Finite Group Modular Data

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Abstract

In a remarkable variety of contexts appears the modular data associated to finite groups. And yet, compared to the well-understood affine algebra modular data, the general properties of this finite group modular data has been poorly explored. In this paper we undergo such a study. We identify some senses in which the finite group data is similar to, and different from, the affine data. We also consider the data arising from a cohomological twist, and write down, explicitly in terms of quantities associated directly with the finite group, the modular S and T matrices for a *general* twist, for what appears to be the first time in print.

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1 Mathematical and Physical Origin

Throughout this paper, let G denote any finite group (good references to finite group and character theory are provided by [1, 2, 3]).

Physically, the modular data we will describe in the next section arise in several ways. It is a (2+1)-dimensional Chern-Simons theory with finite gauge group G [4, 5]. As Witten showed, 3-dimensional topological field theory corresponds to 2-dimensional conformal field theory (CFT), and the corresponding CFTs here are orbifolds by symmetry group G of a holomorphic CFT [6, 4, 7, 8] (e.g. the E_8 level 1 WZW orbifolded by any finite subgroup of the compact simply-connected Lie group $E_8(\mathbb{R})$). This CFT incarnation is important to us, as it provides some motivation for specific investigations we will perform. Nevertheless, both of these incarnations are probably of direct value only as toy models.

Note that more generally, to each G we will obtain a finite-dimensional representation of the mapping class group $\Gamma_{g,n}$ for the genus g surfaces with n punctures [9]. In this paper though we will consider only the modular group $\mathrm{SL}_2(\mathbb{Z})$, corresponding to $\Gamma_{1,0}$, and in particular the matrices S and T.

It is clear that many different CFTs can realise the same modular data – e.g. all 71 of the $c = 24$ holomorphic theories have $S = T = (1)$. So any given G will correspond to several different CFTs. Likewise, all we can say about the central charge c associated to G is that it will be a positive multiple of 8.

Associated with the RCFT we expect to have some sort of quantum-group which captures the modular data and representation theory of the chiral algebra (so e.g. the fusion coefficients are tensor product coefficients of irreducible modules). This was done for arbitrary finite G in [7], and is the quantum-double of the group algebra $\mathbb{C}[G]$ (see also [10, 11, 12]).

There should also be a vertex operator algebra (VOA) interpretation to this data, assigning a VOA to each finite group. One way to do this is to start with the VOA associated with any even self-dual Euclidean lattice Λ for which $G \leq \mathrm{Aut}\,\Lambda$. Orbifolding it by G should yield our data (with the value $c = \dim \Lambda$). For example, any finite subgroup of $\mathrm{SU}_2(\mathbb{C})$ or $\mathrm{SU}_3(\mathbb{C})$ works with the $\Lambda = E_8$ root lattice. By Cayley's Theorem, such a lattice Λ can always be found for a given group G: e.g. embed G in some $\mathfrak{S}_n$ and take Λ to be the orthogonal direct sum of n copies of E_8. In [13, 10], this VOA interpretation is addressed.

Although these holomorphic orbifolds are perhaps too artificial to be of direct interest, it can be expected that they provide a good hint of the behaviour of more general orbifolds. Indeed this is the case – e.g. they can be seen in the theory of permutation orbifolds [14].

Perhaps the most physical incarnation of this modular data is in (2+1)-dimensional quantum field theories where a continuous gauge group has been spontaneously broken to a finite group [15]. Non-abelian anyons (i.e. particles whose statistics are governed by the braid group rather than the symmetric group) arise as topological excitations. The effective field theory describing the long-distance physics is governed by the quantum-group of [7]. Adding a Chern-Simons term corresponds to the cohomological twist to be discussed shortly.

Actually, this modular data arose originally in mathematics, in an important but technical way in Lusztig's determination of the irreducible characters of the finite groups of Lie type [16, 17]. In describing some of these, the so-called 'unipotent' characters, he was led to consider this modular data for the groups $G \in \{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2\ , \mathfrak{S}_3\ , \mathfrak{S}_4\ , \mathfrak{S}_5\ \}$. For example, $\mathfrak{S}_5$ arises in groups of type E_8. Our primary fields Φ parametrise the unipotent characters associated to a given 2-sided cell in the Weyl group. [16] interprets the fusion algebra as the Grothendieck ring for G-equivariant vector bundles. This Lusztig interpretation is significant as it indicates the richness of the purely group theoretic side of the data we explore below.

An intriguing application to quantum computing‡ has been suggested by Kitaev [18]. One of the main challenges of actually implementing an effective quantum computer is decoherence: interaction with the environment makes quantum superpositions very unstable. The standard approaches to this involve quantum error correction, but Kitaev's proposal is to incorporate into the hardware the non-abelian anyons of [15]. The resulting computer would operate quite robustly amidst localised disturbances.

From many of these approaches, we also expect to have characters $\mathrm{ch}_a(\tau)$ which realise this modular data as in (2.1) and (2.2) below. These appear to be given in [8], at least when G is a subgroup of $E_8(\mathbb{C})$ (see Theorem 4.3 there).

In summary, what we get is a set of modular data (i.e. matrices S and T) for any choice of finite group G. Now, much information about a group can be recovered easily from its character table: e.g. whether it is abelian, simple, solvable, nilpotent, ... (see e.g. Ch.22 of [1]). For instance, G is simple iff for all irreducible $\chi \not\equiv 1$, $\chi(a) = \chi(e)$ only for $a = e$. Thus it can be expected that our modular data, which probably includes the character table, should tell us a lot about the group, i.e. be sensitive to a lot of the group-theoretic properties of G.

It does not appear to be known yet whether the modular data associated to distinct G will always be distinct. A tempting guess would be that this modular data is a function of the group algebra, but that it succeeds in distinguishing groups which have different character tables. An easy result along these lines is: Given any groups G and H with equal matrix S, if G is abelian then $G \cong H$. (The character table of an abelian group also uniquely determines it.) The non-abelian order-8 groups $\mathfrak{D}_4$ and $\mathfrak{Q}_4$ have identical character tables, but their matrices S and T are both different. More generally, we will see below that the matrices S for the groups with at most 50 primary fields are all distinct.

There are two ways to generalise this data. One is by a 'twisting' by some element of a cohomology group [4] (see also [6]). We will look at this twisting in Section 5. This twisting is rather interesting, because completely analogously one can speak of the twisting of modular data associated to Lie groups: the cohomology group there is $\mathbb{Z}$, and 'twisting' by cohomology class $[k]$ gives WZW at level k for that Lie group ! So the untwisted finite group modular data is 'level 0' in this sense. On the other hand, the WZW models at level 0 are trivial. Another difference with WZW is that the cohomology group here is finite, so there are only finitely many different possible 'levels' to this finite group data.

The twisting was incorporated in the quantum-group picture in [7]. How to obtain topological (e.g. oriented knot) invariants from this twisted data was explained in [20]. It turns out that these knot invariants are functions of the "knot group" (i.e. the fundamental group of the complement of the knot). Though non-isotopic knots can have the same knot group, this does not imply that these finite group topological invariants are uninteresting. For instance [21], these invariants can distinguish a knot from its inverse (i.e. the knot with opposite orientation), unlike the more familiar topological invariants coming from affine algebras (or quantum groups $U_q(\mathfrak{g})$).

This twisting is reminiscent, though independent, of *discrete torsion* [19]. The twist

‡ We thank Sander Bais for informing us of this.

changes the modular data, while discrete torsion changes the modular invariant partition function but leaves unchanged the modular data. We use discrete torsion in §4.1.

A second way to generalise this data is suggested by the subfactor interpretation of RCFT. There is a von Neumann subfactor and from this a fusion ring, associated to any group-subgroup pair $G \leq H$ [22]. Our data comes from the diagonal embedding of G in $G \times G$. The fusion ring for generic pairs $G \leq H$ is non-commutative (so e.g. Verlinde's formula will not work), but it is commutative for much more than just $G \leq G \times G$ and so we can expect here a significant generalisation of this finite group data corresponding in some way to physics. To our knowledge, such a physical interpretation has not been developed.

The finite group modular data behaves very differently from the affine modular data. Yet finite group theory is certainly richer than nontwisted affine Lie theory, so its modular data should be explored. It should be interesting both from the group theory and RCFT sides. In particular, it should be interesting to RCFT for the same reasons as the affine data: Affine modular data is regarded as physically interesting primarily because it serves as a toy model, and because of the GKO coset construction; similarly, the finite group modular data is equally a toy model, and is involved in the orbifold construction.

As always, there are unexpected surprises. One of the differences we have found between the finite group data and the affine data is the large number of modular invariants for the former – in contrast the affine modular invariants are surprisingly rare and generally are associated with Dynkin diagram symmetries. There is one family of exceptions to this rule: the orthogonal algebras at level 2 have a rich collection of modular invariants [33]. This was rather mysterious. But what we find is that the only overlap between the affine data and nonabelian finite group data occurs with the orthogonal algebras at level 2 ! This clearly provides a partial resolution of that mystery.

Our main intention with this paper is to make the mathematical physics community more familiar with this modular data, by developing and illustrating some of the basic ideas and constructions. As a large and relatively unexplored class of RCFTs, studying it can correct some of the prejudices we have developed with our rather artificial preoccupation with the affine data. Finally, this material suggests the thought: "There's some kind of analogy relating affine algebras and finite groups" – after all, both are directly associated to topological groups – and we try to work out some of its aspects. See the Table below. Of course there are differences as well as similarities, and identifying these will help us develop a better understanding of generic modular data.

List of Notation

G	a group with finitely many elements
$\mathbb{N}$	the nonnegative integers
ch_a	RCFT characters, see e.g. (2.1), (2.8)
ξ_m	the m-th root of unity $\exp[2\pi\mathrm{i}/m]$
$\mathbb{Z}_n$	the cyclic group (the integers modulo n)
$\mathfrak{S}_n$	the symmetric group of order $n!$
$\mathfrak{A}_n$	the alternating group of order $n!/2$

$\mathfrak{D}_n$	the (binary) dihedral group (3.3) of order $2n$
$\mathfrak{Q}_{2n}$	the (generalised) quaternion group (3.4) of order $4n$
e	the identity element in the group
$\|G\|$	the order (cardinality) of G
K_a	the conjugacy class of a, i.e. $\{gag^{-1} \mid g \in G\}$
R	a set of representatives for each conjugacy class K_a
$\chi, \psi, \ldots$	irreducible characters
$\mathrm{Irr}(G)$	the set of all irreducible characters of G
$G(a,b)$	see below (2.12)
$\deg(\chi)$	dimension of corresponding representation
$C_G(a)$	the centraliser of a in G, i.e. $\{g \in G \mid ga = ag\}$
Φ	set of "primary fields" (a, χ) where $\chi \in \mathrm{Irr}(C_G(a))$
$e(G)$	the exponent of G (smallest n such that $g^n = e$ for all $g \in G$)
$Z(G)$	the centre of G
G'	the commutator subgroup $\langle ghg^{-1}h^{-1} \rangle$
$k(G)$	the class number of G, i.e. the number of conjugacy classes
$H \leq G$	H is a subgroup of G
$H \triangleleft G$	H is a normal subgroup of G
χ_H^G	the induced character of $\chi \in \mathrm{Irr}(H)$, where $H \leq G$
$M(G)$	the Schur multiplier $H^2(G, U(1))$
β	a $U(1)$-valued 2-cocycle
$r(G, \beta)$	the number of inequivalent projective β-representations of G
β-$\mathrm{Irr}(G)$	the set of all irreducible projective β-characters of G
$\tilde{\chi}, \tilde{\psi}, \ldots$	irreducible projective characters
α	a 3-cocycle in $H^3(G, U(1))$
CT	a 3-cocycle α such that all 2-cocycles β_a are coboundaries (see §5.2)
S^α, T^α	the modular data twisted by a 3-cocycle α
$\mathbb{F}_q$	the finite field with q elements

The affine algebra versus finite group analogy

primary fields in Φ	r-tuples $\lambda \in P_+(X_{r,k})$	pairs (g, χ)
level	a number $k \in \mathbb{N}$	a twist $\alpha \in H^3(G, U(1))$
simple currents	extended Dynkin diagram symmetry (except $\hat{E}_{8,2}$)	$Z(G) \times G/G'$ (untwisted case)
conjugation	unextended Dynkin symmetry	involutive Galois permutations
conformal embedding	$X_{r,k} \subset Y_{s,1}$	perhaps $H \triangleleft G$
central charge	$c = \frac{k \dim X_r}{k + h^\vee}$	$c \in 8\mathbb{Z}$
Chern-Simons theory	gauge group is Lie gp	gauge group is finite group
CFT	WZW model	holomorphic orbifold
quantum-group	$U_q(X_r)$, $q^N = 1$	quantum double $D^\alpha(G)$
VOA	Frenkel-Zhu	Dong-Mason
rank-level duality	$\widehat{\mathrm{su}}(n)_k \leftrightarrow \widehat{\mathrm{su}}(k)_n, \ldots$	electric/magnetic duality(??)

2 Untwisted modular data

2.1 Modular data for any RCFT

In this paper we explore what we call the modular data of G, from the perspective of RCFT. This subsection is intended as a quick review of the basic RCFT data (see e.g. [23] for a more comprehensive treatment), and we end it with a brief description of the modular data associated with affine algebras.

Let Φ denote the (finite) set of primary fields in the RCFT. One of these primaries, the "vacuum" '0', is privileged. By the *modular data* we mean the modular matrices S and T, whose entries S_{ab} and T_{ab} are parametrised by $a, b \in \Phi$. These matrices are unitary, T is diagonal, S is symmetric, and together they define a representation of $\mathrm{SL}_2(\mathbb{Z})$ realised by the modular action on the RCFT characters $\mathrm{ch}_a(\tau)$:

$$\mathrm{ch}_a(-1/\tau) = \sum_{b\in\Phi} S_{ab}\,\mathrm{ch}_b(\tau), \tag{2.1}$$
$$\mathrm{ch}_a(\tau+1) = \sum_{b\in\Phi} T_{ab}\,\mathrm{ch}_b(\tau), \tag{2.2}$$

so $(ST)^3 = S^2 =: C$ commutes with S and T and is a permutation matrix called *charge conjugation*. Note that $S^* = SC$ and $C0 = 0$. These ch_a are assumed to be linearly independent and in general will depend on other variables than τ, but it is conventional to write only τ.

All RCFTs in this paper are *unitary*. This implies that the entries $S_{a0} = S_{0a}$ will all be strictly positive. The ratio $\frac{S_{a0}}{S_{00}}$ is called the *quantum dimension* of a. The *fusion coefficients* $N_{ab}^c \in \mathbb{N}$ of the theory can be obtained from S using *Verlinde's formula*

$$N_{ab}^c = \sum_{d\in\Phi} \frac{S_{ad}\,S_{bd}\,S_{cd}^*}{S_{0d}}. \tag{2.3}$$

A *simple current* [24] can be defined as any $j \in \Phi$ with quantum-dimension 1: i.e. $S_{j0} = S_{00}$. To any simple current j is associated a "charge" $Q_j : \Phi \to \mathbb{Q}$, an integer R_j, and a permutation J of Φ, such that $J0 = j$,

$$S_{Ja,b} = \exp[2\pi\mathrm{i}\,Q_j(b)]\,S_{ab}, \tag{2.4}$$
$$T_{Ja,Ja}T_{aa}^* = \exp[2\pi\mathrm{i}\,(R_j\,\tfrac{n-1}{2n} - Q_j(a))], \tag{2.5}$$
$$N_{j,b}^c = \delta_{c,Jb}, \tag{2.6}$$

where n in (2.5) is the order of J. Note that $nQ_j \in \mathbb{Z}$. For instance $j = 0$ is a simple current, corresponding to the identity permutation. Simple currents define an abelian group given by composition of the corresponding permutations.

The matrix S also obeys another important symmetry, called *Galois* [25]. In particular, the entries S_{ab} will lie in some cyclotomic field $\mathbb{Q}(\xi_m)$, where ξ_m throughout this paper denotes the mth root of unity $\exp[2\pi\mathrm{i}/m]$. This means that each S_{ab} can be written as a polynomial in ξ_m with rational coefficients. The Galois group $\mathrm{Gal}(\mathbb{Q}(\xi_m)/\mathbb{Q})$ consists of all automorphisms of the field $\mathbb{Q}(\xi_m)$, and is isomorphic to the multiplicative group $\mathbb{Z}_m^\times$ of integers coprime to m. In particular, to any $\ell \in \mathbb{Z}_m^\times$ we get an automorphism $\sigma_\ell \in \mathrm{Gal}(\mathbb{Q}(\xi_m)/\mathbb{Q})$ sending ξ_m to ξ_m^ℓ. Applying it to entries of S, we get

$$\sigma(S_{ab}) = \epsilon_\sigma(a)\,S_{\sigma(a),b} = \epsilon_\sigma(b)\,S_{a,\sigma(b)}, \tag{2.7}$$

where each $\epsilon_\sigma(a)$ is a sign ± 1, and where $a \mapsto \sigma(a)$ is a permutation of Φ. The simplest example of this Galois action is charge conjugation C: σ_{-1} is complex conjugation; the corresponding $\epsilon_{\sigma_{-1}}$ is identically $+1$, while the permutation $a \mapsto \sigma_{-1}a$ is given by C.

The Galois and simple current permutations respect each other: $\sigma_\ell(Ja) = J^\ell(\sigma_\ell(a))$. Also, $\epsilon_\sigma(Ja) = \epsilon_\sigma(a)$ and $Q_j(\sigma_\ell(a)) \equiv \ell\, Q_j(a) \pmod 1$.

The one-loop partition function $\mathcal{Z}(\tau)$ of the RCFT is a modular invariant sesquilinear combination of RCFT characters:

$$\mathcal{Z}(\tau) = \sum_{a,b\in\Phi} M_{ab}\, \mathrm{ch}_a(\tau)\, \mathrm{ch}_b(\tau)^*. \tag{2.8}$$

Modular invariance means that the coefficient matrix M commutes with S and T. In addition, $M_{00} = 1$, and each coefficient is a non-negative integer: $M_{ab} \in \mathbb{N}$. Any such $\mathcal{Z}$ or M is called a *physical invariant* or *modular invariant* (such an M may or may not be realised as a partition function for an RCFT). $M = I$ is always a physical invariant. For a given choice of modular data, there will only be finitely many physical invariants.

A special family of physical invariants are the *automorphism invariants*, where M is a permutation matrix. An example is $M = C$. The product MM' of any automorphism invariant M with any physical invariant M' will also be a physical invariant.

One of the uses of both simple currents and this Galois action is that they can be used to construct physical invariants. The simplest construction (originally due to [26] but since generalised considerably, see [27] and references therein) starts with any simple current j, with order n say. Then

$$M(j)_{ab} = \sum_{i=1}^{n} \delta_{J^i a,b}\, \delta^1(Q_j(a) + \frac{i}{2n} R_j), \tag{2.9}$$

where $\delta^1(x) = 1$ if $x \in \mathbb{Z}$ and $= 0$ otherwise. For instance the choice $j = 0$ yields the identity matrix $M(0) = I$. $M(j)$ will be a physical invariant iff $T_{jj}T^*_{00}$ is an nth root of 1. $M(j)$ will be an automorphism invariant iff $T_{jj}T^*_{00}$ is a *primitive* nth root of 1.

Given any order-two Galois automorphism σ with the properties that $\sigma(0) = 0$ and $T_{\sigma a,\sigma a} = T_{a,a}$, then $M(\sigma)_{ab} := \delta_{b,\sigma a}$ defines an automorphism invariant (this construction was originally due to [28] but was since generalised). Again, C is an example.

The most familiar example of modular data comes from the affine nontwisted algebras. The literature on affine modular data is very extensive (indeed this affine $\leftrightarrow$ finite group imbalance is a primary motivation for this paper) and it certainly is not our intention to review it here. See [29, 23] for a deeper treatment.

Choose any finite-dimensional simple Lie algebra X_r, and any *level* $k \in \mathbb{N}$. $X_r^{(1)} = \hat{X}_r$ denotes the infinite-dimensional nontwisted affine Kac-Moody algebra [29]. Its integrable highest-weight modules at level k are parametrised by $(r+1)$-tuples $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_r)$, where each $\lambda_i \in \mathbb{N}$, and $\sum_{i=0}^r a_i^\vee \lambda_i = k$. The positive constants $a_i^\vee$ are called colabels – e.g. for $A_r^{(1)}$ or $C_r^{(1)}$ all $a_i^\vee = 1$. The (finite) collection of these highest-weights is denoted P_+^k and is the set of primary fields Φ. The "vacuum" 0 is $(k, 0, 0, \ldots, 0)$. The matrices S and T are given in Theorem 13.8 of [29]: S is related to the corresponding Lie group characters at elements of finite order, while T is related to the eigenvalues of the quadratic Casimir. A useful parameter is the dual Coxeter number $h^\vee := \sum_{i=0}^r a_i^\vee$, and a useful $(r+1)$-tuple is the Weyl vector $\rho := (1,1,1,\ldots,1)$. The affine fusion coefficients are given combinatorially by what is usually called the Kac-Walton formula [29, 30], though it has other co-discoverers.

The simplest example is the choice $X_r = A_1$ (i.e. $\mathrm{sl}_2(\mathbb{C})$) for which $h^\vee = 2$. For any level $k \in \mathbb{N}$, one has $P_+^k = \{(k,0),(k-1,1),\ldots,(0,k)\}$. The modular data is then

$$S_{\lambda\mu} = \sqrt{\tfrac{2}{k+2}}\,\sin(\pi\tfrac{(\lambda_1+1)(\mu_1+1)}{k+2}), \tag{2.10}$$

$$T_{\lambda\mu} = \exp[\pi\mathrm{i}\tfrac{(\lambda_1+1)^2}{2(k+2)} - \tfrac{\pi\mathrm{i}}{4}]\,\delta_{\lambda,\mu}. \tag{2.11}$$

The charge conjugation of affine modular data corresponds to an order-one or -two symmetry of the unextended Dynkin diagram. For $A_1^{(1)}$ it is trivial, but for the other $A_r^{(1)}$ it is non-trivial. The simple currents were classified in [31], and with one exception ($E_8^{(1)}$ level 2) correspond to symmetries of the extended Dynkin diagram. $A_1^{(1)}$ has one non-trivial such symmetry, corresponding to the interchange of the nodes labeled '0' and '1'. This yields a permutation J of P_+^k given by $J(\lambda_0,\lambda_1) = (\lambda_1,\lambda_0)$, and corresponding to primary field $j = J0 = (0,k)$. Note that $Q_j(\lambda) = (-1)^{\lambda_1}$ and $R_j = k$. Thus $M(j)$ is a physical invariant whenever k is even. It is an automorphism invariant whenever $k \equiv 2 \pmod 4$.

The Galois action can be understood geometrically in terms of the affine Weyl group [25]. In general $\sigma(0) \neq 0$ and the signs ϵ_σ are both positive and negative. The S entries lie in the cyclotomic field $\mathbb{Q}(\xi_{f\,(k+h^\vee)})$, where f is a number depending only on the choice of algebra – e.g. $f = 4$ for A_1, and $f = 3,4,1,4,3$ for $X_r = E_6, E_7, E_8, F_4, G_2$, respectively (we will use this in the proof of Theorem 1 below). Also, $h^\vee = 12, 18, 30, 9, 4$ for those algebras.

A surprising feature is that almost all of the affine physical invariants can be understood in terms of the symmetries of the extended Dynkin diagram. For example, the classification for $A_1^{(1)}$ was done in [32]: apart from $M = M(0) = I$ (for all k) and $M = M(j)$ (for all even k), there are only 3 other physical invariants (at $k = 10, 16, 28$). The physical invariant classification for general affine algebras is still open; see [33] and references therein for a list of the algebras and levels for which it has been completed.

2.2 Finite group modular data

After this brief review of modular data in general RCFTs, and of affine modular data as specific examples, we turn to the modular data associated with a finite group G.

Fix a set R of representatives of each conjugacy class of G. So the identity e of G is in R, and more generally the centre $Z(G)$ of G is a subset of R.

For any $a \in G$, let K_a be the conjugacy class containing a. By $C_G(a)$ we mean the *centraliser* of a in G, i.e. the set of all elements in G which commute with a. $C_G(a)$ is a subgroup of G, and in fact $|G| = |K_a| \cdot |C_G(a)|$.

The primary fields of the G modular data are labelled by pairs (a,χ), where $a \in R$, and where χ is an irreducible character of $C_G(a)$. We will write $\Phi = \Phi(G)$ for the set of all these pairs. In this set, the vacuum '0' corresponds to $(e,1)$, with 1 the character of the trivial representation of $G = C_G(e)$. It will be convenient at times to identify (a,χ) with each $(g^{-1}ag, \chi^g)$, where $\chi^g(h) = \chi(ghg^{-1})$ is an irreducible character of $C_G(g^{-1}ag)$.

We set

$$S_{(a,\chi),(b,\chi')} = \frac{1}{|C_G(a)|\,|C_G(b)|} \sum_{g\in G(a,b)} \chi(gbg^{-1})^*\, \chi'(g^{-1}ag)^* \tag{2.12}$$

$$= \frac{1}{|G|} \sum_{g\in K_a, h\in K_b\cap C_G(g)} \chi(xhx^{-1})^*\, \chi'(ygy^{-1})^*, \tag{2.13}$$

$$T_{(a,\chi),(a',\chi')} = \delta_{a,a'}\delta_{\chi,\chi'}\frac{\chi(a)}{\chi(e)}, \tag{2.14}$$

where $G(a,b) = \{g \in G \mid agbg^{-1} = gbg^{-1}a\}$, and where x, y are any solutions to $g = x^{-1}ax$ and $h = y^{-1}by$. The equivalence of (2.12) and (2.13), and the fact that (2.13) does not depend on the choice of x, y, are easy arguments.

Note that the strange set $G(a,b)$ is precisely the set of all g for which $g^{-1}ag \in C_G(b)$ and $gbg^{-1} \in C_G(a)$. If there are no such g, then $G(a,b)$ is empty and the sum (and the matrix entry) would be equal to 0. A special case of this is Proposition 1(a) below.

S is symmetric and unitary, and gives rise (via Verlinde's formula) to non-negative integer fusion coefficients. The fusion coefficient $N_{(a,\chi_1),(b,\chi_2)}^{(c,\chi_3)}$ in fact has algebraic interpretations. For example, let ρ_i be representations corresponding to characters χ_i. Write $T(a)$ for a set of representatives of the left cosets of $C_G(a)$. Define the space

$$X = \bigoplus (\rho_1 x) \otimes (\rho_2 y), \tag{2.15}$$

where the sum is over all $x \in T(a)$, $y \in T(b)$ for which $(x^{-1}ax)\,(y^{-1}by) = c$, and where we interpret '$\rho_1 x$', '$\rho_2 y$' as belonging to the induced representations $(\rho_1)C_G(a)^G$, $(\rho_2)^G_{C_G(b)}$, respectively (we'll say more on induced representations below). X carries a representation of $C_G(c)$, using the usual coproduct. The fusion coefficient equals the multiplicity of ρ_3 in X [10]. (There are other interpretations of these fusion coefficients – see e.g. [16, 6, 7].)

Since a is in the centre of $C_G(a)$, $\chi(a)/\chi(e)$ in (2.14) will be an nth root of 1, where n is the order of a. Because $T_{00} = 1$, the given normalisation of T corresponds to the central charge c being a multiple of 24; we are free to multiply (2.14) by any third root of 1, permitting c to be other multiples of 8.

In order to more fully exploit the formula (2.12), it is important to understand the notion of *induced character* (or representation) (see e.g. [3]). Given a representation $\rho : H \to W$ of a subgroup H of G, of character χ, we call a representation $\rho' : G \to V$ of G an induced representation if V equals the direct sum of vector spaces W_a, for each coset $aH \in G/H$, where W_a is defined by $\rho'(aH)W = W_a$. An induced representation always exists and is unique up to isomorphism, and is denoted ρ_H^G. In terms of characters, we get the important formula (somewhat reminiscent of (2.12)):

$$\chi_H^G(g) = \frac{1}{|H|} \sum_{\substack{a\in G \\ a^{-1}ga\in H}} \chi(a^{-1}ga). \tag{2.16}$$

Suppose that $\chi' \in \mathrm{Irr}(C_G(b))$ is the restriction to $C_G(b)$ of some character $\overline{\chi}'$ defined on the group $G(a|b) := \langle G(a,b), C_G(b)\rangle$ (this happens fairly often in practice – see e.g. Ch.27 of

[1] for a relevant discussion). Then provided $G(a,b) \neq \emptyset$, (2.12) collapses to

$$S_{(a,\chi),(b,\chi')} = \frac{1}{|C_G(b)|} \overline{\chi}'(a)^* \, \chi^G_{C_G(a)}(b)^*. \tag{2.17}$$

Equation (2.17) for instance applies whenever χ' is identically 1. Another important instance of (2.17) is when z lies in the centre $Z(G)$, since then $C_G(z) = G$ and we get

$$S_{(a,\chi),(z,\chi')} = \frac{\chi(z)^*}{|C_G(a)|} \chi'(a)^*. \tag{2.18}$$

For instance, the quantum dimension of $(a,\chi) \in \Phi$ is

$$\frac{S_{(a,\chi),(e,1)}}{S_{(e,1),(e,1)}} = |K_a| \deg(\chi), \tag{2.19}$$

which amazingly enough is always a positive *integer* ! In fact from (2.16), (2.19) has a simple group-theoretic interpretation: it is the degree of the induced character $\chi^G_{C_G(a)}$. This integrality is very unusual, and shows that generically this finite group modular data is qualitatively different from affine data.

Equation (2.17) says we can expect many 0's in S. Any character $\chi \in \mathrm{Irr}(H)$ with degree $\chi(e) > 1$ will have a zero – in fact $\chi(h)$ will be zero for at least $|Z(H)|\,(\chi(e)^2 - 1)$ group elements (see e.g. Ch. 23 of [1]). Hence the (z,χ) row will have 0's iff χ is not of degree 1. Dually, for any $b \in H$ there will be at least $k(H) - |C_H(b)|$ different $\chi' \in \mathrm{Irr}(H)$ with $\chi'(b) = 0$, where $k(H)$ is the class number of H, so for instance the $(a,1)$ row will have 0's whenever e.g. there are elements in $C_G(a)$ whose centraliser in $C_G(a)$ is small. This seems to also be different from the affine case, where 0's are quite rare and fairly generically tend to be due to simple current fixed points.

Equation (2.18) also says

$$\frac{S_{(e,\chi),(a,\chi')}}{S_{(e,1),(a,\chi')}} = \chi(a)^*. \tag{2.20}$$

As long as we could identify by looking at S and T the primaries of the form $(e,\star)$, then the matrix S will contain the character table of G, and we would know that groups with different character tables would necessarily have different matrices S and T.

From (2.19), we see that the simple currents are precisely the pairs (z,φ), where z lies in the centre $Z(G)$ of G, and φ is a degree-1 character of G. Thus the group of simple currents is isomorphic to the direct product $Z(G) \times G/G'$, where G' is the commutator subgroup of G. The simple current $j = (z,\varphi)$ corresponds to permutation $J_{(z,\varphi)}(a,\chi) = (za, \varphi\chi) \in \Phi$, and charge $e^{2\pi \mathrm{i} Q} = \varphi(a)^*\chi(z)^*/\chi(1)$, which is always a root of unity as it should be. We get that $M(z,\varphi)$ (in the notation of the previous subsection) is always a physical invariant; it is an automorphism invariant iff z and φ have the same order.

Generic groups have many simple currents. A group for which $G = G'$ is called *perfect*; a group will have no non-trivial simple currents iff it is perfect and has trivial centre. For example this happens whenever G is non-cyclic simple. All perfect groups with small orders have been classified [34], and using this we can list all groups G with order $|G| < 688128$ which have no non-trivial simple currents. The orders under 2000 of these groups are (see also Prop. 1(g)): 60, 168, 360, 504, 660, 960 (twice), 1092, 1344 (twice), and 1920.

When (and only when) G is abelian, all primaries will be simple currents, and hence the modular data will be rather trivial and uninteresting. This can also happen with the affine algebras: namely, the simply-laced algebras $A_r^{(1)}, D_r^{(1)}, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}$ at level 1.

Charge conjugation takes (a,χ) to (a^{-1},χ^*), where the complex conjugate χ^* is the character of the contragredient representation. Now a^{-1} may not lie in our set R of conjugacy class representatives: recall that by (a^{-1},χ^*) we really mean $(g^{-1}a^{-1}g,\chi^{g\,*})\in\Phi$ where $g^{-1}a^{-1}g\in R$. This is not so trivial as it may seem: if a and a^{-1} are conjugate, (a,χ) and (a^{-1},χ^*) may be identified even though χ is complex-valued. An example is $\mathfrak{S}_3$, $a=(123)$, given below. Note though from (2.20) that the field $\mathbb{Q}(S)$ contains the field generated over $\mathbb{Q}$ by all character values $\chi(a)$, $\chi\in\mathrm{Irr}(G)$, $a\in G$, so if some characters of G are complex, the C won't be trivial.

The Galois symmetry is also straightforward. The character values of any group lie in the cyclotomic field $\mathbb{Q}(\xi_e)$ where $e=e(G)$ is the exponent of the group (the least common multiple of the orders of all group elements) – see Thm.8.7 of [3]. Hence the entries of S and T lie in the cyclotomic field $\mathbb{Q}(\xi_e)$, so the relevant Galois group is the multiplicative group $\mathbb{Z}_e^\times$. The Galois automorphism σ_ℓ takes (a,χ) to $(a^\ell,\sigma_\ell\circ\chi)$: $\sigma_\ell\circ\chi$ is the function obtained by applying σ_ℓ to each complex number $\chi(a)$; it is an irreducible character (of degree equal to that of χ) iff χ is [3]. Note another curiousity here: the Galois parities ϵ_ℓ are all identically equal to 1 ! This is very different from the generic affine situation. Note also that every Galois permutation fixes the vacuum $(e,1)$ (since every quantum dimension is rational), so large numbers of automorphism invariants will arise generically: whenever $\ell^2\equiv 1 \pmod{e(G)}$, the permutation σ_ℓ on Φ will define an automorphism invariant. We will return to this in §4.1.

Let us collect a few of the observations we have made here.

Proposition 1. (a) Choose any $(a,\chi),(a',\chi')\in\Phi$. If the order of a does not divide the exponent of $C_G(a')$, or if the order of a' does not divide the exponent of $C_G(a)$, then $S_{(a,\chi),(a',\chi')}=0$.

(b) The order of T *equals* (and not merely divides) the exponent of G.

(c) If $S(G)=S(H)$, then $|G|=|H|$.

(d) If $a^{-1}\notin K_a$, then the charge conjugate $C(a,\chi)\neq(a,\chi)$ for all $\chi\in\mathrm{Irr}(C_G(a))$. If $\chi\in\mathrm{Irr}(G)$ is not real-valued, then $C(z,\chi)\neq(z,\chi)$ for all $z\in Z(G)$.

(e) The quantum dimensions $S_{(a,\chi),(e,1)}/S_{(e,1),(e,1)}$ are always integers.

(f) The Galois parities $\epsilon_\sigma(a,\chi)$ are always $+1$, and the vacuum $(e,1)$ is fixed by all σ.

(g) The groups $G\neq\{e\}$ with at most 75 primaries, which have no non-trivial simple currents, are the simple groups $G=\mathfrak{A}_5,\mathrm{PSL}_2(\mathbb{F}_7),\mathfrak{A}_6,\mathrm{SL}_2(\mathbb{F}_8),\mathrm{PSL}_2(\mathbb{F}_{11})$, and $\mathfrak{A}_7$, with $(|G|,|\Phi|)=(60,22),(168,35),(360,44),(504,74),(660,58)$, and (2520,74).

(h) For any group $G\neq\{e\}$, there will be at least 5 physical invariants. If $G/Z(G)$ is nonabelian, then there will be at least $a+(b+c)^2$ physical invariants, where $a=|Z(G)|\,|G/G'|$, b equals the number of subgroups of $Z(G)$, and c equals the number of subgroups of G/G'.

The proof for (b) is the following. Choose any group H, and any $a \in Z(H)$ with prime-power order p^m. Then any irreducible representation of H will send a to the multiple of the identity matrix by some p^mth root of 1, i.e. $\chi(a)/\chi(e)$ will be a p^mth root of 1 for all $\chi \in \mathrm{Irr}(H)$. Now, that root of 1 must be primitive for some χ, as otherwise $a^{p^{m-1}}$ and e would have identical character values. Applying that fact to $H = C_G(a)$ gives us the desired order of T.

The proof of (c) is the comparison of $S_{(e,1),(e,1)}$ for G and H.

A constructive proof of (h) is given in §4.2. Incidentally, this bound can probably be significantly improved. $G = \mathbb{Z}_2$, with 6 physical invariants, is probably the lowest number. By comparison, affine algebras have relatively few physical invariants: e.g. both $B_\ell^{(1)}$ and $C_\ell^{(1)}$ at generic levels are expected to have only 2. ■

We have already remarked on several occasions above, that the finite group modular data are very different from affine modular data. At this stage (we will have more to say when we come to general twisted finite group modular data), this statement can be substantiated by noting that the two sets have a very small intersection.

Theorem 1. Let S and T be the Kac-Peterson matrices corresponding to an affine algebra $X_r^{(1)}$ at some level $k \geq 1$ (where X_r is simple). Let G be a finite group with $S(G) = S$ and $T(G) = \varphi T$ for some third root φ of 1. Then either

(i) $(X_r, k) = (E_8, 1)$ and $G = \{e\}$, or

(ii) $(X_r, k) = (D_{8n}, 1)$, and $G = \mathbb{Z}_2$.

Sketch of proof Write $n = k + h^\vee$. Recall that there is a Galois automorphism for any ℓ coprime to fn, where f for the exceptional algebras was given in §2.1. A consequence of the affine Weyl interpretation of the affine Galois permutation σ_ℓ of P_+^k is that the vectors $\sigma_\ell(\lambda + \rho)$ and $\ell\,(\lambda + \rho)$ have the same norm mod $2n$.

A nice way to handle the exceptional algebras is to check that $\ell\rho$ and ρ have the same norm (mod $2n$) for any ℓ coming from Galois (since all Galois automorphisms here will fix the vacuum 0). For instance, for E_7 we get $\rho^2 = 399/2$, so we see that $2n$ must divide $(\ell^2 - 1)\,399/2$. Now, the "Definition of 24" says that $\ell^2 - 1$ here can be replaced with 24: more precisely, the gcd of all numbers $\ell^2 - 1$, for ℓ coprime to fn, will equal $\gcd(24, (fn)^\infty)$. Hence, n must divide $2 \cdot 3^2 \cdot 7 \cdot 19$.

Now, $n > h^\vee = 18$. Also, if n is too big (i.e. if there is an ℓ coprime to fn such that $(h^\vee - 1)\ell < n$), then $\sigma_\ell(0)$ will equal $(m, \ell - 1, \ell - 1, \ldots, \ell - 1)$, where $m = n - 1 - \ell\,(h^\vee - 1)$, violating the result that 0 is fixed by all Galois automorphisms. For E_7, this is the condition that no ℓ, $1 < \ell < \frac{n}{17}$, can be coprime to $2n$.

We can now write down the possibilities for n: they are $19, 21, 38, 42, 57, 63$. Subtracting 18 gives the possible levels. $n = 19$ fails, since it would have to correspond to an abelian group with order $\frac{1}{S_{00}} = \sqrt{2}$. The remaining 5 possibilities all have non-integral quantum dimensions.

The other exceptional algebras are all done similarly. The only surviving n for G_2 are $n = 7, 8, 14$; for F_4 are $n = 12, 13, 18, 36, 39$; for E_6 are $n = 18, 24, 26, 36, 52$; and for E_8 are $n = 40, 48, 60, 62, 80, 120$. In all these cases, a weight with non-integral quantum dimension is easily found.

The best way to handle the classical algebras is to compute the quantum dimension of any weight in the Galois orbit of 0, and show it must be larger than 1, for some σ_ℓ. For now consider $k > 1$ for A_r and C_r, and $k > 2$ for B_r and D_r. Up to a sign, the quantum dimension of $\sigma_\ell(0)$ for the algebras A_r, B_r, C_r, D_r is, respectively,

$$\prod_{a=1}^{r} \frac{\sin(\pi\ell a/n)^{r+1-a}}{\sin(\pi a/n)^{r+1-a}}, \tag{2.21}$$

$$\prod_{a=0}^{r-1} \frac{\sin(\pi\ell\,(a+1/2)/n)}{\sin(\pi\,(a+1/2)/n)} \prod_{b=1}^{2r-2} \frac{\sin(\pi\ell b/n)^{[\frac{2r-b}{2}]}}{\sin(\pi b/n)^{[\frac{2r-b}{2}]}}, \tag{2.22}$$

$$\prod_{a=1}^{r-1} \frac{\sin(\pi\ell a/n)^{r-a}\,\sin(\pi\ell\,(a-1/2)/n)^{r-a}}{\sin(\pi a/n)^{r-a}\,\sin(\pi\,(a-1/2)/n)^{r-a}} \prod_{b=r}^{2r-1} \frac{\sin(\pi\ell b/2n)}{\sin(\pi b/2n)}, \tag{2.23}$$

$$\prod_{a=1}^{r-1} \frac{\sin(\pi\ell a/n)^{[\frac{2r-a+1}{2}]}}{\sin(\pi a/n)^{[\frac{2r-a+1}{2}]}} \prod_{b=r}^{2r-3} \frac{\sin(\pi\ell b/n)^{[\frac{2r-b-1}{2}]}}{\sin(\pi b/n)^{[\frac{2r-b-1}{2}]}}, \tag{2.24}$$

where $[x]$ here denotes the greatest integer not more than x. The absolute value of each of these is quickly seen to be greater than 1 unless $\ell \equiv \pm 1 \pmod{n}$ (for A_r or D_r) or $\ell \equiv \pm 1 \pmod{2n}$ (for B_r and C_r). This exhausts the possible ℓ coprime to fn only if the Euler totient $\varphi(n) \leq 2$ (for A_r and D_r) or $\varphi(2n) \leq 2$ (for B_r and C_r). By definition $\varphi(m)$ is the number of h, $1 \leq h \leq m$, coprime to m; it is less than 3 only for $m = 1, 2, 3, 4, 6$. So only A_1 level 2, A_2 level 3, A_1 level 4, and A_3 level 2 survive, but their central charges aren't multiples of 8.

The series A_r, D_{odd} and D_{even} at $k = 1$ possess only simple currents (so would have to correspond to an abelian group G), and have the fusion groups $\mathbb{Z}_{r+1}, \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2$ respectively. Abelian G has fusion group $G \times G$, so that leaves only D_{even} and $G = \mathbb{Z}_2$. D_n level 1 has $c = n$, concluding the argument. B_r level 1 has only 2 primaries so is covered e.g. by Theorem 2 below. The modular data of C_r level 1 is identical with that of A_1 level r.

Level 2 for D_r can be handled by requiring $|G| = \frac{1}{S_{00}} = 2\sqrt{2r}$ and the quantum dimension $\frac{S_{\Lambda r,0}}{S_{00}} = \sqrt{r}$ to both be integers (see [33] for the necessary S entries). For B_r at level 2, first read off from [33] that $|G| = \frac{1}{S_{00}} = 2\sqrt{2r+1}$, so $2r + 1 = s^2$ for some odd integer s. Now, $T_{\Lambda_1,\Lambda_1} = \mathrm{i}\,\exp[-\pi\mathrm{i}\,\frac{1}{2s^2}]$ is a primitive s^2-root of 1, but $T(G)$ will have order dividing $|G| = 2s$, and so those matrices cannot be equal. ■

For which groups will the number of primaries be low? Consider the formula

$$|\Phi(G)| = \sum_{a\in R} k(C_G(a)), \tag{2.25}$$

where $k(H)$ is the class number of H, i.e. the number of conjugacy classes in, or irreducible representations of, H. Note that the smaller $k(G)$ is, the fewer summands there will be in (2.25), the larger each conjugacy class $|K_a|$ will tend to be, so the smaller the centralisers $|C_G(a)| = \frac{|G|}{|K_a|}$ will tend to be, and the smaller the $k(C_G(a))$ in (2.25) will tend to be. Thus, we should expect $|\Phi|$ to grow with $k(G)$. The groups with class number less than 13 are classified [35]. This allows all G with at most 77 primaries to be listed. We make this argument precise in the proof of Thm. 2 below.

When G is abelian, it has $|G|^2$ primaries – this is the extreme case. The number of primaries for the even dihedral groups $\mathfrak{D}_{2n}$, and the quaternion group $\mathfrak{Q}_{2n}$, are both $2n^2+14$, which grows like $|G|^2/8$. The odd dihedral groups $\mathfrak{D}_m$, m odd, have $\frac{m^2+7}{2}$ primaries (see below). By comparison (using the data given in Ch.20 of [3]), $\mathrm{SL}_2(\mathbb{F}_q)$ has q^2+8q+9 primaries for q any power of an odd prime, and q^2+q+2 primaries for q any power of 2 – in either case, that number grows like $|G|^{2/3}$. Our computations so far suggest the following rule of thumb: the more abelian the group is, the messier it behaves (i.e. the more its primaries, the more its physical invariants, etc), while the closer the group is to being non-abelian simple, the better behaved it will be.

From this point of view, an interesting measure of how complicated a group is relative to its size, is the ratio

$$\mathcal{N}(G) := \frac{\log|\Phi(G)|}{\log|G|}. \tag{2.26}$$

It ranges from 0 to 2, with 2 achieved iff G is abelian. How low can $\mathcal{N}(G)$ be ? Some small values are $\mathcal{N}(\mathfrak{A}_5)\approx .75$, $\mathcal{N}(\mathfrak{A}_7)\approx .55$ and $\mathcal{N}(M_{11})\approx .49$. Sporadic simple groups like the Monster should have $\mathcal{N}$ very small.

Theorem 2. There are precisely 33 groups G with at most 50 primaries:

- the abelian groups $G=\mathbb{Z}_1,\mathbb{Z}_2,\mathbb{Z}_3,\mathbb{Z}_4,\mathbb{Z}_2\times\mathbb{Z}_2,\mathbb{Z}_5,\mathbb{Z}_6,\mathbb{Z}_7$, with precisely $|G|^2$ primaries;
- the symmetric and alternating groups $\mathfrak{S}_3,\mathfrak{A}_4,\mathfrak{S}_4,\mathfrak{A}_5,\mathfrak{S}_5,\mathfrak{A}_6$ with $(|G|,|\Phi|)=(6,8)$, (12,14),(24,21),(60,22),(120,39),(360,44);
- the (semi)dihedral and quaternion groups $\mathfrak{D}_5,\mathfrak{D}_4,\mathfrak{Q}_4,\mathfrak{D}_7,\mathfrak{D}_9,\mathfrak{D}_8,\mathfrak{Q}_8,S\mathfrak{D}_8$ with $(|G|, |\Phi|)=(10,16),(8,22),(8,22),(14,28),(18,43),(16,46),(16,46),(16,46)$;
- the Frobenius groups $\mathbb{Z}_5\times_f\mathbb{Z}_4$, $\mathbb{Z}_7\times_f\mathbb{Z}_3$, $\mathbb{Z}_3^2\times_f\mathbb{Z}_2$, $\mathbb{Z}_3^2\times_f\mathbb{Z}_4$, $\mathbb{Z}_3^2\times_f\mathfrak{Q}_4$, $\mathbb{Z}_7\times_f\mathbb{Z}_6$, $\mathbb{Z}_{11}\times_f\mathbb{Z}_5$ with $(|G|,|\Phi|)=(20,22),(21,32),(18,44),(36,36),(72,32),(42,44),(55,49)$;
- the remaining groups $\mathbb{Z}_2\times\mathfrak{S}_3$, DC_3, $\mathrm{PSL}_2(\mathbb{F}_7)$, and $\mathrm{SL}_2(\mathbb{F}_3)$, with $(|G|,|\Phi|)=(12,44)$, (12,32), (168,35), (24,42).

The semidihedral group $S\mathfrak{D}_{4m}$ is defined by the presentation

$$S\mathfrak{D}_{4n}=\langle r,s\mid r^{4n}=s^2=e,\, srsr=r^{2n}\rangle. \tag{2.27}$$

and has order $8m$. The other groups are defined in [35]. An interesting consequence of Theorem 2, as mentioned earlier, is that all groups with $|\Phi(G)|\le 50$ are uniquely determined by their matrix S.

A useful quantity for proving Thm. 2 is $h(n)$, the minimum possible class number $k(H)$ for H with order n and non-trivial centre. Knowing $h(n)$ gives a lower bound for $|\Phi(G)|$ once we know the orders $|C_G(a)|$ of its centralisers. [35] can be used for the smaller n, basic results on the classification of finite groups, as well as the congruence $k(H)\equiv|H| \pmod{16}$ when $|H|$ is odd, give us other n. What we find is e.g. $h(n)=n$ for all $n\le 27$, except for $n = 8, 12, 16, 18, 20, 24$ (with $h(n) = 5, 6, 7, 9, 8, 7$, resp). Also useful is the largest value $\ell(k)$ of $|H|$, for H with non-trivial centres and class number $k(H) \leq k$. For instance for $k = 1, 2, \ldots, 8$ we get $\ell = 1, 2, 3, 4, 8, 12, 24, 48$.

Consider the smallest $|\Phi|$ can be if $k(G) \geq 13$. Note that at most one $a \in R$ can have a centraliser $C_G(a)$ with class number 2 — otherwise if there were two then together exactly $|G|/\ell(2) + |G|/\ell(2) > |G| - 1$ elements would be in the (disjoint) conjugacy classes $K_a \cup K_b \subset G - e$. More generally, at most $\ell(k) - 1$ a's could have $k(C_G(a)) \leq k$. Thus (2.25) will be bounded below by $13 + 2 + 3 + 4 + 4 \cdot 5 + 4 \cdot 6 + 7 = 73$. Tightening the argument (e.g. $\frac{1}{2} + \frac{1}{3} + \frac{1}{4}$ is too big) gives $|\Phi| \geq 78$. Thus using the tables of [35] we would be able to find all groups with at most 77 primaries. Similarly, to do $|\Phi| \leq 50$ it is enough to consider $k(G) \leq 9$. In [35] are also given the orders of the centralisers $C_G(a)$. It is now straightforward to get the Theorem. ■

3 Examples

In this section we give a number of explicit examples of untwisted modular data. We also identify their physical invariants in some cases, leading to a perplexing situation we will discuss more fully next section.

Incidentally, a simple construction is direct product: the modular data for the direct product $G \times H$ is easily obtained from that of G and H. For example, $\Phi(G \times H) = \Phi(G) \times \Phi(H)$, $S(G \times H)$ is the Kronecker matrix product $S(G) \otimes S(H)$, etc. Of course, semi-direct product in general will be much more difficult to work out.

3.1 Abelian groups

Abelian G (untwisted) is trivial to work out, but also very uninteresting. Write G in the following canonical way: $G \cong \mathbb{Z}_{d_1} \times \mathbb{Z}_{d_2} \times \cdots \times \mathbb{Z}_{d_s}$ where $d_1 | d_2 | \cdots | d_s$. For convenience, define a bilinear form on $\mathbb{Z}^s$ by $\langle m, n \rangle = \sum_i \frac{m_i n_i}{d_i}$. We can identify Φ here with the $2s$-tuple $(m, n) \in \mathbb{Z}^s \times \mathbb{Z}^s$, where $0 \leq m_i \leq d_i - 1$ and $0 \leq n_i \leq d_i - 1$: in particular, m corresponds to the group element $(m_1, m_2, \ldots, m_s) \in \mathbb{Z}_{d_1} \times \cdots \times \mathbb{Z}_{d_s}$, and n corresponds to the character φ_n of G defined by $\varphi_n(m) = \exp[2\pi \mathrm{i} \langle m, n \rangle]$. The matrices S and T are given by

$$S_{(m,n),(m',n')} = \frac{1}{|G|} \exp[-2\pi \mathrm{i}(\langle m', n \rangle + \langle m, n' \rangle)], \qquad T_{(m,n),(m,n)} = \exp[2\pi \mathrm{i} \langle m, n \rangle]. \tag{3.1}$$

All $(m, n) \in \Phi$ are simple currents, with composition given by pairwise addition. Charge conjugation takes (m, n) to $(-m, -n)$, and more generally the ℓth Galois automorphism (for $\ell \in \mathbb{Z}_{d_s}^\times$) sends (m, n) to $(\ell m, \ell n)$.

G will have many physical invariants, but they can all be most elegantly interpreted using lattices as was explained in [36] (the standard reference for lattice theory is [37]). In particular let Λ be the $4s$-dimensional integral indefinite lattice, given by the orthogonal direct sum $\Lambda = \sqrt{d_1} II_{2,2} \oplus \sqrt{d_2} II_{2,2} \oplus \cdots \oplus \sqrt{d_s} II_{2,2}$, where $II_{2,2} = II_{1,1} \oplus II_{1,1}$ is the unique 4-dimensional even self-dual indefinite lattice. Then there is a natural one-to-one bijection between the physical invariants of G and the even self-dual 'gluings' of Λ, i.e. the even self-dual $4s$-dimensional lattices containing Λ. For instance, for $G = \mathbb{Z}_p$ (p prime), one finds that there are precisely 6 (if $p=2$) or 8 (if $p>2$) physical invariants. Two of these are

$$(\sum_{i=0}^{p-1} \mathrm{ch}_{i0})\,(\sum_{j=0}^{p-1}\mathrm{ch}^*_{0j}) \qquad \text{and} \qquad \sum_{i,j=0}^{p-1} \mathrm{ch}_{ij}\,\mathrm{ch}^*_{-j,-i}\,, \tag{3.2}$$

using obvious notation.

The non-abelian groups are much more interesting, and we turn to them in the next subsection. The number of non-abelian groups of order $n\le 50$ are 1 (for $n=6,10,14,21,22,26,34,38,39,46$), 2 ($n=8,27,28,44$), 3 ($n=12,18,20,30,50$), 5 (for $n=42$), 9 (for $n=16$), 10 (for $n=36$), 11 (for $n=40$), 12 (for $n=24$), 44 (for $n=32$), 47 (for $n=48$), and 0 otherwise.

3.2 Some infinite series

It is not hard to work out S and T for the infinite series $\mathfrak{D}_n$ (dihedral) and $\mathfrak{Q}_{2n}$ (quaternion):

$$\mathfrak{D}_n = \langle r,s \mid r^n = s^2 = e,\; rsrs = e\rangle, \tag{3.3}$$
$$\mathfrak{Q}_{2n} = \langle r,s \mid r^{2n} = e,\; s^2 = r^n,\; rsrs^{-1} = e\rangle, \tag{3.4}$$

The character tables of $\mathfrak{D}_{4n}$ and $\mathfrak{Q}_{4n}$ are identical.

Consider first the even dihedral groups $\mathfrak{D}_{2n}$, of order $4n$. $\mathfrak{D}_{2n}$ has $n+3$ conjugacy classes, with representatives $R=\{e,\, r^n,\, r^k\ (1\le k\le n-1),\, s,\, sr\}$. It has an equal number of irreducible representations; 4 are one-dimensional and $n-1$ are two-dimensional. The characters and centralisers of the various classes are indicated in the following table.

	e	r^n	r^k	s	sr
ψ_0	1	1	1	1	1
ψ_1	1	1	1	-1	-1
ψ_2	1	$(-1)^n$	$(-1)^k$	1	-1
ψ_3	1	$(-1)^n$	$(-1)^k$	-1	1
χ_i	2	$2(-1)^i$	$2\cos\frac{\pi ik}{n}$	0	0
$C_G(g)$	$\mathfrak{D}_{2n}$	$\mathfrak{D}_{2n}$	$\mathbb{Z}_{2n}$	$\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{Z}_2$

It follows that the number of primaries is equal to $|\Phi| = 2n^2+14$. Denote by ψ_i ($0\le i<2n$) the characters of $\langle r\rangle\cong\mathbb{Z}_{2n}$, given by $\psi_i(r^k)=\xi_{2n}^{ik}$. Likewise, denote by ϕ_{ab} and φ_{ab} ($0\le a,b\le 1$) the characters of $\langle s,r^n\rangle\cong\mathbb{Z}_2\times\mathbb{Z}_2$ and $\langle rs,r^n\rangle\cong\mathbb{Z}_2\times\mathbb{Z}_2$, respectively, where $\phi_{ab}(s^kr^{n\ell})=\varphi_{ab}((rs)^kr^{n\ell})=(-1)^{ak+b\ell}$.

The values of the diagonal entries of T follow directly from (2.14). Eq. (2.18) easily computes any S entry involving e or r^n. The remaining non-zero S entries are

$$S_{(r^k,\psi_i),(r^\ell,\psi_j)} = \frac{1}{n}\cos(\pi\frac{\ell i+kj}{n}), \tag{3.5}$$
$$S_{(s,\phi_{ab}),(s,\phi_{cd})} = S_{(rs,\varphi_{ab}),(rs,\varphi_{cd})} = \frac{1}{4}\cdot\begin{cases} (-1)^{a+c}+(-1)^{a+b+c+d} & \text{if } n \text{ even,}\\ (-1)^{a+c} & \text{if } n \text{ odd,}\end{cases} \tag{3.6}$$

and in addition for n odd $S_{(s,\phi_{ab}),(sr,\varphi_{cd})} = \frac{1}{4}(-1)^{a+b+c+d}$.

The odd dihedral groups $\mathfrak{D}_{2n+1}$ can be worked out in the same way. $\mathfrak{D}_{2n+1}$ has $n+2$ conjugacy classes, with representatives $R = \{e,\, r^k\ (1 \le k \le n),\, s\}$. It has two one-dimensional representations, and n two-dimensional representations. The characters and centralisers of the various classes are reproduced in the following table.

	e	r^k	s
ψ_0	1	1	1
ψ_1	1	1	-1
χ_i	2	$2\cos\frac{2\pi ik}{2n+1}$	0
$C_G(g)$	$\mathfrak{D}_{2n+1}$	$\mathbb{Z}_{2n+1}$	$\mathbb{Z}_2$

One finds that the number of primary fields is $|\Phi| = 2n^2 + 2n + 4$. Write ψ_i for the characters of $\langle r\rangle \cong \mathbb{Z}_{2n+1}$ as before, and φ_i for the obvious two characters of $\langle s\rangle \cong \mathbb{Z}_2$. The calculation of S and T proceeds like for the even dihedral groups:

$$S_{(r^k,\psi_i),(r^\ell,\psi_j)} = \frac{2}{2n+1}\cos(2\pi\frac{kj+\ell i}{2n+1}), \tag{3.7}$$

$$S_{(s,\varphi_i),(s,\varphi_j)} = \frac{1}{2}(-1)^{i+j}. \tag{3.8}$$

The semidihedral groups $S\mathfrak{D}_{4m}$ (2.27) have the same matrix T as $\mathfrak{D}_{4m}$, but their matrix S is always complex.

Next turn to the quaternions $\mathfrak{Q}_{2n}$. It has $n+3$ conjugacy classes, with representatives $R = \{e,\, r^n,\, r^k\ (1 \le k \le n-1),\, s,\, sr\}$. It has 4 one-dimensional and $n-1$ two-dimensional representations. The characters and centralisers of the various classes are indicated in the following table (put $\iota = 1$ for n even, and $\iota = \mathrm{i}$ for n odd).

	e	r^n	r^k	s	sr
ψ_0	1	1	1	1	1
ψ_1	1	1	1	-1	-1
ψ_2	1	$(-1)^n$	$(-1)^k$	ι	$-\iota$
ψ_3	1	$(-1)^n$	$(-1)^k$	$-\iota$	ι
χ_i	2	$2(-1)^i$	$2\cos\frac{\pi ik}{n}$	0	0
$C_G(g)$	$\mathfrak{Q}_{2n}$	$\mathfrak{Q}_{2n}$	$\mathbb{Z}_{2n}$	$\mathbb{Z}_4$	$\mathbb{Z}_4$

The number of primaries is equal to $|\Phi| = 2n^2 + 14$. Denote by ψ_i the $2n$ characters of $\langle r\rangle \cong \mathbb{Z}_{2n}$, and by ϕ_a and φ_a the 4 characters of $\langle s\rangle \cong \mathbb{Z}_4$ and $\langle rs\rangle \cong \mathbb{Z}_4$, respectively. The nonzero S entries not involving e or r^n are

$$S_{(r^k,\psi_i),(r^\ell,\psi_j)} = \frac{1}{n}\cos(\pi\frac{\ell i + kj}{n}), \tag{3.9}$$

$$S_{(s,\phi_a),(s,\phi_b)} = S_{(rs,\varphi_a),(rs,\varphi_b)} = \frac{1}{4}\cdot\begin{cases} \mathrm{i}^{a-b} & \text{if } n \text{ odd}, \\ 2\cos(\pi\frac{a+b}{2}) & \text{if } n \text{ even}, \end{cases} \tag{3.10}$$

and in addition for n odd $S_{(s,\phi_a),(sr,\varphi_b)} = \frac{1}{4}\mathrm{i}^{a+b}$.

For our final example in this subsection, we will consider the series of non-abelian simple groups, $\mathrm{SL}_2(\mathbb{F}_q)$ for $q = 2^n$. Note that $\mathrm{SL}_2(\mathbb{F}_2) \cong \mathfrak{S}_3$ and $\mathrm{SL}_2(\mathbb{F}_4) \cong \mathfrak{A}_5$.

The order of $\mathrm{SL}_2(\mathbb{F}_q)$ is $q(q^2-1)$. There are $q+1$ conjugacy classes whose representatives can be chosen in

$$R = \left\{e,\, \iota = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},\, \alpha^a = \begin{pmatrix} s^a & 0 \\ 0 & s^{-a} \end{pmatrix},\, \beta^b \;:\; 1 \leq a \leq \tfrac{q-2}{2} \text{ and } 1 \leq b \leq \tfrac{q}{2}\right\}, \tag{3.11}$$

where s is any generator of the cyclic multiplicative group $\mathbb{F}_q^*$, and where β is an element of order $q+1$ (its exact form is not important). The characters (the labels i and j run from 1 to $\frac{q}{2} - 1$ and $\frac{q}{2}$ respectively) and centralisers are given below, from which one finds the number of primaries equals $|\Phi| = q^2 + q + 2$, as mentioned before.

	e	ι	α^a	β^b
ψ_0	1	1	1	1
ψ_1	q	0	1	-1
χ_i	$q+1$	1	$2\cos\frac{2\pi ia}{q-1}$	0
θ_j	$q-1$	-1	0	$-2\cos\frac{2\pi jb}{q+1}$
$C_G(g)$	$\mathrm{SL}_2(\mathbb{F}_q)$	$\mathbb{Z}_2^n$	$\mathbb{Z}_{q-1}$	$\mathbb{Z}_{q+1}$

Because $2, q-1, q+1$ are pairwise coprime, Proposition 1(a) says that the only potentially nonvanishing $S_{(a,\chi),(b,\phi)}$ entries have $a = e$ or $b = e$, or $a = b = \iota$, or a, b are both powers of α, or a, b are both powers of β. The relevant sets $G(g, h)$ here are $G(\iota, \iota) = \cup_{a=0}^{q-2} \left(\begin{smallmatrix} 1 & 0 \\ * & 1 \end{smallmatrix}\right) \alpha^a$, $G(\alpha^a, \alpha^b) = \langle a \rangle \cup \langle a \rangle \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, and $G(\beta^a, \beta^b) = \langle \beta \rangle \cup \langle \beta \rangle \gamma$ where $\gamma^{-1}\beta\gamma = \beta^{-1}$.

Explicitly writing down its matrix S would require the evaluation of some interesting character sums, something we have not yet done.

3.3 The physical invariants for $\mathfrak{S}_3$

The non-abelian group of smallest order is $\mathfrak{S}_3$. Its character table is

	e	(123)	(12)
ψ_0	1	1	1
ψ_1	1	1	-1
ψ_2	2	-1	0
$C_G(g)$	$\mathfrak{S}_3$	$\mathbb{Z}_3$	$\mathbb{Z}_2$

The modular data for $\mathfrak{S}_3$ will have 8 primary fields: (e,ψ_i) for $i=0,1,2$; $((123),\varphi_k)$, $k=0,1,2$, for the 3 characters $a\mapsto\xi_3^{ak}$ of $\mathbb{Z}_3$; and $((12),\varphi'_k)$, $k=0,1$, for the 2 characters $b\mapsto(-1)^{bk}$ of $\mathbb{Z}_2$. For convenience label these primaries $0,1,\ldots,7$.

Since $\mathfrak{S}_3\cong\mathfrak{D}_3$, we can read $S(\mathfrak{S}_3)$ and $T(\mathfrak{S}_3)$ off from the previous subsection:

$$S=\frac{1}{6}\begin{pmatrix}1&1&2&2&2&2&3&3\\1&1&2&2&2&2&-3&-3\\2&2&4&-2&-2&-2&0&0\\2&2&-2&4&-2&-2&0&0\\2&2&-2&-2&-2&4&0&0\\2&2&-2&-2&4&-2&0&0\\3&-3&0&0&0&0&3&-3\\3&-3&0&0&0&0&-3&3\end{pmatrix}, \tag{3.12}$$

$$T=\mathrm{diag}(1,1,1,1,\xi_3,\xi_3^2,1,-1). \tag{3.13}$$

There is one non-trivial simple current (e,ψ_1) (namely primary #1), identifiable by the 1 in the corresponding entry of the 0th row of S. Since all entries of S are rational, the charge conjugation and the other Galois permutations σ_ℓ are trivial. Incidentally, groups G for which $S(G)$ is rational are rare; that property requires that the exponent of G divides 24 (e.g. the exponent of $\mathfrak{S}_3$ is 6). To see that, apply Prop.1(b) to the equation $(T_{a,a})^{\ell^2}=T_{\sigma_\ell a,\sigma_\ell a}$.

There are precisely 32 physical invariants for $\mathfrak{S}_3$. Write ch_i for the CFT character corresponding to the ith primary. The automorphism invariants are $M=I$, and the one (call it M') which interchanges $2\leftrightarrow 3$ and fixes everything else. Extending by the simple current gives us 3 invariants:

$$|\mathrm{ch}_0+\mathrm{ch}_1|^2+2|\mathrm{ch}_2|^2+2|\mathrm{ch}_3|^2+2|\mathrm{ch}_4|^2+2|\mathrm{ch}_5|^2+k(\mathrm{ch}_2\mathrm{ch}_3^*+\mathrm{ch}_3\mathrm{ch}_2^*-|\mathrm{ch}_2|^2-|\mathrm{ch}_3|^2), \tag{3.14}$$

for $k=0,1,2$. Write $s_1:=\mathrm{ch}_0+\mathrm{ch}_1+\mathrm{ch}_2+\mathrm{ch}_3$, $s_2:=\mathrm{ch}_0+\mathrm{ch}_1+2\mathrm{ch}_2$, $s_3:=\mathrm{ch}_0+\mathrm{ch}_1+2\mathrm{ch}_3$, $s_4:=\mathrm{ch}_0+\mathrm{ch}_2+\mathrm{ch}_6$, and $s_5:=\mathrm{ch}_0+\mathrm{ch}_3+\mathrm{ch}_6$; then $s_is_j^*$ is a physical invariant. The final ones are

$$|\mathrm{ch}_0+\mathrm{ch}_1+\mathrm{ch}_2+\mathrm{ch}_3|^2+k(\mathrm{ch}_2\mathrm{ch}_3^*+\mathrm{ch}_3\mathrm{ch}_2^*-|\mathrm{ch}_2|^2-|\mathrm{ch}_3|^2), \tag{3.15}$$

for $k=\pm1$.

In the next section we manage to identify general constructions yielding most of these physical invariants. Two however remain unexplained: the automorphism invariant M', and the modular invariance of the sum s_1.

Note that in the basis defined by ($\xi = \xi_3 = \exp[2\pi\mathrm{i}/3]$)

$$(e_0, \ldots, e_7) \;=\; (\mathrm{ch}_0 - \mathrm{ch}_1\,,\, \mathrm{ch}_0 + \mathrm{ch}_1 + 2\mathrm{ch}_2\,,\, \mathrm{ch}_0 + \mathrm{ch}_1 - \mathrm{ch}_2\,,\, \mathrm{ch}_3 + \mathrm{ch}_4 + \mathrm{ch}_5\,,\\ \mathrm{ch}_3 + \xi\mathrm{ch}_4 + \xi^2\mathrm{ch}_5\,,\, \mathrm{ch}_3 + \xi^2\mathrm{ch}_4 + \xi\mathrm{ch}_5\,,\, \mathrm{ch}_6 + \mathrm{ch}_7\,,\, \mathrm{ch}_6 - \mathrm{ch}_7), \quad (3.16)$$

the matrices S and T become permutation matrices (both non diagonal), corresponding respectively to the permutations $(06)(23)(45)$ and $(345)(67)$ of $\mathfrak{S}_8$ (written in terms of cycles). From this, it is not difficult to compute the dimension (over $\mathbb{C}$) of the commutant of S and T, which turns out to be 11. Hence the 32 physical invariants are not linearly independent (e.g. $s_2 + s_3 = 2s_1$ and $2s_4 - 2s_5 = s_2 - s_3$).

Incidentally, it is a consequence of [6] that for any G, such a basis can always be found in which S and T are permutation matrices. In particular, the characters $\mathrm{ch}_{(a,\chi)}$, for $(a,\chi) \in \Phi$, can be thought of as a function on $G \times G$ taking any pair (xax^{-1}, xbx^{-1}) to $\chi(b)$ when a and b commute, and sending all other pairs in $G \times G$ to 0. They span a space called $C^0(G_{comm})$ in [12]. Now choose any commuting pair $(a,b) \in G \times G$, and define the function $f_{(a,b)}$ to be identically 1 on the set (xax^{-1}, xbx^{-1}) and 0 elsewhere. Then these functions form another basis for $C^0(G_{comm})$. For this choice of basis, our representation of $\mathrm{SL}_2(\mathbb{Z})$ becomes manifestly a permutation representation: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ acts on the right by sending $f_{(g,h)}$ to $f_{(g^a h^c, g^b h^d)}$. For instance, we recover T by noting that

$$\mathrm{ch}_{(a,\chi)}(a, ab) = \chi(ab) = \frac{\chi(a)}{\chi(e)}\,\chi(b) = T_{(a,\chi),(a,\chi)}\mathrm{ch}_{(a,\chi)}\ . \quad (3.17)$$

(See equation (5.12) of [12] for the corresponding matrix S calculation.) It is a very special property of the (untwisted) finite group modular data that this $\mathrm{SL}_2(\mathbb{Z})$ representation is actually a permutation representation. For general modular data, usually the matrix T won't even be conjugate to a permutation matrix. Two examples are the simplest affine data, i.e. $A_1^{(1)}$ at level 1, and the simplest twisted group data, $G = \mathbb{Z}_2$ with twist α_1 (see §6.1), whose matrices T are respectively

$$\mathrm{diag}\,(\exp[-\pi\mathrm{i}/12], \exp[5\pi\mathrm{i}/12]) \qquad \text{and} \qquad \mathrm{diag}\,(1, 1, \mathrm{i}, -\mathrm{i})\ . \quad (3.18)$$

The first cannot be similar to a permutation matrix because it does not have 1 as an eigenvalue, while the second cannot because an order 4 permutation matrix has -1 as an eigenvalue.

Incidentally, the analogue of this permutation representation for the twisted finite group data is known [11], and will be discussed in §5.3.

A total of 32 physical invariants for such a small number of primaries is completely unprecedented from the more familiar WZW situation. The orthogonal algebras at level 2 are the worst behaved affine cases, but e.g. $D_{16}^{(1)}$ at level 2 has 23 primaries but only 22 physical invariants. But $\mathfrak{S}_3$, being a dihedral group, is nearly abelian so should be especially bad in this respect.

It seems that the twisted data is better behaved in this respect than the untwisted. For example, we find, combining the discussion of §6.3 with the results of [33], that the twisted modular data for $\mathfrak{S}_3$ has only 9 physical invariants.

4 Making sense of all the physical invariants

We learned in the previous section that there is a surprising number of physical invariants associated to finite group modular data. In this section we try to tame the zoo!

The conventional wisdom in RCFT is that all physical invariants are constructed in two ways: as extensions of chiral algebras, and as automorphisms of those chiral algebras (called automorphism invariants in the unextended case). In the affine case, the physical invariants are almost always 'obvious' in hindsight, as symmetries of the appropriate Dynkin diagram (e.g. simple currents and charge conjugation) are directly responsible for almost all WZW physical invariants. It would be nice to find the analogous statement for the finite group modular data.

A rich supply of physical invariants always comes from simple currents [26, 27]. As these are well-understood, we won't say any more than we did in §2.

4.1 Automorphism invariants

The outer automorphism group $\mathrm{Out}(G)$ provides a systematic way of constructing automorphism invariants. Choose any $\pi \in \mathrm{Out}(G)$. Then π induces an invertible homomorphism $C_G(a) \to C_G(\pi a)$ between centralisers. Define χ^π by the formula $\chi^\pi(g) = \chi(\pi^{-1}g)$ — if $\chi \in \mathrm{Irr}(C_G(a))$ then $\chi^\pi \in \mathrm{Irr}(C_G(\pi a))$. Consider the permutation of Φ defined by $(a,\chi) \mapsto (\pi a, \chi^\pi)$. Then it commutes with S and T and hence defines an automorphism invariant.

Many (but not all) outer automorphisms can be interpreted in the following way: Any time a given group G is a normal subgroup of another group $\widehat{G}$, then any element $\hat{g} \in \widehat{G}$ defines an automorphism of G by conjugation: $g \mapsto \hat{g}g\hat{g}^{-1}$.

Examples of these groups are $\mathrm{Out}(\mathbb{Z}_n) \cong \mathbb{Z}_n^\times$; $\mathrm{Out}(\mathfrak{D}_{2k+1}) \cong \mathbb{Z}_{2k+1}^\times/\{\pm 1\}$; for $n \geq 3$, $\mathrm{Out}(\mathfrak{S}_n) = \{1\}$ and $\mathrm{Out}(\mathfrak{A}_n) \cong \mathbb{Z}_2$, except $\mathrm{Out}(\mathfrak{S}_6) \cong \mathbb{Z}_2$ and $\mathrm{Out}(\mathfrak{A}_6) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Another systematic source are the Galois automorphism invariants — there is one of these for every $\sigma_\ell \in \mathrm{Gal}(\mathbb{Q}(S)/\mathbb{Q})$ with $\ell^2 \equiv 1$ modulo the exponent $e(G)$ of G, and the permutation is simply given by the Galois permutation, which we may write $\sigma_\ell(a,\chi) = (a^\ell, \sigma_\ell\chi)$. If there are s distinct primes which divide $e(G)$, then there will be precisely

$$\begin{cases} 2^{s-1} & \text{if } e(G) \equiv 2 \pmod 4 \\ 2^{s+1} & \text{if } e(G) \equiv 0 \pmod 8 \\ 2^s & \text{otherwise} \end{cases} \tag{4.1}$$

such ℓ, and hence that number of Galois automorphism invariants (though these won't necessarily be distinct, if $\mathbb{Q}(S)$ is smaller than $\mathbb{Q}(\xi_{e(G)})$).

A final source of automorphism invariants is *discrete torsion* [19]. This involves the language of cohomology — see §5.1 for the appropriate definitions. Take any 2-cocycle $\beta \in Z^2(G, U(1))$ for G. This β is entirely independent of the 2-cocycles β_a we discuss in §5.2. For each conjugacy class representative $a \in R$, define $e_a(b) := \beta(a,b)\beta(b,a)^{-1}$. This will be a 1-dimensional representation of $C_G(a)$. Define the permutation of primary fields $(a,\chi) \in \Phi$ sending (a,χ) to $(a, e_a\chi)$. It is easy to check that this commutes with S and T and thus defines an automorphism invariant.

When is this permutation trivial ? Iff each $e_a(b) = 1$, for all $b \in C_G(a)$. In other words, iff each $a \in G$ is β-regular (see §5.1 for the definition). One thing this means is that cohomologous β give the same discrete torsion. So the possibilities for discrete torsion are given by the classes in the finite abelian group $H^2(G, U(1)) = M(G)$, known as the Schur multiplier of G. Each class in $M(G)$ will usually but not always give rise to a different automorphism invariant. Incidentally, this construction applies to untwisted, CT twisted, or non-CT twisted, modular data (see next section for the twisted modular data).

So discrete torsion will always give $|M(G)|$ automorphism invariants, although these are not necessarily all distinct. They are all distinct for $\mathbb{Z}_n^2$, $\mathbb{Z}_n^3$, and $\mathfrak{D}_{even}$.

4.2 Chiral extensions

Making sense of the large number of physical invariants also requires finding the "generic" chiral extensions. Many will come from simple currents. It is tempting to suspect that another rich source could be from certain normal subgroups $N \triangleleft G$.

In particular, Clifford theory (which is concerned with the theory of induced and restricted characters) can be used to explicitly relate the modular data of a group G to that of its normal subgroups H. The formulas we have obtained however are complicated enough that at this point we have been unable to determine explicit relations between the physical invariants of G and H. Nevertheless, the situation is sufficiently analogous to that of conformal embeddings [43] in affine algebras that we expect this to likewise be a rich source of chiral extensions. Incidentally, there are also relations between the modular data of G and G/N.

Nevertheless we have found some interesting generic chiral extensions. Choose any central elements $z, z' \in Z(G)$ and any degree-1 characters $\varphi, \psi \in \mathrm{Irr}(G)$. Define the combinations

$$s_{z,z'} := \sum_{\chi \in \mathrm{Irr}(G)} \chi(z')\, \mathrm{ch}_{(z,\chi)} \tag{4.2}$$

$$s'_{\varphi,\psi} := \sum_{g \in R} \psi(g)\, \mathrm{ch}_{(g,\varphi)} \tag{4.3}$$

Then it is easy to verify, using (2.17), Frobenius reciprocity, and the orthogonality relations for characters, that $s_{z,z'}(-1/\tau) = s_{z',z^{-1}}(\tau)$ and $s'_{\varphi,\psi}(-1/\tau) = s'_{\psi,\varphi^*}(\tau)$, while $s_{z,z'}(\tau+1) = s_{z,zz'}(\tau)$ and $s'_{\varphi,\psi}(\tau+1) = s'_{\varphi,\varphi\psi}(\tau)$. To see this, consider the coefficient of $\mathrm{ch}_{(a,\chi)}(\tau)$ in $s'_{\varphi,\psi}(-1/\tau)$: it will be

$$\sum_{g \in R} \psi(g)\, \frac{1}{|G|}\, \varphi(a)^* \chi^G_{C_G(a)}(g)^* \tag{4.4}$$

which is $\varphi(a)^*$ times the coefficient of ψ in $\chi^G_{C_G(a)}$, i.e. $\varphi(a)^*$ times the coefficient of χ in $\psi|_{C_G(a)}$, i.e. $\varphi(a)^* \delta_{\psi,\chi}$.

Choose any subgroups $A \leq Z(G)$, $B \leq G/G'$, and define the sums

$$s(A) = \frac{1}{|A|} \sum_{z,z' \in A} s_{z,z'} = \sum_{z \in A} \sum_{\chi : \ker(\chi) \geq A} \chi(e)\, \mathrm{ch}_{z,\chi} \tag{4.5}$$

$$s'(B) = \frac{1}{|B|} \sum_{\varphi,\psi \in B} s'_{\varphi,\psi} = \sum_{\psi \in B} \sum_{g \in \ker(B) \cap R} \mathrm{ch}_{g,\psi} \tag{4.6}$$

By $\ker(\chi)$ we mean all $g \in G$ for which $\chi(g) = \chi(e)$, and by $\ker(B)$ we mean the intersection of all $\ker(\psi)$, where $\psi \in B$ is identified with a degree-1 character of G. Note that the sums $s(A), s'(B)$ are all invariant under S and T. Thus *for any* G, A, A', B, B' we get the remarkable physical invariants

$$s(A)^*\, s(A'), \quad s(A)^* s'(B), \quad s'(B)^* s(A), \quad s'(B)^*\, s'(B'). \tag{4.7}$$

In the case of $\mathfrak{S}_3$ which we worked out in §3.3, $s(e) = s_2$, $s'(\psi_0) = s_5$, and $s'(\psi_0, \psi_1) = s_3$. This alone accounts for 9 of the 32 physical invariants for $\mathfrak{S}_3$.

Incidentally, taking A and B to be the trivial groups completes the proof of Proposition 1(h): the fifth physical invariant of course is the diagonal sum. For the second claim there, there is a natural degree-preserving bijection (see §3.5 of [3]) between the $\chi \in \mathrm{Irr}(G)$ with $\ker(\chi) \geq Z(G)$, and $\mathrm{Irr}(G/Z(G))$, so if $G/Z(G)$ is nonabelian there will always be degree > 1 characters appearing in each $s(A)$ and so they will be different from any $s'(B)$.

5 Twisting, Group Cohomology, and Projective Representations

As advertised in the introduction, one way to generalize the group data described in Section 2 is by introducing some "twisting". This twisting has a cohomological origin, as in the theory of affine algebras [29], where the infinitely many possible twists are labelled by the level k, an integer. In constrast, the twisting of the finite group modular data offers but a finite number of possibilities.

The twisting of this modular data was first described in the most generality by [4, 7, 20] (see also [6]), although to our knowledge explicit expressions for the modular matrices S and T in the most general case have not appeared until now (though the most general fusions appear in (6.44) of [4]). We will recall here their construction, as concretely as possible, and then we will compute explicit examples, making contact with known structures (most notably affine algebras).

5.1 Cohomological preliminaries

As a twisting of the finite group modular data is effected by elements of cohomology groups $H^i(G, U(1)) \cong H^i(G, \mathbb{C}^\times) \cong H^{i+1}(G, \mathbb{Z})$, we will first review the relevant properties and give some examples. For our purposes it is not very important to know how these are defined. We refer the reader to standard textbooks (like [38]) for a more complete treatment. For projective representations, see [2].

For all $i > 0$, $H^i(G, U(1))$ is a finite abelian group, which we will usually write additively, obeying $|G| H^i(G, U(1)) = 0$ (i.e. all elements have order dividing the order of G). For any G, $H^0(G, U(1)) = 0$, while $H^1(G, U(1)) = G/G'$ is the group of one-dimensional representations of G (G' is the commutator subgroup). The next group $M(G) := H^2(G, U(1))$, called the Schur multiplier (or multiplicator), classifies the (projectively) inequivalent projective representations, and for that reason, it will play a central role in the whole construction. The only other group we will be interested in is $H^3(G, U(1))$.

The following results can be useful: the square of the exponent of $M(G)$ divides $|G|$; if G is a p-group (i.e. its order is a power of a prime), then $H^3(G, U(1)) \neq 0$; if e is the exponent of G and e_i is the exponent of $H^i(G, U(1))$, then ee_2, $e_1 e_2$ and $e_2 e_3$ all divide $|G|$.

Computing cohomology groups, even $M(G)$, is usually difficult. They are known however for several familiar groups. For a cyclic group, one has $M(\mathbb{Z}_n) = 0$ and $H^3(\mathbb{Z}_n, U(1)) = \mathbb{Z}_n$. For products of identical cyclic groups,

$$M(\mathbb{Z}_n^k) = \mathbb{Z}_n^{k(k-1)/2}, \quad H^3(\mathbb{Z}_n^k, U(1)) = \mathbb{Z}_n^{k(k^2+5)/6}. \tag{5.1}$$

More generally, the Schur multiplier for any abelian group is as follows: write $G = \mathbb{Z}_{d_1} \times \mathbb{Z}_{d_2} \times \cdots \times \mathbb{Z}_{d_s}$ where $d_1|d_2|\cdots|d_s$, then

$$M(G) = \prod_{j=1}^{s} \mathbb{Z}_{d_j}^{j-1}. \tag{5.2}$$

Other known groups are (see e.g. Ch.6 of [3])

$$M(\mathfrak{S}_3) = 0, \quad H^3(\mathfrak{S}_3, U(1)) = \mathbb{Z}_6, \quad M(\mathfrak{S}_n) = \mathbb{Z}_2 \quad \text{for } n \geq 4, \tag{5.3}$$
$$M(\mathfrak{A}_n) = \mathbb{Z}_2 \quad \text{for } n \geq 4,\ n \neq 6,7, \quad M(\mathfrak{A}_n) = \mathbb{Z}_6 \quad \text{for } n = 6,7, \tag{5.4}$$
$$M(\mathfrak{D}_n) = 0, \quad H^3(\mathfrak{D}_n, U(1)) = \mathbb{Z}_{2n} \qquad \text{for } n \text{ odd}, \tag{5.5}$$
$$M(\mathfrak{D}_n) = \mathbb{Z}_2, \quad H^3(\mathfrak{D}_n, U(1)) = \mathbb{Z}_n \times \mathbb{Z}_2 \times \mathbb{Z}_2 \qquad \text{for } n \text{ even}, \tag{5.6}$$
$$M(\mathfrak{Q}_n) = 0, \quad H^3(\mathfrak{Q}_4, U(1)) = \mathbb{Z}_8, \tag{5.7}$$
$$M(\mathrm{SL}_n(\mathbb{F}_q)) = 0, \tag{5.8}$$
$$M(\mathrm{PSL}_n(\mathbb{F}_q)) = \mathbb{Z}_{\gcd(q-1,n)}, \tag{5.9}$$

where in (5.8) and (5.9), $(n,q) \notin \{(2,4),(2,5),(2,7),(2,9),(3,2),(3,3),(3,4),(4,2)\}$.

For odd prime p, there are precisely two different non-abelian groups of order p^3: one of these (a split extension of cyclic groups, the $p \neq 2$ analogue of $\mathfrak{D}_4$) has $M = 0$ and $H^3 = \mathbb{Z}_p \times \mathbb{Z}_p$; the other has $M = \mathbb{Z}_p \times \mathbb{Z}_p$ and $H^3 = \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$. (All groups of order p^2 are abelian.)

This gives some taste of what $M(G)$ and $H^3(G)$ look like for nice groups. The Schur multiplier is known for all simple groups. For instance it is trivial for the Monster.

As mentioned before, the Schur multiplier plays a central role in the theory of projective representations. Indeed a normalised 2-cocycle $\beta \in Z^2(G, U(1))$ is a map $G \times G \to U(1)$ satisfying $\beta(x,1) = \beta(1,x) = 1$ and the cocycle condition $\beta(x,y)\,\beta(xy,z) = \beta(y,z)\,\beta(x,yz)$ for all $x,y,z \in G$. For any such β, one may consider the projective β-representations, i.e. the maps $\tilde{\rho} : G \to \mathrm{GL}(V)$ obeying $\tilde{\rho}(x)\tilde{\rho}(y) = \beta(x,y)\tilde{\rho}(xy)$. The cocycle condition corresponds to associativity. If β is identically 1, $\tilde{\rho}$ will be an ordinary representation and will be called *linear*.

If β, β' are cohomologous (i.e. $\beta^{-1}\beta'$ is a 2-coboundary – a *2-coboundary* is any $\beta \in Z^2$ of the form $\beta = \gamma(x)\gamma(y)\gamma(xy)^{-1}$), then any β'-representation will be projectively equivalent to some β-representation, and there will be the same number of k-dimensional β-representations as k-dimensional β'-representations, for each $k = 1, 2, \ldots$. Moreover, a β-representation can be one-dimensional only if β is a coboundary.

A natural question is to classify the projective representations belonging to a given cocycle. So let $r(G,\beta)$ denote the number of linearly inequivalent irreducible β-representations of G. We know that $r(G,\beta)$ is a cohomology class invariant. In order to compute this number, one introduces the notion of a β-regular group element: $g \in G$ is β-regular if $\beta(g,h) = \beta(h,g)$

for all h in $C_G(g)$. One may check that g is β-regular if and only if all its conjugates are, so that a whole conjugacy class is β-regular or not. Then

$$r(G,\beta) = \text{number of } \beta\text{−regular conjugacy classes of } G. \tag{5.10}$$

As an immediate consequence, one has the inequality $r(G,\beta) \leq k(G) = r(G,1)$ for any 2-cocycle β, with equality if (but not only if) β is a coboundary.

The projective characters $\tilde{\chi} \in \beta$-Irr(G) (defined as usual by the trace of the representation) share many properties with the usual characters, except that they are in general not class functions. Schur's lemma still holds, from which orthogonality and completeness relations follow: for any $\tilde{\chi}, \tilde{\chi}' \in \beta$-Irr$(G)$ and any β-regular $a \in G$,

$$\frac{1}{|G|} \sum_{g \in G} \tilde{\chi}(g)^* \, \tilde{\chi}'(g) = \delta_{\tilde{\chi},\tilde{\chi}'}, \tag{5.11}$$

$$\frac{|K_a|}{|G|} \sum_{\tilde{\chi} \ \beta-\mathrm{Irr}} \tilde{\chi}(a)^* \, \tilde{\chi}(b) = \delta_{b \in K_a} \,. \tag{5.12}$$

The dimensions of the irreducible projective representations divide the order of the group, and their squares sum up to $|G|$. All β-characters vanish at non-regular classes.

Examples of projective representation data include the following. Writing $H^2(\mathbb{Z}_n^2, U(1)) = \langle \beta_1 \rangle \cong \mathbb{Z}_n$ and $H^2(\mathbb{Z}_n^3, U(1)) = \langle \beta_1, \beta_2, \beta_3 \rangle \cong \mathbb{Z}_n^3$, one finds

$$r(\mathbb{Z}_n^2, \beta_1^x) = [\mathrm{GCD}(x,n)]^2, \qquad r(\mathbb{Z}_n^3, \beta_1^x \, \beta_2^y \, \beta_3^z) = n \cdot [\mathrm{GCD}(x,y,z,n)]^2. \tag{5.13}$$

for any $1 \leq x, y, z \leq n$. For the dihedral groups, one has

$$r(\mathfrak{D}_{2n+1}, \beta) = n+2, \qquad r(\mathfrak{D}_{2n}, \beta) = \begin{cases} n+3 & \text{if } \beta \text{ is coboundary,} \\ n & \text{otherwise.} \end{cases} \tag{5.14}$$

These numbers are known also for $\mathfrak{S}_n$ and $\mathfrak{A}_n$.

5.2 General construction

The possible twistings are parametrised by the finitely many elements $\alpha \in H^3(G, U(1))$. The 3-cocycle condition is $\alpha(g,h,k)\,\alpha(g,hk,\ell)\,\alpha(h,k,\ell) = \alpha(gh,k,\ell)\,\alpha(g,h,k\ell)$. Different 3-cocycles give rise to different modular data, with in general different numbers of primary fields (never more than in the untwisted case), and with matrices S^α and T^α.

The construction starts from a normalised[§] element α of $H^3(G, U(1))$. We can and will assume for convenience that the values of α are always roots of 1 – in fact if α has cohomological order n, then we can require α to take the values of nth roots of 1 (proof: Write $\alpha^n = \delta\beta$; choose any $\gamma = \beta^{-1/n}$, then $\alpha\gamma$ is cohomologous to α and has the desired property.) Because $|G|\, H^3(G, U(1)) = 0$, n will necessarily divide $|G|$.

For all $a, g, h \in G$, we define auxiliary quantities

$$\beta_a(h,g) = \alpha(a,h,g)\,\alpha(h,h^{-1}ah,g)^{-1}\,\alpha(h,g,(hg)^{-1}ahg). \tag{5.15}$$

[§]That is, it satisfies $\alpha(e,h,g) = \alpha(h,e,g) = \alpha(h,g,e) = 1$. This implies that all 2-cocycles are accordingly normalised, $\beta_e(h,g) = \beta_h(e,g) = \beta_h(g,e) = 1$ for all h, g.

It follows from this definition that the β_a's are normalised twisted cocycles on G, namely they satisfy

$$\beta_a(x,y)\,\beta_a(xy,z) = \beta_a(x,yz)\,\beta_{x^{-1}ax}(y,z), \qquad \forall x,y,z \in G. \tag{5.16}$$

Furthermore, the restriction of each β_a to $C_G(a)$ clearly is a normalised 2-cocycle. As such they define projective representations of $C_G(a)$.

The primary fields Φ^α in the model twisted by a given 3-cocycle α will consist of all pairs $(a,\tilde{\chi})$ where $a \in R$ (as before) and $\tilde{\chi} \in \beta_a\text{-Irr}(C_G(a))$. As a consequence of the inequality $r(H,\beta) \leq k(H)$, we find

$$|\Phi^\alpha| = \sum_{a\in R} r(C_G(a),\beta_a) \leq |\Phi|. \tag{5.17}$$

A simplification seems to occur when the 3-cocycle α is such that each 2-cocycle β_a is a coboundary on $C_G(a)$, which is the case considered in [6]. We call the resulting twisting "CT" (cohomologically trivial). Note that when $M(C_G(a)) = 0$ for all a, any twist α will automatically be CT. By no means though are CT twistings restricted to the groups whose centralisers have trivial Schur multipliers — see §6.2 for a class of examples. A CT-twisted theory has the same number of primaries as the untwisted one considered in the previous sections (but different modular matrices).

For CT twistings, [6, 4] proposed explicit formulae for the modular matrices that resemble the untwisted case except with additional phases. Each β_a being a coboundary on $C_G(a)$ by hypothesis, (5.16) can be used to show¶ that one can find 1-cochains $\epsilon_a : C_G(a) \to U(1)$ for which $\epsilon_a(e) = 1$ and both

$$\beta_a(h,g) = (\delta\epsilon_a)(h,g) = \epsilon_a(h)\,\epsilon_a(g)\,\epsilon_a(hg)^{-1}, \tag{5.18}$$

$$\epsilon_{x^{-1}ax}(x^{-1}hx) = \frac{\beta_a(x,x^{-1}hx)}{\beta_a(h,x)}\epsilon_a(h), \tag{5.19}$$

for all $g,h \in C_G(a)$ and $x \in G$. Now, if $\tilde{\rho}$ is a β_a-representation with character $\tilde{\chi}$, then clearly $\rho(g) = \epsilon_a^{-1}(g)\tilde{\rho}(g)$ is a linear representation with character $\chi = \epsilon_a^{-1}\tilde{\chi}$. The modular matrices then become [6]

$$S^\alpha_{(a,\chi),(a',\chi')} = \frac{1}{|C_G(a)|\cdot|C_G(a')|}\sum_{g\in G(a,a')} \chi^*(ga'g^{-1})\,\chi'^*(g^{-1}ag)\,\sigma^*(a|ga'g^{-1}), \tag{5.20}$$

$$T^\alpha_{(a,\chi),(a',\chi')} = \delta_{a,a'}\delta_{\chi,\chi'}\frac{\chi(a)}{\chi(e)}\,\epsilon_a(a), \tag{5.21}$$

where the function $\sigma(\cdot|\cdot)$ is

$$\sigma(h|g) = \epsilon_h(g)\,\epsilon_g(h). \tag{5.22}$$

(Since $g \in C_G(h)$ if and only if $h \in C_G(g)$, this definition makes sense.) It is easy to see that (5.20),(5.21) are (essentially, i.e. up to a relabelling of the characters χ) independent of the choice of 1-cochains ϵ_a in (5.18),(5.19). These equations permit CT twists to be analysed as thoroughly as the untwisted data.

The interpretation involving 3-cocycles was developed in [4]. Though they give the fusion coefficients for arbitrary α, neither they nor to our knowledge anyone else has given S^α and

¶ We thank Peter Bántay for correspondence on this point.

T^α for nonCT α, *explicitly* in terms of quantities directly associated with G (the closest is [11], which gives S^α in terms of $D^\alpha(G)$ characters). The formulae in the most general case can be obtained by going back to first principles.

The quantum double $D^\alpha(G)$ is a finite dimensional quasi-Hopf quasi triangular algebra. Its representations form a braided monoidal modular category with explicitly known universal braiding morphisms R_{12} , R_{21}. The point is that (up to normalisation) the mapping class group representations, in our case the matrices S and T, can be identified with Markov traces of intertwiners defined from coloured ribbon links. In particular, S_{ij} corresponds to the i,j coloured Hopf link and T_i to the twist θ (or v^{-1} in the [20] convention). We then obtain (as explained in the Appendix)

$$\begin{aligned}
S^\alpha_{(a,\tilde\chi),(b,\tilde\chi')} &= S^\alpha_{(e,1),(e,1)}\ \mathrm{Tr}_{(a,\tilde\chi),(b,\tilde\chi')}\ \Big(R_{21}R_{12}\Big)^* \\
&= \frac{1}{|G|}\sum_{g\in K_a,g'\in K_b\cap C_G(g)} \Big(\frac{\beta_g(g',x^{-1})\beta_{g'}(g,y^{-1})}{\beta_g(x^{-1},h)\beta_{g'}(y^{-1},h')}\Big)^*\ \tilde\chi(h)^*\,\tilde\chi'(h')^*, \\
&= \frac{1}{|G|}\sum_{g\in K_a,g'\in K_b\cap C_G(g)} \Big(\frac{\beta_a(x,g')\beta_a(xg',x^{-1})\beta_b(y,g)\beta_b(yg,y^{-1})}{\beta_a(x,x^{-1})\beta_b(y,y^{-1})}\Big)^*\ \tilde\chi(h)^*\,\tilde\chi'(h')^*,
\end{aligned} \tag{5.23}$$

where $g=x^{-1}ax=y^{-1}h'y,\ g'=y^{-1}by=x^{-1}hx,\ h\in C_G(a),\ h'\in C_G(b)$, and

$$T^\alpha_{(a,\tilde\chi),(b,\tilde\chi')} = \delta_{a,b}\delta_{\tilde\chi,\tilde\chi'}\frac{\tilde\chi(a)}{\tilde\chi(e)}\,T^\alpha_{(e,1),(e,1)}, \tag{5.24}$$

where $T^\alpha_{(e,1),(e,1)}$ can equal any third root of 1 (i.e. c is a multiple of 8), as for the untwisted data. The normalisation $S^\alpha_{(e,1),(e,1)}=\frac{1}{|G|}$ can be obtained from the orthogonality relations (5.11),(5.12) of projective characters.

The phases β_g here are defined by (5.15). Note that they are evaluated in (5.23) on elements which are not in $C_G(a)$ or $C_G(b)$. Unfortunately this makes the derivation of (5.20) from (5.23) more difficult — see question (1) in §7.

If we let n be the cohomological order of α, then the β_a will also have nth roots of 1 as its values. Thm. 6.5.15 in [2] then implies that the projective β_a-characters will have values in $\mathbb{Q}[\xi_{n\,e(G)}]$. Hence the entries of S^α and T^α will also lie in that field. Note that this field specialises to $\mathbb{Q}[\xi_{e(G)}]$ in the untwisted ($n=1$) case, which was our previous result.

That S^α and T^α depend only on the cohomology class $\alpha\in H^3(G,U(1))$, is clear from the $D^\alpha(G)$ interpretation. A direct derivation of this for CT α is sketched in [4].

5.3 Analysis of twisted modular data

It is important to realise that the 2-cocycle β_e in (5.15) is identically 1. Thus $(e,\chi)\in\Phi^\alpha$ for any $\chi\in\mathrm{Irr}(G)$, and we obtain the useful formula

$$S^\alpha_{(e,\chi),(b,\tilde\chi')} = \frac{1}{|C_G(b)|}\tilde\chi'(e)\,\chi(b)^* \tag{5.25}$$

We immediately see from this that once again all quantum dimensions will be integers, and $(a, \tilde{\chi}) \in \Phi^\alpha(G)$ will be a simple current iff $a \in Z(G)$ and $\tilde{\chi}$ is degree-one (which implies β_a is coboundary).

Rationality of the entries $S_{(e,1),(b,\tilde{\chi}')}$ implies all Galois parities $\epsilon_\sigma(a, \tilde{\chi}) = +1$, and also that the vacuum $(e, 1)$ is fixed by all Galois automorphisms, exactly as before.

From (5.25) we also learn about the Galois action on arbitrary primaries. Choose $\ell \in \mathbb{Z}^\times_{|G|^2}$, then $\sigma_\ell(a, \tilde{\chi}) = (a_\ell, \tilde{\chi}_\ell)$, where a_ℓ denotes the element in $K_{a^\ell} \cap R$ as before, and where $\tilde{\chi}_\ell$ is some projective β_{a_ℓ}-character with dimension equal to that of $\tilde{\chi}$. To see this, consider

$$\chi(a_\ell)^* = \sigma_\ell \frac{S^\alpha_{(e,\chi),(a,\tilde{\chi})}}{S^\alpha_{(e,1),(a,\tilde{\chi})}} = \sigma_\ell \chi(a)^* = \chi(a^\ell)^* \tag{5.26}$$

for all $\chi \in \mathrm{Irr}(G)$, and hence $a_\ell \in K_{a^\ell}$. In particular, specialising to $\ell = -1$ tells us about charge conjugation.

Proposition 1(a),(c),(e),(f) are thus exactly as before. As mentioned before, the order of T^α will divide ne, where n is the order of α, so in particular the order of T^α will always divide $|G|^2$. We also see directly from (5.23) that $S^\alpha_{(a,\tilde{\chi}),(b,\tilde{\chi}')} = 0$ unless $K_b \cap C_G(a)$ has β_a-regular elements and $K_a \cap C_G(b)$ has β_b-regular elements. Thm. 1 becomes

Theorem 3. Let S and T be the Kac-Peterson matrices corresponding to an affine algebra $X_r^{(1)}$ at some level $k \geq 1$ (where X_r is simple). Let G be a finite group with $S^\alpha(G) = S$ and $T^\alpha(G) = \varphi T$ for some third root φ of 1. Then either:

(i) as before, either $(X_r, k) = (E_8, 1)$ and $G = \{e\}$, or $(X_r, k) = (D_{8n}, 1)$ and $G = \mathbb{Z}_2$;

(ii) $(X_r, k) = (A_{n^2-1}, 1)$ and $G = \mathbb{Z}_n$, n odd, for a specific twist;

(iii) $(X_r, k) = (B_{(m^2-1)/2}, 2)$ and $G = \mathfrak{D}_m$, m odd, for a specific twist.

The proof is very similar to that of Theorem 1, the only difference being the specific handling of the finitely many algebras and levels which survive the Galois and quantum dimension arguments. For example, use the formulae $S_{00} = \frac{1}{\sqrt{r+1}}$ and $c = r$ for $A_r^{(1)}$ level 1. The demonstration of *(ii)* and *(iii)* is made explicit in section 6.

We would also like an analogue of Thm.2. This is more difficult, but a key observation is that $|\Phi^\alpha| \geq 2k(G) - 1$. Indeed, $\beta_e \equiv 1$ gives us $k(G)$ primaries of the form (e, χ), and the remaining $k(G) - 1$ conjugacy classes K_a will each contribute $r(a, \beta_a) \geq 1$ primaries. Also important is the observation made in §6.2 that β_a will always be coboundary when $C_G(a) \cong \mathbb{Z}_n^2$ for some n.

Theorem 4. The only groups with at most 20 primaries are $\mathbb{Z}_1$, $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathfrak{S}_3$, $\mathfrak{A}_4$, $\mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathfrak{D}_5$, $\mathfrak{S}_4$, $\mathfrak{D}_4$, and the order 48 Frobenius group $\mathbb{Z}_2^4 \times_f \mathbb{Z}_3$ (defined in [35]), with at least 1, 4, 9, 8, 14, 16, 16, 16, 18, 19, and 19 primaries.

The proof is like that of Thm.2, using the tables of [35]. For instance, we know it is sufficient to consider up to class number $k(G) = 8$. Consider e.g. $G = \mathfrak{S}_4$: it has $k(G) = 5$ and centralisers $\mathfrak{S}_4$, $\mathfrak{D}_4$, $\mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, and $\mathbb{Z}_3$. Thus a lower bound for $|\Phi^\alpha|$ is $5 + 2 + 4 + 4 + 3 = 18$.

In §3.3 we gave a basis in terms of which the untwisted S and T become permutation matrices. The analogue of this for the twisted data is as follows [11]. Let $C^\alpha(G_{comm})$ denote the space of all functions $f : G \times G \to \mathbb{C}$ for which $f(a, b) = 0$ unless a and b commute, and which obey the formula

$$f(x^{-1}ax, x^{-1}bx) = \frac{\beta_a(x, x^{-1}bx)}{\beta_a(b, x)} f(a, b) \tag{5.27}$$

Then S and T act on $C^\alpha(G_{comm})$ by $(Sf)(a, b) = \beta_b(a, a^{-1})^* f(b, a^{-1})$ and $(Tf)(a, b) = \beta_a(a, b)\, f(a, ab)$. See Questions 5 and 7 of §7. Incidentally, this action is completely natural in the case of twisted partition functions which involve projective representations of the symmetry group in some twisted sectors, and follows from the action of the modular group on the homological cycles of the torus — see [39] for examples in WZW models.

6 Twisted examples

We give in this final section a few examples of twisted finite group modular data, in order to give a flavour as to how they differ from the untwisted ones.

6.1 Abelian cyclic groups

One can easily illustrate the previous formalism in the case of a cyclic group $G = \mathbb{Z}_n$. Write $\langle\cdot\rangle : \mathbb{Z} \to \{0, 1, \ldots, n-1\}$ for reduction modulo n. One has $H^3(\mathbb{Z}_n, U(1)) = \mathbb{Z}_n$, and the following explicit representatives

$$\alpha_q(g_1, g_2, g_3) = \exp\Big\{2\mathrm{i}\pi q g_1(g_2 + g_3 - \langle g_2 + g_3\rangle)/n^2\Big\}, \tag{6.1}$$

where $q \in \mathbb{Z}_n$ parametrizes the different classes.

One easily computes that $\beta_a(h, g) = \alpha_q(a, h, g)$ is a coboundary for every q (expected since $M(\mathbb{Z}_n) = 0$). One finds $\sigma(h|g) = e^{4\mathrm{i}\pi qhg/n^2}$.

The linear characters of $\mathbb{Z}_n$ are $\chi_\ell(a) = e^{2\mathrm{i}\pi a\ell/n}$ for $\ell \in \mathbb{Z}_n$, and the modular matrices take the simple forms

$$S^{\alpha_q}_{(a,\chi_\ell),(a',\chi_{\ell'})} = \frac{1}{n}\,\exp\Big\{-2\mathrm{i}\pi[2qaa' + n(a\ell' + a'\ell)]/n^2\Big\}, \tag{6.2}$$

$$T^{\alpha_q}_{(a,\chi_\ell),(a,\chi_\ell)} = \exp\Big\{2\mathrm{i}\pi[qa^2 + na\ell]/n^2\Big\}. \tag{6.3}$$

Note that the charge conjugation $C = S^2$ is given $C(0, \chi_\ell) = (0, \chi_{-\ell})$ and $C(a, \chi_\ell) = (n - a, \chi_{-\ell-2q})$ if $a \neq 0$.

The identity corresponds to $(0, \chi_0)$, and as in the untwisted case, all primary fields are simple currents. The Verlinde formula yields the fusion coefficients

$$N^{(a'',\chi_{\ell''})}_{(a,\chi_\ell),(a',\chi_{\ell'})} = \delta_{a'',\langle a+a'\rangle}\,\delta_{\ell'',\ell+\ell'+2q(a+a'-a'')/n}. \tag{6.4}$$

This is in agreement with the charge conjugation, if one thinks of the conjugate of a field ϕ as the unique field ϕ^* such that the fusion $\phi \times \phi^*$ contains the identity. The fusion group (the group of simple currents) is isomorphic to $\mathbb{Z}_f \times \mathbb{Z}_{n^2/f}$ with $f = \mathrm{GCD}(2q, n)$.

The entries of S and T lie in $\mathbb{Q}(\xi_{n^2})$, which has Galois group $\mathbb{Z}_{n^2}^\times$. It is not difficult to see that the Galois action on the primaries is by fusion powers (as is always the case for simple currents):

$$\sigma_h(a, \chi_\ell) = (a, \ell)^{\times h}, \qquad h \in \mathbb{Z}_{n^2}^\times. \tag{6.5}$$

The physical invariants are known in all cases, since all primaries are simple currents. Their number varies much with the value of q (of f). Two extreme cases are $f = 1$ ('maximal' twisting) for which the number of physical invariants is equal to $\sigma_0(n^2)$, the number of divisors of n^2 [40], and $f = n$ (no twisting), for which their number is $2(n+1)$ if n is odd prime [27].

We close this simple example by showing that it gives, for a specific twisting, the affine modular data of su(n^2), level 1, if n is an odd integer.

One may first make a few simple observations. The central charge of su$(n^2)_1$ is equal to $c \equiv n^2 - 1 \equiv 0 \pmod 8$ for n odd. The affine quantum dimensions $S_{0,j}/S_{0,0} = 1$ all equal 1, so all affine primaries are simple currents, and form a group isomorphic to $\mathbb{Z}_{n^2}$. Thus one may hope for a relation with twisted $G = \mathbb{Z}_n$ data where the twist obeys $f = \mathrm{GCD}(q, n) = 1$.

The affine primaries can be labelled by integers $j = 0, 1, 2, \ldots, n^2 - 1$ modulo n^2. The affine matrices S and T are

$$S^{\rm aff}_{j,j'} = \frac{1}{n}\, e^{2{\rm i}\pi jj'/n^2}, \tag{6.6}$$

$$T^{\rm aff}_{j,j} = T_{0,0}\, e^{2{\rm i}\pi j(n^2-j)/2n^2}, \tag{6.7}$$

where $T_{0,0} = e^{2{\rm i}\pi c/24}$ is some third root of unity, which we will ignore.

It is now a simple matter to see that the two sets of matrices exactly coincide provided one chooses the twisting parameter as $q = \frac{n^2-1}{2}$. The bijection $j \leftrightarrow (a, \chi_\ell)$ between the two sets of primary fields is given by $j = a - n\ell$. Note that for this specific value of q, the fusion coefficients (6.4) amount to the addition modulo n^2.

6.2 Abelian non-cyclic groups

The simplest non-cyclic group is $G = \mathbb{Z}_n^2$, but it leads to nothing really new. The cohomology group $H^3(\mathbb{Z}_n^2, U(1)) = \mathbb{Z}_n^3$ has three generators, but all 3-cocycles α lead to β's which are all coboundaries. Thus all twistings are CT, despite the fact that the Schur multiplier $M(\mathbb{Z}_n^2) = \mathbb{Z}_n$ is not trivial. (This fact was important for the proof of Thm.4.)

More interesting is the case $G = \mathbb{Z}_n^3$, for which $M(\mathbb{Z}_n^3) = \mathbb{Z}_n^3$ and $H^3(\mathbb{Z}_n^3, U(1)) = \mathbb{Z}_n^7$. Following [41, 42], the generators of H^3 can be taken to be (same notations as above, the group elements are triplets $a = (a_1, a_2, a_3)$)

$$\alpha_I^{(j)}(a, b, c) = \exp\Big\{2{\rm i}\pi a_j(b_j + c_j - \langle b_j + c_j\rangle)/n^2\Big\}, \qquad 1 \le j \le 3, \tag{6.8}$$

$$\alpha_{II}^{(jk)}(a, b, c) = \exp\Big\{2{\rm i}\pi a_j(b_k + c_k - \langle b_k + c_k\rangle)/n^2\Big\}, \qquad 1 \le j < k \le 3, \tag{6.9}$$

$$\alpha_{III}(a, b, c) = \exp\Big\{2{\rm i}\pi a_1 b_2 c_3/n\Big\}. \tag{6.10}$$

An arbitrary 3-cocycle is a monomial in the generators, but only those which involve a non-trivial power of α_{III} define non-CT twistings. In other words, all α which contain a fixed cocycle of type III give rise to 2-cocycles β_a which are cohomologically equivalent, and hence lead to theories with the same number of primaries. In order to give a first feeling for non-CT twistings, we will compute the number of primary fields. It is sufficient to take α of type III, namely $\alpha = \alpha^q_{III}$, for $q \in \mathbb{Z}_n$. The 2-cocycles one obtains are then

$$\beta_a(b,c) = \exp\Big\{2\mathrm{i}\pi q(a_1b_2c_3 - b_1a_2c_3 + b_1c_2a_3)/n\Big\}. \tag{6.11}$$

Given a, we want to count the number of classes b (elements here) which are β_a-regular, i.e. which satisfy $\beta_a(b,c) = \beta_a(c,b)$ for all c. Taking successively $c = (1,0,0), (0,1,0)$ and $(0,0,1)$, the β_a-regular elements b are those which satisfy

$$a_2b_3 - a_3b_2 \equiv a_1b_3 - a_3b_1 \equiv a_1b_2 - a_2b_1 \equiv 0 \pmod f, \tag{6.12}$$

where $f = n/\mathrm{GCD}(q,n)$. The number of solutions $(b_1,b_2,b_3) \in \mathbb{Z}_n^3$ to this modular linear system is equal to $n^3 \cdot [\mathrm{GCD}(a_1,a_2,a_3,f)/f]^2$, which is the result announced in (5.13).

It remains to sum those numbers for all a to obtain the number of primaries. The result is an arithmetical function, best expressed in terms of the prime decomposition of $f = \prod_p p^{k_p}$:

$$|\Phi^\alpha| = \frac{n^6}{f^3} \prod_{\substack{p|f \\ p \text{ prime}}} \Big[(p^{k_p} - 1)(1 + p^{-1} + p^{-2}) + 1\Big]. \tag{6.13}$$

The modular matrices can be given quite explicitly in the general case, for all n and for any type of 3-cocycle. However, they are complicated arithmetic functions of the various parameters, something that obscures the structure. To simplify, we consider here the case when n is an odd prime number, and when the 3-cocycle is α_{III}.

When G is abelian, all factors in the formula (5.23) for S^α that involve the cocycles drop out, and we are left with the simple expressions:

$$S^\alpha_{(a,\tilde\chi),(b,\tilde\chi')} = \frac{1}{|G|}\tilde\chi^*(b)\,\tilde\chi'^*(a), \qquad T^\alpha_{(a,\tilde\chi),(b,\tilde\chi')} = \delta_{a,b}\,\delta_{\tilde\chi,\tilde\chi'}\,\frac{\tilde\chi(a)}{\tilde\chi(e)}, \tag{6.14}$$

where $\tilde\chi$ and $\tilde\chi'$ are respectively β_a- and β_b-projective characters, for the cocycles given above in (6.11) with $q = 1$.

It remains to compute the projective characters. To simplify, consider n an odd prime. One then finds n inequivalent irreducible β_a-projective representations of dimension n if a is not the identity, while there are of course n^3 representations of dimension 1 if $a = e$. Depending on the value of $a = (a_1,a_2,a_3)$, the characters are given in the following table, where it is implicit that the element $g = (g_1,g_2,g_3)$ must be β_a-regular for the character not to vanish. In the first three cases, the character label u runs over $\mathbb{Z}_n$, and in the last column, $\vec u$ takes all values in $\mathbb{Z}_n^3$.

	$a_1 \neq 0$	$a_1 = 0,\ a_2 \neq 0$	$a_1 = a_2 = 0,\ a_3 \neq 0$	$a_1 = a_2 = a_3 = 0$
$\tilde\chi(g)$	$n\,\xi_n^{a_1^{-1}ug_1 - a_1^{-1}a_2a_3g_1^2/2}$	$n\,\xi_n^{a_2^{-1}ug_2}$	$n\,\xi_n^{a_3^{-1}ug_3}$	$\xi_n^{\vec u\cdot\vec g}$

The condition that g must be β_a-regular makes the components of a and g play a symmetrical role. If a_2 is also invertible for instance, then (6.12) yields $a_1^{-1}g_1 = a_2^{-1}g_2$, so that the first character value is also equal to $\tilde{\chi}(g) = n\,\xi_n^{a_2^{-1}ug_2 - a_1a_2^{-1}a_3g_2^2/2}$.

The primary fields are thus $(e, \chi_{\vec{u}})$ and $(a, \tilde{\chi}_u)$, for a total of $|\Phi^\alpha| = n^3 + (n^3-1)n = n^4 + n^3 - n$.

The formulae for S and T are now straightforward to establish. Taking the condition of β-regularity into account, one finds that S is almost block-diagonal:

$$S^\alpha_{(a,\tilde{\chi}_u),(b,\tilde{\chi}_{u'})} = \frac{1}{n}\begin{pmatrix} \frac{1}{n^2} & \frac{1}{n}\xi_n^{-u_1b_1-u_2b_2-u_3b_3} & \frac{1}{n}\xi_n^{-u_2b_2-u_3b_3} & \frac{1}{n}\xi_n^{-u_3b_3} \\ \frac{1}{n}\xi_n^{-u'_1a_1-u'_2a_2-u'_3a_3} & \begin{matrix}\xi_n^{-ua_1^{-1}b_1-u'b_1^{-1}a_1+(a_2a_3b_1+b_2b_3a_1)/2} \\ \times\delta(b_2-a_1^{-1}a_2b_1)\,\delta(b_3-a_1^{-1}a_3b_1)\end{matrix} & 0 & 0 \\ \frac{1}{n}\xi_n^{-u'_2a_2-u'_3a_3} & 0 & \begin{matrix}\xi_n^{-ua_2^{-1}b_2-u'b_2^{-1}a_2} \\ \times\delta(b_3-a_2^{-1}a_3b_2)\end{matrix} & 0 \\ \frac{1}{n}\xi_n^{-u'_3a_3} & 0 & 0 & \xi_n^{-ua_3^{-1}b_3-u'b_3^{-1}a_3} \end{pmatrix}, \tag{6.15}$$

where the blocks correspond to the subsets $\{a = e\}$, $\{a_1 \neq 0\}$, $\{a_1 = 0,\ a_2 \neq 0\}$, and $\{a_1 = a_2 = 0,\ a_3 \neq 0\}$.

The T matrix is particularly simple

$$T_{(a,\tilde{\chi}),(a,\tilde{\chi})} = \begin{cases} 1 & \text{for } a = e = (0,0,0), \\ \xi_n^{u-a_1a_2a_3/2} & \text{otherwise.} \end{cases} \tag{6.16}$$

One may check that they satisfy the expected relations $S^2 = (ST)^3 = C$ with the charge conjugation given by $C(e,\chi_{\vec{u}}) = (e,\chi_{-\vec{u}})$ and $C(a,\tilde{\chi}_u) = (a^{-1},\tilde{\chi}_{u-a_1a_2a_3})$ for $a \neq e$. More generally, the Galois transformations on the primary fields take a somewhat unusual form[‖]

$$\sigma_\ell(a,\tilde{\chi}_u) = \begin{cases} (e,\chi_{\ell\vec{u}}) & \text{for } a = e, \\ (a^\ell, \tilde{\chi}_{\ell^2u+a_1a_2a_3\ell^2(\ell-1)/2}) & \text{for } a \neq e, \end{cases} \qquad \ell \in \mathbb{Z}_n^\times. \tag{6.17}$$

All Galois parities are equal to $+1$. One also checks the relation $\sigma_{\ell^2}T_{(a,\tilde{\chi}),(a,\tilde{\chi})} = T_{\sigma_\ell(a,\tilde{\chi}),\sigma_\ell(a,\tilde{\chi})}$.

Finally the fusion algebra can be computed, which shows a structure radically different from the untwisted or CT case. Here too, the fusion coefficients generically involve the group law and the structure constants for the irreducible characters. Since the latter have degree 1 or n, the fusion of two primary fields may contain 1, n or n^2 primary fields, with possible multiplicities (they turn out to be 1 or n only). The resulting formulae are tedious to write in full form as various cases need be distinguished. As an illustration, we give the coefficients

[‖] The unusual factor ℓ^2 in front of u (instead of ℓ) is a consequence of the insertion of factors a_i^{-1} in front of u in the projective characters. Those insertions are purely conventional and help make the symmetry in the components a_i manifest.

when $a, b, c \neq e$:

$$N_{(a,\tilde\chi_u),(b,\tilde\chi_{u'})}^{(c,\tilde\chi_{u''})} = \delta(a{+}b{-}c) \begin{cases} 1 \quad \text{for } c_1 = 0 \text{ and } a_1, b_1 \neq 0, \\ \Big[n\,\delta(a \text{ is } \beta_b\text{-reg})\,\delta(a_1^{-1}u + b_1^{-1}u' + a_2 b_3 - c_1^{-1}u'') \\ \qquad + \delta(a \text{ is not } \beta_b\text{-reg})\Big] \quad \text{for } c_1 \neq 0 \text{ and } a_1, b_1 \text{ not both } 0, \\ \Big[n\,\delta(a \text{ is } \beta_b\text{-reg})\,\delta(a_2^{-1}u + b_2^{-1}u' - c_2^{-1}u'') \\ \qquad + \delta(a \text{ is not } \beta_b\text{-reg})\Big] \quad \text{for } a_1, b_1 = 0 \text{ and } a_2, b_2 \neq 0, \\ \Big[n\,\delta(a \text{ is } \beta_b\text{-reg})\,\delta(a_3^{-1}u + b_3^{-1}u' - c_3^{-1}u'') \\ \qquad + \delta(a \text{ is not } \beta_b\text{-reg})\Big] \quad \text{for } a_1, a_2, b_1, b_2 = 0 \text{ and } a_3, b_3 \neq 0. \end{cases} \tag{6.18}$$

An intriguing observation made in [42] (what he called *electric/magnetic duality*) is that the modular data for $\mathbb{Z}_2^3$ twisted by α_{III} *equals* that of untwisted $\mathfrak{D}_4$ (for an appropriate identification of primary fields), while the twist $\alpha_I^{(1)}\alpha_{III}$ yields the $\mathfrak{Q}_4$ modular data. We will address this again in Question **(8)** in §7.

6.3 Odd dihedral groups

The simplest non-abelian groups are the dihedral groups $\mathfrak{D}_m$. Their Schur multipliers are equal to 0 or $\mathbb{Z}_2$ for m odd or even respectively. For m odd, all centralisers of elements of $\mathfrak{D}_m$ also have trivial Schur multipliers, implying that all twistings of the modular data will be CT. Despite that fact, we will precisely consider m odd here**, since we want to show that a particular CT twisting of the $\mathfrak{D}_m$ modular data yields nothing but the affine modular data of the odd orthogonal series B_ℓ at level 2, and for $m = \sqrt{2\ell+1}$.

The relevant group theoretic data for $\mathfrak{D}_m$ have been recalled in section 3.2. The number of primary fields is the same for all twistings, and given by $|\Phi| = \frac{m^2+7}{2}$.

The third cohomology group $H^3(\mathfrak{D}_m, U(1)) = \mathbb{Z}_{2m}$ is cyclic and so all 3-cocycles are powers of some generator. They have been very explicitly determined in [41]. Write the elements of $\mathfrak{D}_m$ as $g = s^A\, r^a$ for $A = 0, 1$ and $a = 0, 1, \ldots, m-1$. The group law then takes the form

$$(A, a)(B, b) = (\langle A + B\rangle_2, \langle(-1)^B a + b\rangle_m) \tag{6.19}$$

where the notation $\langle x\rangle_n$ means taking the residue of x modulo n between 0 and $n-1$. Then the 3-cocycles are given by [41]

$$\alpha((A,a),(B,b),(C,c)) = \exp\left\{ -\frac{2\mathrm{i}\pi p}{m^2}\Big[(-1)^{B+C}a[(-1)^C b + c - \langle(-1)^C b + c\rangle_m] + \frac{m^2}{2}ABC\Big]\right\}, \tag{6.20}$$

where $p = 0, 1, 2, \ldots, 2m-1$ labels the cohomology classes.

We now proceed to compute the matrices S and T from the formulas (5.20) and (5.21). We first compute

$$G(e,g) = \mathfrak{D}_m, \quad G(r^k, r^l) = \mathfrak{D}_m, \quad G(s,s) = \{e, s\}, \quad G(s, r^a) = \emptyset. \tag{6.21}$$

**The specific case $m = 3$ has been treated in [6].

In particular, the last equality implies $S^\alpha_{(s,\chi),(r^k,\chi')} = 0$. The remaining entries of S^α only require knowing the following values of σ, which easily follow from (5.15), (5.18) and (5.22),

$$\sigma(e|g) = 1, \qquad \sigma(s|s) = (-1)^p, \qquad \sigma(r^a|r^b) = \exp\left(\frac{-4\mathrm{i}\pi pab}{m^2}\right). \tag{6.22}$$

The rest is just a matter of a few calculations. We will write the two modular matrices with their rows and columns indexed by

$$(e,\psi_0), (e,\psi_1), (s,1), (s,\chi\neq 1), (e,\chi_i), (r^k,\chi_\gamma), \tag{6.23}$$

where χ_γ are characters of the group $\mathbb{Z}_m$, with values $\chi_\gamma(a) = \mathrm{e}^{-2\mathrm{i}\pi a\gamma/m}$. The indices i and γ run from 1 to $\frac{m-1}{2}$ and m respectively.

One finds

$$S^\alpha = \frac{1}{2m}\begin{pmatrix} 1 & 1 & m & m & 2 & 2 \\ 1 & 1 & -m & -m & 2 & 2 \\ m & -m & (-1)^p m & (-1)^{p+1}m & 0 & 0 \\ m & -m & (-1)^{p+1}m & (-1)^p m & 0 & 0 \\ 2 & 2 & 0 & 0 & 4 & 4\cos\frac{2\pi i\gamma}{m} \\ 2 & 2 & 0 & 0 & 4\cos\frac{2\pi i\gamma}{m} & 4\cos\frac{2\pi(2pkk'+\gamma k'+\gamma' k)}{m^2} \end{pmatrix}, \tag{6.24}$$

and

$$T^\alpha = \mathrm{diag}\Big[1, 1, \mathrm{e}^{4\mathrm{i}\pi p/8}, -\mathrm{e}^{4\mathrm{i}\pi p/8}, 1, \mathrm{e}^{2\mathrm{i}\pi(pk^2-mk\gamma)/m^2}\Big]. \tag{6.25}$$

On the affine side, the alcôve of B_ℓ level 2 contains $\ell+4$ primary fields, corresponding to the weights: 0, $2\omega^1$, $\omega^1+\omega^\ell$, ω^ℓ and ν^j, defined as $\nu^j = \omega^j$ for $1 \leq j \leq \ell-1$, and $\nu^\ell = 2\omega^\ell$. The conformal central charge is equal to 2ℓ, which is multiple of 8 if ℓ is a multiple of 4. For simplicity, we will use the height variable, $n = 2\ell+1$.

The affine S matrix, with the primary fields labelled as above, is equal to [43]

$$S = \frac{1}{2\sqrt{n}}\begin{pmatrix} 1 & 1 & \sqrt{n} & \sqrt{n} & 2 \\ 1 & 1 & -\sqrt{n} & -\sqrt{n} & 2 \\ \sqrt{n} & -\sqrt{n} & \sqrt{n} & -\sqrt{n} & 0 \\ \sqrt{n} & -\sqrt{n} & -\sqrt{n} & \sqrt{n} & 0 \\ 2 & 2 & 0 & 0 & 4\cos\frac{2\pi jk}{n} \end{pmatrix}, \tag{6.26}$$

while the affine T is

$$T = T_{0,0}\cdot\mathrm{diag}\Big[1, 1, -\mathrm{e}^{2\mathrm{i}\pi\ell/8}, \mathrm{e}^{2\mathrm{i}\pi\ell/8}, \mathrm{e}^{\mathrm{i}\pi j(n-j)/n}\Big]. \tag{6.27}$$

The comparison of the two pictures is now easy. As far as the first four primary fields are concerned, the two sets of modular data coincide provided $m = \sqrt{n} = \sqrt{2\ell+1}$ and $p \equiv 0 \pmod 2$ (the individual identifications among the third and fourth fields depend on the value of $\ell \pmod 8$). It implies that the finite group and the affine theories have the same number of primaries, $\ell + 4 = \frac{m^2+7}{2}$.

The remaining ℓ fields, on the affine side, are the ν^j. They can be set in correspondence with the finite group primary fields (e, χ_i) and (r^k, χ_γ), depending on whether j is a multiple of m or not, via the following formula:

$$\nu^{im} \sim (e, \chi_i), \qquad \nu^{k+m\gamma} \sim (r^k, \chi_\gamma). \tag{6.28}$$

When i, k and γ run over their domain, the index j indeed takes all integer values from 1 to $\ell = \frac{n-1}{2}$.

One may then check that the two sets of modular matrices coincide for a specific choice of p. For $j = i\sqrt{n}$, one has $j(n-j) \equiv 0 \pmod{2n}$, while for $j = k + \sqrt{n}\gamma$, one finds

$$j(n-j) \equiv \tfrac{n-1}{2}k^2 - k\gamma\sqrt{n} \equiv pk^2 - mk\gamma \pmod{2n}, \tag{6.29}$$

provided one makes the choice $p \equiv \frac{n-1}{2} \pmod{2n}$. It gives the unique twisting for which the finite group modular data and the affine data coincide. That the S matrices are equal is done in the same way.

Hence infinitely many physical invariant classifications for CT-twisted nonabelian G were done in [33]. If we let $d = \sigma_0(m^2) - 1$ denote the number of divisors $d' \leq m$ of m^2, then the number of physical invariants of $\mathfrak{D}_m$ for the given twist α is $d(d+3)/2 + 4$. Since $\mathfrak{D}_3 \cong \mathfrak{S}_3$, we get 'only' 9 physical invariants for twisted $\mathfrak{S}_3$ – still a large number considering the small number of primaries, but dwarfed by the 32 ones for the untwisted data.

7 Questions

We conclude by collecting a small number of the questions raised in this paper.

(1)[††] We have been unable to derive the CT formula (5.20), appearing in e.g. [6], from our general formula (5.23), except in special cases such as when G is abelian. Though this isn't strictly necessary, it would make a nice consistency check of the modular data. Our formula (5.23) for S^α has the strength that it is given explicitly in terms of quantities directly associated with G, but it has the weakness that it involves the 2-cocycles β_a away from $C_G(a)$, and this is what complicates our attempt to derive (5.20). The derivations in [4, 11] involve considering the β_a's at the centralisers only.

(2) It should be possible to generalise the observations in §4.2 by considering the relations of the modular data for G with that of normal subgroups and quotient groups. This should yield an analogue of "conformal embeddings" for this data.

(3) Can we recover the character table from the matrices S and T? Do non-isomorphic groups have different S and T? These are big questions. Probably the answer to the first is yes, and to the second is no. It turns out to be very difficult to identify a group from its character table together with nice additional information – perhaps the modular data provides a means? A natural place to look for nonisomorphic groups with identical modular data are the order 16 groups with identical group algebras, or Brauer pairs (i.e. groups with identical character tables and identical power maps $K_a \mapsto K_{a^n}$).[‡‡] Perhaps relevant is [44],

[††] We thank Peter Bántay for correspondence regarding this point.

[‡‡] We thank John McKay for bringing Brauer pairs to our attention.

which proves that two groups have the same character table iff their group algebras are isomorphic as quasi-Hopf algebras.

(4) We see in places a tantalising hint of some sort of duality between the group element component of the primary field, and its character component (see e.g. §4.2). Is there any way to make this precise?

(5) Does the $\mathrm{SL}_2(\mathbb{Z})$ representation obtained from the finite group modular data, factor through a congruence subgroup? Presumably the answer is always yes. Certainly it is true for untwisted data. The easiest way to see this uses the permutation-like representation of [6, 11, 12], so a natural starting point for the twisted data would involve the basis given at the end of §5.3.

(6) An important fact for affine modular data is the Kac-Peterson formula, which provides a useful interpretation for the eigenvalues $S_{\lambda\mu}/S_{0\mu}$ of the fusion matrices. It would be highly desirable to find out what form if any that takes here.

(7) What is the analogue for the affine modular data of the basis discussed at the end of §5.3, in which S and T appear as generalised permutation matrices?

(8) As mentioned at the end of §6.2, twisted $\mathbb{Z}_2^3$ modular data yields [42] that of untwisted $\mathfrak{D}_4$ and $\mathfrak{Q}_4$ modular data. It is natural to expect that there are many more such examples: e.g. circumstantial evidence (such as quantum dimensions and numbers of primaries) suggests that appropriately twisted $\mathbb{Z}_p^3$ would yield the modular data for the Heisenberg group (consisting of all 3×3 upper triangular matrices with 1's down the diagonal and entries in $\mathbb{Z}_p$), or the order p^3 generalisation of $\mathfrak{D}_4$ (presentation $\langle a,b\,|\,a^{p^2}=b^p=b^{-1}aba^{-1-p}=e\rangle$). This is probably the analogue of "rank-level duality" for this finite group data. Note that it is not a coincidence that $\mathbb{Z}_2^3$ and $\mathfrak{D}_4$ both have the same order — if the matrices S are to be equal, their entries $S_{0,0}$ must agree and hence the two groups must have equal order. Again, perhaps relevant is [44], which may supply a starting point for a general theory.

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A Appendix

We sketch here the derivation of formula (5.24) for T^α. A basis of $D^\alpha(G)$ is labelled by pairs (g,x) of group elements with algebra product

$$(g,x)\cdot(k,y)\ :=\delta_{g,xkx^{-1}}\ \beta_g(x,y)\ (g,xy). \tag{A.1}$$

Now, $D^\alpha(G)$ is semi-simple and finite-dimensional over $\mathbb{C}$, so its irreducibles $R(a,\tilde\rho)$ are subrepresentations of the regular one. They can be described as follows [7]. The representation space for $R_{(a,\tilde\rho)}$ is spanned by vectors $|x_j\rangle\otimes|w\rangle$, and the action is give by

$$R_{(a,\tilde\rho)}(g,x)|x_j\rangle\otimes|w\rangle=\delta_{g,x_kax_k^{-1}}\ \frac{\beta_g(x,x_j)}{\beta_g(x_k,h)}\ |x_j\rangle\otimes\ \tilde\rho(h)|w\rangle, \tag{A.2}$$

where x_j are coset representatives for $G/C_G(a)$ and $(x_k, h) \in G/C_G(a) \times C_G(a)$ are uniquely defined by $x\ x_j = x_k h$. Here $\tilde{\rho}$ denotes a projective irreducible β_a-representation of $C_G(a)$ obtained by restriction of the regular representation of $D^\alpha(G)$.

Contact with the regular representation has been made by Altschuler and Coste [20]. Let us give here an account of it. The central element $v^{-1} = \sum_{k \in G}(k, k)$ is easily diagonalised: For each conjuguacy class K_a of G denote by $e(a)$ the common order of its elements, then

$$\omega_a := \prod_{j=0}^{e(a)-1} \alpha(g, g^j x, x^{-1} g x) \tag{A.3}$$

is independent of the choice of $g \in K_a$ and $x \in G$.

Then all eigenvalues of v^{-1} are the $e(a)$-th roots of the ω_a's. For each such root λ, each $g \in K_a$ and each class $\{g^j x\}$ of $G/\langle g \rangle$, an eigenvector $\psi_\lambda[g, x]$ is easily constructed.

That these eigenvalues of projective β_a-representations are given in terms of (roots of) values of a 3-cocycle raises the interesting group theoretic question: is this a clever way of classifying or putting together all projective representations of a finite group in terms of 3-cocycles for extensions inside which it is a centraliser ?

The central element v is a constant on each $R_{(a,\tilde{\rho})}$, for convenience let us rather focus on v^{-1}, which has a simpler expression and on its eigenvalues: they are necessarily $e(a)$-th roots of the important quantities ω_a, numbers which should be considered as tabulation data of any 3-cocycle. All of their $e(a)$-th roots are eigenvalues of v^{-1}.

The following nice identity satisfied by the twisted non abelian 2-cocycle and valid for any $a, x \in G$, is due to Altschuler:

$$\beta_{xax^{-1}}(xax^{-1},\ x) = \beta_{xax^{-1}}(x,\ a) = \alpha(xax^{-1}, x, a) \tag{A.4}$$

and is proved by direct evaluation in terms of the 3-cocycle α. From it we get the remarkably simple expression: $R_{(a,\tilde{\rho})}(v^{-1})\ \ |x_j\rangle \otimes |w\rangle =\ |x_j\rangle \otimes\ \tilde{\rho}(a)|w\rangle = \lambda\ |x_j\rangle \otimes |w\rangle$ (here $|w\rangle$ spans the representation space of $\tilde{\rho}$. Because of the expression of v^{-1} the quotient of cocycles which appear in $R_{(a,\tilde{\rho})}$ becomes here independent of $x = x_j = x_k$ and in fact equal to 1.

Since for $h \in C_G(a)$,

$$\beta_a(h,\ a) = \beta_a(a,\ h) = \alpha(a, h, a) \tag{A.5}$$

we easily get, using an induction on j for $h = a^j$: $\tilde{\alpha}(a)^{e(a)} = \omega_a I$, $\tilde{\alpha}(a)$ central and equal to λI.

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