Growth-optimal investments and numeraire portfolios under transaction costs: An analysis based on the von Neumann-Gale model*

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Abstract

The aim of this work is to extend the capital growth theory developed by Kelly, Breiman, Cover and others to asset market models with transaction costs. We define a natural generalization of the notion of a numeraire portfolio proposed by Long and show how such portfolios can be used for constructing growth-optimal investment strategies. The analysis is based on the classical von Neumann-Gale model of economic dynamics, a stochastic version of which we use as a framework for the modelling of financial markets with frictions.

Key words: capital growth theory, transaction costs, numeraire portfolios, random dynamical systems, convex multivalued operators, von Neumann–Gale model, rapid paths

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1 Introduction

How to invest in order to achieve the maximum growth rate of wealth in the long run? This question has been in the focus of studies by Kelly [24], Breiman [4], Thorp [36], Algoet and Cover [2], Hakansson and Ziemba [17], Platen and Heath [28], and many others[1]. For the most part, results available in the literature on capital growth pertain to markets without transaction costs. Up to now, only some specialized models of markets with friction have been analyzed in this field; see e.g. Taksar, Klass and Assaf [35], Iyengar and Cover [18], Akian, Sulem and Taksar [1], and Iyengar [19]. The goal of the present work is to develop a capital growth theory within a general discrete-time framework taking into account proportional transaction costs. Our main tool in this study is one of the fundamental models in mathematical economics—the von Neumann-Gale model of economic growth.

The mathematical framework of the von Neumann-Gale model is a special class of multivalued dynamical systems possessing certain properties of convexity and homogeneity. The original theory of such systems (von Neumann [37], Gale [16], Rockafellar [33], Makarov and Rubinov [26]) aimed basically at the modeling of economic dynamics. Initially, this theory was purely deterministic; it did not reflect the influence of random factors on economic growth. The importance of taking these factors into account was realized early on. First attempts of constructing stochastic analogues of the von Neumann–Gale model were undertaken in the 1970s by Dynkin [9, 10, 11], Radner [29, 30] and their research groups. However, the first attack on the problem left many questions unanswered. Studies in this direction faced serious mathematical difficulties. To overcome these difficulties, new mathematical techniques were required, that were developed only during the last decade—see [13, 15] and [3].

In a recent work of Dempster, Evstigneev and Taksar [7], it has been observed that stochastic analogues of von Neumann-Gale dynamical systems provide a natural and convenient framework for the analysis of some fundamental problems in finance (asset pricing and hedging under transaction

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[1] To the list of those who contributed to this line of research, one has to add the name of Claude Shannon—the famous founder of the mathematical theory of information. Although he did not publish on investment-related issues, his ideas expressed in his lectures on investment problems in the 1950s and 60s should be regarded as one of the main sources for that strand of literature which we cite here. For the history of these ideas and the related discussion see Cover [5].

costs). This paper focuses on a different area of applications of such systems in finance. It demonstrates how methods and concepts developed in the context of von Neumann-Gale dynamics can be applied to the analysis of growth optimal investments under transaction costs. A central notion related to von Neumann-Gale dynamical systems, that of a *rapid path*, plays a crucial role in this work. We show that it yields a generalization of the concept of a *numeraire portfolio* (Long [25]) suitable for the analysis of markets with transaction costs and trading constraints. We obtain results on the existence of asymptotically optimal trading strategies in markets with transaction costs by using results [3, 15] on the existence of rapid paths in von Neumann-Gale systems.

The theory of von Neumann-Gale dynamical systems is one of the highlights of mathematical economics. The results we refer to combine advanced methods of ergodic theory, stochastic processes and functional analysis. In this paper, we concentrate only on the modelling issues and the applications in finance. The reader is referred to the literature cited for the proofs of the mathematical results employed in this work. The main goal of this article is to attract attention of theorists and practitioners working in quantitative finance to new powerful methods developed in the field.

The paper is organized as follows. In Section 2 we describe the dynamic securities market model we deal with. Section 3 introduces the basic concepts and results related to the von Neumann-Gale dynamical systems. In Section 4 we apply these results to the analysis of capital growth under transaction costs. Section 5 concludes the paper.

2 Dynamic securities market model.

Let $s_0, s_1, ...$ be a stochastic process with values in a measurable space S. The process $(s_t)_{t=0}^{+\infty}$ models random factors influencing the market: the random element s_t represents the "state of the world" at date $t = 0, 1, ...$ We denote by $s^t := (s_0, s_1, ..., s_t)$ the history of the process (s_t) up to date t.

There are n *assets* traded in the market. A (*contingent*) *portfolio* of assets held by an investor at date t is represented by a vector

$$x_t(s^t) = (x_t^1(s^t), ..., x_t^n(s^t))$$

whose coordinates (portfolio positions) describe the holdings of assets $i = 1, 2, ..., n$. The positions can be described either in terms of "physical units" of assets or in terms of their market values. A contingent portfolio $x_t(s^t)$ depends generally on the whole history s^t of the process (s_t), which means that the investor can select his/her portfolio at date t based on information available by that date. In the applications which we will deal with (capital growth), the standard models, e.g. [2, 4, 24, 36], exclude short selling. Negative portfolio positions might lead to infinite negative values of logarithmic functionals, playing a central role in the present context. Following this approach, we will assume that all contingent portfolios $x_t(s^t)$ are represented by non-negative vector functions. All functions of s^t will be assumed to be measurable and those representing contingent portfolios essentially bounded.

Any sequence of contingent portfolios $x_0(s^0), x_1(s^1), x_2(s^2), ...$ will be called a *trading strategy.* Trading strategies describe possible scenarios of investors' actions at the financial market influenced by random factors. In the model, we are given sets $G_t(s^t) \subseteq \mathbb{R}^n_+ \times \mathbb{R}^n_+$ specifying the *self-financing* (*solvency*) constraints. The main focus of the study is on self-financing trading strategies. A strategy $x_0(s^0), x_1(s^1), x_2(s^2), ...$ is called *self-financing* if

$$(x_{t-1}(s^{t-1}), x_t(s^t)) \in G_t(s^t) \tag{1}$$

almost surely (a.s.) for all $t \geq 1$. The inclusion $(x_{t-1}(s^{t-1}), x_t(s^t)) \in G_t(s^t)$ means that the portfolio $x_{t-1}(s^{t-1})$ can be rebalanced to the portfolio $x_t(s^t)$ at date t in the random situation s^t under transaction costs and trading constraints. The rebalancing of a portfolio excludes inflow of external funds, but it may take into account dividends paid by the assets.

It is assumed that for each $t \geq 1$, the set $G_t(s^t)$ is a *closed convex cone* depending measurably[2] on s^t. This assumption means that the model takes into account *proportional* transaction costs. We give examples of the cones $G_t(s^t)$ below.

Example 1. *No transaction costs.* Let

$$q_t(s^t) = (q_t^1(s^t), ..., q_t^n(s^t)),\ q_t^i(s^t) > 0,$$

be the vector of the market prices of assets $i = 1, 2, ..., n$ at date t. Suppose that portfolio positions are measured in terms of the market values of assets. Define

$$G_t(s^t) := \{(a, b) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ : \sum_{i=1}^n b^i \leq \sum_{i=1}^n \frac{q_t^i(s^t)}{q_{t-1}^i(s^{t-1})} a^i\}. \tag{2}$$

A portfolio $a = (a^1, ..., a^n)$ can be rebalanced to a portfolio $b = (b^1, ..., b^n)$ (without transaction costs) if and only if $(a, b) \in G_t(s^t)$.

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[2] A closed set $G(s) \subseteq \mathbb{R}^n$ is said to depend measurably on a parameter s if the distance to this set from each point in $\mathbb{R}^n$ is a measurable function of s.

Example 2. *Proportional transaction costs: single currency.* Let $G_t(s^t)$ be the set of those $(a,b) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+$ for which

$$\sum_{i=1}^n (1+\lambda^+_{t,i}(s^t))\,(b^i - \frac{q^i_t(s^t)}{q^i_{t-1}(s^{t-1})}a^i)_+ \leq$$

$$\sum_{i=1}^n (1-\lambda^-_{t,i}(s^t))\,(\frac{q^i_t(s^t)}{q^i_{t-1}(s^{t-1})}a^i - b^i)_+\,, \tag{3}$$

where $r_+ := \max\{r, 0\}$ for the real number r. The transaction cost rates for buying and selling are given by the numbers $\lambda^+_{t,i}(s^t) \geq 0$ and $1 > \lambda^-_{t,i}(s^t) \geq 0$, respectively. A portfolio $a = (a^1, ..., a^n)$ can be rebalanced to a portfolio $b = (b^1, ..., b^n)$ (with transaction costs) if and only if the pair of vectors (a, b) belongs to the cone $G_t(s^t)$. Here, we again assume that the coordinates a^i and b^i of the portfolio vectors indicate the current market values of the asset holdings. The inequality in (3) expresses the fact that purchases of assets are made only at the expense of sales of other assets. The approach based on relations (3) is standard in the analysis of transaction costs; see e.g. Jouini and Kallal [20], Cvitanić and Karatzas [6], and Pham and Touzi [27].

Example 3. *Proportional transaction costs: several currencies.* Consider an asset market where n currencies are traded. Suppose that for each $t = 1, 2, ...$ a matrix

$$\mu^{ij}_t(s^t) \text{ with } \mu^{ij}_t > 0 \text{ and } \mu^{ii}_t = 1$$

is given, specifying the exchange rates of the currencies $i = 1, 2, ..., n$ (including transaction costs). The number $\mu^{ij}_t(s^t)$ shows how many units of currency i can be obtained by exchanging one unit of currency j. A portfolio $a = (a^1, ..., a^n)$ of currencies can be exchanged to a portfolio $b = (b^1, ..., b^n)$ at date t in the random situation s^t if and only if there exists a nonnegative matrix (d^{ji}_t) (*exchange matrix*) such that

$$a^i \geqslant \sum_{j=1}^n d^{ji}_t,\ 0 \leqslant b^i \leqslant \sum_{j=1}^n \mu^{ij}_t(s^t) d^{ij}_t.$$

Here, d^{ij}_t $(i \neq j)$ stands for the amount of currency j exchanged into currency i. The amount d^{ii}_t of currency i is left unexchanged. The second inequality says that at time t the ith position of the portfolio cannot be greater than the sum $\sum_{j=1}^n \mu^{ij}_t d^{ij}_t$ obtained as a result of the exchange. The model we deal with here is a version of the multicurrency models considered by Kabanov, Stricker and others (see e.g. [22], [23] and [21]). In spite of some similarity, it cannot be included into the framework developed in the above papers. Note that in this example asset holdings are expressed in terms "physical units" of assets (currencies).

An important class of dynamic securities market models is formed by *stationary models*. They are defined as follows. A model is called stationary if the stochastic process (s_t) is stationary[3] and the given cones $G_t(s^t)$ (specifying the solvency constraints) are of the following form:

$$G_t(s^t) = G(s^t), \tag{4}$$

where for each s^t the set $G(s^t)$ is a closed convex cone in $\mathbb{R}^n_+ \times \mathbb{R}^n_+$ depending measurably on s^t. Assumption (4) expresses the fact that the solvency constraints do not explicitly depend on time: their structure depends only on the current and previous states of the world—on the history s^t of the underlying stochastic process. In the stationary context it is convenient to assume that s_t is defined for each $t = 0, \pm 1, \pm 2, ...$, and in this case the notation s^t refers to the infinite history $s^t = (..., s_{t-1}, s_t)$. This convention will always apply when we shall deal with stationary models.

If the stochastic process (s_t) is stationary, then the models considered in Examples 1 and 2 are stationary if the asset returns $R_t(s^t) := q^i_t(s^t)/q^i_{t-1}(s^{t-1})$ and the transaction cost rates $\lambda^-_{t,i}(s^t)$ and $\lambda^+_{t,i}(s^t)$ do not explicitly depend on t:

$$R_t(s^t) = R(s^t),\ \lambda^{\pm}_{t,i}(s^t) = \lambda^{\pm}_i(s^t).$$

The analogue of this assumption in the Example 3 is the condition that the exchange rates do not explicitly depend on t: $\mu^{ij}_t(s^t) = \mu^{ij}(s^t)$.

In the analysis of stationary models, we will consider a class of trading strategies called *balanced.* A strategy $x_0, x_1, x_2, ...$ is termed balanced if there exist a vector function $x(s^0) \in \mathbb{R}^n_+$ and scalar function $\alpha(s^0) > 0$ such that

$$x_0(s^0) = x(s^0);\ x_t(s^t) = \alpha(s^t)...\alpha(s^1)x(s^t),\ t \geq 1, \tag{5}$$

and $|x(s^0)| = 1$. (We write $|\cdot|$ for the sum of the absolute values of the coordinates of a vector). According to (5), portfolios $x_t(s^t)$ grow with stationary proportions defined by the random vector process $x(s^0), x(s^1), ...$ and at a stationary rate $\alpha(s^1), \alpha(s^2), ...$ The results of capital growth theory pertaining to stationary models (see Section 3) will be stated in terms of balanced trading strategies.

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[3] Recall that a stochastic process (s_t) is called stationary if for any $m = 1, 2, ...$ and any measurable function ϕ on the product of m copies of the space $S \times ... \times S$, the distribution of the random variable $\phi_t := \phi(s_{t+1}, ..., s_{t+m})$ does not depend on t.

3 Von Neumann–Gale dynamical systems

Von Neumann-Gale dynamical systems are defined in terms of multivalued (set-valued) operators possessing properties of convexity and homogeneity. States of such systems are represented by elements of convex cones X_t $(t = 0, 1, ...)$ in linear spaces. Possible one-step transitions from one state to another are described in terms of given operators $A_t(x)$, assigning to each $x \in X_{t-1}$ a convex subset $A_t(x) \subseteq X_t$. It is assumed that the graphs $Z_t := \{(x, y) \in X_{t-1} \times X_t : y \in A_t(x)\}$ of the operators $A_t(x)$ are convex cones. *Paths* (*trajectories*) of the von Neumann-Gale dynamical system are sequences $x_0 \in X_0, x_1 \in X_1, ...$ such that $x_t \in A_t(x_{t-1})$.

In this work we consider stochastic von Neumann-Gale dynamical systems in which a stochastic process (s_t) and a sequence of random closed convex cones $G_t(s^t) \subseteq \mathbb{R}^n_+ \times \mathbb{R}^n_+$ $(t = 1, 2, ...)$ are given. The random elements s_t of a measurable space S are defined either for all non-negative integers t or for all integers t. In the former case $s^t := (s_0, ..., s_t)$ and in the latter $s^t := (...s_{t-1}, s_t)$. We denote by $\mathcal{X}_t$ the cone of measurable essentially bounded vector functions $x(s^t)$ with values in $\mathbb{R}^n_+$ and we put

$$Z_t = \{(x, y) \in \mathcal{X}_{t-1} \times \mathcal{X}_t : (x(s^{t-1}), y(s^t)) \in G_t(s^t) \text{ (a.s.)}\}, \tag{6}$$

$$A_t(x) := \{y \in \mathcal{X}_t : (x, y) \in Z_t\}. \tag{7}$$

The multivalued operators $x \mapsto A_t(x)$ $(t = 1, 2, ...)$ transforming elements of $\mathcal{X}_{t-1}$ into subsets of $\mathcal{X}_t$ define the von Neumann-Gale dynamical system we deal with. Paths of this system are sequences of vector functions $x_t(s^t)$ such that $x_t \in \mathcal{X}_t$ and $x_t \in A_t(x_{t-1})$. In the applications we have in mind, these paths are self-financing investment strategies in the dynamic securities market model described in the previous section and $G_t(s^t)$ are the solvency cones in this model.

It is assumed that the cone $G_t(s^t)$ depends measurably on s^t, and for all t the following basic conditions hold:

(**G.1**) for any $a \in \mathbb{R}^n_+$, the set $\{b : (a, b) \in G_t(s^t)\}$ is non-empty;

(**G.2**) the set $G_t(s^t)$ is contained in $\{(a, b) : |b| \leq M_t|a|\}$, where M_t is a constant independent of s^t;

(**G.3**) there exist a strictly positive constant $\gamma_t > 0$ and a pair of essentially bounded vector functions $(\check{a}_{t-1}(s^t), \check{b}_t(s^t))$ such that $(\check{a}_{t-1}(s^t), \check{b}_t(s^t)) \in G_t(s^t)$ for all s^t and $\check{b}_t(s^t) \geq \gamma_t e$, where $e = (1, ..., 1)$;

(**G.4**) if $(a, b) \in G_t(s^t)$, $a' \geq a$ and $0 \leq b' \leq b$, then $(a, b) \in G_t(s^t)$ ("free disposal hypothesis").

All inequalities between vectors, strict and non-strict, are understood coordinatewise.

Define

$$G_t^{\times}(s^t) = \{(c,d) \geq 0 : db - ca \leq 0 \text{ for all } (a,b) \in G_t(s^t)\}, \tag{8}$$

where ca and db denote the scalar products of the vectors. Let $\mathcal{P}_t$ denote the set of measurable vector functions $p(s^t)$ with values in $\mathbb{R}_+^n$ such that $E|p(s^t)| < \infty$. A *dual path* (*dual trajectory*) is a finite or infinite sequence $p_1(s^t), p_2(s^t), ...$ such that $p_t \in \mathcal{P}_t$ $(t \geq 1)$ and

$$(p_t(s^t), E_t p_{t+1}(s^t)) \in G_t^{\times}(s^t) \text{ (a.s.)} \tag{9}$$

for all $t \geq 1$. We write $E_t(\cdot) = E(\cdot|s^t)$ for the conditional expectation given s^t. By virtue of (8) and (9), $E_t(p_{t+1}y) \leq p_t x$ (a.s.) for any $(x,y) \in Z_t$. This inequality shows that for any path $x_0, x_1, ...$ the sequence of random variables $p_1 x_0, p_2 x_1, ...$ is a supermartingale with respect to the given filtration in the underlying probability space generated by s^t.

A dual path $p_1, p_2, ...$ is said to *support* a path $x_0, x_1, ...$ if

$$p_t x_{t-1} = 1 \text{ (a.s.)} \tag{10}$$

for all $t \geq 1$. A trajectory is called *rapid* if there exists a dual trajectory supporting it. The term "rapid" is motivated by the fact that

$$\frac{E_t(p_{t+1}y_t)}{p_t y_{t-1}} \leq \frac{E_t(p_{t+1}x_t)}{p_t x_{t-1}} = 1 \text{ (a.s.)}$$

for each path $y_0, y_1, ...$ with $p_t y_{t-1} > 0$ (see (9) and (10)). This means that the path $x_0, x_1, ...$ maximizes the conditional expectation of the *growth rate* at each time t, the maximum being equal to 1. Growth rates are measured in terms of the random linear functions $p_t a$ of $a \in \mathbb{R}_+^n$. If states x_t of the von Neumann-Gale system represent portfolios whose positions are expressed in terms of units of assets, then p_t can be interpreted as asset price vectors. If the ith coordinate x_t^i of the vector x_t stands for the market value of the ith position of the portfolio, then p_t^i may be regarded as a discount factor for the market price of the ith asset. Another motivation of the term "rapid path" lies in the fact that any rapid path is asymptotically optimal—see the next section.

4 Capital growth theory and von Neumann-Gale dynamical systems

From the point of view of capital growth, those investment strategies are of primary interest for which investor's wealth grows at an asymptotically optimal rate. There are various approaches to the notion of asymptotic optimality. In the definition below, we follow essentially Algoet and Cover [2].

Definition 1. Let $x_0, x_1, ...$ be an investment strategy. It is called *asymptotically optimal* if for any other investment strategy $y_0, y_1, ...$ there exists a supermartingale ξ_t such that

$$\frac{|y_t|}{|x_t|} \le \xi_t,\ t = 0, 1, ...\ \text{(a.s.)}.$$

Recall that for a vector $b = (b^1, ..., b^n)$ we write $|b| = |b^1| + ... + |b^n|$. If $b \ge 0$, then $|b| = b^1 + ... + b^n$, and if the vector b represents a portfolio whose positions are measured in terms of the market values of assets, then $|b|$ is the market value of this portfolio. Note that the above property remains valid if $|b|$ is replaced by any function $\psi_t(s^t, b)$ (possibly random and depending on t) which satisfies

$$l|b| \le \psi_t(s^t, b) \le L|a|, \tag{11}$$

where $0 < l < L$ are non-random constants. As an example of such a function, we can consider the *liquidation value* (or *net asset value*) of the portfolio

$$\psi_t(s^t, b) = \sum_{i=1}^{n} (1 - \lambda^-_{t,i}(s^t)) b^i$$

within the model defined by (3). This is the amount of money the investor gets if he/she decides to liquidate the portfolio (sell all the assets) at date t. Clearly condition (11) holds if the random variables $1 - \lambda^-_{t,i} > 0$ are uniformly bounded away from zero.

The strength of the above definition, which might seem not immediately intuitive, is illustrated by the following implications of asymptotic optimality. As long as $|y_t|/|x_t| \le \xi_t$, $t = 0, 1, ...$ (a.s.), where ξ_t is a supermartingale, the following properties hold.

(a) With probability one

$$\sup_t \frac{|y_t|}{|x_t|} < \infty\ ,$$

i.e. for no strategy wealth can grow asymptotically faster than for $x_0, x_1, ...$ (a.s.).

(b) The strategy $x_0, x_1, ...$ a.s. maximizes the exponential growth rate of wealth

$$\limsup_{t \to \infty} \frac{1}{t} \ln |x_t|.$$

(c) We have

$$\sup_t E\frac{|y_t|}{|x_t|} < \infty \quad \text{and} \quad \sup_t E \ln \frac{|y_t|}{|x_t|} < \infty.$$

Assertion (a) follows from a.s. convergence of non-negative supermartingales; (b) is immediate from (a); the first part of (c) holds because ξ_t is a non-negative supermartingale; the second part of (c) is obtained by using Jensen's inequality and the supermartingale property: $E(\ln \xi_{t+1}|s^t) \leq \ln E(\xi_{t+1}|s^t) \leq \ln \xi_t$.

This work aims at obtaining results on optimal growth in the model with transaction costs described in Section 2. The main results are concerned with the existence of asymptotically optimal strategies in the general (non-stationary) version of the model and the existence of asymptotically optimal balanced strategies in its stationary version. Our main tool for analyzing the questions of asymptotic optimality is the concept of a rapid path in the stochastic von Neumann-Gale system (see the previous section).

Definition 2. A self-financing trading strategy $x_0, x_1, ...$ is called *rapid* if it forms a rapid path in the underlying von Neumann-Gale dynamical system which defines the asset market model.

When dealing with the dynamic securities market model defined in terms of a von Neumann-Gale dynamical system, we will use the terms "paths" and "self-financing trading strategies" interchangeably.

In the context of the present model, rapid paths may be regarded as analogues of *numeraire portfolios* (Long [25]). As we have noticed, the price system (or the system of discount factors) (p_t) involved in the definition of a rapid path is such that the value $p_{t+1}x_t$ of the portfolio x_t is always equal to one, while for any other feasible sequence (y_t) of contingent portfolios (self-financing trading strategy), the values $p_{t+1}y_t$ form a supermartingale. In the classical case when transaction costs are absent, these conditions hold for the price vectors $p_{t+1} := \lambda_t q_t$, where q_t are the market prices and $\lambda_t^{-1} = q_t x_t$ is the market value of the numeraire portfolio x_t. The latter is defined so that the normalized prices $q_t/q_t x_t$ form a supermartingale. (Long [25] considered a model with unlimited short selling, and in that context one can speak of martingales rather than supermartingales.)

The results are based on assumption (**G.5**) below.

(**G.5**) There exists an integer $l \geq 1$ such that for every $t \geq 0$ and $i = 1, ..., n$ there is a path $y_{t,i}, ..., y_{t+l,i}$ over the time interval $[t, t+l]$ satisfying

$$y_{t,i} = e_i, ..., y_{t+l,i} \geq \gamma e,$$

where $e_i = (0, 0, ..., 1, ..., 0)$ (the ith coordinate is 1 while the others are 0) and γ is a strictly positive non-random constant.

Proposition 1. *If the constants M_t in condition (**G.2**) do not depend on t and assumption **(G.5)** holds, then any rapid path is asymptotically optimal.*

Thus in order to prove the existence of asymptotically optimal strategies it is sufficient to establish the existence of infinite rapid paths. For a proof of Proposition 1 see Evstigneev and Flåm [12], Proposition 2.5. In specific dynamic securities market models, condition (**G.5**) holds typically with $l = 1$. Then it means a possibility of buying some fixed strictly positive amounts of all the assets by selling one unit of any asset $i = 1, ..., n$ (or if portfolio positions are measured in terms of the market values of assets—by selling the amount of asset i worth a unit of cash).

The main results of this paper are collected in the following theorem.

Theorem 1. *(i) Let $x_0(s^0)$ be a vector function in $\mathcal{X}_0$ such that $ce \leq x_0(s^0) \leq Ce$ for some constants $0 < c \leq C$. Then there exists an infinite rapid path with initial state $x_0(s^0)$. (ii) If the model is stationary and (**G.5**) holds, then there exists a balanced rapid path. (iii) If the constants M_t in condition (**G.2**) do not depend on t and assumption **(G.5)** holds, then the rapid paths whose existence is claimed in (i) and (ii) are asymptotically optimal.*

Assertion (iii) is immediate from Proposition 1. Statement (i) of the above theorem is proved in [3], where the existence of infinite rapid paths with the given initial state is established. The proof in [3] is conducted by passing to the limit from finite time horizons, for which the existence of rapid paths is obtained in [12]. The passage to the limit is based on a compactness principle involving Fatou's lemma in several dimensions (Schmeidler [34]).

Assertion (ii) follows from the results of papers [14, 15], where not only the existence of a rapid path is proved, but also it is shown that there exists a balanced rapid path supported by a dual trajectory with the following special structure:

$$p_1(s^1) = p(s^1),\ p_t(s^t) = \frac{p(s^t)}{\alpha(s^{t-1})...\alpha(s^1)},\ t = 2, 3, ..., \tag{12}$$

where $\alpha(s^1) > 0$ and $p(s^1) \geq 0$ are scalar and vector functions such that $E|p(s^1)| < \infty$ (*balanced dual trajectory*). The triplet of functions $\alpha(\cdot)$, $p(\cdot)$, $x(\cdot)$ involved in (5) and (12) is called a *von Neumann equilibrium*. It can be shown that if $\alpha(\cdot)$, $p(\cdot)$, $x(\cdot)$ is a von Neumann equilibrium, then the balanced trajectory defined by (5) maximizes $E \ln \alpha$ among all such trajectories. This means by definition that (5) is a *von Neumann path*. The existence of a von Neumann equilibrium established in [14, 15] is a deep result solving a problem that remained open for more than three decades. In the former of the two papers [14, 15], a version of the existence theorem for a von Neumann equilibrium is obtained which deals with an extended model defined in terms of *randomized paths*. In the latter paper, the final result is derived by using the method of elimination of randomization (Dvoretzky, Wald and Wolfowitz [8]).

To use Theorem 1 in specific models, one has to verify assumptions (**G.1**)–(**G.5**) (note that (**G.3**) is a consequence of (**G.5**) with $l = 1$). In Example 1, these conditions follow from the assumption that the the asset returns $R_t^i(s^t) := q_t^i(s^t)/q_{t-1}^i(s^{t-1})$ are uniformly bounded and uniformly bounded away from zero. To obtain (**G.1**)–(**G.5**) in Example 3 it is sufficient to assume that the exchange rates $\mu_t^{ij}(s^t)$ are uniformly bounded away from zero and infinity. In Example 2, all the conditions needed can be obtained if the above assumption regarding $R_t^i(s^t)$ holds and the following requirement regarding the transaction costs is fulfilled: the random variables $\lambda_{t,i}^+(s^t)$ are uniformly bounded and the random variables $1 - \lambda_{t,i}^-(s^t) > 0$ are uniformly bounded away from zero. In all the three cases, (**G.2**) holds with constants M_t independent of t.

5 Conclusion

This work deals with an important class of multivalued random dynamical systems originally studied in mathematical economics—von Neumann–Gale dynamical systems. We show how they can be applied to the analysis of some fundamental issues in finance. This approach allows us to establish a link with the classical von Neumann and Gale economic growth models, which makes it possible to use concepts, techniques and results from mathematical economics to obtain new theoretical results in finance. Even though one would think that models of economic dynamics are the "next of kin" to dynamic security market models, surprisingly they have not been analyzed from this angle for quite a while, and interconnections between these two types of modeling frameworks have not been examined in as much detail as they deserve. In a previous paper [7], this approach was applied to questions of asset pricing and hedging under transaction costs and portfolio constraints. In the present study we show how it can be employed to develop a theory of capital growth under proportional transaction costs.

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References

[1] Akian, M., Sulem, A. and Taksar, M.I., Dynamic optimization of long-term growth rate for a mixed portfolio with transactions costs, *Mathematical Finance* **11** (2001), 153–188.

[2] Algoet, P.H. and Cover, T.M., Asymptotic optimality and asymptotic equipartition properties of log-optimum investment, *Annals of Probability* **16** (1988), No. 2, 876–898.

[3] Bahsoun, W., Evstigneev, I.V. and Taksar, M.I., Rapid paths in von Neumann-Gale dynamical systems, *Stochastics* **80** (2008), 129–142.

[4] Breiman, L., Optimal gambling systems for favorable games, *Proceedings of the 4th Berkeley Symposium on Mathematical Statistics and Probability,* Vol. I, 1961, University of California Press, Berkeley, pp. 65–78.

[5] Cover, T.M., Shannon and investment, *IEEE Information Theory Society Newsletter*, Summer 1998, Special Golden Jubilee Issue, 10-11.

[6] Cvitanić, J. and Karatzas, I., Hedging and portfolio optimization under transaction costs: A martingale approach, *Mathematical Finance* **6** (1996) 133–165.

[7] Dempster, M.A.H., Evstigneev, I.V. and Taksar, M.I., Asset pricing and hedging in financial markets with transaction costs: An approach based on the von Neumann-Gale model, *Annals of Finance* **2** (2006), 327-355.

[8] Dvoretzky, A., Wald, A., and Wolfowitz, J., Elimination of randomization in certain problems of statistics and of the theory of games, *Proceedings of the National Academy of Sciences of the USA*, **36** (1950), 256–260.

[9] Dynkin, E.B., Some probability models for a developing economy, *Soviet Mathematics Doklady* **12** (1971), 1422–1425.

[10] Dynkin, E.B., Stochastic concave dynamic programming, *USSR Mathematics Sbornik* **16** (1972), 501–515.

[11] Dynkin, E.B. and Yushkevich, A.A., *Controlled Markov Processes and their Applications*, 1979, Springer, N.Y.

[12] Evstigneev, I.V. and Flåm, S.D., Rapid growth paths in multivalued dynamical systems generated by homogeneous convex stochastic operators, *Set-Valued Analysis* **6** (1998), 61–82.

[13] Evstigneev, I.V. and Schenk-Hoppé, K.R., The von Neumann-Gale growth model and its stochastic generalization, in: Dana, R.-A., Le Van, C., Mitra, T. and Nishimura, K. (Eds.), *Handbook on Optimal Growth*, Vol. I, Chapter 2, 2006, Springer, N.Y, pp. 337–383.

[14] Evstigneev, I.V., and Schenk-Hoppé, K.R., Pure and randomized equilibria in the stochastic von Neumann-Gale model, *Journal of Mathematical Economics* **43** (2007), 871–887.

[15] Evstigneev, I.V. and Schenk-Hoppé, K.R., Stochastic equilibria in von Neumann-Gale dynamical systems, *Transactions of the American Mathematical Society*, **360** (2008), 3345–3364.

[16] Gale, D., A closed linear model of production, in: Kuhn, H.W. et al. (Eds.), *Linear Inequalities and Related Systems*, 1956, Princeton University Press, Princeton, pp. 285–303.

[17] Hakansson, N.H. and Ziemba, W.T., Capital growth theory, in: Jarrow, R., Maksimovic, A.V. and Ziemba, W.T. (Eds.), *Handbooks in Operations Research and Management Science*, Vol. 9, Finance, Chapter 3, 1995, Elsevier, Amsterdam, 65–86.

[18] Iyengar, G. and Cover, T.M., Growth optimal investment in horse race markets with costs, *IEEE Transactions on Information Theory* **46** (2000), 2675–2683.

[19] Iyengar, G., Universal investment in markets with transaction costs, *Mathematical Finance* **15** (2005), 359–371.

[20] Jouini, E. and Kallal, H., Martingales and arbitrage in securities markets with transaction costs, *Journal of Economic Theory* **66** (1995) 178–197.

[21] Kabanov, Yu.M., The arbitrage theory, in: *Handbooks in Mathematical Finance: Option Pricing, Interest Rates and Risk Management*, E. Jouini, J. Cvitanić and M. Musiela (Eds.), 2001, Cambridge University Press, Cambridge, pp. 3-42.

[22] Kabanov Yu.M., Hedging and liquidation under transaction costs in currency markets, *Finance and Stochastics* **3** (1999), 237–248.

[23] Kabanov, Yu.M. and Stricker, C., The Harrison–Pliska arbitrage pricing theorem under transaction costs, *Journal of Mathematical Economics* **35** (2001), 185–196.

[24] Kelly, J.L., A new interpretation of information rate, *Bell System Technical Journal* **35** (1956), 917-926.

[25] Long, J.B., The numeraire portfolio, *Journal of Financial Economics* **26** (1990), 29–69.

[26] Makarov, V.L. and Rubinov, A.M., *Mathematical Theory of Economic Dynamics and Equilibria*, 1977, Springer-Verlag, Berlin.

[27] Pham, H. and Touzi, N., The fundamental theorem of asset pricing under cone constraints, *Journal of Mathematical Economics* **31** (1999) 265–279.

[28] Platen, E. and Heath, D., *A Benchmark Approach to Quantitative Finance*, 2006, Springer, N.Y.

[29] Radner, R., Balanced stochastic growth at the maximum rate, in: *Contributions to the von Neumann Growth Model* (Conference Proceedings, Institute for Advanced Studies, Vienna, 1970), Zeitschrift für Nationalökonomie, Suppl. No. 1, pp. 39–53.

[30] Radner, R., Optimal steady-state behavior of an economy with stochastic production and resources, in: Day, R.H. and Robinson, S.M. (Eds.), *Mathematical Topics in Economic Theory and Computation*, 1972, SIAM, Philadelphia, pp. 99–112.

[31] Radner, R., Optimal stationary consumption with stochastic production and resources, Journal of Economic Theory **6** (1973), 68–90.

[32] Radner, R., Equilibrium under uncertainty, in: Arrow, K.J. and Intrilligator, M.D. (Eds.), *Handbook of Mathematical Economics*, 1982, North-Holland, Amsterdam, 923–1006.

[33] Rockafellar, R.T., *Monotone Processes of Convex and Concave Type,* Memoirs of the American Mathematical Society, Volume 77, 1967, American Mathematical Society, Providence, RI.

[34] Schmeidler, D., Fatou's lemma in several dimensions, *Proceedings of the American Mathematical Society* **24** (1970), 300–306.

[35] Taksar, M.I., Klass, M.J. and Assaf, D., Diffusion model for optimal portfolio selection in the presence of brokerage fees, *Mathematics of Operations Research* **13** (1988), 277–294.

[36] Thorp, E.O., Portfolio choice and the Kelly criterion, in: Ziemba, W.T. and Vickson, R.G. (Eds.), *Stochastic Models in Finance*, 1971, Academic Press, N.Y., pp. 599–619.

[37] Von Neumann, J., Über ein ökonomisches Gleichungssystem und eine Verallgemeinerung des Brouwerschen Fixpunktsatzes, in: *Ergebnisse eines Mathematischen Kolloquiums*, **8** (1937), 1935–1936 (Leipzig and Wien: Franz-Deuticke), pp. 73–83. [Translated: A model of general economic equilibrium, *Review of Economic Studies* **13** (1945-1946), pp. 1–9.]