arXiv:1011.3030v1 [math.PR] 12 Nov 2010

L^p-solutions of backward doubly stochastic differential equations

Auguste Aman*
U.F.R Mathématiques et informatique, Université de Cocody,
582 Abidjan 22, Côte d'Ivoire

Abstract

The goal of this paper is to solve backward doubly stochastic differential equation (BDSDE, in short) under weak assumptions on the data. The first part is devoted to the development of some new technical aspects of stochastic calculus related to BDSDEs. Then we derive a priori estimates and prove existence and uniqueness of solutions in L^p, $p \in (1,2)$, extending the work of pardoux and Peng (see Probab. Theory Related Fields 98 (1994), no. 2).

MSC 2000: 60H05, 60H20.
Key Words: Backward doubly stochastic differential equations; monotone generator; p-integrable data.

1 Introduction

In this paper, we are concerned with backward doubly stochastic differential equations (BDSDEs, in short):

$$Y_t = \xi + \int_t^T f(r,Y_r,Z_r)dr + \int_t^T g(r,Y_r,Z_r)\overleftarrow{dB}_r - \int_t^T Z_r dW_r,\ 0 \leq t \leq T, \quad (1.1)$$

which involves both a standard (forward) stochastic Itô integral driven by dW_t and a backward stochastic itô integral driven by $\overleftarrow{dB}_t$. The random variable ξ and functions f and g are data, while the pair of processes $(Y_t,Z_t)_{t\in[0,T]}$ is the unknowns.

The theory of nonlinear backward doubly SDE have been firstly introduced in [8] by Pardoux and Peng . Among other they proved existence and uniqueness result under Lipschitz continuous and square integrable assumptions on the data. They also showed that in the markovian framework, BDSDEs give the representation to quasi-linear stochastic partial differential equations (SPDEs). Indeed, under stonger conditions (f, g are C^3) they proved that $u(t,x) = Y_t^{t,x}$ is classical solution of the SPDE (f,g). This generalize the well-know Feymann-Kac formula to SPDEs. Since this first existence and uniqueness result, many other works have been devoted to existence and/or uniqueness results for BDSDEs under weaker assumptions. For scalar BDSDEs case, N'zi and Owo [5] deal with discontinuous

*augusteaman5@yahoo.fr

coefficients by using the comparison theorem establish in [10]. There is no comparison theorem for multidimensional BDSDEs. To overcome this difficulty, a monotonicity assumption on the generator f with respect y uniformly on z is used. This condition appear in the paper by Peng and Shi [9] and N'zi and Owo [6].

However, in all the above works the data are supposed to be at least square integrable. This condition is too restrictive to be assumed in many applications. For example, the pricing problem of an American claim is equivalent to solving the linear BDSDE

$$-dY_t = (r_tY_t + \theta_tZ_t)dt + c_tY_t\overleftarrow{dB}_t - Z_tdW_t,\ \ Y_T = \xi, \tag{1.2}$$

where r_t is the interest rate, θ_t is the risk premium vector and c_t is the market exterior volatility factor. In general all of this coefficients are unbounded and the terminal condition ξ is only integrable. Consequently the result of Pardoux and Peng in [8] and all above paper may be invalid.

The aim of this present paper is to correct this gap and prove existence and uniqueness result for BDSDEs in $\mathbb{R}^k$ when $\xi,\ f(t,0,0)$ and $g(t,0,0)$ are p-integrable, $p\in(1,2)$, with f only monotone. To our knowledge, this result do not exists in literature, therefore it is new.

The paper is organized as follows. In Section 2, we give all notations and basic identities of this paper. The Section 3 contains essential a priori estimates. In Section 4, we prove existence and uniqueness result.

2 Preliminaries

2.1 Assumptions and basic notations

Let $\mathbb{R}^{k\times d}$ be identified to the space of real matrices with k rows and d columns; hence for each $z\in\mathbb{R}^{k\times d},\ |z|^2 = trace(zz^*)$.

In throughout this paper, we consider the probability space $(\Omega,\mathcal{F},\mathbb{P})$ and a real positive constant T. We define on $(\Omega,\mathcal{F},\mathbb{P})$ two mutually independent standard Brownian motion processes $\{W_t, 0\le t\le T\}$ and $\{B_t, 0\le t\le T\}$ taking values in $\mathbb{R}^d$ and $\mathbb{R}^\ell$ respectively. Let $\mathcal{N}$ denote the class of $\mathbb{P}$-null sets of $\mathcal{F}$ and set

$$\mathcal{F}_t = \mathcal{F}_t^{B}\otimes\mathcal{F}_{t,T}^{W}\vee\mathcal{N},\ \ 0\le t\le T$$

defined by $\mathcal{F}_{s,t}^{\eta} = \sigma\{\eta_r - \eta_s, s\le r\le t\}$ for any η_t, and $\mathcal{F}_t^{\eta} = \mathcal{F}_{0,t}^{\eta}$.

We emphasize that the collection $\{\mathcal{F}_t, t\in[0,T]\}$ is not filtration. Indeed, it is neither increasing nor decreasing. For any real $p>0$, we also denote by $\mathcal{S}^p(\mathbb{R}^n)$ the set of jointly measurable processes $\{X_t\}_{t\in[0,T]}$ taking values in $\mathbb{R}^n$ such that

(i)

$$\|X\|_{\mathcal{S}^p} = \mathbb{E}\left(\sup_{0\le t\le T}|X_t|^p\right)^{1\wedge\frac{1}{p}} < +\infty;$$

(ii) X_t is $\mathcal{F}_t$-measurable, for any $t\in[0,T]$.

and $\mathcal{M}^p(\mathbb{R}^n)$) the set of (classes of $d\mathbb{P}\times dt$ a.e. equal) n-dimensional jointly measurable processes such that

(i)

$$\|X\|_{\mathcal{M}^p} = \mathbb{E}\left[\left(\int_0^T |X_t|^2 dt\right)^{\frac{p}{2}}\right]^{1\wedge\frac{1}{p}} < +\infty.$$

(ii) X_t is $\mathcal{F}_t$-measurable, for a.e. $t \in [0,T]$.

If $p \geq 1$, $(\mathcal{S}^p(\mathbb{R}^n), \|X\|_{\mathcal{S}^p})$ and $(\mathcal{M}^p(\mathbb{R}^n), \|X\|_{\mathcal{M}^p})$ are Banach spaces.

Let

$$f : \Omega \times [0,T] \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \to \mathbb{R}^k;\ g : \Omega \times [0,T] \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \to \mathbb{R}^{k\times \ell}$$

be jointly measurable such that for any $(y,z) \in \mathbb{R}^k \times \mathbb{R}^{k\times d}$. We have

(H1) $f(.,y,z) \in \mathcal{M}^p(0,T,\mathbb{R}^k)$, $g(.,y,z) \in \mathcal{M}^p(0,T,\mathbb{R}^{k\times\ell})$

(**H2**) There exist constants $\mu \in \mathbb{R}$, $\lambda > 0$ and $0 < \alpha < 1$ such that for any $t \in [0,T]$; $(y_1,z_1),(y_2,z_2) \in \mathbb{R}^k \times \mathbb{R}^{k\times d}$,

$$\begin{cases} (i)\ |f(t,y_1,z_1) - f(t,y_1,z_2)| \leq \lambda|z_1 - z_2|, \\ \\ (ii)\ \langle y_1 - y_2, f(t,y_1,z_1) - f(t,y_2,z_1)\rangle \leq \mu|y_1 - y_2|^2, \\ \\ (iii)\ |g(t,y_1,z_1) - g(t,y_2,z_2)|^2 \leq \lambda|y_1 - y_2|^2 + \alpha|z_1 - z_2|^2. \end{cases}$$

Given a $\mathbb{R}^k$-valued $\mathcal{F}_T$-measurable random vector ξ, we consider the backward doubly stochastic differential equation:

$$Y_t = \xi + \int_t^T f(s,Y_s,Z_s)ds + \int_t^T g(s,Y_s,Z_s)\overleftarrow{dB}_s - \int_t^T Z_s\, dW_s,\ 0 \leq t \leq T. \tag{2.1}$$

Now we recall what we mean by a solution to the BDSDE (2.1).

Definition 2.1. A solution of BDSDE (2.1) is a pair $(Y_t,Z_t)_{0\leq t\leq T}$ of jointly measurable processes taking values in $\mathbb{R}^k \times \mathbb{R}^{k\times d}$ and satisfying (2.1) such that: $\mathbb{P}$ a.s., $t \mapsto (Z_t, g(t,Y_t,Z_t))$ belongs in $L^2(0,T)$, $t \mapsto f(t,Y_t,Z_t)$ belongs in $L^1(0,T)$.

2.2 Generalized Tanaka formula

As explained in the introduction, we want to deal with BDSDEs with data in L^p, $p \in (1,2)$ like the works of Pardoux et al. (see [3]) which treat BSDEs case i.e $g \equiv 0$. We start by Tanaka formula relative to BDSDEs, which is the critical tool in this paper. For this, we note $\hat{x} = |x|^{-1}x\mathbf{1}_{\{x\neq 0\}}$.

Lemma 2.2. *Let $\{K_t\}_{t\in[0,T]}$, $\{H_t\}_{t\in[0,T]}$ and $\{G_t\}_{t\in[0,T]}$ be jointly measurable such that $K \in \mathcal{M}^p(0,T,\mathbb{R}^k)$, $H \in \mathcal{M}^p(0,T,\mathbb{R}^{k\times d})$, $G \in \mathcal{M}^p(0,T,\mathbb{R}^{k\times\ell})$. We consider the $\mathbb{R}^k$-valued semi martingale $\{X_t\}_{t\in[0,T]}$ defined by*

$$X_t = X_0 + \int_0^t K_s\, ds + \int_0^t G_s\, \overleftarrow{dB}_s + \int_0^t H_s\, dW_s,\ 0 \leq t \leq T. \tag{2.2}$$

Then, for any $p \geq 1$, we have

$$\begin{aligned}
|X_t|^p - \mathbf{1}_{\{p=1\}} L_t &= |X_0|^p + p\int_0^t |X_s|^{p-1}\langle \hat{X}_s, K_s\rangle ds \\
&\quad + p\int_0^t |X_s|^{p-1}\langle \hat{X}_s, G_s \overleftarrow{dB}_s\rangle + p\int_0^t |X_s|^{p-1}\langle \hat{X}_s, H_s dW_s\rangle \\
&\quad - \frac{p}{2}\int_0^t |X_s|^{p-2}\mathbf{1}_{\{X_s\neq 0\}}\{(2-p)(|G_s|^2 - \langle \hat{X}_s, G_s G_s^* \hat{X}_s\rangle) + (p-1)|G_s|^2\} ds \\
&\quad + \frac{p}{2}\int_0^t |X_s|^{p-2}\mathbf{1}_{\{X_s\neq 0\}}\{(2-p)(|H_s|^2 - \langle \hat{X}_s, H_s H_s^* \hat{X}_s\rangle) + (p-1)|H_s|^2\} ds,
\end{aligned}$$

where $\{L_t\}_{t\in[0,T]}$ is a continuous process with $L_0 = 0$, which varies only on the boundary of the random set $\{t \in [0,T], X_t = 0\}$.

Proof. Since the function $x \mapsto |x|^p$ is not smooth enough, for $p \in (1,2)$, we approximate it by the function $u_\varepsilon(x) = (|x|^2 + \varepsilon^2)^{1/2}, \ \forall\ \varepsilon > 0$. The function u^ε is actually smooth and setting by I the identity matrix of $\mathbb{R}^k$, we have

$$\nabla u_\varepsilon^p(x) = p u_\varepsilon^{p-2}(x) x; \quad D^2 u_\varepsilon^p(x) = p u_\varepsilon^{p-2}(x) I + p(p-2) u_\varepsilon^{p-4}(x)(x \otimes x).$$

Therefore we get by Itô's formula the equality

$$\begin{aligned}
u_\varepsilon^p(X_t) &= u_\varepsilon^p(X_0) + p\int_0^t u_\varepsilon^{p-2}(X_s)\langle X_s, K_s\rangle ds \\
&\quad + p\int_0^t u_\varepsilon^{p-2}(X_s)\langle X_s, G_s \overleftarrow{dB}_s\rangle + p\int_0^t u_\varepsilon^{p-2}(X_s)\langle X_s, H_s dW_s\rangle \\
&\quad - \frac{1}{2}\int_0^t trace(D^2 u_\varepsilon^p(X_s) G_s G_s^*) ds + \frac{1}{2}\int_0^t trace(D^2 u_\varepsilon^p(X_s) H_s H_s^*) ds. \quad (2.3)
\end{aligned}$$

The rest of this proof is essentially to pass to the limit when $\varepsilon \to 0$ in (2.3). To do this, we remark first that

$$\int_0^t u_\varepsilon^{p-2}(X_s)\langle X_s, K_s\rangle ds \to \int_0^t |X_s|^{p-1}\langle \widehat{X}_s, K_s\rangle ds, \ \mathbb{P}\text{-a.s.}$$

We also have

$$\int_0^t u_\varepsilon^{p-2}(X_s)\langle X_s, G_s \overleftarrow{dB}_s\rangle \to \int_0^t |X_s|^{p-1}\langle \widehat{X}_s, G_s \overleftarrow{dB}_s\rangle$$

and

$$\int_0^t u_\varepsilon^{p-2}(X_s)\langle X_s, H_s dW_s\rangle \to \int_0^t |X_s|^{p-1}\langle \widehat{X}_s, H_s dW_s\rangle;$$

in $\mathbb{P}$-probability uniformly on $[0,T]$. The convergence of the stochastic integrals follows from the following convergence:

$$\int_0^T |X_s|^2 \mathbf{1}_{\{X_s\neq 0\}} |G_s|^2 (|X_s|^{p-2} - u_\varepsilon^{p-2}(X_s))^2 ds \to 0$$

and

$$\int_0^T |X_s|^2 \mathbf{1}_{\{X_s\neq 0\}} |H_s|^2 (|X_s|^{p-2} - u_\varepsilon^{p-2}(X_s))^2 ds \to 0,$$

which is provided by the dominated convergence theorem.

It remains to study the convergence of the term including the second derivative of u_ε. It is shown in [3] that

$$\begin{aligned} trace(D^2 u_\varepsilon^p(X_s) G_s G_s^*) &= p(2-p)(|X_s| u_\varepsilon^{-1}(X_s))^{4-p} |X_s|^{p-2} \mathbf{1}_{\{X_s\neq 0\}} (|G_s|^2 - \langle \widehat{X}_s, G_s G_s^* \widehat{X}_s\rangle) \\ &\quad + p(p-1)(|X_s| u_\varepsilon^{-1}(X_s))^{4-p} |X_s|^{p-2} \mathbf{1}_{\{X_s\neq 0\}} |G_s|^2 + p\varepsilon^2 |G_s|^2 u_\varepsilon^{p-4}(X_s) \end{aligned}$$

and

$$\begin{aligned} trace(D^2 u_\varepsilon^p(X_s) H_s H_s^*) &= p(2-p)(|X_s| u_\varepsilon^{-1}(X_s))^{4-p} |X_s|^{p-2} \mathbf{1}_{\{X_s\neq 0\}} (|H_s|^2 - \langle \widehat{X}_s, H_s H_s^* \widehat{X}_s\rangle) \\ &\quad + p(p-1)(|X_s| u_\varepsilon^{-1}(X_s))^{4-p} |X_s|^{p-2} \mathbf{1}_{\{X_s\neq 0\}} |H_s|^2 + p\varepsilon^2 |H_s|^2 u_\varepsilon^{p-4}(X_s). \end{aligned}$$

One has also

$$\begin{aligned} |G_s|^2 &\geq \langle \widehat{X}_s, G_s G_s^* \widehat{X}_s\rangle \\ |H_s|^2 &\geq \langle \widehat{X}_s, H_s H_s^* \widehat{X}_s\rangle \end{aligned} \tag{2.4}$$

and

$$\frac{|X_s|}{u_\varepsilon(X_s)} \nearrow \mathbf{1}_{\{X_s\neq 0\}}$$

as $\varepsilon \to 0$. Hence by monotone convergence, as $\varepsilon \to 0$,

$$\int_0^t (|X_s| u_\varepsilon^{-1}(X_s))^{4-p} |X_s|^{p-2} \mathbf{1}_{\{X_s\neq 0\}} \{(2-p)(|G_s|^2 - \langle \widehat{X}_s, G_s G_s^* \widehat{X}_s\rangle) + (p-1)|G_s|^2\} ds$$

converge to

$$\int_0^t |X_s|^{p-2} \mathbf{1}_{\{X_s\neq 0\}} \{(2-p)(|G_s|^2 - \langle \widehat{X}_s, G_s G_s^* \widehat{X}_s\rangle) + (p-1)|G_s|^2\} ds$$

and

$$\int_0^t (|X_s| u_\varepsilon^{-1}(X_s))^{4-p} |X_s|^{p-2} \mathbf{1}_{\{X_s\neq 0\}} \{(2-p)(|H_s|^2 - \langle \widehat{X}_s, H_s H_s^* \widehat{X}_s\rangle) + (p-1)|H_s|^2\} ds$$

converge to

$$\int_0^t |X_s|^{p-2} \mathbf{1}_{\{X_s\neq 0\}} \{(2-p)(|H_s|^2 - \langle \widehat{X}_s, H_s H_s^* \widehat{X}_s\rangle) + (p-1)|H_s|^2\} ds,$$

$\mathbb{P}$-a.s., for all $0 \leq t \leq T$.

Let denote

$$L_t^\varepsilon(p) = \int_0^t C_s^\varepsilon(p) ds,$$

where $C^\varepsilon_s(p) = \frac{p}{2}\varepsilon^2 u_\varepsilon^{p-4}(X_s)(|H_s|^2 - |G_s|^2)$. Then it follows from (2.3) that $L^\varepsilon(p)$ converges to a continuous process $L(p)$ as $\varepsilon \to 0$; moreover, $L(p) \equiv 0$ for $p > 1$. Indeed, for $p \geq 4$, $L(p) \equiv 0$ since $C^\varepsilon_s(p)$ converges to 0 in $L^1(0,T)$. Next, if $p \in (1,4)$, by setting $\theta = (4-p)/3 \in (0,1)$ we get

$$L^\varepsilon_t = \frac{p}{2}\int_0^t \left(\varepsilon^2(|H_s|^2 - |G_s|^2)u_\varepsilon^{-3}(X_s)\right)^\theta \left(\varepsilon^2(|H_s|^2 - |G_s|^2)\right)^{1-\theta} ds.$$

Hence Hölder's inequality provide that

$$L^\varepsilon_t(p) \leq p(L^\varepsilon_t(1))^\theta \left(\int_0^T \varepsilon^2(|H_s|^2 - |G_s|^2)ds\right)^{1-\theta}$$

which tends to 0 as $\varepsilon \to 0$ for each $t \in [0,T]$. For $p = 1$, let set $L(1) = L$ and remark that L_t can be decomposed in two continuous and increasing functions L^1_t and L^2_t, which are limit of processes $\frac{1}{2}\int_0^t \varepsilon^2|H_s|^2 u_\varepsilon^{-3}(X_s)$ and $\frac{1}{2}\int_0^t \varepsilon^2|G_s|^2 u_\varepsilon^{-3}(X_s)$ respectively. As it is shown in [3], L^1_t and L^2_t increase only on the boundary of the random set $\{t \in [0,T], X_t = 0\}$. Therefore L_t varies only in this case. □

Remark 2.3. Since the process L_t is neither increasing nor decreasing, we can not apply the similarly argument used in [3]. Therefore the following corollary works only in the case $p \in (1,2)$, which correspond to our framework.

Corollary 2.4. *Let $p \in (1,2)$ and denote $c(p) = p(p-1)/2$ and $\bar{c}(p) = p(3-p)/2$. If (Y,Z) is a solution of the BDSDE* (2.1)*, then for* $0 \leq t \leq T$

$$\begin{aligned} & |Y_t|^p + c(p)\int_t^T |Y_s|^{p-2}\mathbf{1}_{\{Y_s\neq 0\}}|Z_s|^2 ds \\ \leq\ & |Y_T|^p + p\int_t^T |Y_s|^{p-1}\langle \hat{Y}_s, f(s,Y_s,Z_s)\rangle ds \\ & +\bar{c}(p)\int_t^T |Y_s|^{p-2}\mathbf{1}_{\{Y_s\neq 0\}}|g(s,Y_s,Z_s)|^2 ds \\ & +p\int_t^T |Y_s|^{p-1}\langle \hat{Y}_s, g(s,Y_s,Z_s)\overleftarrow{dB}_s\rangle - p\int_0^t |Y_s|^{p-1}\langle \hat{Y}_s, Z_s dW_s\rangle. \end{aligned}$$

Proof. The proof follows from Lemma 2.2. Indeed, recall that (Y,Z) is solution of BDSDE (2.1) and replace (X,K,H,G) by $(Y, f(.,Y,Z), Z, g(.,Y,Z))$, it follows that

$$\begin{aligned} & |Y_t|^p + \frac{p}{2}\int_t^T |Y_s|^{p-2}\mathbf{1}_{\{Y_s\neq 0\}}\{(2-p)(|Z_s|^2 - \langle \hat{Y}_s, Z_s Z_s^* \hat{Y}_s\rangle) + (p-1)|Z_s|^2\}ds \\ =\ & |Y_T|^p + p\int_t^T |Y_s|^{p-1}\langle \hat{Y}_s, f(s,Y_s,Z_s)\rangle ds + p\int_t^T |Y_s|^{p-1}\langle \hat{Y}_s, g(s,Y_s,Z_s)\overleftarrow{dB}_s\rangle - p\int_t^T |Y_s|^{p-1}\langle \hat{Y}_s, Z_s dW_s\rangle \\ & +\frac{p}{2}\int_t^T |Y_s|^{p-2}\mathbf{1}_{\{Y_s\neq 0\}}\{(2-p)(|g(s,Y_s,Z_s)|^2 - \langle \hat{Y}_s, g(s,Y_s,Z_s)g^*(s,Y_s,Z_s)\hat{Y}_s\rangle) + (p-1)|g(s,Y_s,Z_s)|^2\}ds. \end{aligned} \tag{2.5}$$

Since $p \in (1,2)$, it follows from (2.4) that

$$\begin{aligned} & (p-1)\int_t^T |Y_s|^{p-2}\mathbf{1}_{\{Y_s\neq 0\}}|Z_s|^2 ds \\ \leq\ & \int_t^T |Y_s|^{p-2}\mathbf{1}_{\{Y_s\neq 0\}}\{(2-p)(|Z_s|^2 - \langle \hat{Y}_s, Z_s Z_s^* \hat{Y}_s\rangle) + (p-1)|Z_s|^2\}ds. \end{aligned} \tag{2.6}$$

and

$$\begin{aligned}
&\int_t^T |Y_s|^{p-2}\mathbf{1}_{\{Y_s\neq 0\}}\{(2-p)(|g(s,Y_s,Z_s)|^2-\langle \hat{Y}_s, g(s,Y_s,Z_s)g^*(s,Y_s,Z_s)\hat{Y}_s\rangle)+(p-1)|g(s,Y_s,Z_s)|^2\}ds\\
\leq\ & (3-p)\int_t^T |Y_s|^{p-2}\mathbf{1}_{\{Y_s\neq 0\}}|g(s,Y_s,Z_s)|^2ds. \qquad (2.7)
\end{aligned}$$

Therefore putting (2.6) and (2.7) to (2.5) we obtain

$$\begin{aligned}
&|Y_t|^p + c(p)\int_t^T |Y_s|^{p-2}\mathbf{1}_{\{Y_s\neq 0\}}|Z_s|^2ds\\
\leq\ & |Y_T|^p + p\int_t^T |Y_s|^{p-1}\langle \hat{Y}_s, f(s,Y_s,Z_s)\rangle ds + p\int_t^T |Y_s|^{p-1}\langle \hat{Y}_s, g(s,Y_s,Z_s)\overleftarrow{dB}_s\rangle - p\int_t^T |Y_s|^{p-1}\langle \hat{Y}_s, Z_s dW_s\rangle\\
&+\bar{c}(p)\int_t^T |Y_s|^{p-2}\mathbf{1}_{\{Y_s\neq 0\}}|g(s,Y_s,Z_s)|^2ds.
\end{aligned}$$

which proved the result. □

3 A priori estimates

In this section, we state some estimation concerning solution to BDSDE (2.1). These estimates are very useful for the study of existence and uniqueness of solutions. In what follows, we are two difficulty. The function f is not Lipschitz continuous and we desire estimate in L^p-sense, $p \in (1,2)$.

We begin by derive the following result which permit us to control the process Z by the data and the process Y.

Lemma 3.1. *Let assumptions* (**H1**)-(**H2**) *hold and* (Y,Z) *be a solution of BDSDE* (2.1). *If* $Y \in \mathcal{S}^p$ *then* Z *belong to* $\mathcal{M}^p$ *and there exists a real constant* $C_{p,\lambda}$ *depending only on* p, T *and* λ *such that*

$$\mathbb{E}\left[\left(\int_0^T |Z_r|^2dr\right)^{p/2}\right] \leq C_p\mathbb{E}\left\{\sup_{0\leq t\leq T}|Y_t|^p + \left(\int_0^T |f_r^0|dr\right)^p + \left(\int_0^T |g_r^0|^2dr\right)^{p/2}\right\}.$$

Proof. For each integer n, let us define

$$\tau_n = \inf\left\{t\in[0,T], \int_0^t |Z_r|^2dr \geq n\right\}\wedge T.$$

The sequence $(\tau_n)_{n\geq 0}$ is stationary since the process Z belongs to $L^2(0,T)$ and then $\int_0^T |Z_s|^2ds < \infty$, $\mathbb{P}$- a.s.

For arbitrary real a, using Itô's formula, we have

$$\begin{aligned}
&|Y_0|^2 + \int_0^{\tau_n} e^{ar}|Z_r|^2dr\\
=\ & e^{a\tau_n}|Y_{\tau_n}|^2 + 2\int_0^{\tau_n} e^{ar}\langle Y_r, f(r,Y_r,Z_r)-aY_r\rangle dr + \int_0^{\tau_n} e^{ar}|g(r,Y_r,Z_r)|^2dr\\
&+2\int_0^{\tau_n} e^{ar}\langle Y_r, g(r,Y_r,Z_r)\overleftarrow{dB}_r\rangle - 2\int_0^{\tau_n} e^{ar}\langle Y_r, Z_r dW_r\rangle. \qquad (3.1)
\end{aligned}$$

But, it follows from assumptions (**H1**)-(**H2**) and inequality $2bd \leq \frac{1}{\varepsilon}b^2 + \varepsilon d^2$ that, for any arbitrary positive real constant ε and ε',

$$\begin{aligned}
2\langle Y_r, f(r,Y_r,Z_r) - aY_r\rangle &\leq 2|Y_r||f_r^0| + 2\mu|Y_r|^2 + 2\lambda|Y_r||Z_r| - a|Y_r|^2 \\
&\leq 2|Y_r||f_r^0| + (2\mu + 2\lambda + \varepsilon^{-1}\lambda^2 - a)|Y_r|^2 + \varepsilon|Z_r|^2, \\
\|g(r,Y_r,Y_r)\|^2 &\leq (1+\varepsilon')\lambda|Y_r|^2 + (1+\varepsilon')\alpha|Z_r|^2 + (1+\frac{1}{\varepsilon'})|g_r^0|^2.
\end{aligned}$$

Thus, since $\tau_n \leq T$, taking ε, ε' such that $\varepsilon + (1+\varepsilon')\alpha < 1$ and $2\mu + (3+\varepsilon')\lambda + \varepsilon^{-1}\lambda^2 - a \leq 0$, we deduce

$$\begin{aligned}
\left(\int_0^{\tau_n} |Z_r|^2 dr\right)^{p/2} &\leq C_{p,\lambda}\left\{\sup_{0\leq t\leq \tau_n}|Y_t|^p + \left(\int_0^{\tau_n}|f_r^0|dr\right)^p + \left(\int_0^{\tau_n}|g_r^0|^2 dr\right)^{p/2}\right. \\
&\left.+\left|\int_0^{\tau_n} e^{ar}\langle Y_r, g(s,Y_r,Z_r)\overleftarrow{dB}_r\rangle\right|^{p/2} + \left|\int_0^{\tau_n} e^{ar}\langle Y_r, Z_r dW_r\rangle\right|^{p/2}\right\}.
\end{aligned} \tag{3.2}$$

But thanks to BDG's inequality, we have

$$\begin{aligned}
\mathbb{E}\left(\left|\int_0^{\tau_n} e^{ar}\langle Y_r, Z_r dW_r\rangle\right|^{p/2}\right) &\leq d_p\mathbb{E}\left[\left(\int_0^{\tau_n}|Y_r|^2|Z_r|^2 dr\right)^{p/4}\right] \\
&\leq \bar{C}_p\mathbb{E}\left[\sup_{0\leq t\leq\tau_n}|Y_t|^{p/2}\left(\int_0^{\tau_n}|Z_r|^2 dr\right)^{p/4}\right] \\
&\leq \frac{\bar{C}_p^2}{\eta_1}\mathbb{E}\left(\sup_{0\leq t\leq\tau_n}|Y_t|^p\right) + \eta_1\mathbb{E}\left(\int_0^{\tau_n}|Z_r|^2 dr\right)^{p/2}.
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}\left(\left|\int_0^{\tau_n} e^{ar}\langle Y_r, g(s,Y_r,Z_r)\overleftarrow{dB}_r\rangle\right|^{p/2}\right) &\leq d_p\mathbb{E}\left[\left(\int_0^{\tau_n}|Y_r|^2|g(r,Y_r,Z_r)|^2 dr\right)^{p/4}\right] \\
&\leq \bar{C}_p\mathbb{E}\left[\sup_{0\leq t\leq\tau_n}|Y_t|^{p/2}\left(\int_0^{\tau_n}|g(r,Y_r,Z_r)|^2 dr\right)^{p/4}\right] \\
&\leq \frac{\bar{C}_p^2}{\eta_2}\mathbb{E}\left(\sup_{0\leq t\leq\tau_n}|Y_t|^p\right) + \eta_2\mathbb{E}\left(\int_0^{\tau_n}|g(r,Y_r,Z_r)|^2 dr\right)^{p/2} \\
&\leq C_p\mathbb{E}\left(\sup_{0\leq t\leq\tau_n}|Y_t|^p + \left(\int_0^{\tau_n}|g_r^0|^2\right)^{p/2}\right) \\
&\quad +(1+\eta_3)\eta_2\alpha\mathbb{E}\left(\int_0^{\tau_n}|Z_r|^2 dr\right)^{p/2}.
\end{aligned}$$

Let us take η_1, η_2 and η_3 small enough such that coming back to (3.2), we obtain, for each $n \in \mathbb{N}$,

$$\mathbb{E}\left[\left(\int_0^{\tau_n}|Z_r|^2 dr\right)^{p/2}\right] \leq C_p\mathbb{E}\left\{\sup_{0\leq t\leq T}|Y_t|^p + \left(\int_0^T|f_r^0|dr\right)^p + \left(\int_0^T|g_r^0|^2 dr\right)^{p/2}\right\},$$

which by Fatou's lemma implies

$$\mathbb{E}\left[\left(\int_0^T |Z_r|^2 dr\right)^{p/2}\right] \;\leq\; C_p\mathbb{E}\left\{\sup_{0\leq t\leq T}|Y_t|^p + \left(\int_0^T |f_r^0|dr\right)^p + \left(\int_0^T |g_r^0|^2 dr\right)^{p/2}\right\},$$

the desired result. □

We keep on this study by stating the estimate which is the main tool to derive existence and uniqueness result in our context. The difficulty comes from the fact that f is non-Lipschitz in y and the function $y \mapsto |y|^p$ is not $\mathcal{C}^2$ since we work with $p \in (1,2)$.

Lemma 3.2. *Assume (**H1**)-(**H2**). Let (Y,Z) be a solution of the backward doubly SDE associated to the data (ξ, f, g) where Y belong to $\mathcal{S}^p$. Then there exists a constant $C_{p,\lambda}$ depending only on p and λ such that*

$$\begin{aligned}\mathbb{E}\left\{\sup_{0\leq t\leq T}|Y_t|^p + \left(\int_0^T |Z_s|^2 ds\right)^{p/2}\right\} \;\leq\; & C_{p,\lambda}\mathbb{E}\left\{|\xi|^p + \left(\int_0^T |f_s^0|ds\right)^p\right.\\ & \left.+\left(\int_0^T |g_s^0|^2 ds\right)^{p/2} + \int_0^T |Y_s|^{p-2}\mathbf{1}_{\{Y_s\neq 0\}}|g_s^0|^2 ds\right\}.\end{aligned}$$

Proof. Applying Corollary 2.1 we have, for any $a > 0$ and any $0 \leq t \leq u \leq T$:

$$\begin{aligned}& e^{apt}|Y_t|^p + c(p)\int_t^u e^{aps}|Y_s|^{p-2}\mathbf{1}_{\{Y_s\neq 0\}}|Z_s|^2 ds\\ \leq\; & e^{apu}|Y_u|^p - ap\int_t^u e^{aps}|Y_s|^p ds + p\int_t^u e^{aps}|Y_s|^{p-1}\langle \widehat{Y}_s, f(s,Y_s,Z_s)\rangle ds\\ & +\overline{c}(p)\int_t^u e^{aps}|Y_s|^{p-2}\mathbf{1}_{\{Y_s\neq 0\}}|g(s,Y_s,Z_s)|^2 ds + p\int_t^u e^{aps}|Y_s|^{p-1}\langle \widehat{Y}_s, g(s,Y_s,Z_s)\overleftarrow{dB}_s\rangle\\ & -p\int_t^u e^{aps}|Y_s|^{p-1}\langle \widehat{Y}_s, Z_s dW_s\rangle.\end{aligned}$$

The assumption on f and g yields

$$\begin{aligned}\langle \hat{y}, f(s,y,z)\rangle &\leq |f_s^0| + \mu|y| + \lambda|z|\\ |g(s,y,z)|^2 &\leq (1+\varepsilon)\lambda|y|^2 + (1+\varepsilon)\alpha|z|^2 + (1+\frac{1}{\varepsilon})|g_s^0|^2,\end{aligned}$$

for any arbitrary $\varepsilon > 0$. Therefore for all $t \in [0,u]$, we get with probability one:

$$\begin{aligned}& e^{apt}|Y_t|^p + c(p)\int_t^u e^{aps}|Y_s|^{p-2}\mathbf{1}_{\{Y_s\neq 0\}}|Z_s|^2 ds\\ \leq\; & e^{apu}|Y_u|^p + [p(\mu-a) + \overline{c}(p)(1+\varepsilon)\lambda]\int_t^u e^{aps}|Y_s|^p ds\\ & +p\int_t^u e^{aps}|Y_s|^{p-1}|f_s^0|ds + \overline{c}(p)(1+\varepsilon^{-1})\int_t^u e^{aps}|Y_s|^{p-2}\mathbf{1}_{\{Y_s\neq 0\}}|g_s^0|^2 ds\\ & +\overline{c}(p)(1+\varepsilon)\alpha\int_t^u e^{aps}|Y_s|^{p-2}\mathbf{1}_{\{Y_s\neq 0\}}|Z_s|^2 ds + p\lambda\int_t^u e^{aps}|Y_s|^{p-1}|Z_s|ds\\ & +p\int_t^u e^{aps}|Y_s|^{p-1}\langle \widehat{Y}_s, g(s,Y_s,Z_s)\overleftarrow{dB}_s\rangle - p\int_t^u e^{p\alpha s}|Y_s|^{p-1}\langle \widehat{Y}_s, Z_s dW_s\rangle. \qquad (3.3)\end{aligned}$$

We deduce from the previous inequality that, $\mathbb{P}$-a.s.,

$$\int_0^T e^{aps}|Y_s|^{p-2}\mathbf{1}_{\{Y_s\neq 0\}}|Z_s|^2ds<\infty.$$

Moreover, we have

$$p\lambda|Y_s|^{p-1}|Z_s|\leq\gamma^{-1}\frac{p\lambda^2}{2(p-1)}|Y_s|^p+\gamma c(p)|Y_s|^{p-1}\mathbf{1}_{\{Y_s\neq 0\}}|Z_s|^2,$$

for any arbitrary $\gamma>0$.

Next for γ and ε small enough and a large enough such that $\alpha'=[(1-\gamma)c(p)-(1+\varepsilon)\alpha\overline{c}(p)]>0$ and $c(a,\varepsilon,\gamma)=p[\mu+\frac{(3-p)(1+\varepsilon)}{2}\lambda+\frac{\gamma^{-1}}{2(p-1)}\lambda^2-a]\leq 0$, we have

$$\begin{aligned}
&|Y_t|^p+\alpha'\int_t^T e^{aps}|Y_s|^{p-2}\mathbf{1}_{\{Y_s\neq 0\}}|Z_s|^2ds\\
\leq\;& e^{apT}|\xi|^p+p\int_t^T e^{aps}|Y_s|^{p-1}|f_s^0|ds+\overline{c}(p)(1+\varepsilon^{-1})\int_t^T e^{aps}|Y_s|^{p-2}\mathbf{1}_{\{Y_s\neq 0\}}|g_s^0|^2ds\\
&+p\int_t^T e^{aps}|Y_s|^{p-1}\langle\widehat{Y}_s,g(s,Y_s,Z_s)\overleftarrow{dB}_s\rangle-p\int_t^T e^{aps}|Y_s|^{p-1}\langle\widehat{Y}_s,Z_sdW_s\rangle\\
\leq\;& X+p\int_t^T e^{aps}|Y_s|^{p-1}\langle\widehat{Y}_s,g(s,Y_s,Z_s)\overleftarrow{dB}_s\rangle-p\int_t^T e^{aps}|Y_s|^{p-1}\langle\widehat{Y}_s,Z_sdW_s\rangle, \qquad (3.4)
\end{aligned}$$

where

$$X=e^{apT}|\xi|^p+p\int_0^T e^{aps}|Y_s|^{p-1}|f_s^0|ds+c(p)(1+\varepsilon^{-1})\int_0^T e^{aps}|Y_s|^{p-2}\mathbf{1}_{\{Y_s\neq 0\}}|g_s^0|^2ds.$$

On can show that $M_t=\int_t^T e^{aps}|Y_s|^{p-1}\langle\widehat{Y}_s,g(s,Y_s,Z_s)\overleftarrow{dB}_s\rangle$ and $N_t=\int_t^T e^{p\alpha s}|Y_s|^{p-1}\langle\widehat{Y}_s,Z_sdW_s\rangle$ are uniformly integrable martingale. Indeed, Using BDG inequality and then Young's inequality we have,

$$\begin{aligned}
\mathbb{E}\langle M,M\rangle_T^{1/2}&\leq\mathbb{E}\left[\sup_{0\leq t\leq T}|Y_t|^{p-1}\left(\int_0^T|Z_r|^2dr\right)^{1/2}\right]\\
&\leq\frac{p-1}{p}\mathbb{E}\left(\sup_{0\leq t\leq T}|Y_t|^p\right)+\frac{1}{p}\mathbb{E}\left[\left(\int_0^T|Z_r|^2dr\right)^{p/2}\right] \qquad (3.5)
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}\langle N,N\rangle_T^{1/2}&\leq\mathbb{E}\left[\sup_{0\leq t\leq T}|Y_t|^{p-1}\left(\int_0^T|g(r,Y_r,Z_r)|^2dr\right)^{1/2}\right]\\
&\leq\frac{p-1}{p}\mathbb{E}\left(\sup_{0\leq t\leq T}|Y_t|^p\right)+\frac{1}{p}\mathbb{E}\left[\left(\int_0^T|g(r,Y_r,Z_r)|^2dr\right)^{p/2}\right]. \qquad (3.6)
\end{aligned}$$

The last term of (3.5) and (3.6) being finite since Y and $g(.,Y,Z)$ belong to $\mathcal{S}^p$ and $\mathcal{M}^p$ respectively, and then Z belongs to $\mathcal{M}^p$ by Lemma 3.1.

Return to (3.4), we get both

$$\begin{aligned}
\mathbb{E}\left(e^{apt}|Y_t|^p\right) &\leq \mathbb{E}(X), \\
\alpha'\mathbb{E}\int_0^T e^{aps}|Y_s|^{p-2}\mathbf{1}_{\{Y_s\neq 0\}}|Z_s|^2 ds &\leq C_p\mathbb{E}(X), \qquad (3.7)\\
\mathbb{E}\left(\sup_{0\leq t\leq T} e^{apt}|Y_t|^p\right) &\leq \mathbb{E}(X)+k_p\mathbb{E}\langle N,N\rangle_T^{1/2}+h_p\mathbb{E}\langle M,M\rangle_T^{1/2}.
\end{aligned}$$

On the other hand, we also have

$$k_p\mathbb{E}\langle M,M\rangle_T^{1/2} \leq \frac{1}{4}\mathbb{E}\left(\sup_{0\leq t\leq T} e^{apt}|Y_t|^p\right)+4h_p^2\mathbb{E}\int_0^T e^{aps}|Y_s|^{p-2}\mathbf{1}_{\{Y_s\neq 0\}}|Z_s|^2 ds$$

and

$$\begin{aligned}
h_p\mathbb{E}\langle N,N\rangle_T^{1/2} &\leq \frac{1}{4}\mathbb{E}\left(\sup_{0\leq t\leq T} e^{apt}|Y_t|^p\right)+4k_p^2\mathbb{E}\int_0^T e^{aps}|Y_s|^{p-2}\mathbf{1}_{\{Y_s\neq 0\}}|Z_s|^2 ds \\
&\quad +d_p\mathbb{E}\left(\int_0^T e^{aps}|Y_s|^p ds+\int_0^T e^{aps}|Y_s|^{p-2}\mathbf{1}_{\{Y_s\neq 0\}}|g_s^0|^2 ds\right).
\end{aligned}$$

Therefore from (3.7), we obtain

$$\mathbb{E}\left(\sup_{0\leq t\leq T} e^{apt}|Y_t|^p\right)\leq C_p\mathbb{E}(X).$$

Applying once again Young's inequality, we get

$$pC_p\int_0^T e^{aps}|Y_s|^{p-1}|f_s^0|ds \leq \frac{1}{2}\sup_{0\leq s\leq T}|Y_s|^p+C_p'\left(\int_0^T e^{aps}|f_s^0|ds\right)^p$$

from which we deduce, in view of X, that

$$\mathbb{E}\left(\sup_{0\leq t\leq T} e^{apt}|Y_t|^p\right)\leq C_p\mathbb{E}\left[|\xi|^p+\left(\int_0^T e^{aps}|f_s^0|ds\right)^p+\int_0^T e^{aps}|Y_s|^{p-2}\mathbf{1}_{\{Y_s\neq 0\}}|g_s^0|^2 ds\right].$$

The result follows from Lemma 3.1. □

4 Existence and uniqueness of a solution

This section is devoted to derive existence and uniqueness result to BDSDE (ξ, f, g) in L^p-sense, $(p\in(1,2))$. We use above a priori estimates and L^∞-approximation. We work under (**H1**)-(**H2**) and the additional assumptions.

(**H3**) For $p\in(1,2)$,

$$\begin{cases}
(i)\ \mathbb{E}[|\xi|^p]<\infty, \\
(ii)\ \mathbb{P}\,a.s.\ \forall(t,z)\in[0,T]\times\mathbb{R}^{k\times d},\ y\mapsto f(t,y,z)\ \text{is continuous}, \\
(iii)\ g(.,0,0)\equiv 0, \\
(iv)\ \forall r>0,\ \psi_r(t)=\sup_{|y|<r}|f(t,y,0)-f_t^0|\in L^1([0,T],m\otimes\mathbb{P}).
\end{cases}$$

Firstly, we generalize the result of Pardoux and Peng (see Theorem 1.1, [8]) to monotone case. To do this, let assume this assumption which appear in [7].

(H4) $\mathbb{P}-a.s. \ \forall (t,y)\in[0,T]\times\mathbb{R}^k, \ |f(t,y,0)|\leq|f(t,0,0)|+\varphi(|y|)$.

Theorem 4.1. *Under assumptions* (**H1**)-(**H4**)*, BDSDE (2.1) has a unique solution in* $\mathcal{S}^2(\mathbb{R}^k)\times\mathcal{M}^2(\mathbb{R}^{k\times d})$.

Proof. It follows by combining argument of Pardoux (see Theorem 2.2 [7]) with one used in Pardoux and Peng (see Theorem 1.1, [8]). Therefore, we will give the main line.
Uniqueness
Let (Y,Z) and (Y',Z') be two solutions of BDSDE (ξ,f,g) verify above assumptions. It follows from Itô formula that

$$\begin{aligned}
&\mathbb{E}(|Y_t-Y'_t|^2)+\mathbb{E}\left(\int_t^T|Z_s-Z'_s|^2ds\right)\\
=\ &2\mathbb{E}\left(\int_t^T\langle Y_s-Y'_s, f(s,Y,Z_s)-f(s,Y'_s,Z'_s)\rangle ds\right)+\mathbb{E}\left(\int_t^T|g(s,Y_s,Z_s)-g(s,Y'_s,Z'_s)|^2ds\right)\\
\leq\ &\mathbb{E}\left(\int_t^T[(2\mu+\lambda)|Y_s-Y'_s|^2+2\lambda|Y_s-Y'_s||Z_s-Z'_s|+\alpha|Z_s-Z'_s|^2]ds\right)\\
\leq\ &(2\mu+\lambda+\gamma^{-1}\lambda^2)\mathbb{E}\left(\int_t^T|Y_s-Y'_s|^2ds\right)+(\alpha+\gamma)\mathbb{E}\left(\int_t^T|Z_s-Z'_s|^2ds\right).
\end{aligned}$$

Hence, taking γ small enough such that $1-\alpha-\gamma>0$, we have

$$\mathbb{E}(|Y_t-Y'_t|^2)\ \leq\ C\mathbb{E}\left(\int_t^T|Y_s-Y'_s|^2ds\right),$$

which provide with Gronwall's lemma that $\mathbb{E}(|Y_t-Y'_t|^2)=0$, for all $t\in[0,T]$, and then $\mathbb{E}\left(\int_t^T|Z_s-Z'_s|^2ds\right)=0$.

Existence
Firstly, let state this result which is proved similarly as Proposition 2.4 (see [7]) with additional computations due to backward stochastic integral with respect Brownian motion B, so we omit it.

Proposition 4.2. *Given* $V\in\mathcal{M}^2(0,T,\mathbb{R}^{k\times d})$ *and assume* (**H1**)-(**H4**)*, there exists a unique measurable processes* $(Y_t,Z_t)_{\{0\leq t\leq T\}}$ *with values in* $\mathbb{R}^k\times\mathbb{R}^{k\times d}$ *satisfies*

$$Y_t=\xi+\int_t^T f(s,Y_s,V_s)ds+\int_t^T g(s,Y_s,Z_s)\overleftarrow{dB}_s-\int_t^T Z_sdW_s,\ \ 0\leq t\leq T.$$

Using Proposition (4.2), we consider $\Phi:\mathcal{S}^2(\mathbb{R}^k)\times\mathcal{M}^2(0,T,\mathbb{R}^{k\times d})\to\mathcal{S}^2(\mathbb{R}^k)\times\mathcal{M}^2(0,T,\mathbb{R}^{k\times d})$ defined by $(Y,Z)=\Phi(U,V)$ is the unique solution of the BDSDE

$$Y_t=\xi+\int_t^T f(s,Y_s,V_s)ds+\int_t^T g(s,Y_s,Z_s)\overleftarrow{dB}_s-\int_t^T Z_sdW_s,\ \ 0\leq t\leq T.$$

Let (U,V), (U',V') belong in $\mathcal{S}^2(\mathbb{R}^k)\times\mathcal{M}^2(0,T,\mathbb{R}^{k\times d})$, $(Y,Z)=\Phi(U,V)$ and $(Y',Z')=\Phi(U',V')$. Setting $(\overline{U},\overline{V})=(U-U',V-V')$ and $(\overline{Y},\overline{Z})=(Y-Y',Z-Z')$, it follows from Itô formula that for $\gamma\in\mathbb{R}$,

$$\begin{aligned}
&e^{\gamma t}\mathbb{E}|\overline{Y}_t|^2+\mathbb{E}\int_t^T e^{\gamma s}(\gamma\left|\overline{Y}_s\right|^2+|\overline{Z}_s|^2)ds\\
=\;\;& 2\mathbb{E}\int_t^T\langle\overline{Y}_s,f(s,Y_s,V_s)-f(s,Y',V')\rangle ds+\mathbb{E}\int_t^T|g(s,Y_s,Z_s)-g(s,Y',Z')|^2ds\\
\leq\;\;& \mathbb{E}\int_t^T(2\mu+\frac{\lambda^2}{\varepsilon}+\lambda)|\overline{Y}_s|^2+\varepsilon|\overline{V}_s|^2+\alpha|\overline{Z}_s|^2)ds.
\end{aligned}$$

Hence, if we choose $\gamma=\frac{\lambda^2}{\varepsilon}+\lambda+1-\alpha$, we have

$$\mathbb{E}\int_t^T e^{\gamma s}(\left|\overline{Y}_s\right|^2+|\overline{Z}_s|^2)ds\quad\leq\quad\frac{\varepsilon}{1-\alpha}\left(\mathbb{E}\int_t^T e^{\gamma s}(\left|\overline{U}_s\right|^2+\left|\overline{V}_s\right|^2)ds\right)$$

Taken $\varepsilon<1-\alpha$, Φ is a strict contraction on $\mathcal{S}^2(\mathbb{R}^k)\times\mathcal{M}^2((0,T);\mathbb{R}^{k\times d})$ equipped with the norm

$$\|(Y,Z)\|^2=\mathbb{E}\int_t^T e^{\gamma s}\left(|Y_s|^2+|Z_s|^2\right)ds.$$

Its unique fixed point solves BDSDE (ξ,f,g) in $\mathcal{S}^2(\mathbb{R}^k)\times\mathcal{M}^2(\mathbb{R}^{k\times d})$. □

We are now ready to state existence and uniqueness result of BDSDEs in L^p-sense.

Theorem 4.3. *Under assumptions* (**H1**)-(**H3**), *BDSDE* (2.1) *has a unique solution in* $\mathcal{S}^p(\mathbb{R}^k)\times\mathcal{M}^p(\mathbb{R}^{k\times d})$, $p\in(1,2)$.

Proof. **Uniqueness**
Let us consider (Y,Z) and (Y',Z') two solutions of BDSDE (ξ,f,g). Setting $\overline{Y}=Y-Y'$ and $\overline{Z}=Z-Z'$, then the process $(\overline{Y},\overline{Z})$ solves BDSDE

$$\overline{Y}_t=\int_t^T\varphi(s,\overline{Y}_s,\overline{Z}_s)ds+\int_t^T\psi(s,\overline{Y}_s,\overline{Z}_s)\overleftarrow{dB}_s-\int_t^T\overline{Z}_sdW_s,\;\;0\leq t\leq T,$$

where

$$\varphi(s,y,z)=f(s,y+Y'_s,z+Z'_s)-f(s,Y'_s,Z'_s)\;\text{ and }\;\psi(s,y,z)=g(s,y+Y'_s,z+Z'_s)-g(s,Y'_s,Z'_s).$$

Thanks to (**H2**), the process $(\overline{Y},\overline{Z})$ satisfies Lemma 3.2 with $\varphi^0=\psi^0=0$. Thus, $(\overline{Y},\overline{Z})=(0,0)$ immediately.

Existence

It split in two steps.

Step 1. For positive real r, we suppose ξ, $\sup|f_t^0|$ are bounded random variables such that

$$e^{(1+\lambda^2)T}(|\xi|+T\|f^0\|_\infty)\quad<\quad r.$$

For such r, we define θ_r a smooth function such that $0 \le \theta_r \le 1$ and

$$\theta_r(y) = \begin{cases} 1 \text{ for } |y| \le r \\ \\ 0 \text{ for } |y| \ge r+1. \end{cases}$$

For each $n \in \mathbb{N}^*$, we denote

$$h_n(t,y,z) \quad = \quad \theta_r(y)(f(t,y,q_n(z)) - f_t^0)\frac{n}{\pi_{r+1}(t) \vee n} + f_t^0,$$

where $q_n(z) = z\frac{n}{|z| \vee n}$.

As it is shown in [3], for each $n \in \mathbb{N}$, (ξ, h_n, g) satisfies assumptions (**H1**)-(**H4**) with μ positive and, hence, there exist a unique process (Y^n, Z^n) solution of BDSDE (ξ, h_n, g) in $\mathcal{S}^2(\mathbb{R}^k) \times \mathcal{M}^2(\mathbb{R}^{k\times d})$.

Moreover, we show combining argument of Briand and Carmona (see Lemma 2.2, [2]) and remark 2.2 in [1] that $\|Y^n\|_\infty \le r$ which together with Lemma 3.1 provide

$$\|Z^n\|_{\mathcal{M}^2} \le r', \tag{4.1}$$

where r' is another constant depending on r.

As a byproduct if we denote

$$f_n(t,y,z) = (f(t,y,q_n(z)) - f_t^0)\frac{n}{\pi_{r+1}(t) \vee n} + f_t^0,$$

then (Y^n, Z^n) still solution to BDSDE $(\xi, \underline{f_n}, g)$.

For $i \in \mathbb{N}$, setting $\bar{Y}^{n,i} = Y^{n+i} - Y^n$, $\bar{Z}^{n,i} = Z^{n+i} - Z^n$, we have

$$\begin{aligned} & e^{at}|\bar{Y}_t^{n,i}|^2 + (1-\varepsilon-\alpha)\int_t^T e^{as}|\bar{Z}_s^{n,i}|^2 ds \\ \le\ & 2\int_t^T e^{as}\langle \bar{Y}_s^{n,i}, f_{n+i}(s,Y_s^n,Z_s^n) - f_n(s,Y_s^n,Z_s^n)\rangle ds \\ & +(2\mu + \frac{1}{\varepsilon}\lambda^2 + \lambda - a)\int_t^T e^{as}|\bar{Y}_s^{n,i}|^2 ds \\ & +2\int_t^T e^{as}\langle \bar{Y}_s^{n,i}, (g(s,Y_s^{n+i},Z_s^{n+i}) - g(s,Y_s^n,Z_s^n))dB_s\rangle \\ & -2\int_t^T e^{as}\langle \bar{Y}_s^{n,i}, \bar{Z}_s^{n,i} dW_s\rangle, \end{aligned}$$

for any $a > 0$ and $\varepsilon > 0$.

Next, since $\|\bar{Y}^{n,i}\|_\infty \le 2r$ and setting $\gamma = 1-\varepsilon-\alpha > 0$ and $(2\mu + \frac{1}{\varepsilon}\lambda^2 + \lambda - a) \le 0$, we obtain

$$\begin{aligned} & e^{at}|\bar{Y}_t^{n,i}|^2 + \gamma\int_t^T e^{as}|\bar{Z}_s^{n,i}|^2 ds \\ \le\ & 4r\int_t^T e^{as}|f_{n+i}(s,Y_s^n,Z_s^n) - f_n(s,Y_s^n,Z_s^n)| ds \\ & +2\int_t^T e^{as}\langle \bar{Y}_s^{n,i}, (g(s,Y_s^{n+i},Z_s^{n+i}) - g(s,Y_s^n,Z_s^n))dB_s\rangle \\ & -2\int_t^T e^{as}\langle \bar{Y}_s^{n,i}, \bar{Z}_s^{n,i} dW_s\rangle. \end{aligned}$$

Therefore, combining rigorously Gronwall's and BDG inequality, there exist a constant C depending only on λ, α and T such that

$$\mathbb{E}\left[\sup_{0\le t\le T}|\bar{Y}_t^{n,i}|^2+\int_0^T|\bar{Z}_s^{n,i}|^2ds\right]\le Cr\mathbb{E}\left[\int_0^T|f_{n+i}(s,Y_s^n,Z_s^n)-f_n(s,Y_s^n,Z_s^n)|ds\right].$$

But $\|Y^n\|_\infty\le r$ so that

$$|f_{n+i}(s,Y_s^n,Z_s^n)-f_n(s,Y_s^n,Z_s^n)|\quad\le\quad 2\lambda|Z_s^n|\mathbf{1}_{\{|Z_s^n|\;>n\}}+2\lambda|Z_s^n|\mathbf{1}_{\{\pi_{r+1}(s)>n\}}+2\pi_{r+1}(s)\mathbf{1}_{\{\pi_{r+1}(s)>n\}},$$

from which, (Y^n,Z^n) is a cauchy sequence in Banach space $\mathcal{S}^2(\mathbb{R}^k)\times\mathcal{M}^2(\mathbb{R}^{k\times d})$. Hence, (Y^n,Z^n) admit a limit $(Y,Z)\in\mathcal{S}^2(\mathbb{R}^k)\times\mathcal{M}^2(\mathbb{R}^{k\times d})$, which solves BDSDE (2.1).

Step 2. In the general case, let us define for each $n\in\mathbb{N}^*$,

$$\xi_n=q_n(\xi),\quad f_n(t,y,z)=f(t,y,z)-f_t^0+q_n(f_t^0).$$

Thank to the Step 1, BDSDE (ξ_n,f_n,g) has a unique solution $(Y^n,Z^n)\in L^2$, but also in L^p far all $p\in(1,2)$ according to Lemma 3.1. Moreover, from Lemma 3.2, for $(i,n)\in\mathbb{N}\times\mathbb{N}^*$, there exist $C(T,\alpha,\lambda)$ such that

$$\begin{aligned}&\mathbb{E}\left\{\sup_{0\le t\le T}|Y_t^{n+i}-Y_t^n|^p+\left(\int_0^T|Z_s^{n+i}-Z_s^n|^2ds\right)^{p/2}\right\}\\ \le\;& C\mathbb{E}\left\{|\xi_{n+i}-\xi_n|^p+\left(\int_0^T|q_{n+i}(f_s^0)-q_n(f_s^0)|ds\right)^p\right\}.\end{aligned}\tag{4.2}$$

The right-hand side of (4.2) tends to 0, as $n\to\infty$, uniformly in i; thus (Y^n,Z^n) is a Cauchy sequence in $\mathcal{S}^p(\mathbb{R}^k)\times\mathcal{M}^p(\mathbb{R}^{k\times d})$ and its limit (Y,Z) solves BDSDE (2.1). □

References

[1] Boufoussi, B.; Van Casteren, J.; Mrhardy, N. Generalized backward doubly stochastic differential equations and SPDEs with nonlinear Neumann boundary conditions. *Bernoulli* **13** (2007), no. 2, 423-446.

[2] Briand PH.; Carmona R. BSDE with polynomial growth generators. *J. Appl. Math. Stochastic. Anal.* **13** (2000), no. 3, 207 – 238.

[3] Briand, PH.; Deylon, D., Hu, Y; Pardoux, E.; Stoica L. L^p-solution of Backward stochastic differential equations. *Stochastic Process. Appl.* **108** (2003), no. 1, 109-129.

[4] Nualart, D.; Pardoux, É. Stochastic calculus with anticipating integrands. *Probab. Theory Related Fields* **78** (1988), no. 4, 535-581.

[5] N'zi, M.; Owo, J. M. Backward doubly stochastic differential equations with discontinuous coefficients. *Statist. Probab. Lett.* (2008), doi:10.1016/j.spl.2008.11.011.

[6] N'zi, M.; Owo, J. M. Backward doubly stochastic differential equations with non-lipschitz coefficients. *Random Oper. Stochastic Equations* **16** (2008), no. 307-324

[7] Pardoux, E., BSDEs, weak convergence and homogenization of semilinear PDEs. *Nonlinear analysis, differential equations and control (Montreal, QC, 1998)*, 503-549, NATO Sci. Ser. C Math. Phys. Sci., 528, *Kluwer Acad. Publ., Dordrecht*, 1999.

[8] Pardoux, E.; Peng, S. Backward doubly stochastic differential equations and systems of quasilinear SPDEs. *Probab. Theory Related Fields* **98** (1994), no. 2, 209-227.

[9] Peng, S.; Shi, Y. A type of time-symmetric forward-backward stochastic differential equations. *C. R. Acad. Sci. Paris* Ser. I **336** (2003), no. 1, 773-778.

[10] Shi, Y.; Gu, Y.; Liu, K. Comparison theorem of backward doubly stochastic differential equations and applications. *Stoch. Ana. Appl.* **23** (1998), no. 1, 1-14.