HARMONIC MAPPINGS AND DISTANCE FUNCTION

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ABSTRACT. We prove the following theorem: every quasiconformal harmonic mapping between two plane domains with $C^{1,\alpha}$ ($\alpha < 1$), respectively $C^{1,1}$ compact boundary is bi-Lipschitz. The distance function with respect to the boundary of the image domain is used. This in turn extends a similar result of the author in [10] for Jordan domains, where stronger boundary conditions for the image domain were needed.

1. Introduction and statement of the main result

We say that a real function $u : D \to \mathbf{R}$ is ACL (absolutely continuous on lines) in the region D, if for every closed rectangle $R \subset D$ with sides parallel to the x and y-axes, u is absolutely continuous on a.e. horizontal and a.e. vertical line in R. Such a function has of course, partial derivatives u_x, u_y a.e. in D. A homeomorphism $f\colon D \mapsto G$, where D and G are subdomains of the complex plane $\mathbf{C}$, is said to be K-quasiconformal (K-q.c), $K \geq 1$, if f is ACL and

$$|\nabla f(z)| \leq K l(\nabla f(z)) \quad \text{a.e. on } D, \tag{1.1}$$

where

$$|\nabla f(x)| := \max_{|h|=1} |\nabla f(x)h| = |f_z| + |f_{\bar z}|$$

and

$$l(\nabla f(z)) := \min_{|h|=1} |\nabla f(z)h| = |f_z| - |f_{\bar z}|$$

(cf. [1, p.23–24] and [21]). Note that, the condition (1.1) can be written as

$$|f_{\bar z}| \leq k|f_z| \quad \text{a.e. on } D \text{ where } k = \frac{K-1}{K+1} \text{ i.e. } K = \frac{1+k}{1-k}$$

or in its equivalent form

$$|\nabla f(z)|^2 \leq K J_f(z),\ z \in \mathbb{U}, \tag{1.2}$$

where J_f is the jacobian of f.

A function w is called *harmonic* in a region D if it has form $w = u + iv$ where u and v are real-valued harmonic functions in D. If D is simply-connected, then there are two analytic functions g and h defined on D such that w has the representation

$$w = g + \overline{h}.$$

If w is a harmonic univalent function, then by Lewy's theorem (see [22]), w has a non-vanishing Jacobian and consequently, according to the inverse mapping theorem, w is a diffeomorphism.

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Let

$$P(r,x)=\frac{1-r^2}{2\pi(1-2r\cos x+r^2)}$$

denote the Poisson kernel. Then every bounded harmonic function w defined on the unit disc $\mathbf{U}:=\{z:|z|<1\}$ has the following representation

$$w(z)=P[F](z)=\int_0^{2\pi}P(r,x-\varphi)F(e^{ix})dx, \tag{1.3}$$

where $z=re^{i\varphi}$ and F is a bounded integrable function defined on the unit circle S^1.

In this paper we continue to study q.c. harmonic mappings. See [24] for the pioneering work on this topic and see [8] for related earlier results. In some recent papers, a lot of work have been done on this class of mappings ([3], [10]-[20], [28], [27], [23] and [25]). In these papers it is established the Lipschitz and the co-Lipschitz character of q.c. harmonic mappings between plane domains with certain boundary conditions. In [31] it is considered the same problem for hyperbolic harmonic quasiconformal selfmappings of the unit disk. Notice that, in general, quasi-symmetric self-mappings of the unit circle do not provide quasiconformal harmonic extension to the unit disk. In [24] it is given an example of C^1 diffeomorphism of the unit circle onto itself, whose Euclidean harmonic extension is not Lipschitz. Alessandrini and Nessi proved in [2] the following proposition:

Proposition 1.1. *Let $F:S^1\to\gamma\subset\mathbb{C}$ be an orientation preserving diffeomorphism of class C^1 onto a simple closed curve. Let D be the bounded domain such that $\partial D=\gamma$. Let $w=P[F]\in C^1(\overline{\mathbb{U}};\mathbb{C})$. The mapping w is a diffeomorphism of $\mathbb{U}$ onto D if and only if*

$$J_w>0 \text{ everywhere on } S^1. \tag{1.4}$$

In view of the inequalities (1.2) and (1.4), we easily see that.

Corollary 1.2. *Under the condition of Proposition 1.1, the harmonic mapping w is a diffeomorphism if and only if it is K quasiconformal for some $K\ge 1$.*

In contrast to the Euclidean metric, in the case of hyperbolic metric, if $f:S^1\mapsto S^1$ is C^1 diffeomorphism, or more general if $f:S^{n-1}\mapsto S^{m-1}$ is a mapping with the non-vanishing energy, then its hyperbolic harmonic extension is C^1 up to the boundary ([4]) and ([5]).

To continue we need the definition of $C^{k,\alpha}$ Jordan curves ($k\in\mathbb{N}$, $0<\alpha\le 1$). Let γ be a rectifiable curve in the complex plane. Let l be the length of γ. Let $g:[0,l]\mapsto\gamma$ be an arc-length parametrization of γ. Then $|\dot g(s)|=1$ for all $s\in[0,l]$. We will write that the curve $\gamma\in C^{k,\alpha}$, $k\in\mathbb{N}$, $0<\alpha\le 1$ if $g\in C^k$, and $M(k,\alpha):=\sup_{t\neq s}\frac{|g^{(k)}(t)-g^{(k)}(s)|}{|t-s|^\alpha}<\infty$. Notice this important fact, if $\gamma\in C^{1,1}$ then γ has the curvature κ_z for a.e. $z\in\gamma$ and $\operatorname{ess\,sup}\{|\kappa_z|:z\in\gamma\}\le M(1,1)<\infty$.

This definition can be easily extended to arbitrary $C^{k,\alpha}$ compact $1-$ dimensional manifold (not necessarily connected).

The starting point of this paper is the following proposition.

Proposition 1.3. *Let $w=f(z)$ be a K quasiconformal harmonic mapping between a Jordan domain Ω_1 with $C^{1,\alpha}$ boundary and a Jordan domain Ω with $C^{1,\alpha}$ (respectively $C^{2,\alpha}$) boundary. Let in addition $b\in\Omega_1$ and $a=f(b)$. Then w is Lipschitz* *(co-Lipschitz). Moreover there exists a positive constant $c = c(K, \Omega, \Omega_1, a, b) \geq 1$ such that*

$$|f(z_1) - f(z_2)| \leq c|z_1 - z_2|, \quad z_1, z_2 \in \Omega_1 \tag{1.5}$$

and

$$\frac{1}{c}|z_1 - z_2| \leq |f(z_1) - f(z_2)|, \quad z_1, z_2 \in \Omega_1, \tag{1.6}$$

respectively.

See [13] for the first part of Proposition 1.3 and [10] for its second part.

In [10], it was conjectured that the second part of Proposition 1.3 remains hold if we assume that Ω has $C^{1,\alpha}$ boundary only.

Notice that the proof of Proposition 1.3 relied on Kellogg-Warschawski theorem ([32], [33], [6]) from the theory of conformal mappings, which asserts that if w is a conformal mapping of the unit disk onto a domain $\Omega \in C^{k,\alpha}$, then $w^{(k)}$ has a continuous extension to the boundary ($k \in \mathbb{N}$). It also depended on the Mori's theorem from the theory of q.c. mappings, which diels with Hölder character of q.c. mappings between plane domains (see[1] and [30]). In addition, Lemma 3.2 were needed.

Using a different approach, we extend the second part of Proposition 1.3 to the class of image domains with $C^{1,1}$ boundary. Its extension is Theorem 1.4. The proof of Theorem 1.4, given in the last section, is different form the proof of second part of Proposition 1.3, and the use of Kellogg-Warschawski theorem for the second derivative ([33]) is avoided. The distance function is used and thereby a "weaker" smoothness of the boundary of image domain is needed.

Theorem 1.4 (The main theorem)**.** *Let $w = f(z)$ be a K quasiconformal harmonic mapping of the unit disk $\mathbb{U}$ and a Jordan domain Ω with $C^{1,1}$ boundary. Let in addition $a = f(0)$. Then w is co-Lipschitz. More precisely there exists a positive constant $c = c(K, \Omega, a) \geq 1$ such that*

$$\frac{1}{c}|z_1 - z_2| \leq |f(z_1) - f(z_2)|, \quad z_1, z_2 \in \Omega. \tag{1.7}$$

Since the composition of a q.c. harmonic and a conformal mapping is itself q.c. harmonic, using Theorem 1.4 and Kellogg's theorem for the first derivative we obtain:

Corollary 1.5. *Let $w = f(z)$ be a K quasiconformal harmonic mapping between a plane domain Ω_1 with $C^{1,\alpha}$ compact boundary and a plane domain Ω with $C^{1,1}$ compact boundary. Let in addition $a_0 \in \Omega_1$ and $b_0 = f(a_0)$. Then w is bi-Lipschitz. Moreover there exists a positive constant $c = c(K, \Omega, \Omega_1, a_0, b_0) \geq 1$ such that*

$$\frac{1}{c}|z_1 - z_2| \leq |f(z_1) - f(z_2)| \leq c|z_1 - z_2|, \quad z_1, z_2 \in \Omega_1. \tag{1.8}$$

Proof of corollary 1.5. Let $b = f(a) \in \partial\Omega$. As $\partial\Omega \in C^{1,1}$, it follows that there exists a $C^{1,1}$ Jordan curve $\gamma_b \subset \overline{\Omega}$, whose interior D_b lies in Ω, and $\partial\Omega \cap \gamma_b$ is a neighborhood of b. See [13, Theorem 2.1] for an explicit construction of such Jordan curve. Let $D_a = f^{-1}(D_b)$, and take a conformal mapping g_a of the unit disk onto D_a. Then $f_a = f \circ g_a$ is a q.c. harmonic mapping of the unit disk onto the $C^{1,1}$ domain D_b. According to Theorem 1.4 it follows that f_a is bi-Lipschitz. According to Kellogg's theorem, it follows that $f = f_a \circ g_a^{-1}$ and its inverse f^{-1} are Lipschitz in some small neighborhood of a and of $b = f(a)$ respectively. This means that ∇f is bounded in some neighborhood of a. Since $\partial\Omega_1$ is a compact, we obtain that ∇f is bounded in $\partial\Omega_1$. The same hold for ∇f^{-1} with respect to $\partial\Omega$. This in turn implies that f is bi-Lipschitz. □

2. Auxiliary results

Let Ω be a domain in $\mathbb{R}^2$ having non-empty boundary $\partial\Omega$. The distance function is defined by

$$d(x) = \mathrm{dist}\,(x, \partial\Omega). \tag{2.1}$$

Let Ω be bounded and $\partial\Omega \in C^{1,1}$. The conditions on Ω imply that $\partial\Omega$ satisfies the following condition: at a.e. point $z \in \partial\Omega$ there exists a disk $D = D(w_z, r_z)$ depending on z such that $\overline{D} \cap (\mathbb{C} \setminus \Omega) = \{z\}$. Moreover $\mu := \operatorname{ess\,inf}\{r_z, z \in \partial\Omega\} > 0$. It is easy to show that μ^{-1} bounds the curvature of $\partial\Omega$, which means that $\frac{1}{\mu} \geq \kappa_z$, for $z \in \partial\Omega$. Here κ_z denotes the curvature of $\partial\Omega$ at $z \in \partial\Omega$. Under the above conditions, we have $d \in C^{1,1}(\Gamma_\mu)$, where $\Gamma_\mu = \{z \in \overline{\Omega} : d(z) < \mu\}$ and for $z \in \Gamma_\mu$ there exists $\omega(z) \in \partial\Omega$ such that

$$\nabla d(z) = \nu_{\omega(z)}, \tag{2.2}$$

where $\nu_{\omega(z)}$ denotes the inner normal vector to the boundary $\partial\Omega$ at the point $\omega(z)$. See [7, Section 14.6] for details.

Lemma 2.1. *Let $w : \Omega_1 \mapsto \Omega$ be a K q.c. and $\chi = -d(w(z))$. Then*

$$|\nabla \chi| \leq |\nabla w| \leq K|\nabla \chi| \tag{2.3}$$

in $w^{-1}(\Gamma_\mu)$ for $\mu > 0$ such that $1/\mu > \kappa_0 = \mathrm{ess}\ \sup\{|\kappa_z| : z \in \partial\Omega\}$.

Proof. Observe first that ∇d is a unit vector. From $\nabla\chi = -\nabla d \cdot \nabla w$ it follows that

$$|\nabla\chi| \leq |\nabla d||\nabla w| = |\nabla w|.$$

For a non-singular matrix A we have

$$\begin{aligned} \inf_{|x|=1} |Ax|^2 &= \inf_{|x|=1} \langle Ax, Ax\rangle = \inf_{|x|=1} \left\langle A^T Ax, x\right\rangle \\ &= \inf\{\lambda : \exists x \neq 0, A^T Ax = \lambda x\} \\ &= \inf\{\lambda : \exists x \neq 0, AA^T Ax = \lambda Ax\} \\ &= \inf\{\lambda : \exists y \neq 0, AA^T y = \lambda y\} = \inf_{|x|=1} |A^T x|^2. \end{aligned} \tag{2.4}$$

Next we have that $(\nabla\chi)^T = -(\nabla w)^T \cdot (\nabla d)^T$ and therefore for $x \in w^{-1}(\Gamma_\mu)$, we obtain

$$|\nabla\chi| \geq \inf_{|e|=1} |(\nabla w)^T\, e| = \inf_{|e|=1} |\nabla w\, e| = l(w) \geq K^{-1}|\nabla w|.$$

The proof of (2.3) is completed.

□

Lemma 2.2. *Let $\{e_1, e_2\}$ be the natural basis in the space $\mathbf{R}^2$. Let $w : \Omega_1 \mapsto \Omega$ be a twice differentiable mapping and let $\chi = -d(w(z))$. Then*

$$\Delta\chi(z_0) = \frac{\kappa_{w_0}}{1 - \kappa_{w_0} d(w(z_0))} |(O_{z_0} \nabla w(z_0))^T e_1|^2 - \langle(\nabla d)(w(z_0)), \Delta w\rangle\,, \tag{2.5}$$

where $z_0 \in w^{-1}(\Gamma_\mu)$, $\omega_0 \in \partial\Omega$ with $|w(z_0) - \omega_0| = \mathrm{dist}(w(z_0), \partial\Omega)$, $\mu > 0$ such that $1/\mu > \kappa_0 = \operatorname{ess\,sup}\{|\kappa_z| : z \in \partial\Omega\}$ and O_{z_0} is an orthogonal transformation.

Proof. Let ν_{ω_0} be the inner unit normal vector of γ at the point $\omega_0 \in \gamma$. Let O_{z_0} be an orthogonal transformation that takes the vector e_2 to ν_{ω_0}. In complex notations:

$$O_{z_0} w = -i\nu_{\omega_0} w.$$

Take $\tilde{\Omega} := O_{z_0}\Omega$. Let $\tilde{d}$ be the distance function with respect to $\tilde{\Omega}$. Then

$$d(w) = \tilde{d}(O_{z_0} w) = \text{dist }(O_{z_0} w, \partial\tilde{\Omega}).$$

Therefore $\chi(z) = -\tilde{d}(O_{z_0}(w(z)))$.
Furthermore

$$\begin{aligned}\Delta\chi(z) = &-\sum_{i=1}^{2}(D^2\tilde{d})(O_{z_0}(w(z)))(O_{z_0}\nabla w(z)e_i, O_{z_0}\nabla w(z)e_i)\\ &-\langle \nabla d(w(z)), \Delta w(z)\rangle .\end{aligned} \tag{2.6}$$

To continue, we make use of the following proposition.

Proposition 2.3. [7, Lemma 14.17] *Let Ω be bounded and $\partial\Omega \in C^{1,1}$. Then under notation of Lemma 2.2 we have*

$$(D^2\tilde{d})(O_{z_0} w(z_0)) = \text{diag}(\frac{-\kappa_{\omega_0}}{1-\kappa_{\omega_0} d}, 0) = \begin{pmatrix} \frac{-\kappa_{\omega_0}}{1-\kappa_{\omega_0} d} & 0 \\ 0 & 0 \end{pmatrix}, \tag{2.7}$$

where κ_{ω_0} denotes the curvature of $\partial\Omega$ at $\omega_0 \in \partial\Omega$.

Applying (2.7) we have

$$\begin{aligned}&\sum_{i=1}^{2}(D^2\tilde{d})(O_{z_0}(w(z_0)))(O_{z_0}(\nabla w(z_0))e_i, O_{z_0}(\nabla w(z_0))e_i)\\ &=\sum_{i=1}^{2}\sum_{j,k=1}^{2} D_{j,k}\tilde{d}(O_{z_0}(w(z_0)))\, D_i(O_{z_0}w)_j(z_0)\cdot D_i(O_{z_0}w)_k(z_0)\\ &=\sum_{j,k=1}^{2} D_{j,k}\tilde{d}(O_{z_0}(w(z_0)))\left\langle (O_{z_0}\nabla w(z_0))^T e_j, (O_{z_0}\nabla w(z_0))^T e_k\right\rangle\\ &=\frac{-\kappa_{\omega_0}}{1-\kappa_{\omega_0}\tilde{d}}|(O_{z_0}\nabla w(z_0))^T e_1|^2.\end{aligned} \tag{2.8}$$

Finally we obtain

$$\Delta\chi(z_0) = \frac{\kappa_{\omega_0}}{1-\kappa_{\omega_0}\tilde{d}}|(O_{z_0}\nabla w(z_0))^T e_1|^2 - \langle (\nabla d)(w(z_0)), \Delta w\rangle .$$

□

3. The proof of the main theorem

The main step in proving the main theorem is the following lemma.

Lemma 3.1. *Let $w = f(z)$ be a K quasiconformal mapping of the unit disk onto a $C^{1,1}$ Jordan domain Ω satisfying the differential inequality*

$$|\Delta w| \le B|\nabla w|^2, \ B \ge 0 \tag{3.1}$$

for some $B \geq 0$*. Assume in addition that* $w(0) = a_0 \in \Omega$*. Then there exists a constant* $C(K, \Omega, B, a) > 0$ *such that*

$$|\frac{\partial w}{\partial r}(t)| \geq C(K, \Omega, B, a_0) \text{ for almost every } t \in S^1. \tag{3.2}$$

Proof. Let us find $A > 0$ so that the function $\varphi_w(z) = -\frac{1}{A} + \frac{1}{A}e^{-Ad(w(z))}$ is subharmonic on $\{z : d(w(z)) < \frac{1}{2\kappa_0}\}$, where

$$\kappa_0 = \operatorname{ess\,sup}\{|\kappa_w| : w \in \gamma\}.$$

Let $\chi = -d(w(z))$. Combining (2.3), (2.5) and (3.1) we get

$$|\Delta\chi| \leq 2\kappa_0|\nabla w|^2 + B|\nabla w|^2 \leq (2\kappa_0 + B)K^2|\nabla\chi|^2. \tag{3.3}$$

Take

$$g(t) = -\frac{1}{A} + \frac{1}{A}e^{At}.$$

Then $\varphi_w(z) = g(\chi(z))$. Thus

$$\Delta\varphi_w = g''(\chi)|\nabla\chi|^2 + g'(\chi)\Delta\chi. \tag{3.4}$$

Since

$$g'(\chi) = e^{-Ad(w(z))} \tag{3.5}$$

and

$$g''(\chi) = Ae^{-Ad(w(z))}, \tag{3.6}$$

it follows that

$$\Delta\varphi_w \geq (A - (2\kappa_0 + B)K^2)|\nabla\chi|^2e^{-Ad(u(z))}. \tag{3.7}$$

In order to have $\Delta\varphi_w \geq 0$, it is enough to take

$$A = (2\kappa_0 + B)K^2. \tag{3.8}$$

Choosing

$$\varrho = \max\{|z| : \operatorname{dist}(w(z), \gamma) = \frac{1}{2\kappa_0}\},$$

then φ_w satisfies the conditions of the following generalization of E. Hopf lemma ([9]):

Lemma 3.2. [10] *Let* φ *satisfies* $\Delta\varphi \geq 0$ *in* $R_\varrho = \{z : \varrho \leq |z| < 1\}$, $0 < \varrho < 1$, φ *be continuous on* $\overline{R_\varrho}$, $\varphi < 0$ *in* R_ϱ, $\varphi(t) = 0$ *for* $t \in S^1$*. Assume that the radial derivative* $\frac{\partial\varphi}{\partial r}$ *exists almost everywhere at* $t \in S^1$*. Let* $M(\varphi, \varrho) = \max_{|z|=\varrho} \varphi(z)$*. Then the inequality*

$$\frac{\partial\varphi(t)}{\partial r} > \frac{2M(\varphi, \varrho)}{\varrho^2(1 - e^{1/\varrho^2-1})}, \text{ for a.e. } t \in S^1, \tag{3.9}$$

holds.

We will make use of (3.9), but under some improvement for the class of q.c. harmonic mappings. The idea is to make the right hand side of (3.9) independent on the mapping w for $\varphi = \varphi_w$.

We will say that a q.c. mapping $f : \mathbf{U} \mapsto \Omega$ is normalized if $f(1) = w_0$, $f(e^{2\pi/3i}) = w_1$ and $f(e^{-2\pi/3i}) = w_2$, where w_0w_1, w_1w_2 and w_2w_0 are arcs of $\gamma = \partial\Omega$ having the same length $|\gamma|/3$.

In what follows we will prove that, for the class $\mathcal{H}(\Omega, K, B)$ of normalized K q.c. mappings, satisfying (3.1) for some $B \geq 0$, and mapping the unit disk onto the domain Ω, the inequality (3.9) holds uniformly (see (3.10)).

Let

$$\varrho := \sup\{|z| : \mathrm{dist}(w(z),\gamma) = \frac{1}{2\kappa_0}, w \in \mathcal{H}(\Omega, K, B)\}.$$

Therefore there exists a sequence $\{w_n\}$, $w_n \in \mathcal{H}(\Omega, K, B)$ such that

$$\varrho_n = \max\{|z| : \mathrm{dist}(w_n(z),\gamma) = \frac{1}{2\kappa_0}\},$$

and

$$\varrho = \lim_{n\to\infty} \varrho_n.$$

Notice now that, if w_n is a sequence of normalized K-q.c. mappings of the unit disk onto Ω, then, up to some subsequence, w_n is a locally uniform convergent sequence converging to some q.c. mapping $w \in \mathcal{H}(\Omega, K, B)$. Under the condition on the boundary of Ω, by [26, Theorem 4.4] this sequence is uniformly convergent on $\mathbb{U}$. Then there exists a sequence $z_n : \mathrm{dist}(w_n(z_n),\gamma) = \frac{1}{2\kappa_0}$, such that, $\lim_{n\to\infty} z_n = z_0$ and $\varrho = |z_0|$. Since w_n converges uniformly to w, it follows that, $\lim_{n\to\infty} w_n(z_n) = w(z_0)$, and $\mathrm{dist}(w(z_0),\gamma) = \frac{1}{2\kappa_0}$. This infers $\varrho < 1$.
Let now

$$M(\varrho) := \sup\{M(\varphi_w, \varrho), w \in \mathcal{H}(\Omega, K, B)\}.$$

Using the similar argument as above, we obtain that there exists a uniformly convergent sequence w_n, converging to a mapping w_0, such that

$$M(\varrho) = \lim_{n\to\infty} M(\varphi_{w_n}, \varrho) = M(\varphi_{w_0}, \varrho).$$

Thus

$$M(\varrho) < 0.$$

Setting $M(\varrho)$ instead of $M(\varrho,\varphi)$ and φ_w instead of φ in (3.9), we obtain

$$\frac{\partial \varphi_w(t)}{\partial r} > \frac{2M(\varrho)}{\varrho^2(1-e^{1/\varrho^2-1})} := C(K,\Omega,B), \text{ for a.e. } t \in S^1. \tag{3.10}$$

To continue observe that

$$\frac{\partial \varphi_w(t)}{\partial r} = e^{Ad(w(z))}|\nabla d| \left|\frac{\partial w}{\partial r}(t)\right| = e^{Ad(w(z))}\left|\frac{\partial w}{\partial r}(t)\right|.$$

Combining (3.8) and (3.10) we obtain for a.e. $t \in S^1$

$$\left|\frac{\partial w}{\partial r}(t)\right| = e^{-Ad(w(z))}\frac{\partial \varphi_w(t)}{\partial r} \geq e^{-K^2}\frac{2M(\varrho)}{\varrho^2(1-e^{1/\varrho^2-1})}.$$

The Lemma 3.1 is proved for normalized mapping w. If w is not normalized, then we take the corresponding composition of w and the corresponding Möbius transformation, in order to obtain the desired inequality. The proof of Lemma 3.1 is completed. □

The finish of proof of Theorem 1.4. In this setting w is harmonic and therefore $B = 0$.

Assume first that "$w \in C^1(\overline{\mathbf{U}})$".

Let $l(\nabla w)(t) = ||w_z(t)| - |w_{\bar z}(t)||$. As w is K q.c., according to (3.2) we have

$$l(\nabla w)(t) \geq \frac{|\nabla w(t)|}{K} \geq \frac{|\frac{\partial w}{\partial r}(t)|}{K} \geq \frac{C(K,\Omega,0,a_0)}{K} \tag{3.11}$$

for $t \in S^1$. Therefore, having in mind Lewy's theorem ([22]), which states that $|w_z| > |w_{\bar z}|$ for $z \in \mathbb{U}$, we obtain for $t \in S^1$ that $|w_z(t)| \neq 0$ and hence:

$$\frac{1}{|w_z|}\frac{C(K,\Omega,0,a_0)}{K} + \frac{|w_{\bar z}|}{|w_z|} \le 1,\ \ t \in S^1.$$

As $w \in C^1(\overline{\mathbf{U}})$, it follows that the functions

$$a(z) := \frac{\overline{w_{\bar z}}}{w_z}, \quad b(z) := \frac{1}{w_z}\frac{C(K,\Omega,0,a_0)}{K}$$

are well-defined holomorphic functions in the unit disk having a continuous extension to the boundary. As $|a| + |b|$ is bounded on the unit circle by 1, it follows that it is bounded on the whole unit disk by 1 because

$$|a(z)| + |b(z)| \le P[|a|_{S^1}](z) + P[|b|_{S^1}](z) = P[|a|_{S^1} + |b|_{S^1}](z), \quad z \in \mathbb{U}.$$

This in turn implies that for every $z \in \mathbf{U}$

$$l(\nabla w)(z) \ge \frac{C(K,\Omega,0,a_0)}{K} =: C(\Omega, K, a_0). \tag{3.12}$$

This infers that

$$C(K,\Omega,a_0) \le \frac{|w(z_1) - w(z_2)|}{|z_1 - z_2|}, \qquad z_1, z_2 \in \mathbf{U}.$$

Assume now that "$w \notin C^1(\overline{\mathbf{U}})$". We begin by this definition.

Definition 3.3. *Let G be a domain in $\mathbb{C}$ and let $a \in \partial G$. We will say that $G_a \subset G$ is a neighborhood of a if there exists a disk $D(a,r) := \{z : |z-a| < r\}$ such that $D(a,r) \cap G \subset G_a$.*

Let $t = e^{i\beta} \in S^1$, then $w(t) \in \partial\Omega$. Let γ be an arc-length parametrization of $\partial\Omega$ with $\gamma(s) = w(t)$. Since $\partial\Omega \in C^{1,1}$, there exists a neighborhood Ω_t of $w(t)$ with $C^{1,1}$ Jordan boundary such that,

$$\Omega_t^\tau := \Omega_t + i\gamma'(s)\cdot\tau \subset \Omega, \text{ and } \partial\Omega_t^\tau \subset \Omega \text{ for } 0 < \tau \le \tau_t\ \ (\tau_t > 0)\ . \tag{3.13}$$

An example of a family Ω_t^τ such that $\partial\Omega_t^\tau \in C^{1,1}$ and with the property (3.13) has been given in [13].

Let $a_t \in \Omega_t$ be arbitrary. Then $a_t + i\gamma'(s)\cdot\tau \in \Omega_t^\tau$. Take $U_\tau = f^{-1}(\Omega_t^\tau)$. Let η_t^τ be a conformal mapping of the unit disk onto U_τ such that $\eta_t^\tau(0) = f^{-1}(a_t + i\gamma'(s)\cdot\tau)$, and $\arg \frac{d\eta_t^\tau}{dz}(0) = 0$. Then the mapping

$$f_t^\tau(z) := f(\eta_t^\tau(z)) - i\gamma'(s)\cdot\tau$$

is a harmonic K quasiconformal mapping of the unit disk onto Ω_t satisfying the condition $f_t^\tau(0) = a_t$. Moreover

$$f_t^\tau \in C^1(\overline{\mathbf{U}}).$$

Using the case "$w \in C^1(\overline{\mathbf{U}})$", it follows that

$$|\nabla f_t^\tau(z)| \ge C(K, \Omega_t, a_t).$$

On the other hand

$$\lim_{\tau\to 0+} \nabla f_t^\tau(z) = \nabla(f\circ\eta_t)(z)$$

on the compact sets of $\mathbf{U}$ as well as

$$\lim_{\tau\to 0+} \frac{d\eta_t^\tau}{dz}(z) = \frac{d\eta_t}{dz}(z),$$

where η_t is a conformal mapping of the unit disk onto $U_0 = f^{-1}(\Omega_t)$ with $\eta_t(0) = f^{-1}(a_t)$. It follows that

$$|\nabla f_t(z)| \geq C(K, \Omega_t, a_t).$$

By using the Schwarz's reflexion principle to the mapping η_t, and using the formula

$$\nabla(f \circ \eta_t)(z) = \nabla f \cdot \frac{d\eta_t}{dz}(z)$$

it follows that in some neighborhood $\tilde{U}_t$ of $t \in S^1$ with smooth boundary ($D(t, r_t) \cap \mathbf{U} \subset \tilde{U}_t$ for some $r_t > 0$), the function f satisfies the inequality

$$|\nabla f(z)| \geq \frac{C(K, \Omega_t, a_t)}{\max\{|\eta_t'(\zeta)| : \zeta \in \overline{\tilde{U}_t}\}} =: \tilde{C}(K, \Omega_t, a_t) > 0. \tag{3.14}$$

Since S^1 is a compact set, it can be covered by a finite family $\partial\tilde{U}_{t_j} \cap S^1 \cap D(t, r_t/2)$, $j = 1, \ldots, m$. It follows that the inequality

$$|\nabla f(z)| \geq \min\{\tilde{C}(K, \Omega_{t_j}, a_{t_j}) : j = 1, \ldots, m\} =: \tilde{C}(K, \Omega, a_0) > 0, \tag{3.15}$$

there holds in the annulus

$$\tilde{R} = \left\{ z : 1 - \frac{\sqrt{3}}{2} \min_{1 \leq j \leq m} r_{t_j} < |z| < 1 \right\} \subset \bigcup_{j=1}^{m} \tilde{U}_{t_j}.$$

This implies that the subharmonic function $S = |a(z)| + |b(z)|$ is bounded in $\mathbf{U}$. According to the maximum principle, it is bounded by 1 in the whole unit disk. This in turn implies again (3.12) and consequently

$$\frac{C(K, \Omega, a_0)}{K} |z_1 - z_2| \leq |w(z_1) - w(z_2)|, \quad z_1, z_2 \in \mathbf{U}.$$

□

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