Reflected generalized backward doubly SDEs driven by Lévy processes and Applications*

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Abstract

In this paper, we study reflected generalized backward doubly stochastic differential equations driven by Teugels martingales associated with Lévy process (RGBDSDELs, in short) with one continuous barrier. Under uniformly Lipschitz coefficients, we prove existence and uniqueness result by means of the penalization method and the fixed point theorem. As an application, this study allows us to give a probabilistic representation for the solutions to a class of reflected stochastic partial differential integral equations (SPDIEs, in short) with a nonlinear Neumann boundary condition.

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1 Introduction

The theory of nonlinear backward stochastic differential equations (BSDEs, in short) have been first introduced by Pardoux and Peng [16]. They proved existence and uniqueness of the adapted processes (Y,Z) solution of the following equation:

$$Y_t \quad = \quad \xi+\int_t^T f(s,Y_s,Z_s)ds-\int_t^T Z_s dW_s,\ 0\leq t\leq T, \tag{1.1}$$

when the terminal value ξ is square integrable and the coefficient f is Lipschitz in (y,z) uniformly in (t,ω).

Mainly motivated by financial problems, stochastic control, stochastic games and probabilistic interpretation for solutions to nonlinear partial differential equations (PDE, in short), the theory of BSDEs was developed at high speed during the 1990. We refer the

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reader to survey article by El Karoui et al. [5], Hamadène and Lepeltier [7], Pardoux and Peng [17], Pardoux and Zhang [19] and references therein.

In this dynamic, El Karoui et al. firstly introduced in [4] the notation of a solution of reflected backward stochastic differential equations (RBSDEs, in short) with a continuous barrier. A solution for such equation associated with (ξ, f, S), is a triple $(Y_t, Z_t, K_t)_{0\leq t\leq T}$, which satisfies

$$Y_t \;\; = \;\; \xi + \int_t^T f(s, Y_s, Z_s)ds + K_T - K_t - \int_t^T Z_s dW_s,\;\; 0 \leq t \leq T,$$

and $Y_t \geq S_t$ a.s. for any $t \in [0,T]$. The process $(K_t)_{0\leq t\leq T}$ is non decreasing continuous whose role is to push upward the process Y, in order to keep it above S. And it satisfies Skorokhod condition

$$\int_0^T (Y_s - S_s)dK_s = 0.$$

As shown in [4], RBSDE's are a useful tool for the pricing of American options and the probabilistic representation for solutions to PDE's obstacle problem. Recall that many assumptions have been made to relax the assumption on the coefficient f and the barrier; for instance, in [12] Matoussi established the existence of a solution for RBSDE's with continuous and linear growth coefficient. Moreover, in [6, 8] RBSDEs with discontinuous barrier and double barrier with continuous coefficients have been studied respectively. Also many authors studied RBSDEs replacing brownian motion by jumps process (see Hamadène and Ouknine [9] and references therein).

On the other hand, Pardoux and Peng study in [18] the so-called backward doubly stochastic differential equations (BDSDEs, in short):

$$Y_t \;\; = \;\; \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)dB_s - \int_t^T Z_s dW_s,\;\; 0 \leq t \leq T. \tag{1.2}$$

where dW is a forward Itô integral and dB the backward one. They prove among other a probabilistic representation for a class of quasi linear stochastic partial differential equations (SPDEs, in short).

In this paper, we study reflected generalized BDSDEs driven by Teugel martingale with respect to Lévy process (RGBDSDEL, in short) under Lipschiz coefficient, motivated by it application to obstacle problem for stochastic partial differential integral equations (SPDIEs, in short) and inspired by [24].

The theory of BSDEs driven by Teugels martingales associated with Lévy process have been intensively study since Nualart and Schoutens prove in [14] the martingale representation theorem associated to Lévy process. They also derive in [15] an existence and uniqueness result to BSDEs driven by Teugels martingales associated with Lévy process. Since then, many others results have been derived. We refer the reader to [3], [22], [10] and reference therein. Note that all those studies were important from a pure mathematical point of view as well as in the world of finance. It could be used for the purpose of option pricing in a Lévy market and related PDIEs which provided an analogue of the famous Black and Scholes formula.

Roughly speaking, the present paper have two goal: first the existence and uniqueness of the solution to RGBDSDEL

$$\begin{aligned} Y_t &= \xi + \int_t^T f(s,Y_{s^-},Z_s)ds + \int_t^T \phi(s,Y_{s^-})dA_s + \int_t^T g(s,Y_{s^-},Z_s)\,dB_s \\ &\quad - \sum_{i=1}^{m}\int_t^T Z_s^{(i)}dH_s^{(i)} + K_T - K_t,\ 0\le t\le T, \end{aligned} \tag{1.3}$$

is derived by means of the penalization method and the fixed point theorem when the terminal value ξ is square integrable and the coefficients f and g is Lipschitz in (y,z) uniformly in (t,ω). Furthermore, using this result we get a probabilistic representation for the solution of reflected SPDIE.

Due to the fact that the solution should be adapted to a family $(\mathcal{F}_t)$ which is not a filtration, the usual technics used in the classical reflected BSDEs (see e.g. [4]) does not work. Indeed, the section theorem cannot be easily used to derive that the solution stays above the obstacle for all time.

We give here a method which allows us to overcome this difficulty. The idea consists to start from the basic RGBDSDEL with g independent from (y,z). We transform it to a RGBDSDEL with $g=0$, for which we prove the existence and uniqueness of solutions by a penalization method. The section theorem is then used in this simple context $(g=0)$ to prove that the solution of the RGBDSDEL with $g=0$, stays above the obstacle at each time. The case where the coefficients g depend on (y,z) is then deduced by using a Banach fixed point theory.

The rest of paper is organized as follows. In Section 2, we state some notations, needed assumptions and the definition of solution to RGBDSDELs. Section 3, is devoted to give our main results for existence and uniqueness for RGBDSDEL. Finally Section 4 point out a probabilistic representation of solutions to a class of reflected SPDIEs with a nonlinear Neumann boundary condition.

2 Notations, assumptions and definitions

The scalar product of the space $\mathbb{R}^d(d\ge 2)$ will be denoted by $<.>$ and the associated Euclidian norm by $\|.\|$.

In what follows let us fix a positive real number $T>0$. Let $(\Omega,\mathcal{F},\mathbb{P},\mathcal{F}_t,B_t,L_t : t\in[0,T])$ be a complete Wiener-Lévy space in $\mathbb{R}\times\mathbb{R}\backslash\{0\}$, with Levy measure ν, i.e. $(\Omega,\mathcal{F},\mathbb{P})$ is a complete probability space, $\{\mathcal{F}_t : t\in[0,T]\}$ is a right-continuous increasing family of complete sub σ-algebras of $\mathcal{F}$, $\{B_t : t\in[0,T]\}$ is a standard Wiener process in $\mathbb{R}$ with respect to $\{\mathcal{F}_t : t\in[0,T]\}$ and $\{L_t : t\in[0,T]\}$ is a $\mathbb{R}$-valued Lévy process independent of $\{B_t : t\in[0,T]\}$ and has only m jumps size with non Brownian associated to a standard Lévy measure ν satisfying the following conditions: $\int_{\mathbb{R}}(1\wedge y)\nu(dy)<\infty$,

Let $\mathcal{N}$ denote the totality of $\mathbb{P}$-null sets of $\mathcal{F}$. For each $t\in[0,T]$, we define

$$\mathcal{F}_t = \mathcal{F}_t^{L}\vee\mathcal{F}_{t,T}^{B} \text{ and } \widetilde{\mathcal{F}}_t = \mathcal{F}_t^{L}\vee\mathcal{F}_T^{B}$$

where for any process $\{\eta_t\}$, $\mathcal{F}_{s,t}^{\eta} = \sigma(\eta_r-\eta_s, s\le r\le t)\vee\mathcal{N}$, $\mathcal{F}_t^{\eta} = \mathcal{F}_{0,t}^{\eta}$.

We remark that $\mathbf{F} = \{\mathcal{F}_t,\ t \in [0,T]\}$ is neither increasing nor decreasing so that it does not a filtration. However $\widetilde{\mathbf{F}} = \{\widetilde{\mathcal{F}}_t,\ t \in [0,T]\}$ is a filtration.

We denote by $(H^{(i)})_{i\geq 1}$ the Teugels Martingale associated with the Lévy process $\{L_t : t \in [0,T]\}$. More precisely

$$H^{(i)} = c_{i,i}Y^{(i)} + c_{i,i-1}Y^{(i-1)} + \cdots + c_{i,1}Y^{(1)}$$

where $Y_t^{(i)} = L_t^i - \mathbb{E}(L_t^i) = L_t^i - t\mathbb{E}(L_t^1)$ for all $i \geq 1$ and L_t^i are power-jump processes. That is $L_t^1 = L_t$ and $L_t^i = \sum_{0<s<t}(\Delta L_s)^i$ for all $i \geq 2$, where $X_{t^-} = \lim_{s\nearrow t} X_s$ and $\Delta X_t = X_t - X_{t^-}$. It was shown in Nualart and Schoutens [14] that the coefficients $c_{i,k}$ correspond to the orthonormalization of the polynomials $1, x, x^2, \ldots$ with respect to the measure $\mu(dx) = x^2 d\nu(x) + \sigma^2\delta_0(dx)$:

$$q_{i-1}(x) = c_{i,i}x^{i-1} + c_{i,i-1}x^{i-2} + \cdots + c_{i,1}.$$

We set

$$p_i(x) = xq_{i-1}(x) = c_{i,i}x^i + c_{i,i-1}x^{i-1} + \cdots + c_{i,1}x^1.$$

The martingale $(H^{(i)})_{i=1}^m$ can be chosen to be pairwise strongly orthonormal martingale.

In the sequel, let $\{A_t,\ 0 \leq t \leq T\}$ be a continuous, increasing and $\mathbf{F}$-measurable real valued with bounded variation on $[0,T]$ such that $A_0 = 0$.

For any $m \geq 1$, we consider the following spaces of processes:

1. $\mathcal{M}^2(\mathbb{R}^m)$ denote the space of real valued, square integrable and $\mathcal{F}_t$-measurable processes $\varphi = \{\varphi_t : t \in [0,T]\}$ such that

   $\|\varphi\|^2_{\mathcal{M}^2} = \mathbb{E}\int_0^T \|\varphi_t\|^2 dt < \infty.$

2. $\mathcal{S}^2(\mathbb{R})$ is the subspace of $\mathcal{M}^2(\mathbb{R})$ formed by the $\mathcal{F}_t$-measurable processes $\varphi = \{\varphi_t : t \in [0,T]\}$ right continuous with left limit (rcll) such that

   $$\|\varphi\|^2_{\mathcal{S}^2} = \mathbb{E}\left(\sup_{0\leq t\leq T} |\varphi_t|^2\right) < \infty.$$

3. $\mathcal{A}^2(\mathbb{R})$ is the set of $\mathcal{F}_t$-measurable, continuous, real-valued, increasing process $\varphi = \{\varphi_t : t \in [0,T]\}$ such that $\varphi_0 = 0$, $\mathbb{E}|\varphi_T|^2 < \infty$

Finally $\mathcal{E}^{2,m} = \mathcal{S}^2(\mathbb{R}) \times \mathcal{M}^2(\mathbb{R}^m) \times \mathcal{A}^2(\mathbb{R})$ endowed with the norm

$$\|(Y,Z,K)\|^2_{\mathcal{E}} = \mathbb{E}\left(\sup_{0\leq t\leq T} |Y_t|^2 + \int_0^T \|Z_t\|^2 dt + |K_T|^2\right).$$

is a Banach space.

Next, we consider needed assumptions

($\mathbf{H1}$) ξ is a square integrable random variable which is $\mathcal{F}_T$-measurable such that for all $\mu > 0$

$$\mathbb{E}\left(e^{\mu A_T}|\xi|^2\right) < \infty.$$

$(\mathbf{H2})$ $f:\Omega\times[0,T]\times\mathbb{R}\times\mathbb{R}^m\to\mathbb{R}$ and $\phi:\Omega\times[0,T]\times\mathbb{R}\to\mathbb{R}$, such that

(a) There exist $\mathcal{F}_t$-measurable processes $\{f_t,\, \phi_t,\, 0\le t\le T\}$ with values in $[1,+\infty)$, and constants $\mu>0$ and $K>0$ such that for any $(t,y,z)\in[0,T]\times\mathbb{R}\times\mathbb{R}^m$ we have:

$$\begin{cases} f(t,y,z) \text{ and } \phi(t,y) \text{ are } \mathcal{F}_t\text{-measurable processes}, \\ |f(t,y,z)|\le f_t+K(|y|+\|z\|), \\ |\phi(t,y)|\le \phi_t+K|y|, \\ \mathbb{E}\left(\int_0^T e^{\mu A_t} f_t^2 dt+\int_0^T e^{\mu A_t}\phi_t^2 dA_t\right)<\infty. \end{cases}$$

(b) There exist constants $c>0, \beta<0$ and $0<\alpha<1$ such that for any (y_1,z_1), $(y_2,z_2)\in\mathbb{R}\times\mathbb{R}^m$,

$$\begin{cases} (i)\, |f(t,y_1,z_1)-f(t,y_2,z_2)|^2\le c(|y_1-y_2|^2+\|z_1-z_2\|^2), \\ (ii)\ \langle y_1-y_2, \phi(t,y_1)-\phi(t,y_2)\rangle\le\beta|y_1-y_2|^2, \\ (iv)\ |\phi(t,y_1)-\phi(t,y_2)|\le c|y_1-y_2|^2, \end{cases}$$

$(\mathbf{H3})$ $g:\Omega\times[0,T]\times\mathbb{R}\times\mathbb{R}^m\to\mathbb{R}$, such that

(a) There exist $\mathcal{F}_t$-measurable process $\{g_t: 0\le t\le T\}$ with values in $[1,+\infty)$, constants $\mu>0$ and $K>0$ such that for any $(t,y,z)\in[0,T]\times\mathbb{R}\times\mathbb{R}^m$ we have:

$$\begin{cases} g(t,y,z) \text{ is } \mathcal{F}_t\text{-measurable processes}, \\ |g(t,y,z)|\le g_t+K(|y|+|z|), \\ \mathbb{E}\left(\int_0^T e^{\mu A_t} g_t^2 dt\right)<\infty. \end{cases}$$

(b) There exist constants $c>0$ and $0<\alpha<1$ such that for any (y_1,z_1), $(y_2,z_2)\in\mathbb{R}\times\mathbb{R}^m$,
$|g(t,y_1,z_1)-g(t,y_2,z_2)|^2\le c|y_1-y_2|^2+\alpha\|z_1-z_2\|$.

(H4) The obstacle $\{S_t, 0\le t\le T\}$, is a $\mathcal{F}_t$-measurable real-valued process satisfying

(i) $\mathbb{E}\left(\sup_{0\le t\le T}\left|S_t^+\right|^2\right)<\infty$,

(ii) $S_T\le\xi$ a.s.

Definition 2.1. We call solution of the RGBDSDEL (1.3), a $(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}+)$-valued process (Y,Z,K) which satisfied (1.3) such that the following holds $\mathbb{P}$-a.s

(i) $(Y,Z,K) \in \mathcal{E}^{2,m}$

(ii) $Y_t \geq S_t, \;\; 0 \leq t \leq T,$

(iii) $\int_0^T (Y_{t^-} - S_t)\, dK_t = 0.$

3 Main results

Lemma 3.1. *(Comparison theorem see [22]) Let ξ^1 and ξ^2 be two square integrable and $\widetilde{\mathcal{F}}_T$-measurable random variables, $f^1, f^2 : [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ and $\phi : [0,T] \times \Omega \times \mathbb{R} \to \mathbb{R}$ be three measurable functions. For $k = 1,2$, let (Y^k, Z^k) be a unique solution of the following BSDE:*

$$\begin{cases} Y_t^k = \xi^k + \int_t^T f^k(s, Y_{s^-}^k, Z_s^k) ds + \int_t^T \phi(s, Y_{s^-}^k) dA_s - \sum_{i=1}^m \int_t^T Z_s^{k(i)} dH_s^{(i)} \\ \\ \mathbb{E}\left(\sup_{0 \leq t \leq T} |Y_t^k|^2 + \int_0^T \|Z_s^k\|^2 ds \right) \end{cases}$$

We assume that

- $\xi^1 \geq \xi^2$, $\mathbb{P}$-*a.s.*,
- $f^1(t, Y^2, Z^2) \geq f^2(t, Y^2, Z^2)$, $\mathbb{P}$-*a.s.*,
- $\beta_t^i = \dfrac{f^1(t, Y_{t^-}^2, \widetilde{Z}_t^{(i-1)}) - f^1(t, Y_{t^-}^2, \widetilde{Z}_t^{(i)})}{Z_t^{1(i)} - Z_t^{2(i)}} \mathbf{1}_{\left\{Z_t^{1(i)} \neq Z_t^{2(i)}\right\}}$,

where

$$\widetilde{Z}^{(i)} = \left(Z^{2(1)}, Z^{2(2)}, \ldots, Z^{2(i)}, Z^{1(i+1)}, \ldots, Z^{1(m)} \right)$$

satisfying $\sum_{i=1}^m \beta_t^i \Delta H_t^{(i)} > -1$, $dt \otimes d\mathbb{P}$-*a.s.*

Then, we have $Y_t^1 \geq Y_t^2$, a.s., $\forall t \in [0,T]$. Moreover, if $\xi^1 > \xi^2$ or $f^1(t,Y^2,Z^2) > f^2(t,Y^2,Z^2)$ or $\phi^1(t,Y^2) > \phi^2(t,Y^2)$, a.s., we have $Y_t^1 > Y_t^2$, a.s., $\forall t \in [0,T]$.

Proof. Let define

$$a_t = [f^1(t, Y_{t^-}^1, Z_t^1) - f^1(t, Y_{t^-}^2, Z_t^1)]/(Y_{t^-}^1 - Y_{t^-}^2)\mathbf{1}_{\{Y_{t^-}^1 \neq Y_{t^-}^2\}}$$

$$b_t = [\phi(t, Y_{t^-}^1) - \phi(t, Y_{t^-}^2)]/(Y_{t^-}^1 - Y_{t^-}^2)\mathbf{1}_{\{Y_{t^-}^1 \neq Y_{t^-}^2\}};$$

We note that $(a_t)_{t \in [0,T]}$ and $(b_t)_{t \in [0,T]}$ are bounded measurable processes.

For $0 \leq s \leq t \leq T$, let $\Gamma_{s,t} = 1 + \int_s^t \Gamma_{s,r^-} dX_r$, where

$$X_t = \int_0^t a_r dr + \int_0^t b_r dA_r + \sum_{i=1}^m \int_0^t \beta_r^i dH_r^{(i)}.$$

Then, we have (cf. Doléans-Dade exponential formula)

$$\Gamma_{s,t} = \exp\left(\int_s^t dX_r - \frac{1}{2} \int_s^t \|\beta_r\|^2 dr \right) \prod_{s < r \leq t} (1 + \Delta X_r) \exp(-\Delta X_r) \tag{3.1}$$

with $\Delta X_t = \sum_{i=1}^{m} \beta_t^i \Delta H_t^{(i)} > -1$. Thus, for all $0 \le s \le t \le T,\ \Gamma_{s,t} > 0$.

Let denote

$$\bar{\xi} = \xi^1 - \xi^2, \quad \bar{Y}_t = Y_t^1 - Y_t^2, \quad \bar{Z}_t = Z_t^1 - Z_t^2$$

$$\bar{f}_t = f^1(t, Y_{t^-}^2, Z_t^2) - f^2(t, Y_{t^-}^2, Z_t^2)$$

and

$$\bar{\phi}_t = \phi^1(t, Y_{t^-}^2) - \phi^2(t, Y_{t^-}^2).$$

Then

$$\bar{Y}_t \quad = \quad \bar{\xi} + \int_t^T [a_s \bar{Y}_{s^-} + \sum_{i=1}^{m} \beta_s^i \bar{Z}_s^{(i)} + \bar{f}_s] ds + \int_t^T [b_s \bar{Y}_{s^-} + \bar{\phi}_s] dA_s - \sum_{i=1}^{m} \int_t^T \bar{Z}_s^{(i)} dH_s^{(i)}, \qquad t \in [0,T]$$

Applying Itô's formula to $\Gamma_{s,r} Y_r$ from $r = t$ to $r = T$, it follows that

$$\begin{aligned}
\Gamma_{s,t} \bar{Y}_t &= \Gamma_{s,T} \bar{\xi} - \int_t^T \Gamma_{s,r^-} d\bar{Y}_r - \int_t^T \bar{Y}_{r^-} d\Gamma_{s,r} - \int_t^T d[\Gamma_{s,.}, \bar{Y}]_r \\
&= \Gamma_{s,T} \bar{\xi} + \int_t^T \Gamma_{s,r^-} [\sum_{i=1}^{m} \beta_r^i \bar{Z}_r^{(i)} + \bar{f}_r] dr + \int_t^T \Gamma_{s,r^-} \bar{\phi}_r dA_r - \sum_{i=1}^{m} \int_t^T \Gamma_{s,r^-} \bar{Z}_s^{(i)} dH_s^{(i)} \\
&\quad + \sum_{i=1}^{m} \int_t^T \bar{Y}_{r^-} \Gamma_{s,r^-} \beta_r^i dH_r^{(i)} - \sum_{i,j=1}^{m} \int_t^T \Gamma_{s,r^-} \beta_r^i \bar{Z}_r^{(j)} d[H^i, H^j]_r \\
&= \Gamma_{s,T} \bar{\xi} + \int_t^T \Gamma_{s,r^-} \bar{f}_r dr + \int_t^T \Gamma_{s,r^-} \bar{\phi}_r dA_r - \sum_{i=1}^{m} \int_t^T \Gamma_{s,r^-} \bar{Z}_s^{(i)} dH_s^{(i)} + \sum_{i=1}^{m} \int_t^T \bar{Y}_{r^-} \Gamma_{s,r^-} \beta_r^i dH_r^{(i)}.
\end{aligned}$$

Taking conditional expectation w.r.t. $\widetilde{\mathcal{F}}_s$, is not hard to see that for $s = t$

$$\bar{Y}_t \quad = \quad \mathbb{E}\left(\Gamma_{t,T} \bar{\xi} + \int_t^T \Gamma_{t,r^-} \bar{f}_r dr + \int_t^T \Gamma_{s,r^-} \bar{\phi}_r dA_r \mid \widetilde{\mathcal{F}}_t \right).$$

Therefore $\bar{Y}_t \ge 0$, i.e. $Y_t^1 \ge Y_t^2$, a.s. Moreover if $\bar{\xi} > 0$ or $\bar{f}_t > 0$, a.s., then $\bar{Y}_t > 0$, i.e. $Y_t^1 > Y_t^2$, a.s. $\square$

Now we state existence and uniqueness result.
Firstly, we suppose g independent from (Y, Z) and consider RGBDSDEL:

$$\begin{cases}
Y_t = \xi + \int_t^T f(s, Y_{s^-}, Z_s) ds + \int_t^T \phi(s, Y_{s^-}) dA_s + \int_t^T g(s)\, dB_s - \sum_{i=1}^{m} \int_t^T Z_s^{(i)} dH_s^{(i)} + K_T - K_t, \\
Y_t \ge S_t, \quad 0 \le t \le T, \\
(K_s)_{0 \le s \le T} \text{ is increasing, continuous and satisfies } \int_0^T (Y_{s^-} - S_s)\, dK_s = 0.
\end{cases} \tag{3.2}$$

Proposition 3.2. *Under assumptions* (**H1**), (**H2**) *and* (**H4**), *the basic RGBDSDEL* (3.2) *has a unique solution.*

Proof. In the sequel, C denotes a strictly positive and finite constant which may take different values from line to line.
Existence. For each $n \in \mathbb{N}^*$, we set

$$f_n(s,y,z) = f(s,y,z) + n(y - S_s)^-. \tag{3.3}$$

By [10], let (Y^n, Z^n) be a unique pair of process with values in $\mathbb{R} \times \mathbb{R}^m$ satisfying: $(Y^n, Z^n) \in S^2 \times \mathcal{M}^2$ and

$$\begin{aligned} Y_t^n &= \xi + \int_t^T f_n(s, Y_{s^-}^n, Z_s^n) ds + \int_t^T \phi(s, Y_{s^-}^n) dA_s \\ &\quad + \int_t^T g(s)\, dB_s - \sum_{i=1}^m \int_t^T (Z_s^n)^{(i)} dH_s^{(i)}. \end{aligned} \tag{3.4}$$

Let

$$K_t^n = n \int_0^t (Y_{s^-}^n - S_s)^- ds \tag{3.5}$$

Step 1: A priori estimate
We have

$$\sup_{n \in \mathbb{N}^*} \mathbb{E}\left(\sup_{0 \le t \le T} |Y_t^n|^2 + \int_t^T \|Z_s^n\|^2 ds + |K_T^n|^2 \right) < C.$$

Indeed, by Itô's formula, we have

$$\begin{aligned} &\mathbb{E}\,|Y_t^n|^2 + \int_t^T \|Z_s^n\|^2 ds \\ &\le |\xi|^2 + 2\mathbb{E} \int_t^T Y_{s^-}^n f(s, Y_{s^-}^n, Z_s^n) ds + 2\mathbb{E} \int_t^T Y_{s^-}^n \phi(s, Y_{s^-}^n) dA_s \\ &+ \mathbb{E} \int_t^T |g(s)|^2 ds + 2\mathbb{E} \int_t^T S_s dK_s^n. \end{aligned}$$

Using $(\mathbf{H2})$ and the elementary inequality $2ab \le \gamma a^2 + \frac{1}{\gamma} b^2,\ \forall \gamma > 0,$

$$\begin{aligned} 2Y_s^n f(s, Y_s^n, Z_s^n) &\le (c\gamma_1 + \frac{1}{\gamma_1})|Y_s^n|^2 + 2c\gamma_1 \|Z_s^n\|^2 + 2\gamma_1 f_s^2, \\ 2Y_s^n \phi(s, Y_s^n) &\le (\gamma_2 - 2|\beta|)|Y_s^n|^2 + \frac{1}{\gamma_2}\phi_s^2. \end{aligned}$$

We choose $\gamma_1 = \dfrac{1}{4c}$, $\gamma_2 = 2|\beta|$ in the previous to obtain for all $\varepsilon > 0$

$$\begin{aligned} &\mathbb{E}\,|Y_t^n|^2 + \frac{1}{2}\mathbb{E} \int_t^T \|Z_s^n\|^2 ds \\ &\le C\mathbb{E}\left\{ |\xi|^2 + \int_t^T |Y_s^n|^2 ds + \int_t^T f_s^2 ds + \int_t^T \phi_s^2 dA_s + \int_t^T |g(s)|^2 ds \right\} \\ &+ \frac{1}{\varepsilon}\mathbb{E}\left(\sup_{0 \le s \le t} (S_s^+)^2 \right) + \varepsilon \mathbb{E}\,(K_T^n - K_t^n)^2. \end{aligned} \tag{3.6}$$

In virtue of (3.4) and (3.5) we have

$$\mathbb{E}(K_T^n-K_t^n)^2 \le C\mathbb{E}\left\{|\xi|^2+\int_t^T f_s^2ds+\int_t^T \phi_s^2 dA_s+\int_t^T |g(s)|^2ds+\int_t^T |Y_s^n|^2ds \right.$$
$$\left. +\mathbb{E}\left(\sup_{0\le s\le t}(S_s^+)^2\right)+\int_0^t |Y_s^n|^2 dA_s+\int_t^T \|Z_s^n\|^2 ds\right\} \qquad (3.7)$$

which, put in (3.6) together with Burkhölder-Davis-Gundy inequality provides

$$\begin{aligned}\mathbb{E}\left\{\sup_{0\le t\le T}|Y_t^n|^2+\int_t^T \|Z_s^n\|^2ds+|K_T^n|^2\right\} &\le C\mathbb{E}\left\{|\xi|^2+\int_0^T f_s^2ds+\int_0^T \phi_s^2dA_s\right.\\ &\left.+\int_0^T |g(s)|^2ds+\sup_{0\le t\le T}(S_t^+)^2\right\},\end{aligned}$$

provided that ε is small enough.

Step 2: $Y_t \ge S_t$, a.s. $\forall\, t\in[0,T]$ where for all $0\le t\le T$, $Y_t=\sup_n Y_t^n$

Let us define

$$\begin{cases} \overline{\xi} := \xi+\int_0^T g(s)\,dB_s \\ \overline{S}_t := S_t+\int_0^t g(s)\,dB_s \\ \overline{Y}_t^n := Y_t^n+\int_0^t g(s)\,dB_s. \end{cases}$$

Hence, according (3.4) we have

$$\overline{Y}_t^n=\overline{\xi}+\int_t^T f(s,Y_{s^-}^n,Z_s^n)\,ds+n\int_t^T\left(\overline{Y}_{s^-}^n-\overline{S_s}\right)^- ds+\int_t^T\phi(s,Y_{s^-}^n)\,dA_s-\sum_{i=1}^m\int_t^T (Z_s^n)^{(i)}dH_s^{(i)}. \qquad (3.8)$$

Let $(\widetilde{Y}^n,\widetilde{Z}^n)$ be a unique solution of the GBDSDEL

$$\widetilde{Y}_t^n = \overline{S}_T+\int_t^T f(s,Y_{s^-}^n,Z_s^n)\,ds+n\int_t^T(\overline{S}_s-\widetilde{Y}_{s^-}^n)ds+\int_t^T\phi(s,Y_{s^-}^n)\,dA_s-\sum_{i=1}^m\int_t^T(\widetilde{Z}_s^n)^{(i)}dH_s^{(i)}.$$

Since $\overline{S}_T\le\overline{\xi}$, the previous comparison theorem shows that for every $n\ge1$, $\overline{Y}_t^n\ge\widetilde{Y}_t^n$ a.s., for all $0\le t\le T$.

Next, let σ be a $\widetilde{\mathcal{F}}_t$-stopping time, and $\nu=\sigma\wedge T$. The sequence of processes $(\widetilde{Y}_\nu^n)$ satisfies the equality

$$\begin{aligned}\widetilde{Y}_\nu^n &= \mathbb{E}^{\widetilde{\mathcal{F}}_\nu}\left\{e^{-n(T-\nu)}\overline{S}_T+\int_\nu^T e^{-n(\nu-s)}f(s,Y_{s^-}^n,Z_s^n)ds+n\int_\nu^T e^{-n(\nu-s)}\overline{S}_sds\right.\\ &\left.+\int_\nu^T e^{-n(\nu-s)}\phi(s,Y_{s^-}^n)dA_s\right\}\end{aligned}$$

and therefore converges to $\overline{S}_\nu$ a.s. This implies that $S_\nu\le Y_\nu$ a.s. It follows from the section theorem ([2], p. 220) that for every $t\in[0,T]$, $Y_t\ge S_t$ a.s.

Step 3: Convergence of (Y^n,Z^n)

Since $Y_t^n\nearrow Y_t$ a.s. for all $0\le t\le T$, using Fatou's lemma and step 1, we have

$$\mathbb{E}\left(\sup_{0\le t\le T}|Y_t|^2\right)<+\infty.$$

Moreover, Lebesgue's dominated convergence theorem provide

$$\mathbb{E}\left(\int_0^T |Y_s^n - Y_s|^2\,ds\right) \longrightarrow 0,\ \text{ as } n \to \infty.$$

Next, in virtue of step 2 we get, for $n \geq p$,

$$\mathbb{E}\left(\sup_{0\leq s\leq T} |Y_s^n - Y_s^p|^2 + \int_0^T \|Z_s^n - Z_s^p\|^2\,ds + \sup_{0\leq s\leq T} |K_s^n - K_s^p|^2\right) \longrightarrow 0,\ \text{ as } n, p \longrightarrow \infty,$$

which provides that the sequence of processes (Y^n, Z^n, K^n) is Cauchy in the Banach space $\mathcal{E}^{2,m}$. Consequently, there exists a triplet $(Y, Z, K) \in \mathcal{E}^{2,m}$ such that

$$\mathbb{E}\left\{\sup_{0\leq s\leq T} |Y_s^n - Y_s|^2 + \int_0^T \|Z_s^n - Z_s\|^2\,ds + \sup_{0\leq s\leq T} |K_s^n - K_s|^2\right) \to 0,\ \text{as } n \to \infty.$$

Step 4: The limit (Y,Z,K) solve RGBDSDEL (3.2)
Since (Y^n, K^n) converge to (Y, K) in probability, the measure dK^n converges to dK weakly in probability, so that $\int_0^T (Y_{s^-}^n - S_s)dK_s^n \to \int_0^T (Y_{s^-} - S_s)dK_s$ in probability as $n \to \infty$. Obviously, $\int_0^T (Y_{s^-} - S_s)dK_s \geq 0$, while, on the other hand, for all $n \geq 0$, $\int_0^T (Y_{s^-}^n - S_s)dK_s^n \leq 0$. Hence $\int_0^T (Y_{s^-} - S_s)dK_s = 0$, a.s. Finally, passing to the limit in (3.4), (Y, Z, K) verifies (1.3).

Uniqueness. Let $(\Delta Y, \Delta Z, \Delta K)$ be the difference between two arbitrary solutions. Since $\int_t^T (\Delta Y - \Delta S_s)d(\Delta K_s) = 0$, the uniqueness follows using the same computation as above. □

Theorem 3.3. *Assume that* (**H1**), (**H2**), (**H3**) *and* (**H4**) *hold. Then, RGBDSDEL* (1.3) *has a unique solution.*

Proof. **Existence.** In light of Proposition 3.2 and for $(\overline{Y}, \overline{Z}) \in \mathcal{S}^2(\mathbb{R}) \times \mathcal{M}^2(\mathbb{R}^m)$, let (Y, Z, K) be a unique solution of RGBDSDEL:

$$\begin{cases} Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T \phi(s, Y_s)dA_s + \int_t^T g(s, \overline{Y}_s, \overline{Z}_s)\,dB_s - \sum_{i=1}^m \int_t^T Z_s^{(i)} dH_s^{(i)} + K_T - K_t, \\ Y_t \geq S_t,\ \text{ a.s.}, \\ \int_0^T (Y_s - S_s)^- dK_s = 0. \end{cases}$$

We consider the mapping

$$\begin{array}{rccc} \Psi: & \mathcal{S}^2(\mathbb{R}) \times \mathcal{M}^2(\mathbb{R}^m) & \longrightarrow & \mathcal{S}^2(\mathbb{R}) \times \mathcal{M}^2(\mathbb{R}^m) \\ & (\overline{Y}, \overline{Z}) & \longmapsto & (Y, Z) = \Psi(\overline{Y}, \overline{Z}). \end{array}$$

Let (Y, Z), (Y', Z'), $(\overline{Y}, \overline{Z})$ and $(\overline{Y'}, \overline{Z'})$ in $\mathcal{S}^2(\mathbb{R}) \times \mathcal{M}^2(\mathbb{R}^m)$ such that $(Y, Z) = \Psi(\overline{Y}, \overline{Z})$ and $(Y', Z') = \Psi(\overline{Y'}, \overline{Z'})$. Putting $\Delta\eta = \eta - \eta'$ for any process η, we have

$$\begin{aligned} &\mathbb{E}e^{-\mu t}|\Delta Y_t|^2 + \mathbb{E}\int_t^T e^{-\mu s}\|\Delta Z_s\|^2 ds \\ &= 2\mathbb{E}\int_t^T e^{-\mu s}\Delta Y_s\left\{f(s, Y_{s^-}, Z_s) - f(s, {Y'}_{s^-}, {Z'}_s)\right\}ds + 2\mathbb{E}\int_t^T e^{-\mu s}\Delta Y_s\left\{\phi(s, Y_{s^-}) - \phi(s, {Y'}_{s^-})\right\}dA_s \\ &+ 2\mathbb{E}\int_t^T e^{-\mu s}\Delta Y_s d(\Delta K_s) + \int_t^T e^{-\mu s}\left|g(s, \overline{Y}_{s^-}, \overline{Z}_s) - g(s, \overline{Y'}_{s^-}, \overline{Z'}_s)\right|^2 ds - \mu\mathbb{E}\int_t^T e^{-\mu s}|\Delta Y_s|^2 ds. \end{aligned}$$

Since $\mathbb{E}\int_t^T e^{-\mu s}\Delta Y_s d(\Delta K_s) \leq 0$ and using (**H2**)-(**H3**), there exists constant $\alpha < \alpha' < 1$ such that

$$
\begin{aligned}
&(\mu-\gamma)\mathbb{E}\int_t^T e^{-\mu s}|\Delta Y_s|^2\,ds + \alpha'\mathbb{E}\int_t^T e^{-\mu s}\|\Delta Z_s\|^2 ds \\
&\leq c\mathbb{E}\int_t^T e^{-\mu s}|\Delta \bar{Y}_s|^2\,ds + \alpha\mathbb{E}\int_t^T e^{-\mu s}|\Delta \bar{Z}_s|^2\,ds,
\end{aligned}
$$

with $\gamma = \frac{c}{1-\alpha'} - 1 + \alpha$.

Choosing $\mu = \gamma + \alpha' c/\alpha$ and set $\bar{c} = \alpha' c/\alpha$, it follows from above that

$$
\begin{aligned}
&\bar{c}\mathbb{E}\int_0^T e^{-\mu s}|\Delta Y_s|^2\,ds + \alpha'\mathbb{E}\int_0^T e^{-\mu s}\|\Delta Z_s\|^2 ds \\
&\leq \frac{\alpha}{\alpha'}\left(\bar{c}\mathbb{E}\int_0^T e^{-\mu s}|\Delta \bar{Y}_s|^2\,ds + \alpha'\mathbb{E}\int_0^T e^{-\mu s}|\Delta \bar{Z}_s|^2\,ds\right).
\end{aligned}
$$

Therefore Ψ is a strict contraction on $\mathcal{S}^2(\mathbb{R}) \times \mathcal{M}^2(\mathbb{R}^m)$ equipped with the norm

$$
\|Y,Z)\|^2 = \bar{c}\mathbb{E}\int_0^T e^{-\mu s}|Y_s|^2\,ds + \alpha'\mathbb{E}\int_0^T e^{-\mu s}\|Z_s\|^2 ds
$$

such that its unique fixed point is the solution of RGBDSDEL (1.3).

Uniqueness. Assume $(Y_t, Z_t, K_t)_{0\leq t\leq T}$ and $(Y'_t, Z'_t, K'_t)_{0\leq t\leq T}$ are two solutions of the RGBDSDEL (ξ, f, g, ϕ, S). We set $\Delta Y_t = Y_t - Y'_t$, $\Delta Z_t = Z_t - Z'_t$ and $\Delta K_t = K_t - K'_t$.

Applying Itô's formula to $|\Delta Y|^2$ on the interval $[t,T]$ and taking expectation on both sides, we have

$$
\begin{aligned}
&\mathbb{E}\,|\Delta Y_t|^2 + \mathbb{E}\int_t^T \|\Delta Z_s\|^2 ds \\
\leq\quad &(4c^2 + c + \frac{1}{2})\mathbb{E}\int_t^T |\Delta Y_s|^2 ds + \frac{1}{2}\mathbb{E}\int_t^T \|\Delta Z_s\|^2 ds.
\end{aligned}
$$

Hence by Gronwall's inequality, we derive $\mathbb{E}|\Delta Y_t|^2 = 0$ i.e $Y_t = Y'_t$ a.s, for all $0 \leq t \leq T$. Therefore $Z_t = Z'_t$ and $K_t = K'_t$. $\square$

4 Connection to reflected stochastic PDIEs with nonlinear Neumann boundary condition

In this section, we aim to show that the adapted solution of RGBDSDEL is the solution of an obstacle problem for SPDIEs with a nonlinear Neumann boundary condition in the Markovian case under a regular assumptions on the coefficients.

We consider the Lévy process L with no Brownian part and bounded jump i.e $L_t = at + \int_{|z|\leq 1} z(N_t(.,dz) - t\nu(dz))$ where $N_t(\omega, dz)$ denotes the random measure such that $\int_\Lambda N_t(.,dz)$ is a Poisson process with parameter $\nu(\Lambda)$ for all set Λ $(0 \notin \Lambda)$. Without lost of generality, we suppose that $\sup_t |\Delta L_t| \leq 1$. Then, for all $p \geq 1$, $\mathbb{E}|L_t|^p < \infty$. For more detail see [21], Theorem 34, page 25.

4.1 A class of reflected diffusion process

We now introduce a class of reflected diffusion process. For $\theta > 0$, let $\Theta = (-\theta, \theta)$ and $e : [-\theta, \theta] \to \mathbb{R}$ such that $e(-\theta) = 1$ and $e(\theta) = -1$.
Let $\sigma : \mathbb{R} \to \mathbb{R}$ be a uniformly bounded function which saisfies:

(i) $|\sigma(x) - \sigma(x)| \leq K|x - x'|$ for every $x, x' \in \overline{\Theta}$,

(ii) $x + y\sigma(x)\mathbf{1}_{\{|y|\leq 1\}} \in \overline{\Theta}$ for every $x \in \overline{\Theta}$ and $y \in \mathbb{R}$,

(iii) $c(x) = c(pr(x))$ for all $x \in \mathbb{R}$, where $pr(.)$ denotes the orthogonal projection on $\overline{\Theta}$

As it shown in [13], for every $(t,x) \in \overline{\Theta}$, the process $(X^{t,x}, \eta^{t,x})$ is a unique solution of reflected SDE:

$$\begin{cases} \mathbb{P}(X_s^{t,x} \in \overline{\Theta}, s \geq t) = 1 \\ \\ X_s^{t,x} = x + \int_t^s \sigma(X_{r^-}^{t,x}) dL_r + \eta_s^{t,x},\ s \geq t, \end{cases} \tag{4.1}$$

with $\eta_s^{t,x} = \int_t^s e(X_r^{t,x}) d|\eta|_r$ with $|\eta^{t,x}|_s = \int_t^s \mathbf{1}_{\{X_r^{t,x} \in \partial\Theta\}} d|\eta|_r$.

For our next purpose, let recall this needed Lemma. We refer the reader to [15] for more detail.

Lemma 4.1. *let $c : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ be a measurable function such that*

$$|c(s,y)| \leq b_s(y^2 \wedge |y|) \ \ a.s.,$$

where $\{b_s, s \in [0,T]\}$ is a non-negative process such that $\mathbb{E}\int_0^T b_s^2 ds < \infty$. Then, for each $0 \leq t \leq T$, we have

$$\sum_{t \leq s \leq T} c(s, \Delta L_s) = \sum_{i=1}^{m} \int_t^T \langle c(s,.), p_i \rangle_{L^2(\nu)} dH_s^{(i)} + \int_t^T \int_{\mathbb{R}} c(s,y) d\nu(y) ds.$$

4.2 Feynman-Kac Formula

Fix $T > 0$ and for all $(t,x) \in [0,T] \times \overline{\Theta}$, let $(X_s^{t,x}, \eta_s^{t,x})_{s \geq t}$ denote the solution of the reflected SDE (4.1). And we suppose now that the data (ξ, f, ϕ, g, S) of the RGBDSDEL take the form

$$\begin{aligned} \xi &= l(X_T^{t,x}), \\ f(s,y,z) &= f(s, X_s^{t,x}, y, z), \\ \phi(s,y) &= \phi(s, X_s^{t,x}, y), \\ g(s,y,z) &= g(t, X_s^{t,x}, y, z), \\ S_t &= h(t, X_s^{t,x}). \end{aligned}$$

And we give the following assumptions:

Firstly, we assume that $h \in C^3([0,T] \times \overline{\Theta}; \mathbb{R})$ and $h(T,x) = l(x), \forall x \in \overline{\Theta},\ l \in C^3(\overline{\Theta}; \mathbb{R})$

Secondly, let $f \in C^3([0,T] \times \overline{\Theta} \times \mathbb{R} \times \mathbb{R}^m; \mathbb{R})$, $\phi \in C^3([0,T] \times \overline{\Theta} \times \mathbb{R}; \mathbb{R})$ and $g \in C^3([0,T] \times \overline{\Theta} \times \mathbb{R} \times \mathbb{R}^m; \mathbb{R})$ satisfy (**H2**) and (**H3**) respectively uniformly in $x \in \overline{\Theta}$.

It follows from the results of the Section 3 that, for all $(t,x) \in [0,T] \times \overline{\Theta}$ there exists a unique triple $(Y_s^{t,x}, Z_s^{t,x}, K_s^{t,x})_{t\leq s\leq T}$ for the solution of the following RGBDSDEL:

$$\begin{cases} (i)\ \mathbb{E}\left[\sup_{t\leq s\leq T}|Y_s^{t,x}|^2 + \int_t^T \|Z_s^{t,x}\|^2 ds\right]; \\ (ii)\ Y_s^{t,x} = l(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr + \int_s^T \phi(r, X_r^{t,x}, Y_r^{t,x})d|\eta^{t,x}|_r + \int_s^T g(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})\,dB_r \\ -\sum_{i=1}^m \int_s^T (Z^{t,x})_r^{(i)} dH_r^{(i)} + K_T^{t,x} - K_s^{t,x},\ t\leq s\leq T; \\ (iii)\ Y_s^{t,x} \geq h(s, X_s^{t,x}),\ \ t\leq s\leq T; \\ (iv)\ (K_s^{t,x})_{t\leq s\leq T} \text{ is increasing, continuous and satisfies } \int_t^T \left(Y_s^{t,x} - h(s,X_s^{t,x})\right) dK_s^{t,x} = 0. \end{cases} \tag{4.2}$$

We now consider the related obstacle problem for SPDIEs with a nonlinear Neumann boundary condition. Roughly speaking, a classic solution of the obstacle problem is a random field such that $u(t,x)$ is $\mathcal{F}_{t,T}$-measurable for each (t,x) and $u \in C^{1,2}([0,T]\times\overline{\Theta};\mathbb{R})$ which satisfies:

$$\begin{cases} \min\Big\{u(t,x) - h(t,x),\ \dfrac{\partial u}{\partial t}(t,x) + a'\sigma(x)\dfrac{\partial u}{\partial x}(t,x) + f(t,x,u(t,x),(u^{(i)}(t,x))_{i=1}^m) \\ \qquad + \int_{\mathbb{R}} u^1(t,x,y)d\nu(y) + g(t,x,u(t,x),(u^{(i)}(t,x))_{i=1}^m)\dot{B}_t\Big\} = 0,\ \ (t,x)\in[0,T]\times\Theta \\ e(x)\dfrac{\partial u}{\partial x}(t,x) + \phi(t,x,u(t,x)) = 0,\ \ (t,x)\in[0,T]\times\{-\theta,\theta\}, \\ u(T,x) = l(x),\quad x\in\Theta, \end{cases} \tag{4.3}$$

where

(i) $a' = a + \int_{\{|y|\geq 1\}} y\nu(dy)$,

(ii) $dB_t = \dot{B}_t$ denotes a white noise

(iii) $u^1(t,x,y) = u(t,x+y) - u(t,x) - \frac{\partial u}{\partial x}(t,x)y$,

(iv) $u^{(1)}(t,x) = \int_{\mathbb{R}} u^1(t,x,y)p_1(y)\nu(dy) + \sigma(x)\frac{\partial u}{\partial x}(t,x)(\int_{\mathbb{R}} y^2\nu(dy))^{1/2}$,

(v) $u^{(i)}(t,x) = \int_{\mathbb{R}} u^1(t,x,y)p_i(y)\nu(dy),\ 2\leq i\leq m.$

We have

Theorem 4.2. *Let u the classic solution of SPDIE* (4.3). *Then the unique adapted solution of (4.7) is given by*

$$\begin{aligned} u(s, X_s^{t,x}) &= Y_s^{t,x} \\ (Z^{t,x})_s^{(1)} &= \int_{\mathbb{R}} u^1(t, X_{s^-}^{t,x}, y)p_1(y)\nu(dy) + \frac{\partial u}{\partial x}\sigma(X_{s^-}^{t,x})\left(\int_{\mathbb{R}} y^2\nu(dy)\right)^{1/2} \\ (Z^{t,x})_s^{(i)} &= \int_{\mathbb{R}} u^1(t, X_{s^-}^{t,x}, y)p_i(y)\nu(dy),\ 2\leq i\leq m. \end{aligned}$$

In particular $u(t,x)=Y_t^{t,x}$

Proof. For each $n\geq 1$, let $\{{}^nY_s^{t,x},{}^nZ_s^{t,x},\ t\leq s\leq T\}$ denote the solution of the GBDSDEL

$$\begin{aligned}{}^nY_s^{t,x} &= l(X_T^{t,x})+\int_s^T f(r,X_{r^-}^{t,x},{}^nY_{r^-}^{t,x},{}^nZ_r^{t,x})dr+n\int_s^T({}^nY_{r^-}^{t,x}-h(r,X_r^{t,x}))^-dr\\ &\quad+\int_s^T\phi(r,X_{r^-}^{t,x},{}^nY_{r^-}^{t,x})d|\eta^{t,x}|_r\int_s^T g(r,X_{r^-}^{t,x},{}^nY_{r^-}^{t,x},{}^nZ_r^{t,x})dB_r-\sum_{i=1}^m\int_s^T{}^n(Z^{t,x})_r^{(i)}dH_r^{(i)}.\end{aligned}$$

As it shown in [10] we have

$$\begin{aligned}{}^nY_s^{t,x} &= u_n(s,X_s^{t,x}),\\ ({}^nZ^{t,x})_t^{(1)} &= \int_{\mathbb{R}}u_n^1(t,X_{t^-},y)p_1(y)\nu(dy)+\frac{\partial u_n}{\partial x}\sigma(X_{s^-}^{t,x})\left(\int_{\mathbb{R}}y^2\nu(dy)\right)^{1/2}\\ ({}^nZ^{t,x})_t^{(i)} &= \int_{\mathbb{R}}u_n^1(t,X_{s^-}^{t,x},y)p_i(y)\nu(dy),\ 2\leq i\leq m,\end{aligned}$$

where u_n is the classical solution of stochastic PDIE:

$$\begin{cases}\dfrac{\partial u_n}{\partial t}(t,x)+a'\sigma(x)\dfrac{\partial u_n}{\partial x}(t,x)+f_n(t,x,u_n(t,x),(u_n^{(i)}(t,x))_{i=1}^m)\\ \qquad+\int_{\mathbb{R}}u_n^1(t,x,y)d\nu(y)+g(t,x,u_n(t,x),(u_n^{(i)}(t,x))_{i=1}^m)\dot{B}_t=0,\ \ (t,x)\in[0,T]\times\Theta\\ e(x)\dfrac{\partial u_n}{\partial x}(t,x)+\phi(t,x,u_n(t,x))=0,\ \ (t,x)\in[0,T]\times\{-\theta,\theta\},\\ u_n(T,x)=l(x),\quad x\in\Theta,\end{cases}$$

where $f_n(t,x,y,z)=f(t,x,y,z)+n(y-h(t,x))^-$.

Applying Itô's formula to $u_n(s,X_s)$, we obtain

$$\begin{aligned}u_n(T,X_T^{t,x})-u_n(s,X_s^{t,x}) &= \int_s^T\frac{\partial u_n}{\partial r}(r,X_{r^-}^{t,x})dr+\int_s^T e(X_r^{t,x})\frac{\partial u_n}{\partial x}(r,X_r^{t,x})d|\eta|_r\\ &\quad+\int_s^T\sigma(X_{r^-})\frac{\partial u_n}{\partial x}(r,X_{r^-}^{t,x})dL_r\\ &\quad+\sum_{s\leq r\leq T}[u_n(r,X_r^{t,x})-u_n(r,X_{r^-}^{t,x})-\frac{\partial u_n}{\partial x}(r,X_{r^-}^{t,x})\Delta X_r^{t,x}].\end{aligned}\tag{4.4}$$

Since $\Delta X_r^{t,x}=\sigma(X_{r^-}^{t,x})\Delta L_r$, applying Lemma 4.1 with

$$c(r,y)=u_n(r,X_{r^-}^{t,x})+\sigma(X_{r^-}^{t,x})y)-u_n(r,X_{r^-}^{t,x})-\frac{\partial u_n}{\partial x}(r,X_{r^-}^{t,x})\sigma(X_{r^-}^{t,x})y,$$

we get

$$\begin{aligned}\sum_{s\leq r\leq T}[u_n(r,X_r^{t,x})-u_n(r,X_{r^-}^{t,x})-\frac{\partial u_n}{\partial x}(r,X_{r^-}^{t,x})\Delta X_r^{t,x}] &= \sum_{i=1}^m\int_s^T\left(\int_{\mathbb{R}}u_n^1(r,X_{r^-}^{t,x},y)p_i(y)\nu(dy)\right)dH_r^{(i)}\\ &\quad+\int_s^T\left(\int_{\mathbb{R}}u_n^1(r,X_{r^-}^{t,x},y)\nu(dy)\right)dr.\end{aligned}\tag{4.5}$$

Let us recall that

$$L_t = Y_t^{(1)} + t\mathbb{E}L_1 = \left(\int_{\mathbb{R}} y^2\nu(dy)\right)^{1/2} H^{(1)} + t\mathbb{E}L_1, \tag{4.6}$$

where $\mathbb{E}L_1 = a + \int_{\{|y|\geq 1\}} y\nu(dy)$. Hence, substituting (4.1), (4.5) and (4.6) into (4.4) together with (4.4) yields

$$\begin{aligned}
& l(X_T^{t,x}) - u_n(t, X_s^{t,x}) \\
= \;& \int_s^T \left[\frac{\partial u_n}{\partial s}(r, X_{r^-}^{t,x}) + (a + \int_{|y|\geq 1} y\nu(dy))\sigma(X_{r^-})\frac{\partial u_n}{\partial x}(r, X_{r^-}^{t,x}) + \int_{\mathbb{R}} u_n^1(r, X_{r^-}^{t,x}, y)\nu(dy)\right] dr \\
& + \int_s^T e(X_r)\frac{\partial u_n}{\partial x}(r, X_r^{t,x})\mathbf{1}_{\{X_r^{t,x}\in\partial\Theta\}} d|\eta^{t,x}|_r \\
& + \int_s^T \left[\int_{\mathbb{R}} u_n^1(r, X_{r^-}^{t,x}, y)p_1(y)\nu(dy) + \sigma(X_{r^-}^{t,x})\frac{\partial u_n}{\partial x}(r, X_{r^-}^{t,x})\left(\int_{\mathbb{R}} y^2\nu(dy)\right)^{1/2}\right] dH_r^{(1)} \\
& + \sum_{i=2}^m \int_s^T \left(\int_{\mathbb{R}} u_n^1(r, X_{r^-}^{t,x}, y)p_i(y)\nu(dy)\right) dH_r^{(i)}. \\
= \;& -\int_s^T f(r, X_{r^-}^{t,x}, u_n(r, X_r^{t,x}), (u_n(r, X_r^{t,x}))_{i=1}^m)dr - n\int_s^T (u_n(r, X_r^{t,x}) - h(r, X_r^{t,x}))^- dr \\
& - \int_s^T g(r, X_{r^-}^{t,x}, u_n(r, X_r^{t,x}), (u_n^{(i)}(t,x))_{i=1}^m)dB_r - \int_s^T \phi(r, X_{r^-}^{t,x}, u_n(r, X_r^{t,x}))d|\eta|_s \\
& + \int_s^T \left[\int_{\mathbb{R}} u_n^1(r, X_{r^-}^{t,x}, y)p_1(y)\nu(dy) + \sigma(X_{r^-}^{t,x})\frac{\partial u_n}{\partial x}(r, X_{r^-}^{t,x})\left(\int_{\mathbb{R}} y^2\nu(dy)\right)^{1/2}\right] dH_r^{(1)} \\
& + \sum_{i=2}^m \int_t^T \left(\int_{\mathbb{R}} u_n^1(r, X_{r^-}^{t,x}, y)p_i(y)\nu(dy)\right) dH_s^{(i)}.
\end{aligned}$$

Passing to the limit and using the previous section we get

$$\begin{aligned}
& u(t, X_s^{t,x}) - l(X_T^{t,x}) \\
= \;& \int_s^T f(r, X_{r^-}^{t,x}, u(r, X_r^{t,x}), (u(r, X_r^{t,x}))_{i=1}^m)dr \\
& + \int_s^T g(r, X_{r^-}^{t,x}, u(r, X_r^{t,x}), (u(r, X_r^{t,x}))_{i=1}^m)dB_r + K_T - K_t + \int_s^T \phi(r, X_{r^-}^{t,x}, u(r, X_r^{t,x}))d|\eta|_s \\
& - \int_s^T \left[\int_{\mathbb{R}} u^1(r, X_{r^-}^{t,x}, y)p_1(y)\nu(dy) + \sigma(X_{r^-}^{t,x})\frac{\partial u}{\partial x}(r, X_{r^-}^{t,x})\left(\int_{\mathbb{R}} y^2\nu(dy)\right)^{1/2}\right] dH_r^{(1)} \\
& - \sum_{i=2}^m \int_t^T \left(\int_{\mathbb{R}} u^1(r, X_{r^-}^{t,x}, y)p_i(y)\nu(dy)\right) dH_s^{(i)},
\end{aligned}$$

which get the desired result of the Theorem. □

In this follows, we give a example of reflected SPDIEs with a nonlinear Neumann boundary condition.

Example 4.3. We consider the very special case where Lévy process L is defined by $L_t = at + N_t - \alpha t$, where N is Poisson processes with parameters $\alpha > 0$. Then we have $H_t^{(1)} = \frac{\beta}{\sqrt{\alpha}}(N_t - \alpha t)$ and $H_t^{(i)} = 0,\ i \geq 2$ (see [15]). Moreover the reflected SPDIE (4.4) reduces to

$$\begin{cases} \min\Big\{u(t,x) - h(t,x),\ \dfrac{\partial u}{\partial t}(t,x) + d'\sigma(x)\dfrac{\partial u}{\partial x}(t,x) + f(t,x,u(t,x),\dfrac{\partial u}{\partial x}(t,x)) \\ \qquad +\alpha u^1(t,x,\beta) + g(t,x,u(t,x),\frac{\partial u}{\partial x}(t,x))\dot{B}_t\Big\} = 0, \quad (t,x) \in [0,T] \times \Theta \\ e(x)\dfrac{\partial u}{\partial x}(t,x) + \phi(t,x,u(t,x)) = 0, \quad (t,x) \in [0,T] \times \{-\theta, \theta\}, \\ u(T,x) = l(x), \quad x \in \Theta. \end{cases}$$

Then,

$$\begin{aligned} Y_s^{t,x} &= u(s, X_s^{t,x}), \\ Z_s^{t,x} &= \alpha u^1(t, X_{s^-}^{t,x}, \beta) p_1(\beta) + \sqrt{\alpha}|\beta|\sigma(X_{s^-}^{t,x})\frac{\partial u}{\partial x}(s, X_{s^-}^{t,x}), \end{aligned}$$

where for each $(t,x) \in [0,T] \times \overline{\Theta}$, $(Y^{t,x}, Z^{t,x}, K^{t,x})$ is the unique solution of the following

$$\begin{cases} (i)\ Y_s^{t,x} = l(X_T^{t,x}) + \int_s^T f(r, X_{r^-}^{t,x}, Y_{r^-}^{t,x}, Z_r^{t,x})dr + \int_s^T \phi(r, X_{r^-}^{t,x}, Y_{r^-}^{t,x})d|\eta^{t,x}|_r + \int_s^T g(r, X_{r^-}^{t,x}, Y_{r^-}^{t,x}, Z_r^{t,x})\, dB_r \\ -\int_s^T (Z^{t,x})_s d\widetilde{N}_r + K_T^{t,x} - K_s^{t,x},\ t \leq s \leq T \\ (ii)\ Y_s^{t,x} \geq h(s, X_s^{t,x}), \quad t \leq s \leq T \\ (iii)\ \ (K_s^{t,x})_{t \leq s \leq T} \text{ is increasing, continuous and satisfies } \int_t^T \left(Y_s^{t,x} - h(s, X_s^{t,x})\right) K_s^{t,x} = 0. \end{cases}$$

with $\widetilde{N}_t = \frac{\beta}{\sqrt{\alpha}}(N_t - \alpha t)$.

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