Topological Defects in 3-d Euclidean Gravity

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By making use of the complete decomposition of $SO(3)$ spin connection, the topological defect in 3-dimensional Euclidean gravity is studied in detail. The topological structure of disclination is given as the combination of a monopole structure and a vortex structure. Furthermore, the Kac-Moody algebra generated by the monopole and vortex is discussed in three dimensional Chern-Simons theory.

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I. INTRODUCTION

An exciting development in cosmology has been the realization that the universe may behave very much like a condensed matter system. Analogous to those found in some condensed matter systems−vertex line in liquid helium, flux tubes in type-II superconductors, or disclination lines in liquid crystals, the topological space-time defects may have been found at phase transitions in the early history of the universe. They may help to explain some of the largest-scale structure seen in the universe today.

As a kind of topological defect, the disclination is caused by inserting solid angles into the flat space-time. In Riemann-Cartan geometry, this effect is showed by the integral of the affine curvature along a closed surface. Duan, Duan and Zhang [1] had discussed the disclinations in deformable material media by applying the gauge field theory and decomposition theory of gauge potential. In their works, the projection of disclination density along the gauge parallel vector was found corresponding to a set of isolated disclinations in the three dimensional sense and being topologically quantized. Furthermore, the space-time disclinations in 4-dimensional with Euclidean signature and Lorentz signature were discussed by Duan and Li [2,3], similar results were obtained..

In this paper, we discuss the disclinations in three dimensional gravity by making use of the decomposition formula of $SO(3)$ gauge potential proposed by Faddeev and Niemi [4]. This decomposition theory provides new tool to study the topological defects. In their decomposition, a complex field ϕ is introduced naturally from gauge transformation. By studying the transformation properties, they showed this complex field ϕ and the projection of the $SU(2)$ gauge potential of a unit vector n^a form a multiplet. Just like the magnetic monopole theory, by introducing the Abelian projection of $SO(3)$ gauge field, we define the topological charge, which is combined by monopoles and vortices. We find it is the complex field ϕ whose zero points behave as the sources of the vortices. The projection of space-time disclination density along the gauge parallel vector is topological quantized with 2π as the unit solid angle. Further, by making using of the three dimensional Chern-Simons gravity theory (see for examples [5–10]), we show the monopole charges and the vorticities can server as the generator of Kac-Moody algebra.

This paper is arranged as follows: In section II, we introduce the definitions of the topological defects in 3-dimensional gravity. In section III, by using the decomposition of $SO(3)$ gauge potential, we discuss the relationship of disclination and topological charge. The local structure of the topological defect is given in section IV. At last, we discuss the algebra generated by monopole and vortex in three dimensional Chern-Simons theory in section V.

II. TOPOLOGICAL DEFECTS IN 3-DIMENSIONAL EUCLIDEAN GRAVITY

As it was shown in [1], the dislocation and disclination continuum can be described by the reference, deformed and natural states. For the natural state there is only an anholonomic rectangular coordinate Z^a $(a = 1, 2, 3)$ and

$$\delta Z^a = e^a_\mu dx^\mu, \tag{1}$$

where e^a_μ is triad. The metric tensor of the Riemann-Cartan manifold of natural state is defined by

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$$g_{\mu\nu} = e^a_\mu e^a_\nu. \tag{2}$$

We have known that the metric tensor $g_{\mu\nu}$ is invariant under the local $SO(3)$ transformation of triad. The corresponding gauge covariant derivative 1-form of a vector field ϕ^a on $\mathbf{M}$ is given as

$$D\phi^a = d\phi^a - \omega^{ab}\phi^c, \tag{3}$$

where ω^{ab} is $SO(3)$ spin connection 1-form

$$\omega^{ab} = -\omega^{ba} \qquad \omega^{ab} = \omega^{ab}_\mu dx^\mu. \tag{4}$$

The affine connection of the Riemann–Cartan space is determined by

$$\Gamma^\lambda_{\mu\nu} = e^{\lambda a} D_\mu e^a_\nu. \tag{5}$$

The torsion tensor is the antisymmetric part of $\Gamma_{\mu\nu}$ and is expressed as

$$T^\lambda_{\mu\nu} = e^{\lambda a} T^a_{\mu\nu}, \tag{6}$$

where

$$T^a_{\mu\nu} = \frac{1}{2}(D_\mu e^a_\nu - D_\nu e^a_\mu). \tag{7}$$

The Riemannian curvature tensor is equivalent to the $SO(3)$ gauge field strength tensor 2-form F^{ab}, which is given by

$$F^{ab} = d\omega^{ab} - \omega^{ac} \wedge \omega^{cb} \qquad F^{ab} = \frac{1}{2} F^{ab}_{\mu\nu} dx^\mu \wedge dx^\nu \tag{8}$$

and relates with the Riemann curvature tensor by

$$F^{ab}_{\mu\nu} = -R^\lambda_{\mu\nu\sigma} e^a_\lambda e^{\sigma b}. \tag{9}$$

The dislocation density is defined by

$$\alpha^a = T^a \qquad T^a = \frac{1}{2} T^a_{\mu\nu} dx^\mu \wedge dx^\nu. \tag{10}$$

Analogous to the definition of the 3-dimensional disclination density in the gauge field theory of condensed matter, we define the space-time disclination density as

$$\theta^a = \frac{1}{2} \varepsilon^{abc} R^\lambda_{\mu\nu\sigma} e^b_\lambda e^{\sigma c} dx^\mu \wedge dx^\nu = -\varepsilon^{abc} F^{bc}. \tag{11}$$

The size of the space-time disclination can be represented by the means of the surface integral of the projection of the space-time disclination density along an unit vector field n^a

$$S = \oint_\Sigma \theta^a n^a = -\oint_\Sigma \varepsilon^{abc} n^a F^{bc} \tag{12}$$

where Σ is a closed surface including the disclinations. The new quantity S defined by (12) is dimensionless. Using the so-called ϕ-mapping method and the decomposition of gauge potential Y. S. Duan et al have proved that the dislocation flux is quantized in units of the Planck length [11] and the disclination in condensed matter is quantized also in similar structure as that of magnetic monopole. In this paper, using the new decomposition formula given by Faddeev and Niemi [4], we will show that apart from the monopole structure, a vortex structure also contributes to the disclination.

III. DECOMPOSITION OF $SO(3)$ GAUGE POTENTIAL AND TOPOLOGICAL CHARGE OF ABELIAN PROJECTION OF $SO(3)$ GAUGE FIELD

For $so(3)$ Lie algebra is homomorphic to $su(2)$ Lie algebra, we can introduce the $SU(2)$ gauge potential in term of $SO(3)$ spin connection by

$$\omega^a = \frac{1}{2}\varepsilon^{abc}\omega^{bc}. \tag{13}$$

The corresponding gauge field 2-form is

$$F^a = \frac{1}{2}\varepsilon^{abc}F^{bc} = dA^a + \frac{1}{2}\varepsilon^{abc}A^b \wedge A^c. \tag{14}$$

The covariant derivative of n^a is

$$Dn^a = dn^a - \varepsilon^{abc}\omega^b n^c. \tag{15}$$

From this equation we can solve the gauge potential 1-form ω^a expressed in term of n^a as

$$\omega^a = An^a + \varepsilon^{abc}dn^b n^c - \varepsilon^{abc}Dn^b n^c, \tag{16}$$

and curvature 2-form is

$$F^a = -\frac{1}{2}\varepsilon^{abc}dn^b \wedge dn^c + \frac{1}{2}\varepsilon^{abc}Dn^b \wedge Dn^c + \varepsilon^{abc}n^b dDn^c + Dn^a \wedge A + n^a dA. \tag{17}$$

Recently Faddeev and Niemi [4] showed the covariant part of ω^a can be expressed as

$$-\varepsilon^{abc}Dn^b n^c = \rho dn^a + \sigma\varepsilon^{abc}dn^b n^c \tag{18}$$

where ρ and σ are coefficients and can be combined into a complex field

$$\phi = \rho + i\sigma. \tag{19}$$

Then the garge potential 1-form is rewritten as

$$\omega^a = An^a + \varepsilon^{abc}dn^b n^c + \rho dn^a + \sigma\varepsilon^{abc}dn^b n^c. \tag{20}$$

Under a $SU(2)$ gauge transformation generated by $\alpha^a = \alpha n^a$ the functional form of above equation remain intact and ϕ transforms as a $U(1)$ vector field

$$\phi \to e^{i\alpha}\phi \tag{21}$$

and A transforms as

$$A \to A + d\alpha,$$

which means the multiplet (A_μ, ϕ) transforms like the field multiplet in abelian Higgs model. We can introduce the $U(1)$ covariant derivative of ϕ

$$\begin{aligned} D\phi &= d\phi - iA\phi \\ &= d\rho + A\sigma + i(d\sigma - A\rho) \\ &= D\rho + iD\sigma. \end{aligned} \tag{22}$$

The curvature 2-form can be rewritten as

$$F^a = n^a(dA - \frac{1}{2}(1 - ||\phi||^2)\varepsilon^{abc}dn^b \wedge dn^c + D\rho \wedge dn^a + \varepsilon^{abc}D\sigma \wedge dn^b n^c. \tag{23}$$

Solving from the equation (22), we get the decomposition of A as

$$A = \frac{1}{2i||\phi||^2}(d\phi\phi^* - \phi d\phi^* - D\phi\phi^* + \phi D\phi^*). \tag{24}$$

By making using of equation (23), we get the disclination density projection as

$$F^a n^a = f - (1 - ||\phi||^2)K, \tag{25}$$

where f is $U(1)$-like curvature corresponding to the projection of $SO(3)$ spin connection

$$f = dA \tag{26}$$

and K is the solid angle density

$$K = \frac{1}{2}\varepsilon^{abc} n^a dn^b \wedge dn^c = d\Omega. \tag{27}$$

Using (25)-(27), we obtain the surface integral (12) as

$$S = -\int_\Sigma f + \int_\Sigma (1 - ||\phi||^2)K. \tag{28}$$

The topological charge of the Abelian projection is defined as a surface integral of induced $U(1)$-like gauge field over a closed surface Σ

$$Q = \frac{1}{4\pi}\int_\Sigma F \tag{29}$$

where F is the topological charge density defined by

$$\begin{aligned} F &= -F^a n^a + \frac{1}{2}\varepsilon^{abc} n^a Dn^b \wedge Dn^c \\ &= \frac{1}{2}\varepsilon^{abc} n^a dn^b \wedge dn^c - dA. \end{aligned} \tag{30}$$

The first term of the right hand of (30) is a 3-dimensional solid angle density which present a monopole structure and the second term presents a vortex structure. The topological charge is then expressed as

$$Q = \frac{1}{4\pi}\int_\Sigma K - \frac{1}{4\pi}\int_\Sigma f, \tag{31}$$

which means the topological charge equals to the sum of the solid angle of the closed surface Σ (or equatively the monopole charge) and half of the vorticity corresponding to the Abelian projection of $SO(3)$ spin connection. This topological characteristic labels the topological property of the $SO(3)$ gauge field by fixing the coset $SO(3)/U(1)$.

Comparing equation (28) with equation (31) we see the term ϕK distinguish different geometries from the same topology. When $||\phi||$ is taken as a constant the disclination is topological quantized.

$$S = (1 - ||\phi||^2)\int_\Sigma K - \int_\Sigma f.$$

Moreover, if we choose the unit vector n^a as a gauge constant

$$Dn^a = 0,$$

we find the disclination is just the topological charge we defined, or in other words, the projection of the disclination on a gauge constant is topological quantized

$$S = \int_\Sigma K - \int_\Sigma f = 4\pi Q.$$

On the other hand, for A_μ and ϕ form a multiplet, the topological charge and topological structure of the vortex corresponding to A_μ can be expressed directly by the complex field ϕ as what will be shown at the following section.

IV. LOCAL TOPOLOGICAL STRUCTURE OF THE TOPOLOGICAL DEFECTS

In this section, we will study in detail the local topological structure of the topological defects.

By making use of the stokes theorem, it seems the topological charge Q is equals to zero for K and f are exact

$$dK = d^2\Omega = 0; \tag{32}$$

$$df = d^2 A = 0. \tag{33}$$

However, it is not so when considering the singularity of the fields. It will be shown in the follows that the equation (32) and (33) are satisfied "almost everywhere" except some singular points. Defects introduce source terms at the right-hand sides for defects present points where derivatives fail to commute.

The unit vector n^a can be expressed as

$$n^a = \frac{\varphi^a}{||\varphi||} \tag{34}$$

where φ^a is the vector along the direction of n^a. There exists the relations

$$dn^a = \frac{1}{||\varphi||}d\varphi^a + \varphi^a d\frac{1}{||\varphi||} \tag{35}$$

and

$$\frac{\partial}{\partial\varphi^a}\frac{1}{||\varphi||} = -\frac{\varphi^a}{||\varphi||^3}. \tag{36}$$

By making use of above formulas we find

$$\begin{aligned} d^2\Omega &= \frac{1}{2}\varepsilon^{abc}dn^a \wedge dn^b \wedge dn^c \\ &= \frac{1}{2}\varepsilon^{abc}d(n^a dn^b \wedge dn^c) \\ &= \frac{1}{2}\varepsilon^{abc}d(\frac{\varphi^a}{||\varphi||^3}) \wedge d\varphi^b \wedge d\varphi^c \\ &= -\frac{1}{2}\varepsilon^{abc}(\frac{\partial}{\partial\varphi^a}\frac{\partial}{\partial\varphi^d}\frac{1}{||\varphi||^3})d\varphi^d \wedge d\varphi^b \wedge d\varphi^c. \end{aligned} \tag{37}$$

Define the Jacobian $J(\frac{\varphi}{x})$

$$\varepsilon^{abc}J(\frac{\varphi}{x}) = \varepsilon^{\mu\nu\lambda}\partial_\mu\varphi^a\partial_\nu\varphi^b\partial_\lambda\varphi^c \tag{38}$$

and make use of the Laplacian relation

$$\frac{\partial}{\partial\varphi^a}\frac{\partial}{\partial\varphi^a}\frac{1}{||\varphi||^3} = -4\pi\delta^3(\varphi). \tag{39}$$

Then finally we get

$$d^2\Omega = 4\pi\delta^3(\varphi)J(\frac{\varphi}{x})d^3x. \tag{40}$$

It means that the solid angle density is exact everywhere except the zeroes of φ. From (40) we see the derivative do fail to commute at the singular points of the unit vector n^a. The integral of the solid angle density give the monopole charges

$$\begin{aligned} Q_m &= \frac{1}{4\pi}\int_\Sigma d\Omega \\ &= \int_V \delta^3(\varphi)J(\frac{\varphi}{x})d^3x \\ &= \int_{\varphi(V)} \delta^3(\varphi)d^3\varphi \\ &= \deg\varphi. \end{aligned} \tag{41}$$

From the δ-function theory we know that if $\varphi(x)$ has l_φ isolated zeros and let the ith zero be z_i, $\delta(\varphi)$ can be expressed as [12]

$$\delta(\varphi) = \sum_{i=1}^{l_\varphi} \frac{\beta_i(\varphi)}{J(\frac{\varphi}{x})}\delta(x - z_i). \tag{42}$$

Then one obtains

$$\delta^3(\varphi)J(\frac{\varphi}{x}) = \sum_{i=1}^{l_\varphi} \beta_i(\varphi)\eta_i(\varphi)\delta^3(x - z_i). \tag{43}$$

where $\beta_i(\varphi)$ is the positive integer (the Hopf index of the ith zero) and $\eta_i(\varphi)$ the Brouwer degree [13,14]

$$\eta_i(\varphi) = signJ(\frac{\varphi}{x})|_{x=z_i} = \pm 1. \tag{44}$$

From above deduction the topological structure of the solid angle density projection is obtained

$$d^2\Omega = 4\pi \sum_{i=1}^{l_\varphi} \beta_i(\varphi)\eta_i(\varphi)\delta^3(x - z_i)d^3x, \tag{45}$$

and the monopole charge is

$$Q_m = \sum_{i=1}^{l_\varphi} \beta_i(\varphi)\eta_i(\varphi). \tag{46}$$

If we denote the complex field ϕ as

$$\phi = ||\phi||e^{i\theta} \tag{47}$$

where

$$\tan\theta = \frac{\sigma}{\rho}. \tag{48}$$

Then we get from (24) and (47)

$$A = d\theta - \frac{1}{2i||\phi||^2}(D\phi\phi^* - \phi D\phi^*). \tag{49}$$

For the topological charges is independent of choice of gauge, here we take the complex field ϕ as a gauge constant

$$D\phi = 0. \tag{50}$$

It is easy to prove under this gauge condition A is the angle density

$$A = d\theta. \tag{51}$$

Noticing the singularity again we find the gauge field corresponding to A is not exact at the singular points. Analogous to the deduction of the local topological structure of monopole density and using the relationship

$$\frac{\phi^A}{||\phi||^2} = \frac{\partial}{\partial\phi^A}\ln||\phi|| \qquad A = 1, 2$$

in which $\phi^1 = \rho$ and $\phi^2 = \sigma$, we find

$$\begin{aligned} d^2\theta &= \varepsilon^{AB} d\frac{\phi^A}{||\phi||^2} \wedge d\phi^B \\ &= \varepsilon^{AB}\frac{\partial}{\partial\phi^A}\frac{\partial}{\partial\phi^C}\ln(||\phi||)d\phi^C \wedge d\phi^B. \end{aligned} \tag{52}$$

Defining Jacobian vector $J^\mu(\frac{\phi}{x})$

$$\varepsilon^{AB}J^\mu(\frac{\phi}{x}) = \varepsilon^{\mu\nu\lambda}\partial_\nu\phi^A\partial_\lambda\phi^B \tag{53}$$

and using 2-dimensional Laplacian relation

$$\frac{\partial}{\partial\phi^a}\frac{\partial}{\partial\phi^a}\ln||\phi|| = 2\pi\delta^2(\phi), \tag{54}$$

we get

$$d^2\theta = 2\pi\delta^2(\phi)J^\mu(\frac{\phi}{x})d\sigma_\mu \tag{55}$$

where $d\sigma_\mu$ is area element of the surface

$$d\sigma_\mu = \frac{1}{2}\varepsilon_{\mu\nu\lambda}dx^\nu \wedge dx^\lambda. \tag{56}$$

The derivative fails to commute again at the zero points of the complex field ϕ and these points behave as the sources of the vortices.

$$\begin{aligned} Q_v &= \frac{1}{2\pi}\int_\Sigma d^2\theta \\ &= \int_\Sigma \delta^2(\phi)J^\mu(\frac{\phi}{x})d\sigma_\mu \\ &= \int_\Sigma \delta^2(\phi)d^2\phi \\ &= \deg\phi. \end{aligned} \tag{57}$$

Let us choose coordinates $y = (u^1, u^2, v)$ such that $u = (u^1, u^2)$ are intrinsic coordinate on $\sum$. Suppose $\phi(x)$ possess l_v isolated zeros and denote the ith zero by w_i and using the $\delta-$function theory we get

$$\delta^2(\phi)J^\mu(\frac{\phi}{x}) = \sum_{i=1}^{l_v}\beta_i(\phi)\eta_i(\phi)\delta^2(u-w_i)\frac{dy_i^\mu}{dv}. \tag{58}$$

Then the local structure of the vortex density is

$$f = 2\pi\sum_{i=1}^{l_v}\beta_i(\phi)\eta_i(\phi)\delta^2(u-w_i)d^2u \tag{59}$$

and the vorticity is

$$Q_v = \sum_{i=1}^{l_v}\beta_i(\phi)\eta_i(\phi). \tag{60}$$

Therefore we obtain the total topological charge

$$\begin{aligned} Q &= Q_m - \frac{1}{2}Q_v \\ &= \sum_{i=1}^{l_m}\beta_i(\varphi)\eta_i(\varphi) - \frac{1}{2}\sum_{i=1}^{l_v}\beta_i(\phi)\eta_i(\phi) \\ &= \deg\varphi - \frac{1}{2}\deg\phi, \end{aligned} \tag{61}$$

which is fractional quantized in terms of $\frac{1}{2}$. The corresponding disclination

$$\begin{aligned} S &= 4\pi Q_m - 2\pi Q_v \\ &= 2\pi(2Q_m - Q_v) \end{aligned} \tag{62}$$

is quantized with 2π as the unit disclination charge.

V. ALGEBRA IN CHERN-SIMONS GRAVITY

In Chern-Simons theory of the three dimensional gravity there exists an affine Kac-Moody algebra (at the boundary). The canonical generator $Q(n)$ associated to a gauge transformation, $\delta A^a = Dn^a = [A^a, Q(n)]$, any three-dimensional Chern-Simons theory is given by [5,7]

$$Q(n) = \frac{k}{4\pi}\int_\Sigma n^a F^a - \frac{k}{4\pi}\int_{\partial\Sigma} n^a A^a, \tag{63}$$

where Σ is a two-dimensional spatial section with boundary $\partial\Sigma$, F is the 2-form curvature and A is the gauge potential. It is easy to check that the boundary term arising when varying the bulk part of (63) is cancelled by the boundary term. The Poisson bracket of two Functions $F(A_i)$ and $H(A_i)$ is computed as

$$\{F, H\} = \frac{4\pi}{k}\int_\Sigma d^2z \frac{\delta F}{\delta A_i^a(z)}\varepsilon_{ij}\frac{\delta H}{\delta A_j^a(z)}. \tag{64}$$

By direct application of the Poisson bracket (64) one can find the algebra of two transformations with parameters n and m

$$[Q(n), Q(m)] = Q([n, m]) + \frac{k}{4\pi}\int_{\partial\Sigma} n^a dm^a, \tag{65}$$

where $[n, m] = \varepsilon^{abc}n^b m^c$. When $n = 0$ at the boundary on can find $Q(n) = G(n)$. Then there exist

$$[G, G] = G; \tag{66}$$

$$[G, Q] = G. \tag{67}$$

Equations (65) (66) and (67) form a Kac-Moody algebra $[6, 5, 7]$ with central charge is

$$\frac{k}{4\pi}\int_{\partial\Sigma} n^a dm^a. \tag{68}$$

For black hole problem the choice of gauge group in Euclidean signature is $SO(3)$. From (63) we see the generator $Q(n)$ just the topological charge amended for the manifold with a boundary

$$\begin{aligned} Q(n) &= \frac{k}{4\pi}\int_\Sigma (dA - d\Omega) - \frac{k}{4\pi}\int A \\ &= -\frac{k}{4\pi}\int_\Sigma d\Omega \\ &= -kQ_m(n) \end{aligned} \tag{69}$$

which means the generator $Q(n)$ is the solid angle of the surface Σ corresponding the singular points of n^a. One should noticed here the monopole charge Q_m need not to be an integral for the surface Σ is not closed. Now the commutative relation (65) is rewritten as

$$[kQ_m(n), kQ_m(m)] = -kQ_m[n.m] + \frac{k}{4\pi}\int_{\partial\Sigma} n^a dm^a. \tag{70}$$

After the gauge is fixed the value of Q given in (63) reduces to the boundary term

$$\hat{Q}(n) = -\frac{k}{4\pi}\int_{\partial\Sigma} n^a A^a. \tag{71}$$

A theorem [6] states that after the gauge is fixed and one works with the induced Poisson bracket (or Dirac bracket), the charge $\hat{Q}$ satisfies the same algebra (65) as it did the full charge Q

$$[\hat{Q}(n), \hat{Q}(m)] = \hat{Q}([n, m]) + \frac{k}{4\pi}\int_{\partial\Sigma} n^a dm^a. \tag{72}$$

From (72), we know the boundary term is just the vortex term of the topological charge (31) which is deduced as

$$\hat{Q}(n) = -\frac{k}{4\pi}\int_{\partial\Sigma} d\theta(n)$$
$$= -\frac{k}{2}Q_v(n). \quad (73)$$

Now we get the commutative relations of the vorticities

$$[\frac{k}{2}Q_v(n), \frac{k}{2}Q_v(m)] = -\frac{k}{2}Q_v([n,m]) + \frac{k}{4\pi}\int_{\partial\Sigma} n^a dm^a. \quad (74)$$

VI. CONCLUSION

We have shown in this contribution that there exist two kinds of topological structures in 3-dimensional Euclidean gravity by making use of the decomposition of $SO(3)$ spin connection, namely the monopole structure and vortex structure. When projection the disclination density onto a covariant constant unit vector, the size of disclination is quantized topologically. The topological charge equals to the sum of degrees of vectors φ and ϕ which comes from the multiplet (A_μ, ϕ) in the decomposition of gauge potential. The Hopf indices and Brouwer degrees of φ and ϕ label the local structure of the topological defect. Due to the singularity of the gauge potential, the topological densities of disclination are found to be δ-function of φ and ϕ. The noncommutative properties of derivatives at the singular points reveal the sources of the topological defect. We also showed the monopole charges and vorticities can serve as the generators of Kac-Moody algebra in 3-dimensional Chern-Simons gravity.

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