L)u(\zeta,t) &= 0 \\ u(\zeta,0) &= f(\zeta), \end{cases}$$

where $f \in \Gamma(M;\mathcal{E})$ is a smooth section of a vector bundle $\mathcal{E}$.

The heat kernel of L is a distribution on $M^2 \times \mathbb{R}^+$ so that the solution to the above problem is given by:

$$u(\zeta,t) = \int_M h(\zeta,\zeta',t) f(\zeta')\, \mathrm{dvol_g}(\zeta').$$

Here h formally satisfies:

$$\begin{cases} (\partial_t - L_\zeta)h(\zeta,\zeta',t) &= 0 \\ h(\zeta,\zeta',0) &= \delta(\zeta-\zeta'), \end{cases} \tag{A.2}$$

We will see that $h = \beta_* H$, where H is a polyhomogeneous distribution on HM^2_e.

The actual construction of this distribution is done for half-densities, so that it makes sense to compose operators. We briefly review Albin's construction of the heat calculus. We define a weighted bundle of half-densities $D := \rho_{00,2}^{-\frac{n}{2}+2} \rho_{11,0}^{-\frac{n+1}{2}} \Omega^{1/2}(HM^2_e)$. Kernels of operators in the heat calculus are elements of

$$K^{k,l}(M,D) := \rho^k_{00,2}\rho^l_{11,0}\dot{C}^\infty_{B_{00,2},B_{11,0}}(HM^2_e;D).$$

[1]See [10] for a discussion and to make the meaning of $\sim$ precise.

The action of a kernel K_A in $K^{k,l}$ on smooth half-densities is given by

$$A(f)(\zeta,t) = \int_M \beta_* K_A(\zeta,\zeta',t)f(\zeta').$$

We'll denote the operator A acting in this manner by $A \in \Psi^{k,l}_{e,Heat}$.

Albin proves:

Theorem A.1. *If L is the scalar Laplacian of a exact edge metric, then $A \in \Psi^{2,0}_{e,Heat}$, where A is the heat operator of $\partial_t - L$.*

We note that Albin's construction is closely modeled on the work of Melrose [22], and generalizes in a straightforward manner to general second order uniformly degenerate elliptic operators. Furthermore, Melrose also considers the case of elliptic operators between bundles [22, Theorem 7.29] with diagonal principal symbol. Thus we have

Theorem A.2. *If L is a uniformly degenerate elliptic operator with diagonal principal symbol then, then $A \in \Psi^{2,0}_{e,Heat}$, where A is the heat operator of $\partial_t - L$.*

We now give a brief indication of the proof of theorem and refer the reader to [1, 22] for further detail. We work in HM^2_e with the ansatz that the solution already vanishes to infinite order at $B_{10,0}, B_{01,0}$, and $B_{00,1}$. To deal with the rest of the equation and boundary hypersurfaces involves three main steps. First, an initial parametrix is constructed by pulling the heat equation back to HM^2_e in coordinates near the blown up diagonal. As $B_{00,2}$ fibres over the diagonal, we find that the equation restricts to a Euclidean type heat equation on each fibre with smooth coefficients in the variables along the fibre. Thus we may progressively solve way the Taylor series at $B_{00,2}$ with control of the asymptotics down to $B_{11,0}$. This handles the initial condition. The second step is to progressively solve away the Taylor series at $B_{11,0}$ using the heat kernel of hyperbolic space (recall 0-metrics are asymptotically hyperbolic). The result of these two steps is a parametrix solving the heat equation to infinite order at all boundary hypersurfaces. To improve the parametrix to an actual inverse requires an argument involving Volterra operators and is given in [22, Proposition 7.17].

A.2. **Mapping properties.** In this section we study the action of the heat kernels in $\Psi^{2,0}_{e,Heat}$ above on functions, using Melrose's pushforward theorem. Figure 2 introduces some important notation.

We identify functions and half-densities on $M^2 \times \mathbb{R}^+$ and the factors $M \times \mathbb{R}^+$ and M by[2]

$$f(x,y,x',y',t) \leftrightarrow f(x,y,x',y',t)x^{-\frac{(n+1)}{2}}(x')^{-\frac{(n+1)}{2}}|dxdydx'dy'dt|^{1/2},$$
$$f(x,y,t) \leftrightarrow f(x,y,t)x^{-\frac{(n+1)}{2}}|dxdydt|^{1/2},$$
$$f(x,y) \leftrightarrow f(x)x^{-\frac{(n+1)}{2}}|dxdy|^{1/2}.$$

From [1, page 11] an element of $A \in \Psi^{2,0}_{e,Heat}$ has an integral kernel that may be written as $\rho_{00,2}^{-\frac{n}{2}}\rho_{11,0}^{-\frac{n+1}{2}}k \cdot \nu$, where k is a function that vanishes to infinite order at $B_{10,0}, B_{01,0}$, and $B_{00,1}$, and is smooth up to the boundary at $B_{00,2}$ and $B_{11,0}$, and ν is a smooth section of $\Omega^{1/2}(HM^2_e)$.

An operator $A \in \Psi^{2,0}_{e,Heat}$ acts on half-densities by

(A.3)
$$(Af)(x,y,t)x^{-\frac{n+1}{2}}|dxdydt|^{1/2} = (\beta_L)_*\left(\rho_{00,2}^{-\frac{n}{2}}\rho_{11,0}^{-\frac{n+1}{2}}k\nu \cdot (\beta_R)^*(f(x',y')(x')^{-\frac{n+1}{2}}|dx'dy'|^{1/2})\right).$$

[2] Here we omit the smooth factor $\sqrt{\det g}$ in the densities that follow.

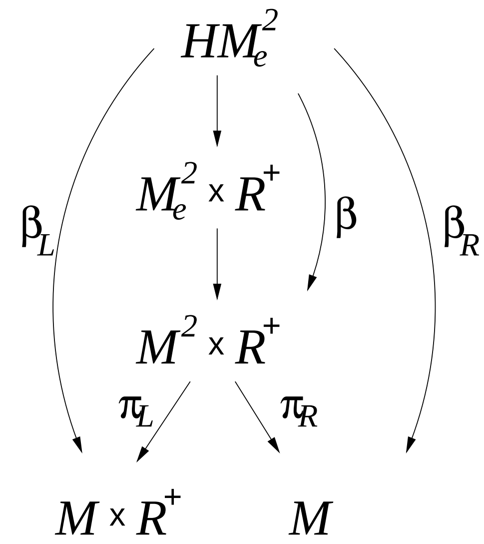

FIGURE 2. Definition of various maps

To relate these half-densities, let us work in coordinates near $B_{11,0}$ and $B_{00,2}$. We may take $\nu = |dSdUdx'dy'd\tau|^{1/2}$. Pulling our standard half-density on $M^2 \times \mathbb{R}^+$ back we find

$$\begin{aligned}
&\beta^*(x^{-\frac{n+1}{2}}(x')^{-\frac{n+1}{2}}|dxdydx'dy'dt|^{1/2}) \\
&\quad = (1+S\tau)^{-\frac{n+1}{2}}(x')^{-\frac{n+1}{2}}(x')^{-\frac{n+1}{2}}(2(x')^{n+1}\tau^{n+2})^{1/2}|dSdUdx'dy'd\tau|^{1/2} \\
&\quad = \sqrt{2}(1+S\tau)^{-\frac{n+1}{2}}(x')^{-\frac{n+1}{2}}\tau^{\frac{n+2}{2}}\nu.
\end{aligned}$$

The factor $\sqrt{2}(1+S\tau)^{-(n+1)/2}$ is smooth and uniformly bounded, so we omit it hereafter.

In order to apply Melrose's push-forward theorem, we must work with smooth b-densities. Here is how to arrange this. We multiply both sides by the half density $x^{-\frac{n+1}{2}}|dxdydt|^{1/2}$, and noting

$$\beta_L^*(x^{-\frac{n+1}{2}}|dxdydt|^{1/2})\beta_R^*((x')^{-\frac{n+1}{2}}|dx'dy'|^{1/2}) = \beta^*(x^{-\frac{n+1}{2}}(x')^{-\frac{n+1}{2}}|dxdydx'dy'dt|^{1/2}),$$

we find the action on smooth densities is given by

$$(Af)(x,y,t)x^{-(n+1)}|dxdydt| = (\beta_L)_*\left(\rho_{00,2}^{1}\rho_{11,0}^{-n-1}k \cdot \beta_R^* f\ \nu^2\right).$$

Finally we introduce a total defining function on both sides of this equation to obtain b-densities, denoted here by the floating b-preindex:

$$\begin{aligned}
&(Af)(x,y,t)xtx^{-(n+1)}\ {}^b|dxdydt| \\
&\quad = (\beta_L)_*\left(\rho_{10,0}\rho_{11,0}^{-n}\rho_{01,1}\rho_{00,1}\rho_{00,2}^{2}k \cdot (\beta_R)^*(f(x',y'))\ {}^b\nu^2\right)
\end{aligned}$$

We now apply this to the following

Proposition A.3. *Let $A \in \Psi^{2,0}_{e,Heat}$. If $f \in \mathcal{A}^{\mathcal{F}}_{phg}(M)$ then $Af \in \mathcal{A}^{(\mathcal{F},0)}_{phg}(M \times \mathbb{R}^+)$.*

Proof. First, let us consider the b-map β_R. As no boundary hypersurface is mapped to a corner of M, β_R is a b-fibration. It is easy to check that the exponent matrix for this map is

	$B_{10,0}$	$B_{11,0}$	$B_{01,0}$	$B_{00,1}$	$B_{00,2}$
∂X	1	1	0	0	0

As a consequence, if $f \in \mathcal{A}^{\mathcal{F}}_{phg}(M)$ with index set $\mathcal{F}$, $\beta_R^* f \in \mathcal{A}^{\{\mathcal{F},\mathcal{F},0,0,0\}}_{phg}(HM_e^2)$, by the pull-back theorem [19, Proposition A.13].

The function k is polyhomogeneous with respect to the index set $\{\emptyset, 0, \emptyset, \emptyset, 0\}$. Accounting for the powers of the defining functions we obtain the index set $\{\emptyset, -n, \emptyset, \emptyset, 2\}$. The index set for the product of this expression with the pull-back of f is then $\mathcal{G} = \{\emptyset, -n+F, \emptyset, \emptyset, 2.\}$.

The map β_L is a b-fibration. The exponent matrix for this map is

	$B_{10,0}$	$B_{11,0}$	$B_{01,0}$	$B_{00,1}$	$B_{00,2}$
$\mathcal{H}_1$	0	1	1	0	0
$\mathcal{H}_2$	0	0	0	1	1

In the above table we have the labeled hypersurfaces of $M \times \mathbb{R}^+$ in the following manner: $\mathcal{H}_1$ represents $t = 0$ and $\mathcal{H}_2$ represents $x = 0$. We now apply Melrose's pushfoward theorem [19, Proposition A.18]. Note that the integrability condition is met at $B_{10,0}$ as $Re(\mathcal{G}(B_{10,0})) > 0$. Now the index set for $\mathcal{H}_1 = \mathcal{G}(B_{11,0})\overline{\cup}\mathcal{G}(B_{01,0}) = -n + \mathcal{F}$ and $\mathcal{H}_2 = \mathcal{G}(B_{00,1})\overline{\cup}\mathcal{G}(B_{00,2}) = 2$. Note that $\mathcal{H}_2 = 2$ is not surprising since $\tau^2 = t$, from the parabolic blow up.

The calculation here provides the index sets needed to for computing the asymptotics of

$$(Af)(x, y, t)xtx^{-(n+1)\ b}|dxdydt|.$$

Canceling the powers of the defining functions, and returning to the identification of densities with functions now shows that $Af \in \mathcal{A}^{(\mathcal{F},0)}_{phg}(M \times \mathbb{R}^+)$. □

We use the above proposition primarily in the form

Corollary A.4. *If $f \in x^\mu C^\infty(\overline{M})$ then $Af \in x^\mu C^\infty(\overline{M}_T)$.*

Proof. The only point that we have to be careful about is that the previous theorem only guarantees an expansion in powers of $\tau = \sqrt{t}$. However we can obtain full smoothness in t by using the fact that Af solves the heat equation and is already smooth in the spatial derivatives. □

Corollary A.5. *If $f \in x^\mu C^\infty(\overline{M}_T)$ and H denotes the time convolution of the heat operator of A, then $Hf \in x^\mu C^\infty(\overline{M}_T)$.*

We conclude this section with a proposition that we will need in the finer regularity analysis of the Ricci flow. This shows that the commutator of a b-vector field with an element of the heat calculus remains in the calculus, and thus has the same mapping properties.

Proposition A.6. *If $A \in \Psi^{2,0}_{e;Heat}(X)$ and V_b is any b-vector field, then $[A, V_b] \in \Psi^{2,0}_{e;Heat}(X)$.*

Proof. The proof is similar to [19, Proposition 3.30], adapted to the heat calculus setting. We sketch the proof here. Return to the action on half-densities, equation (A.3), and suppose that f is a smooth half-density vanishing to all orders at the boundary hypersurfaces. Now suppose for simplicity that $V_b = \partial_y$ is a b-vector field. After an integration by parts, we may write

$$\begin{aligned}&(\partial_y Af)(x, y, t) - (A\partial_y f)(x, y, t)\\ &= (\beta_L)_* \left(\left(\beta_L^*(\partial_y) + \beta_R^*(\partial_{y'}^T)\right) \rho_{00,2}^{-\frac{n}{2}} \rho_{11,0}^{-\frac{n+1}{2}} k\nu \cdot (\beta_R)^*(f(x', y')(x')^{-\frac{n+1}{2}}|dx'dy'|^{1/2}) \right),\end{aligned}$$

where $\partial_{y'}^T$ is the adjoint of $\partial_{y'}$ under the measure. The key now is that while each of ∂_y and $\partial_{y'}$ lifts to a vector field singular near $B_{00,2}$, their sum cancels this behaviour. Indeed, computing in the coordinates defined in equation (A.1), we find that

$$\beta_L^*(\partial_y) = \frac{1}{x'\tau}\partial_U$$

$$\beta_R^*(\partial_{y'}^T) = -\frac{1}{x'\tau}\partial_U + \partial_{y'} + \text{ smooth function.}$$

Consequently, $\beta_L^*(\partial_y) + \beta_R^*(\partial_{y'}^T)$ does not affect the asymptotics of the kernel, and $[A, V_b] \in \Psi^{2,0}_{e;Heat}(X)$. □

We conclude with a discussion of the main existence theorem for the inhomogeneous Cauchy problem:

$$\text{(A.4)} \qquad \begin{cases} (\partial_t - L)u(\zeta, t) &= f(\zeta, t) \\ u(\zeta, 0) &= 0, \end{cases}$$

where $f \in x^\mu C_e^{a,\frac{a}{2}}(M_T)$ and L is a second order uniformly degenerate elliptic operator with coefficients in C_e^a.

Theorem A.7. *Suppose L is a second order uniformly degenerate elliptic operator with time-independent coefficients. For every $f \in x^\mu C_e^{a,\frac{a}{2}}(M_T)$ there is a solution u to (A.4) in $x^\mu C_e^{2+a,\frac{2+a}{2}}(M)$. Moreover, u satisfies the Schauder-type estimate*

$$\text{(A.5)} \qquad ||u||_{x^\mu C_e^{2+a,\frac{2+a}{2}}(M_T)} \leq K||f||_{x^\mu C_e^{a,\frac{a}{2}}(M_T)}.$$

Proof. By Duhamel's principle, a solution to (A.4) is given by

$$\text{(A.6)} \qquad u(\zeta, t) = \int_0^t \int_M h(\zeta, \zeta', t - t') f(\zeta', t') \, \mathrm{dvol_g}(\zeta') dt',$$

where h is the heat kernel of the heat operator e^{tL}, where $e^{tL} \in \Psi^{2,0}_{e,Heat}$.

We now discuss the estimates. The case for nonzero weight μ follows from the unweighted estimate, as to solve the inhomogeneous problem with $u \in x^\mu C_e^{2+a,\frac{2+a}{2}}(M)$ amounts to solving

$$(\partial_t - x^{-\mu} L x^\mu) u'(\zeta, t)) = f'(\zeta, t)$$

for with u' and f' in appropriate unweighted spaces. The kernel of the conjugated operator $x^{-\mu} L x^\mu$ has precisely the same asymptotics as the kernel of L, as may be seen by working in coordinates near the left hand corner. So the mapping properties follow from the $\mu = 0$ case.

The strategy is now to cut up the space HM_e^2. Consider a function ϕ equal to one in a tubular neighbourhood of $B_{00,2}$ and vanishing outside a slightly larger tubular neighbourhood. We may write the heat kernel as

$$h = h_1 + h_2 := \phi h + (1 - \phi) h.$$

To prove (3.2), it will suffice to estimate both

$$u_1(\zeta, t) = \int_0^t \int_M h_1(\zeta, \zeta', t - t') f(\zeta', t') \, \mathrm{dvol_g}(\zeta') dt'$$

and

$$u_2(\zeta, t) = \int_0^t \int_M h_2(\zeta, \zeta', t - t') f(\zeta', t') \, \mathrm{dvol_g}(\zeta') dt'$$

Regarding the estimate for u_2, we view h_2 as a polyhomogeneous distribution vanishing to infinite order at $B_{10,0}, B_{01,0}, B_{00,1}, B_{00,2}$ and smooth up to $B_{11,0}$. The estimates are then straightforward when working in coordinates near the left hand corner.

Regarding the estimate for u_1, we now use the Whitney decomposition from Section 3 to pull back the integral near $B_{00,2}$ to a fixed bounded subset of $\mathbb{R}^{n+1} \times \mathbb{R}^+$. As h_1 shares the asymptotics of the euclidean heat kernel, this reduces the estimate to the classical parabolic case, for example done in [16]. We forgo the lengthy argument. □

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