HYBRID ATLAS MODELS

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Abstract. We study Atlas-type models of equity markets with local characteristics that depend on both name and rank, and in ways that induce a stability of the capital distribution. Ergodic properties and rankings of processes are examined with reference to the theory of reflected Brownian motions in polyhedral domains. In the context of such models, we discuss properties of various investment strategies, including the so-called *growth-optimal* and *universal* portfolios.

Dedicated to Professor J. Michael Harrison on the occasion of his 65th Birthday.

1. Introduction

In modeling equity market behavior, the goal is to construct models that are simple enough to be amenable to mathematical analysis, yet complicated enough to capture the salient characteristics of real equity markets. A particularly salient characteristic of an equity market is its *capital distribution curve*:

$$\log k \;\mapsto\; \log \mu_{(k)}(t)\,, \qquad k = 1, \cdots, n\,, \tag{1.1}$$

i.e., the logarithms of the individual companies' relative capitalizations (market weights) $\mu_{(\cdot)}(t)$ at time t, arranged in descending order, versus the logarithms of their respective ranks from the largest company down to the smallest.

The capital distribution curve for the U.S. equity market has shown remarkable stability over the last century (see, e.g., Figure 5.1 of Fernholz [12]), and this stability has been captured in the Atlas and first-order models introduced in [12] and studied by Banner, Fernholz & Karatzas [3] and

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others. While Atlas and first-order models are able to reproduce the shape and stability of the capital distribution curve, they still fail to provide an accurate representation of market behavior. It was shown in [3] that these models are *ergodic*, i.e., each stock spends about the same proportion of time in each rank over the long term. While ergodicity may be a nice mathematical property, it does not seem to hold for real markets: in real markets the largest stocks seem to retain their status for long periods of time, while most stocks never reach the upper echelons of capitalization. Hence, a more elaborate model is needed.

In this paper we generalize the first-order models by introducing name-based effects of companies in addition to the rank-based effects that are present in the simpler models. The resulting *hybrid model* has more flexibility to describe faithfully the complexity of the whole market; in particular, the model has both stability properties and occupation time properties that are realistic.

Relation to Extant Literature: From a different point of view, the Atlas model can be seen as a physical particle system, with each company represented by a particle diffusing on the positive real line. These individual diffusive motions have drift and volatility coefficients that depend on the entire configuration of particles at any given moment. Recently Pal & Pitman [20] and Chaterjee & Pal [6], [7] studied such systems, specifically when the drift coefficient is a function of the particle's rank and all volatility coefficients are equal to a given constant. Under appropriate conditions on the drift coefficients, the system has a unique invariant measure in a lower-dimensional space; to wit, the system of the n particles is itself not ergodic, but the projected system in a lower-dimensional hyperplane turns out to be ergodic, with invariant measure that has an explicit exponential-product-form probability density function. Moreover, when the number of particles increases to infinity, the system converges weakly to one described by a Poisson-Dirichlet distribution on the real line. These considerations are useful in studying the Atlas model for an equity market, when the volatility coefficients are all equal.

The model is still tractable when its volatility coefficients depend on the rankings. Questions of existence and uniqueness for such systems in this generality are settled through the theory of martingale problem studied by Stroock & Varadhan [23] and Bass & Pardoux [5]. A salient feature is that three or more particles may collide with each other at the same time with positive probability, or even with probability one, under a suitably "uneven" volatility structure. This is a very significant departure from the constant-volatility case. Some sufficient conditions on the volatility coefficients for the occurrence and for the avoidance of triple (or higher-order) collisions, are derived in [16], by comparison with Bessel processes and with help from properties of reflected Brownian motion.

The ranked particle system has a deep relation with the theory of multi-dimensional reflected Brownian motion studied intensively in the context of stochastic network systems by Harrison, Reiman and Williams [13] [14] [15] and their collaborators. Recently Dieker & Moriarty [10] provided necessary and sufficient conditions for the invariant density of semimartingale reflected Brownian motions in a two-dimensional wedge to be written as a finite sum of terms of product-of-exponential form, by extending the geometric considerations on the so-called "skew-symmetry" condition. In the present paper we use this skew-symmetry condition for the n-dimensional reflected Brownian motions to solve the basic adjoint relation introduced in a piece-wise constant drift coefficient structure, and find an invariant density for the ranked process of the hybrid Atlas model in the form of sum of products of exponentials. With this explicit formula we compute the invariant distribution of the capital distribution curve, as well as the long-term average occupation times.

Another interesting system of ranked particles is Dyson's process of non-colliding Brownian motions, which are the ordered eigenvalues of a Brownian motion on the space of Hermitian matrices. Recent work by Warren [24] constructs Dyson's process using Doob's $h-$transform and Brownian motion in the Gelfand-Tsetlin cone, as an extension of Dubédat's work [11] on the relation between reflected Brownian motions on the wedge and a Bessel process of dimension three. The (infinite) ranked particle systems also appear in mean-field spin glass theory of Mathematical Physics. In a further recent development, Arguin & Aizenman [1] analyze robust quasi-stationary competing particle systems with overlapped hierarchical structures where the Poisson-Dirichlet distribution emerges as in [20]. Instead of taking Dyson's process or the spin glass theory as our model for rankings in equity markets, we obtain the ranked particle system through a general formula of Banner & Ghomrasni [4] for continuous semimartingales in the hybrid Atlas model.

Preview: This paper follows the following structure. We describe our model in Section 2, its lower-dimensional ergodic properties in Section 3, the dynamics of its rankings in Section 4, its invariant measure and occupation times in Section 5, and some portfolio analysis in its context in Section 6. In Section 7 we prove some auxiliary results stated in the main sections.

Notation: The following notions and notation are useful to describe rankings as in [3]. We consider a collection $\{Q_k^{(i)}\}_{1\le i,k\le n}$ of polyhedral domains in $\mathbb{R}^n$, where $y = (y_1, \ldots, y_n) \in Q_k^{(i)}$ means that the coordinate y_i is ranked k^{th} among $y_1, \ldots, y_n$, with ties resolved in favor of the lowest index (or "name"). Note that for every index $i = 1, \ldots, n$ and rank $k = 1, \ldots, n$, we have the partition properties $\cup_{\ell=1}^n Q_\ell^{(i)} = \mathbb{R}^n = \cup_{j=1}^n Q_k^{(j)}$.

We shall denote by Σ_n the symmetric group of permutations of $\{1, \ldots, n\}$. For each permutation $\mathbf{p} \in \Sigma_n$ we consider $\mathcal{R}_{\mathbf{p}} := \cap_{k=1}^n Q_k^{(\mathbf{p}(k))}$, the polyhedral chamber consisting of all points $y \in \mathbb{R}^n$ such that $y_{\mathbf{p}(k)}$ is ranked

k^{th} among $y_1, \ldots, y_n$, for every $k = 1, \ldots, n$. The collection of polyhedral chambers $\{\mathcal{R}_{\mathbf{p}}\}_{\mathbf{p}\in\Sigma_n}$ is a partition of all of $\mathbb{R}^n$.

Since for each $y \in \mathbb{R}^n$ there exists a unique $\mathbf{p} \in \Sigma_n$ such that $y \in \mathcal{R}_{\mathbf{p}}$ (because of the way ties are resolved), we shall find it useful to define an *indicator map* $\mathbb{R}^n \ni (x_1, \cdots, x_n)' = x \mapsto \mathfrak{p}^x \in \Sigma_n$ such that $x_{\mathfrak{p}^x(1)} \geq \cdots \geq x_{\mathfrak{p}^x(n)}$. In other words, $\mathfrak{p}^x(k)$ is the index of the coordinate in the vector x that occupies the k^{th} rank among $x_1, \cdots, x_n$.

When using matrices and vectors, the vector norm $\|x\| := \big(\sum_{i=1}^n x_i^2\big)^{1/2}$ and the inner product $\langle x, y\rangle := \sum_{i=1} x_i y_i = x'y$ for $x, y \in \mathbb{R}^n$, where $'$ stands for transposition, are defined in the usual manner. The gradient ∇ and the Laplacian Δ operators on the space C^2 of twice continuously differentiable functions are used in Section 5, as well as the notations $C_c^2(\cdot)$ and $C_b^2(\cdot)$ for the spaces of twice continuously differentiable functions with compact support, and of twice continuously differentiable bounded functions, respectively.

2. Model

We shall study an equity market that consists of n assets (stocks) with capitalizations $\mathfrak{X}(t) = (X_1(t), \ldots, X_n(t))'$ which are positive for all times $0 \leq t < \infty$. The random variable $X_i(t)$ represents the capitalization at time t of the asset with index (name) i.

We shall assume that the log-capitalizations $Y_i(t) := \log X_i(t)$, $i = 1, \ldots, n$ satisfy the system of stochastic differential equations

$$\begin{aligned} d\,Y_i(t) &= \Big(\sum_{k=1}^n g_k \mathbf{1}_{Q_k^{(i)}}(Y(t)) + \gamma_i + \gamma \Big)\, d\,t + \sum_{j=1}^n \rho_{i,j}\; d\,W_j(t) \\ &\quad + \sum_{k=1}^n \sigma_k \mathbf{1}_{Q_k^{(i)}}\big(Y(t)\big)\, d\,W_i(t)\,, \quad Y_i(0) = \bar{y}_i\,; \quad 0 \leq t < \infty \end{aligned} \tag{2.1}$$

with given initial condition $\bar{y} = (\bar{y}_1, \ldots, \bar{y}_n)'$. The process $W(\cdot) := (W_1(\cdot), \ldots, W_n(\cdot))'$ is an n dimensional Brownian motion. As long as the n dimensional process $Y(\cdot) := (\,Y_1(\cdot), \ldots, Y_n(\cdot)\,)'$ of log-capitalizations is in the polyhedron $Q_k^{(i)}$, the $i^{th}-$coordinate $Y_i(\cdot)$ is ranked k^{th} among $Y_1(\cdot), \ldots, Y_n(\cdot)$ and behaves like a Brownian motion with drift $g_k + \gamma_i + \gamma$ and variance $(\sigma_k + \rho_{i,i})^2 + \sum_{j\neq i} \rho_{i,j}^2$. The constants γ, γ_i and g_k represent a common, a name-based, and a rank-based drift (growth rate), respectively; whereas the constants σ_k and $\rho_{i,j}$ represent rank-based volatilities and name-based correlations, respectively.

Assumption: *Throughout this paper we assume that the drift constants satisfy the stability condition*

$$\sum_{k=1}^n g_k + \sum_{i=1}^n \gamma_i = 0\,, \tag{2.2}$$

and that the $(n \times n)$ *matrices*

$$\mathfrak{s}_{\mathbf{p}} := diag\big(\sigma_{\mathbf{p}^{-1}(1)}\,, \dots, \sigma_{\mathbf{p}^{-1}(n)}\big) + \big(\rho_{i,j}\big)_{1 \le i,j \le n} \tag{2.3}$$

are positive-definite for every $\mathbf{p} \in \Sigma_n$ *, with* $\sigma_k > 0$ *for every* $k = 1, \dots, n$ *.*

Equation (2.1) can be cast in vector form as

$$d\,Y(t) = G(Y(t))\,d\,t + S(Y(t))\,d\,W(t)\,, \quad Y(0) = \bar{y} \in \mathbb{R}^n \tag{2.4}$$

for $0 \le t < \infty$, where the functions $G : \mathbb{R}^n \to \mathbb{R}^n$ and $S : \mathbb{R}^n \mapsto \mathbb{R}^{n \times n}$ are

$$G(y) := \sum_{\mathbf{p} \in \Sigma_n} \mathbf{1}_{\mathcal{R}_{\mathbf{p}}}(y) \cdot \big(\, g_{\mathbf{p}^{-1}(1)} + \gamma_1 + \gamma\,, \dots,\, g_{\mathbf{p}^{-1}(n)} + \gamma_n + \gamma \,\big)'$$

$$S(y) := \sum_{\mathbf{p} \in \Sigma_n} \mathbf{1}_{\mathcal{R}_{\mathbf{p}}}(y) \cdot \mathfrak{s}_{\mathbf{p}}\,; \quad y \in \mathbb{R}^n\,.$$

Thus (2.1) is a system of stochastic differential equations with coefficients that are piecewise constant, the same in each polyhedral chamber $\mathcal{R}_{\mathbf{p}}$, $\mathbf{p} \in \Sigma_n$. Under the assumption of positive-definiteness in (2.3), the system (2.1) admits a weak solution (Y, W) on a filtered probability space $(\Omega, \mathcal{F}\,, \{\mathcal{F}_t\}\,, \mathbb{P})$ satisfying the usual conditions. By the martingale-problem theory of Stroock & Varadhan [23] and the results in Bass & Pardoux [5], this weak solution is unique in the sense of the probability distribution.

3. Ergodicity

Thanks to the assumption (2.2) on the drifts, and taking the average of both sides of (2.1), we obtain the average log-capitalization process $\overline{Y}(\cdot) := \sum_{i=1}^n Y_i(\cdot)\,/\,n$ in the form

$$\overline{Y}(t) \;=\; \frac{1}{n}\sum_{i=1}^n y_i + \gamma\,t + \frac{1}{n}\sum_{k=1}^n \sigma_k B_k(t) + \frac{1}{n}\sum_{i,j=1}^n \rho_{i,j}\,d\,W_j(t)\,,$$

$$\text{where} \;\; B_k(t) \;:=\; \sum_{i=1}^n \int_0^t \mathbf{1}_{Q_k^{(i)}}(Y(s))\,W_i(s)\,; \quad k = 1, \dots, n$$

for $0 \le t < \infty$, because of $\cup_{i=1}^n Q_k^{(i)} = \mathbb{R}^n$. Here $B_1(\cdot)\,, \dots, B_n(\cdot)$ are continuous local martingales with quadratic (cross-)variations given as $\langle B_k\,, B_\ell \rangle$ $(t) \;=\; t\,\delta_{k,\ell}$, and hence are independent standard Brownian motions by the F.B. Knight theorem. It follows that the average $\overline{Y}(\cdot)$ of the log-capitalizations $Y_1(\cdot)\,, \dots, Y_n(\cdot)$ grows at a rate determined by the common drift γ, i.e.,

$$\lim_{T \to \infty} \frac{\overline{Y}(T)}{T} \;=\; \gamma \tag{3.1}$$

holds a.s., by the strong law of large numbers for Brownian motion. Let us introduce the column vector $\mathbf{\imath} := (1, \dots, 1)'$ and the subspace

$$\Pi := \{ y \in \mathbb{R}^n \,|\, \mathbf{\imath}' y = 0\,\}.$$

Proposition 1. *In addition to (2.2)-(2.3), let us impose for every* $\mathbf{p} \in \Sigma_n$ *the following stability condition:*

$$\sum_{k=1}^{\ell} \big(g_k + \gamma_{\mathbf{p}(k)} \big) < 0\,; \qquad \ell = 1, \dots, n-1\,. \tag{3.2}$$

Then the deviations $\widetilde{Y}(\cdot) := \big(Y_1(\cdot) - \overline{Y}(\cdot), \dots, Y_n(\cdot) - \overline{Y}(\cdot) \big)$ *of the log-capitalizations* $Y_1(\cdot), \dots, Y_n(\cdot)$ *from their average are* stable in distribution: *there exists a unique invariant probability measure* μ *for the* $\Pi-$*valued Markov process* $\widetilde{Y}(\cdot)$, *and for any bounded, measurable function* $f : \Pi \to \mathbb{R}$ *we have the Strong Law of Large Numbers*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f\big(\widetilde{Y}(t)\big)\, dt \;=\; \int_\Pi f(y)\, \mu(dy)\,, \qquad a.s. \tag{3.3}$$

Proof. From (2.1) and (2.4), we have

$$d\widetilde{Y}(t) \;=\; \widetilde{G}(\widetilde{Y}(t))\, dt \;+\; \widetilde{S}(\widetilde{Y}(t))\, dW(t)\,; \quad \widetilde{Y}(0) = \widetilde{y}\,, \tag{3.4}$$

where $\widetilde{y} := \bar{y} - \mathbf{1}'\bar{y} \cdot \mathbf{1} / n$, $\widetilde{G}(y) := G(y) - \gamma \cdot \mathbf{1}$ and $\widetilde{S}(y) := S(y) - \mathbf{1}\mathbf{1}'S(y) / n$ for $y \in \mathbb{R}^n$. By (2.3) the covariance matrix in (3.4) is uniformly nondegenerate: for all x, $y \in \Pi$ we have

$$x'\widetilde{S}(y)x = x'S(y)x - x'\mathbf{1}\mathbf{1}'S(y)\,x \,/\, n = x'S(y)\,x = \sum_{\mathbf{p}\in\Sigma_n} \mathbf{1}_{\mathcal{R}_\mathbf{p}}(y) \cdot x'\mathfrak{s}_\mathbf{p} x$$

and

$$\lambda_0 \|x\|^2 \le x'\widetilde{S}(y)x \le \lambda_1 \|x\|^2\,, \tag{3.5}$$

where $\lambda_0\,(\lambda_1)$ are the minimum (maximum) of the smallest (largest) eigenvalues of the positive definite matrices $\mathfrak{s}_\mathbf{p}$ over $\mathbf{p} \in \Sigma_n$.

Summation-by-parts, along with (2.2) and (3.2), lead now to

$$\begin{aligned}
y'\widetilde{G}(y) &= \sum_{i=1}^n y_i \left(g_{(\mathfrak{p}^y)^{-1}(i)} + \gamma_i \right) = \sum_{k=1}^n y_{\mathfrak{p}^y(k)} \left(g_k + \gamma_{\mathfrak{p}^y(k)} \right) \\
&= y_{\mathfrak{p}^y(n)} \sum_{k=1}^n \big(g_k + \gamma_{\mathfrak{p}^y(k)} \big) + \sum_{k=1}^{n-1} \big(y_{\mathfrak{p}^y(k)} - y_{\mathfrak{p}^y(k+1)} \big) \sum_{\ell=1}^k \big(g_\ell + \gamma_{\mathfrak{p}^y(\ell)} \big) \\
&\le c\sqrt{n} \sum_{k=1}^n \big(y_{\mathfrak{p}^y(k)} - y_{\mathfrak{p}^y(k+1)} \big) \le c\|y\| < 0\,; \quad y \in \Pi \cap \mathcal{R}_\mathbf{p}
\end{aligned} \tag{3.6}$$

where $c := n^{-1/2} \max_{1\le\ell\le n-1\,,\,\mathbf{p}\in\Sigma_n} \sum_{k=1}^{\ell} (g_k + \gamma_{\mathbf{p}(k)}) < 0$. In the last inequality we have used for $\mathbf{p} \in \Sigma_n$ and $y \in \Pi \cap \mathcal{R}_\mathbf{p}$ the properties $y_{\mathbf{p}(1)} \ge y_{\mathbf{p}(2)} \ge \cdots \ge y_{\mathbf{p}(n)}$, thus also $y_{\mathbf{p}(1)} \ge 0 \ge y_{\mathbf{p}(n)}$ and

$$\|y\|^2 \le\, n \,\max\big(y^2_{\mathbf{p}(1)}, y^2_{\mathbf{p}(n)} \big) \,\le\, n \big(y_{\mathbf{p}(1)} - y_{\mathbf{p}(n)} \big)^2 .$$

Now we consider the one-dimensional process $N(t) := f(\widetilde{Y}(t))$ with $f(y) = (\|y\|^2+1)^{1/2} > \|y\|$ for $y \in \Pi$. An application of Itô's rule gives

$$d\,N(t) = \widetilde{f}(\widetilde{Y}(t))\,d\,t + \big[f(y)\big]^{-1} y'\widetilde{S}(y)\big|_{y=\widetilde{Y}(t)}\,d\,W(t)\,; \quad 0 \le t < \infty\,,$$

$$\widetilde{f}(y) := \big(f(y)\big)^{-1}\Big(y'\widetilde{G}(y) + \frac{1}{2}\mathrm{trace}\big(\widetilde{S}(y)\widetilde{S}(y)'\big)\Big) - \big(f(y)\big)^{-3} y'\widetilde{S}\widetilde{S}(y)'y$$

for $y \in \Pi$. It follows from (3.5), (3.6) and the boundedness of $\widetilde{S}(\cdot)$ that there exists a constant $\kappa > 0$ such that $\widetilde{f}(y) \le c\,/\,2 < 0$ for $\|y\| > \kappa$. The diffusion coefficient $[f(y)]^{-1}y'\widetilde{S}(y)$ of $N(\cdot)$ is a matrix whose entries are uniformly bounded by some constants from (3.5).

Thus $N(\cdot)$ is recurrent with respect to the interval $(0,\kappa)$, and hence so is $\widetilde{Y}(\cdot)$ with respect to $B \cap \Pi$ for some ball $B \subset \mathbb{R}^n$ centered at the origin. Combining this with (3.5) and with the theory of Khas'minskii [18] (Theorem 5.1 on page 121), we obtain the existence of a unique invariant probability measure for the $\Pi-$valued process $\widetilde{Y}(\cdot)$. $\square$

The condition (3.2) ensures that, if $y_1 < y_2 < \cdots < y_n$ and one subdivides at time $t = 0$ the "cloud" of n particles diffusing on the real line according to the dynamics of (2.1), into two "subclouds" – one consisting of the ℓ leftmost, and the other of the $n-\ell$ rightmost, particles – the two subclouds will eventually merge. They will not continue to evolve like separate galaxies, that never again make contact with each other; cf. the Remark following Theorem 4 in Pal & Pitman [20] for an elaboration of this point in the case of the purely rank-based hybrid model with equal variances.

Corollary 1. *Under the assumptions of Proposition 1, the* long-term average occupation time *that company i spends in the k^{th} rank, i.e.,*

$$\theta_{k,i} := \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathbf{1}_{Q_k^{(i)}}(\mathfrak{X}(t))\,d\,t\,, \qquad i,k = 1,\ldots,n\,, \tag{3.7}$$

exists almost surely in $[0,1]$.

The resulting array of numbers $\theta_{k,i} \in [0,1]$ satisfy $\sum_{j=1}^n \theta_{k,j} = \sum_{\ell=1}^n \theta_{\ell,i} = 1$ for each "name" $i = 1,\ldots,n$ and "rank" $k = 1,\ldots,n$, i.e., $\vartheta := (\theta_{k,i})_{1\le k,i\le n}$ is a doubly stochastic matrix. Similarly, the average occupation time $\theta_{\mathbf{p}}$ of the market in the polyhedral chamber $\mathcal{R}_{\mathbf{p}}$, namely,

$$\theta_{\mathbf{p}} := \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathbf{1}_{\mathcal{R}_{\mathbf{p}}}(\mathfrak{X}(t))\,d\,t \tag{3.8}$$

exists a.s. in $[0,1]$ for every $\mathbf{p} \in \Sigma_n$, and we have $\theta_{k,i} = \sum \theta_{\mathbf{p}}$, where the summation is over the set $\{\mathbf{p} \in \Sigma_n \,|\, \mathbf{p}(k) = i\}$, $1 \le i,k \le n$.

By Proposition 1 and in particular (3.3), the quantity of (3.7) satisfies

$$\theta_{k,i} = \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathbf{1}_{Q_k^{(i)}}(\mathfrak{X}(t))\,d\,t = \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathbf{1}_{Q_k^{(i)}}(\widetilde{Y}(t))\,d\,t = \mu(Q_k^{(i)})\,,$$

where μ is the unique invariant measure for the process $\widetilde{Y}(\cdot)$ of (3.4). Since $\cup_{\ell=1}^n Q_\ell^{(i)} = \mathbb{R}^n = \cup_{j=1}^n Q_k^{(j)}$, it is obvious that $\sum_{\ell=1}^n \theta_{\ell,i} = \sum_{j=1}^n \theta_{k,j} = 1$ for $1 \le i, k \le n$. Equation (3.8) is obtained similarly.

4. Rankings

Let us now look at the log-capitalizations of the various companies listed according to rank, namely

$$Z_k(t) := \sum_{i=1}^n \mathbf{1}_{Q_k^{(i)}}(Y(t)) \cdot Y_i(t)\,; \qquad k = 1, \dots, n\,, \quad 0 \le t < \infty\,. \tag{4.1}$$

These are the order statistics $Z_1(\cdot) \ge \cdots \ge Z_n(\cdot)$ for the log-capitalizations $Y_1(\cdot), \cdots, Y_n(\cdot)$, listed from largest down to smallest. We recall the indicator map $\mathfrak{p}^x$ introduced at the end of Section 1, and define the Σ_n-valued *index process*

$$\mathfrak{P}_t \;:=\; \mathfrak{p}^{\mathfrak{X}(t)} \;=\; \mathfrak{p}^{Y(t)}\,, \qquad 0 \le t < \infty\,,$$

so that $X_{\mathfrak{P}_t(1)}(t) \ge \cdots \ge X_{\mathfrak{P}_t(n)}(t)$. We may thus write $Z_k(\cdot) = Y_{\mathfrak{P}.(k)}(\cdot)$ from (4.1).

We shall also introduce the total market capitalization $X(\cdot) := \sum_{i=1}^n X_i(\cdot)$, as well as the market weights (relative capitalizations) for the individual companies and their ranked counterparts, respectively:

$$\mu_i(t) := \frac{X_i(t)}{X(t)}\,,\; i = 1, \cdots, n \qquad \text{and} \qquad \mu_{(k)}(t) := \frac{e^{\,Z_k(t)}}{X(t)}\,,\; k = 1, \cdots, n\,. \tag{4.2}$$

Corollary 2. *Under (2.2)-(2.3) and (3.2), the process of ranked deviations* $\widetilde{Z}(\cdot) := \big(Z_1(\cdot) - \overline{Y}_1(\cdot), \dots, Z_n(\cdot) - \overline{Y}(\cdot)\big)'$ *of the log-capitalizations* $Y_1(\cdot), \cdots, Y_n(\cdot)$ *from their average, is stable in distribution by Proposition 1, and so is the* $\big((\mathbb{R}_+)^{n-1} \times \Sigma_n\big)-$*valued process* $(\Xi(\cdot), \mathfrak{P}.)$, *where* $\Xi(\cdot) := (Z_1(\cdot) - Z_2(\cdot), \dots, Z_{n-1}(\cdot) - Z_n(\cdot))'$ *is the* rank-gap *process of* $Y(\cdot)$.

In fact, since $\widetilde{Z}(\cdot)$ is obtained by permuting the components of $\widetilde{Y}(\cdot)$, the stability in distribution of $\widetilde{Y}(\cdot)$ implies stability in distribution for $\widetilde{Z}(\cdot)$ from Proposition 1. Moreover, the components of the rank-gap process $\Xi(\cdot)$ can be written as linear combinations of those of $\widetilde{Z}(\cdot)$, and the index process $\mathfrak{P}.$ can be seen as $\mathfrak{P}. = \mathfrak{p}^{\widetilde{Z}(\cdot)}$, where the range Σ_n of the mapping $\mathfrak{p}$ is a finite set. Thus, the process $(\Xi(\cdot), \mathfrak{P}.)$ is stable in distribution.

We shall denote by $\Lambda^{k,j}(t) := \Lambda_{Z_k - Z_j}(t)$ the local time accumulated at the origin by the nonnegative semimartingale $Z_k(\cdot) - Z_j(\cdot)$ up to time t for $1 \le k < j \le n$, and set $\Lambda^{0,1}(\cdot) \equiv 0 \equiv \Lambda^{n,n+1}(\cdot)$. Then from Theorem 2.5 of Banner & Ghomrasni [4] it can be shown that we have for $k = 1, \dots, n$,

$0 \le t < \infty$ the dynamics

$$
\begin{aligned}
d\,Z_k(t) &= \sum_{i=1}^{n} \mathbf{1}_{Q_k^{(i)}}\big(Y(t)\big)\,d\,Y_i(t) \\
&\qquad + \big(N_k(t)\big)^{-1}\Big[\sum_{\ell=k+1}^{n} d\,\Lambda^{k,\ell}(t) - \sum_{\ell=1}^{k-1} d\,\Lambda^{\ell,k}(t)\Big]\,.
\end{aligned}
\tag{4.3}
$$

Here $N_k(t)$ is the cardinality of the set of indices of those random variables among $Y_1(t), \cdots, Y_n(t)$ which have the same value as $Z_k(t)$, i.e., $N_k(t) := |\{i\,:\,Y_i(t) = Z_k(t)\}|$. Note that under the assumptions on the coefficients, it can be shown that the finite variation part of the continuous semimartingale $Y(\cdot)$ in (2.1) is absolutely continuous with respect to Lebesgue measure a.s., and it follows from an application of Fubini's theorem and an estimate of Krylov [19] that the Lebesgue measure of the set $\{t : Y_i(t) = Y_j(t)\}$ is zero a.s. for $1 \le i, j \le n$. Thus we can verify the sufficient conditions (2.11-12) of Theorem 2.5 in [4].

Each local time $\Lambda^{k,\ell}(\cdot)$ is flat away from the set $\{0 \le t < \infty \,|\, Z_k(t) = \cdots = Z_\ell(t)\}$; it grows only when the corresponding coordinate processes collide with each other. Examples in [5], [16] study such multiple collisions of order three or higher, and use comparisons with one-dimensional Bessel processes in a crucial manner. Here again, the nonnegative semimartingale $Z_k(\cdot) - Z_\ell(\cdot)$ is compared to an appropriate Bessel process. Since a Bessel process with dimension $\delta > 1$ does not accumulate any local time at the origin (a consequence of Proposition XI.1.5 of [21]), appropriate comparison arguments yield the following result; its proof is in Appendix 7.1.

Lemma 1. *Under (2.3), the local times $\Lambda^{k,\ell}(\cdot)$ generated by triple or higher-order collisions are identically equal to zero, i.e., $\Lambda^{k,\ell}(\cdot) \equiv 0$ for $1 \le k, \ell \le n$ and $|k - \ell| \ge 2$, and (4.3) takes for $k = 1, \dots, n$, $0 \le t < \infty$ the form*

$$
d\,Z_k(t) = \sum_{i=1}^{n} \mathbf{1}_{Q_k^{(i)}}(Y(t))\,d\,Y_i(t) + \frac{1}{2}\Big(d\,\Lambda^{k,k+1}(t) - d\,\Lambda^{k-1,k}(t)\Big)\,. \tag{4.4}
$$

Proposition 2. *Under the assumptions (2.2), (2.3) and (3.2), we obtain the following long-term growth relations, in addition to those of (3.1) :* all log-capitalizations grow at the same rate

$$
\lim_{T\to\infty} \frac{Y_i(T)}{T} = \frac{\log X_i(T)}{T} = \gamma\,; \quad i = 1, \dots, n \tag{4.5}
$$

almost surely. This holds also for the total market capitalization

$$
\lim_{T\to\infty} \frac{1}{T}\log X(T) = \lim_{T\to\infty} \frac{1}{T}\log\big(\sum_{i=1}^{n} X_i(T)\big) = \gamma\,, \quad \text{a.s.}\,, \tag{4.6}
$$

thus the model is coherent: *in the notation of (4.2) we have*

$$
\lim_{T\to\infty} \frac{1}{T}\log \mu_i(T) = 0\,, \quad \text{a.s.}\,; \quad i = 1, \dots, n\,. \tag{4.7}
$$

Proof. It follows from Corollary 2 that

$$\lim_{T\to\infty}\frac{1}{T}\big(Z_k(T)-Z_{k+1}(T)\big)=0\,,\qquad \text{a.s.}\,;\quad k=1,\ldots,n-1\,.$$

Combining this with (2.1), (3.7) and (4.4), we observe

$$\begin{aligned}&\lim_{T\to\infty}\frac{1}{2T}\big(\Lambda^{k-1,k}(T)+\Lambda^{k+1,k+2}(T)-2\Lambda^{k,k+1}(T)\big)\\&=\;g_k+\sum_{i=1}^{n}\gamma_i\theta_{k,i}-\Big(g_{k+1}+\sum_{i=1}^{n}\gamma_i\theta_{k+1,i}\Big)=\mathfrak{g}_k-\mathfrak{g}_{k+1}\,,\qquad\text{a.s.}\end{aligned}$$

where $\mathfrak{g}_k := g_k+\sum_{i=1}^{n}\gamma_i\theta_{k,i}$ for $k=1,\ldots,n-1$. Adding up these equations over $k=\ell,\ldots,n-1$ yields

$$\lim_{T\to\infty}\frac{1}{2T}\big(\Lambda^{\ell-1,\ell}(T)-\Lambda^{\ell,\ell+1}(T)-\Lambda^{n-1,n}(T)\big)=\mathfrak{g}_\ell-\mathfrak{g}_n\,,\qquad\text{a.s.}\tag{4.8}$$

for each $\ell=1,\ldots,n$; and adding up over all these values of ℓ, we obtain

$$\lim_{T\to\infty}\frac{1}{2T}\,\Lambda^{n-1,n}(T)\;=\;\mathfrak{g}_n\,,\qquad\text{a.s.}\tag{4.9}$$

In conjunction with (4.8), we obtain from (4.9) that for $k=1,\ldots,n$:

$$\lim_{T\to\infty}\frac{1}{2T}\,\big(\Lambda^{k-1,k}(T)-\Lambda^{k,k+1}(T)\big)=\mathfrak{g}_k=g_k+\sum_{i=1}^{n}\gamma_i\theta_{k,i}\,,\qquad\text{a.s.}$$

From this, (4.4), and the strong law of large numbers for Brownian motion, we get the long-term average growth rate of ranked log-capitalizations:

$$\lim_{T\to\infty}\frac{Z_k(T)}{T}=\gamma\,,\qquad\text{a.s.}\,;\quad k=1,\ldots,n\,.$$

This yields (4.5), the elementary inequality $\exp\{y_{\mathfrak{p}^y(1)}\}\le\sum_{i=1}^{n}\exp\{y_i\}\le n\exp\{y_{\mathfrak{p}^y(1)}\}$ for $y\in\mathbb{R}^n$ implies (4.6), and equation (4.7) is a direct consequence of (4.5) and (4.6). $\square$

Corollary 3. *Under* (*2.2*), (*2.3*) *and* (*3.2*), *the long-term average occupation times* $\theta_{k,i}$ *of* (*3.7*) *satisfy the equilibrium identity*

$$\sum_{k=1}^{n}\theta_{k,i}\,g_k\;+\;\gamma_i\;=0\;;\quad i=1,\ldots,n\,.\tag{4.10}$$

Indeed, by substituting (4.5) into (2.1) we obtain the a.s. identities

$$\lim_{T\to\infty}\frac{1}{T}\sum_{k=1}^{n}g_k\int_0^T\mathbf{1}_{Q_k^{(i)}}(Y(t))\,dt\;=\;-\gamma_i\qquad\text{a.s.};\quad i=1,\ldots,n$$

and so in conjunction with (3.7) we deduce (4.10).

Example 1. Suppose that the rank-based growth parameters are given as $g_n=(n-1)g$, $g_1=\cdots=g_{n-1}=-g<0$ for some $g>0$. This is the "Atlas configuration", in which the company at the lowest capitalization rank provides all the growth (or support, as with the Titan of mythical lore) for the entire structure. Suppose also that the name-based growth rates $\gamma_1, \cdots, \gamma_n$ satisfy $\sum_{i=1}^n \gamma_i = 0$ and $\max_{1\le i\le n} \gamma_i < g$.

It is then easily checked that (2.2) and (3.2) are satisfied. By Corollary 1, the average occupation times $\{\theta_{k,i}\}$ exist a.s. We shall provide an explicit expression for the $\theta_{k,i}$ under an additional condition (5.7) on the correlation structure, in Section 5.2. For the time being, let us just remark that in this case we get directly from (4.10) the long-term proportions of time

$$\theta_{n,i} = \frac{1}{n}\Big(1 - \frac{\gamma_i}{g}\Big); \qquad i = 1, \dots, n$$

with which the various companies occupy the lowest ("Atlas") rank. □

5. Invariant Measure

5.1. **Reflected Brownian Motions.** Observe now from (4.4) the representation for the vector $\Xi(\cdot) = (\Xi_1(\cdot), \dots, \Xi_{n-1}(\cdot))'$ of gaps in the ranked log-capitalizations $\Xi_k(\cdot) := Z_k(\cdot) - Z_{k+1}(\cdot)$, $k = 1, \cdots, n-1$:

$$\Xi(t) = \Xi(0) + \zeta(t) + \mathfrak{R}\Lambda(t); \quad 0 \le t < \infty. \tag{5.1}$$

Here we have set $\zeta(\cdot) := (\zeta_1(\cdot), \dots, \zeta_{n-1}(\cdot))'$ with

$$\zeta_k(\cdot) = \sum_{i=1}^n \int_0^\cdot \mathbf{1}_{Q_k^{(i)}}(Y(s))\, dY(s) - \sum_{i=1}^n \int_0^\cdot \mathbf{1}_{Q_{k+1}^{(i)}}(Y(s))\, dY(s);$$

and we have introduced the vector $\Lambda(\cdot) := (\Lambda^{1,2}(\cdot), \dots, \Lambda^{n-1,n}(\cdot))' = (\Lambda_{\Xi_1}(\cdot), \dots, \Lambda_{\Xi_{n-1}}(\cdot))'$ of local times, as well as the $((n-1)\times(n-1))$ matrix

$$\mathfrak{R} := \begin{pmatrix} 1 & -1/2 & & & \\ -1/2 & 1 & -1/2 & & \\ & \ddots & \ddots & \ddots & \\ & & -1/2 & 1 & -1/2 \\ & & & -1/2 & 1 \end{pmatrix}. \tag{5.2}$$

This rank-gap process $\Xi(\cdot)$ in (5.1) belongs to a class of processes which Harrison & Williams [14], [15] and Williams [25] call "reflected (or regulated) Brownian motions" (RBM) in polyhedral domains.

The process $\Xi(\cdot)$ has state-space $(\mathbb{R}_+)^{n-1}$ and behaves like the $(n-1)-$dimensional continuous semimartingale $\zeta(\cdot)$ on the interior of $(\mathbb{R}_+)^{n-1}$. When the face $\mathfrak{F}_k := \{(z_1, \dots, z_{n-1})' \in (\mathbb{R}_+)^{n-1} \,|\, z_k = 0\}$, $k = 1, \dots, n-1$ of the boundary is hit, the k^{th} component of $\Lambda(\cdot)$ increases, which causes an instantaneous displacement (reflection) in a continuous fashion. The directions of this reflection are given by the entries in $\mathfrak{r}_k$, the k^{th} column of the matrix $\mathfrak{R}$. For every principal submatrix $\widetilde{\mathfrak{R}}$ of $\mathfrak{R}$, there exists a non-zero vector y such that $\widetilde{\mathfrak{R}} y > 0$, and so the reflection matrix $\mathfrak{R}$ satisfies the so-called *completely*$-\mathcal{S}$ (or "strictly semi-monotone", see Dai & Williams [9] for details) condition for $\mathcal{S} = (\mathbb{R}_+)^{n-1}$.

Let us define the differential operators $\mathcal{A}$ and $\mathcal{D}_k$, acting on $C^2((\mathbb{R}_+)^{n-1})$ functions:

$$\begin{aligned}\big[\mathcal{A} f\big](z,\mathbf{p}) &:= \frac{1}{2}\sum_{k,\ell=1}^{n-1} a_{k,\ell}(\mathbf{p})\frac{\partial^2 f(z)}{\partial z_k \partial z_\ell} + \sum_{k=1}^{n-1} b_k(\mathbf{p})\frac{\partial f}{\partial z_k}(z)\,, \\ \big[\mathcal{D}_k f\big](z) &:= \langle \mathfrak{r}_k\,, \nabla f(z)\rangle\,; \quad k=1,\dots,n-1\,, \quad z\in(\mathbb{R}_+)^{n-1}\,.\end{aligned} \tag{5.3}$$

Here $(a_{k,\ell}(\cdot))_{1\le k,\ell\le n-1}$ is the corresponding covariance matrix with entries

$$\begin{aligned} a_{k,\ell}(\mathbf{p}) := \big(\sigma_k^2+\sigma_{k+1}^2\big)\cdot \mathbf{1}_{\{k=\ell\}} - \sigma_k^2\cdot\mathbf{1}_{\{k=\ell+1\}} - \sigma_{k+1}^2\cdot\mathbf{1}_{\{k=\ell-1\}} \\ + \sum_{m=1}^{n}(\rho_{\mathbf{p}(k),m}-\rho_{\mathbf{p}(k+1),m})(\rho_{\mathbf{p}(\ell),m}-\rho_{\mathbf{p}(\ell+1),m}) \\ + \sum_{(\alpha,\beta)\in\{(k,\ell),(\ell,k)\}}\big\{\sigma_\alpha(\rho_{\mathbf{p}(\beta),\alpha}-\rho_{\mathbf{p}(\beta+1),\alpha})+\sigma_{\alpha+1}(\rho_{\mathbf{p}(\beta+1),\alpha+1}-\rho_{\mathbf{p}(\beta),\alpha+1})\big\}\end{aligned} \tag{5.4}$$

for $k,\ell=1,\dots,n-1\,,\mathbf{p}\in\Sigma_n$; whereas the $((n-1)\times 1)$ vector $\mathfrak{r}_k$ is the k^{th} column of the reflection matrix $\mathfrak{R}$. We also define the $((n-1)\times 1)$ drift coefficient vector $b(\cdot) := (b_1(\cdot),\dots,b_{n-1}(\cdot))'$ for the semimartingale $\zeta(\cdot)$, with components

$$b_k(\mathbf{p}) := g_k+\gamma_{\mathbf{p}^{-1}(k)}-g_{k+1}-\gamma_{\mathbf{p}^{-1}(k+1)}\,; \quad k=1,\dots,n-1\,, \quad \mathbf{p}\in\Sigma_n\,. \tag{5.5}$$

From Corollary 2 we know that there exists an invariant measure $\nu(\cdot\,,\cdot)$ for the $\big((\mathbb{R}_+)^{n-1}\times\Sigma_n\big)-$valued process $(\Xi(\cdot),\mathfrak{P}_\cdot)$. Let us denote by $\nu_0(\cdot)$ the marginal invariant distribution of $\Xi(\cdot)$. As a consequence of Itô's formula and the formulation of the *submartingale problem* studied by Stroock & Varadhan [22] and Harrison & Williams [14], we obtain a characterization of the invariant distribution $\nu(\cdot,\cdot)$ for $(\Xi(\cdot),\mathfrak{P}_\cdot)$.

Lemma 2. *Assume (2.2), (2.3) and (3.2). For each $k=1,\dots,n-1$ there is a finite measure $\nu_{0k}(\cdot)$, absolutely continuous with respect to Lebesgue measure on the k^{th} face $\mathfrak{F}_k$, such that the so-called* Basic Adjoint Relationship (BAR) *holds for any C_b^2-function $f:(\mathbb{R}_+)^{n-1}\to\mathbb{R}$, namely*

$$\int_{(\mathbb{R}_+)^{n-1}\times\Sigma_n}\big[\mathcal{A}f\big](z,\mathbf{p})d\nu(z,\mathbf{p}) + \frac{1}{2}\sum_{k=1}^{n-1}\int_{\mathfrak{F}_k}\big[\mathcal{D}_k f\big](z)\,d\nu_{0k}(z) = 0\,. \tag{5.6}$$

This condition is necessary for the stationarity of $\nu(\cdot,\cdot)$. A proof of Lemma 2 is given in Appendix 7.2. It is not easy to solve (5.6) in general; however, following Harrison & Williams [15], we may obtain an explicit formula for the invariant joint distribution $\nu(\cdot,\cdot)$ under the so-called *skew symmetry condition* between the covariance and reflection matrices; see Proposition 3 and Corollaries 4 & 5.

Lemma 3. *Assume that the rank-based variances $\{\sigma_k^2\}$ grow linearly, and that there are* no name-based correlations *in (2.3), i.e.,*

$$\sigma_2^2-\sigma_1^2=\sigma_3^2-\sigma_2^2=\cdots=\sigma_{n-1}^2-\sigma_n^2\,, \quad \rho_{i,j}=0\,;\ 1\le i,j\le n\,. \tag{5.7}$$

Then the components of covariance matrix $\mathfrak{A} \equiv \big(\mathfrak{a}_{k,\ell}\big)_{1\le k,\ell\le n-1}$ *from (5.4) become*

$$\mathfrak{a}_{k,\ell} = \big(\sigma_k^2 + \sigma_{k+1}^2\big)\cdot \mathbf{1}_{\{k=\ell\}} - \sigma_k^2 \cdot \mathbf{1}_{\{k=\ell+1\}} - \sigma_{k+1}^2 \cdot \mathbf{1}_{\{k=\ell-1\}}$$

and does not *depend on the permutation* $\mathbf{p} \in \Sigma_n$ *. Moreover, it satisfies the so-called* skew symmetry *condition*

$$\big(2\mathfrak{D} - \mathfrak{H}\mathfrak{D} - \mathfrak{D}\mathfrak{H} - 2\mathfrak{A}\big)_{k,\ell} \,=\, 0\,; \quad 1 \le k,\ell \le n-1\,. \tag{5.8}$$

Here we have introduced the diagonal matrix $\mathfrak{D} := diag(\mathfrak{A})$ *, and the* $((n-1)\times(n-1))$ *matrix* $\mathfrak{H} := I - \mathfrak{R}$ *from the reflection matrix* $\mathfrak{R}$ *in (5.2).*

Lemma 3 is proved by straightforward computation; details are in section 5.5 of [16]. Note that under (5.7) the operator (5.3) still depends on the permutation $\mathbf{p}$ through the drift component $b(\mathbf{p})$ for $\mathbf{p} \in \Sigma_n$ in (5.5).

Proposition 3. *Under (2.2), (2.3), (3.2) and (5.7), the invariant joint distribution* $\nu(\cdot)$ *of the* $\big((\mathbb{R}_+)^{n-1}\times\Sigma_n\big)$*−valued process* $(\Xi(\cdot), \mathfrak{P}_\cdot)$ *is*

$$\nu(A\times B) \,:=\, \left(\sum_{\mathbf{q}\in\Sigma_n}\prod_{k=1}^{n-1}\lambda_{\mathbf{q},k}^{-1}\right)^{-1}\sum_{\mathbf{p}\in B}\int_A \exp\big(-\langle\lambda_{\mathbf{p}}\,,z\rangle\big)\,d\,z\,, \tag{5.9}$$

for any $\mathbf{p}\in\Sigma_n$ *where* $\lambda_{\mathbf{p}} := (\lambda_{\mathbf{p},1},\ldots,\lambda_{\mathbf{p},n-1})'$ *is the vector with components*

$$\lambda_{\mathbf{p},k} := \frac{-4\,\big(\sum_{\ell=1}^k g_\ell + \gamma_{\mathbf{p}(\ell)}\big)}{\sigma_k^2+\sigma_{k+1}^2}\,; \quad \mathbf{p}\in\Sigma_n\,,\; 1\le k\le n-1\,, \tag{5.10}$$

for any measurable sets $A \subset (\mathbb{R}_+)^{n-1}$ *and* $B\subset\Sigma_n$ *. In particular, the density* $\wp(\cdot)$ *of the marginal invariant distribution* $\nu_0(\cdot)$ *of* $\Xi(\cdot)$ *has the* sum-of-products-of-exponenentials *form*

$$\wp(z) := \Big(\sum_{\mathbf{q}\in\Sigma_n}\prod_{k=1}^{n-1}\lambda_{\mathbf{q},k}^{-1}\Big)^{-1}\sum_{\mathbf{p}\in\Sigma_n}\exp\big(-\langle\lambda_{\mathbf{p}}\,,z\rangle\big)\,; \quad z\in(\mathbb{R}_+)^{n-1}\,. \tag{5.11}$$

Proof. First, we carry out a linear transformation of the state space to remove the correlation between the components of $\Xi(\cdot)$; this is possible, since the covariance matrix $\mathfrak{A}$ does *not* depend on the index process $\mathfrak{P}_\cdot$, under (5.7) from Lemma 3. Let $\mathfrak{U}$ be the unitary matrix whose columns are the orthogonal eigenvectors of the covariance $\mathfrak{A}$, and let $\mathfrak{L}$ be the corresponding diagonal matrix of eigenvalues such that $\mathfrak{L} = \mathfrak{U}'\mathfrak{A}\,\mathfrak{U}$. Define $\widetilde{\Xi}(\cdot) := \mathfrak{L}^{-1/2}\,\mathfrak{U}\,\Xi(\cdot)$. By this deterministic rotation and scaling, we obtain

$$\widetilde{\Xi}(t) = \widetilde{\Xi}(0) + \widetilde{\zeta}(t) + \widetilde{\mathfrak{R}}\,\Lambda(t)\,; \quad 0\le t<\infty \tag{5.12}$$

from (5.1) where $\widetilde{\zeta}(\cdot) = \mathfrak{L}^{-1/2}\mathfrak{U}\zeta(\cdot)$ is a Brownian motion with drift coefficient $\widetilde{b}(\cdot) := \mathfrak{L}^{-1/2}\,\mathfrak{U}b(\cdot)$ where $b(\cdot)$ is defined in (5.5). We may regard $\widetilde{\Xi}(\cdot)$ as a reflected Brownian motion in a new state space $\mathfrak{S} := \mathfrak{L}^{-1/2}\mathfrak{U}\,(\mathbb{R}_+)^{n-1}$ with faces $\widetilde{\mathfrak{F}}_k := \mathfrak{L}^{-1/2}\mathfrak{U}\,\mathfrak{F}_k$, $k=1,\ldots,n-1$. The transformed reflection matrix $\widetilde{\mathfrak{R}} := \mathfrak{L}^{-1/2}\mathfrak{U}\mathfrak{R}$ can be written $\widetilde{\mathfrak{R}} = (\widetilde{\mathfrak{N}} + \widetilde{\mathfrak{Q}})\mathfrak{C} = (\widetilde{\mathfrak{r}}_1, \ldots, \widetilde{\mathfrak{r}}_{n-1})$, where $\widetilde{\mathfrak{C}} := \mathfrak{D}^{-1/2}$, $\mathfrak{D} = \mathrm{diag}(\mathfrak{A})$, $\widetilde{\mathfrak{N}} = \mathfrak{L}^{1/2}\mathfrak{U}\mathfrak{C} = (\widetilde{\mathfrak{n}}_1, \ldots, \widetilde{\mathfrak{n}}_{n-1})$, $\widetilde{\mathfrak{Q}} := \mathfrak{L}^{-1/2}\mathfrak{U}\mathfrak{R}\widetilde{\mathfrak{C}}^{-1} - \widetilde{\mathfrak{N}} = (\widetilde{\mathfrak{q}}_1, \ldots, \widetilde{\mathfrak{q}}_{n-1})$. The constant vectors $\widetilde{\mathfrak{r}}_k, \widetilde{\mathfrak{q}}_k, \widetilde{\mathfrak{n}}_k$, $k = 1, \ldots, n-1$ are $((n-1) \times 1)$ column vectors.

The corresponding differential operators $\widetilde{\mathcal{A}}, \widetilde{\mathcal{D}_k}$ and their adjoints $\widetilde{\mathcal{A}}^*, \widetilde{\mathcal{D}}^*_k$ are defined by

$$
\begin{aligned}
\big[\widetilde{\mathcal{A}} f\big](z, \mathbf{p}) &:= \frac{1}{2}\Delta f(z) + \langle \widetilde{b}(\mathbf{p}), \nabla f(z) \rangle, \ \big[\widetilde{\mathcal{D}}_k f\big](z) := \langle \widetilde{\mathfrak{r}}_k, \nabla f(z) \rangle, \\
\big[\widetilde{\mathcal{A}}^* f\big](z, \mathbf{p}) &:= \frac{1}{2}\Delta f(z) - \langle \widetilde{b}(\mathbf{p}), \nabla f(z) \rangle, \ \big[\widetilde{\mathcal{D}}^*_k f\big](z) := \langle \widetilde{\mathfrak{r}}^*_k, \nabla f(z) \rangle,
\end{aligned} \tag{5.13}
$$

where $\widetilde{\mathfrak{r}}^*_k := \widetilde{\mathfrak{n}}_k - \widetilde{\mathfrak{q}}_k + \langle \widetilde{\mathfrak{n}}_k, \widetilde{\mathfrak{q}}_k \rangle \widetilde{\mathfrak{n}}_k$ for $k = 1, \ldots, n-1$, $z \in (\mathbb{R}_+)^{n-1}$, $\mathbf{p} \in \Sigma_n$.

With these differential operators as in Lemma 2, we obtain the (BAR) for the process $(\widetilde{\Xi}(\cdot), \mathfrak{P}.)$ and its invariant distribution $\widetilde{\nu}(\cdot,\cdot)$; i.e., for every $k = 1, \ldots, n-1$, there exist a finite measure $\{\widetilde{\nu}_{0k}(\cdot)\}$ which is absolutely continuous with respect to the $(n-2)-$dimensional Lebesgue measure on $\widetilde{\mathfrak{F}}_k$ and such that for any C^2_b-function $f : \mathfrak{S} \mapsto \mathbb{R}$ we have

$$
\int_{\mathfrak{S} \times \Sigma_n} \big[\widetilde{\mathcal{A}} f\big](z, \mathbf{p})\, d\,\widetilde{\nu}(z, \mathbf{p}) + \frac{1}{2}\sum_{k=1}^{n-1} \int_{\widetilde{\mathfrak{F}}_k} \big[\widetilde{\mathcal{D}}_k f\big](z)\, d\,\widetilde{\nu}_{0k}(z) = 0 . \tag{5.14}
$$

Our argument, especially from here onward, relies heavily on the elaborate analysis given by Harrison & Williams [14], [15]. The main distinction between their setting and ours, is in the drift coefficient $b(\cdot)$, which here varies from chamber to chamber as well as within each chamber, and is evaluated along the path of the index process $\mathfrak{P}.$. Here, however, we can use the following.

Lemma 4. *The following two conditions are equivalent* :

(i) *For each collection of constants* $\{g_k, \gamma_i;\ 1 \le i, k \le n\}$, *there are* $(n-1)-$*dimensional vectors* $\widetilde{\lambda}_{\mathbf{p}} := (\widetilde{\lambda}_{\mathbf{p},1}, \ldots, \widetilde{\lambda}_{\mathbf{p},n-1})'$ *for* $\mathbf{p} \in \Sigma_n$, *such that a probability measure in the form of sum of products of exponentials*

$$
\widetilde{\nu}(A \times B) := c \sum_{\mathbf{p} \in B} \int_A \exp\big(\langle \widetilde{\lambda}_{\mathbf{p}}, z \rangle\big)\, d\,z =: \sum_{\mathbf{p} \in B} \int_A \widetilde{\wp}_{\mathbf{p}}(z)\, d\,z \tag{5.15}
$$

for measurable sets $A \subset \mathfrak{S}$ *and* $B \subset \Sigma_n$, *satisfies (5.14) for* $f(\cdot) \in C^2_c(\mathfrak{S})$, *where* c *in (5.15) is a normalizing constant.*

(ii) *The covariance and the direction of reflection satisfy the skew symmetry condition (5.8).*

Indeed, substituting (5.15) into (5.14) and combining the summation over $\mathbf{p} \in \Sigma_n$, we observe that

$$\sum_{\mathbf{p}\in\Sigma_n} \int_{\mathfrak{S}} \big[\widetilde{\mathcal{A}} f\big](z,\mathbf{p}) \cdot \widetilde{\wp}_{\mathbf{p}}(z)\, d\, z + \frac{1}{2}\sum_{k=1}^{n-1} \int_{\widetilde{\mathfrak{F}}_k} \big[\widetilde{\mathcal{D}}_k f\big](z) \sum_{\mathbf{p}\in\Sigma_n} \widetilde{\wp}_{\mathbf{p}}(z)\, d\, z$$
$$= \sum_{\mathbf{p}\in\Sigma_n} \Big\{ \int_{\mathfrak{S}} \big[\widetilde{\mathcal{A}} f\big](z,\mathbf{p}) \cdot \widetilde{\wp}_{\mathbf{p}}(z)\, d\, z + \frac{1}{2}\sum_{k=1}^{n-1} \int_{\widetilde{\mathfrak{F}}_k} \big[\widetilde{\mathcal{D}}_k f\big](z) \cdot \widetilde{\wp}_{\mathbf{p}}(z)\, d\, z \Big\}$$

holds for $f \in C^2_c(\mathfrak{S})$, where the expression in the curly bracket corresponds exactly to the BAR condition studied in [15] with some differences in notation. This way, we may reduce our problem to the case of [15]. Following the proof of Lemma 7.1 in [15], we observe that the condition (i) in Lemma 4 above is equivalent to the following conditions (iii)-(iv), where:
(iii) $\big[\widetilde{\mathcal{A}}^* \widetilde{\wp}_{\cdot}\big](\cdot,\cdot) = 0$ in $\mathfrak{S} \times \Sigma_n$, and
(iv) $\big[\widetilde{\mathcal{D}}^*_k \widetilde{\wp}_{\cdot}\big](\cdot,\cdot) = 2 b_k(\cdot) \widetilde{\wp}_{\cdot}(\cdot,\cdot)$ on $\widetilde{\mathfrak{F}}_k \times \Sigma_n$ for $k = 1, \dots, n-1$.
Here the adjoint operators $\widetilde{\mathcal{A}}^*$, $\widetilde{\mathcal{D}}^*_k$ are defined in (5.13).

Then the same reasoning as in the proof of Theorem 2.1 in [15] yields our Lemma 4, and we obtain $\widetilde{\lambda}_{\mathbf{p}} = 2(I - \widetilde{\mathfrak{N}}\widetilde{\mathfrak{Q}})^{-1} b(\mathbf{p})$ for $\mathbf{p} \in \Sigma_n$ along the way. This gives the invariant distribution $\widetilde{\nu}(\cdot)$ of $\widetilde{\Xi}(\cdot)$ in (5.12). Now transforming back to $\Xi(\cdot)$, we obtain (5.10), (5.9) and then (5.11). □

Example 2. With $\gamma_i = 0$, $\rho_{i,j} = 0$, $1 \le i, j \le n$ and $\sigma_1^2 = \cdots = \sigma_n^2$, we recover the case studied by Banner, Fernholz & Karatzas [3] and Pitman & Pal [20]. Our Proposition 3 is an extension of their results. □

5.2. **Average Occupation Times.** The long-term average occupation time $\theta_{\mathbf{p}}$ of the vector process $\mathfrak{X}(\cdot)$ in the polyhedral chamber $\mathcal{R}_{\mathbf{p}}$ of (3.8) is the probability mass $\nu_1(\mathbf{p}) := \nu\big((\mathbb{R}_+)^{n-1}, \mathbf{p}\big)$ assigned to such a particular chamber by the marginal invariant distribution of the index process $\mathfrak{P}_{\cdot}$, which we can compute directly from (5.9) for $\mathbf{p} \in \Sigma_n$.

Corollary 4. *Under the assumptions of Proposition 3, the long-term average occupation time $\theta_{\mathbf{p}}$ of $\mathfrak{X}(\cdot)$ in the chamber $\mathcal{R}_{\mathbf{p}}$ for $\mathbf{p} \in \Sigma_n$, and the long-term proportion $\theta_{k,i}$ of time spent by company i in the k^{th} rank as in (3.7), are explicitly given by the respective formulae*

$$\theta_{\mathbf{p}} = \Big(\sum_{\mathbf{q}\in\Sigma_n} \prod_{j=1}^{n-1} \lambda^{-1}_{\mathbf{q},j} \Big)^{-1} \prod_{j=1}^{n-1} \lambda^{-1}_{\mathbf{p},j} \qquad \text{and} \qquad \theta_{k,i} = \sum \theta_{\mathbf{p}}\,. \tag{5.16}$$

Here $\lambda_{\mathbf{p}}$ is in (5.10), and the summation for $\theta_{k,i}$ is taken over the set $\{\mathbf{p} \in \Sigma_n \,|\, \mathbf{p}(k) = i\}$ for $1 \le i, k \le n$.

From Corollary 3, the average occupation times $(\theta_{k,i})$ satisfy the equilibrium identity (4.10). As a sanity check, we verify this for the special case (5.16) through some algebraic computations in Appendix 7.3.

Example 3. It should be noted that in the presence of name-based variances, (5.16) can fail significantly. Consider the case where $n = 3$, with $\gamma_i = 0$, for $i = 1, 2, 3$; $\sigma_k = \sigma > 0$, for $k = 1, 2, 3$; $g_3 = g > 0$, $g_2 = 0$, and $g_1 = -g$; all $\rho_{i,j}$ is zero for $i, j = 1, 2, 3$ except $\rho_{3,3} = \rho \gg \sigma$. In this case, $Y_1(\cdot)$ and $Y_2(\cdot)$ will vibrate quietly in the middle with variance rate σ^2, while $Y_3(\cdot)$, with much greater variance rate $(\sigma + \rho)^2$, will be wandering far and wide. From Corollary 1 and (4.10) we obtain

$$\vartheta = \big(\theta_{k,i}\big)_{1 \le i,k \le 3} = \begin{pmatrix} \frac{1-\alpha}{2} & \frac{1-\alpha}{2} & \alpha \\ \alpha & \alpha & 1-2\alpha \\ \frac{1-\alpha}{2} & \frac{1-\alpha}{2} & \alpha \end{pmatrix}, \tag{5.17}$$

where the parameter α is in the interval $(1/3, 1/2)$ for $\rho > 0$. The upper bound $1/2$ is obtained as $\lim_{\rho \to \infty} \theta_{1,3}$. Without name-based variances, i.e., if the $\rho_{i,j}$ were all zero, the $Y_i(\cdot)$ would each spend the same proportion of time in every rank. This gives the lower bound $1/3$. $\square$

Example 4. Let us consider a numerical computation of $(\theta_{k,i})$ for descending name-based drifts γ_i and ascending rank-based drifts g_k, e.g. $n = 10$ and $\sigma_k^2 = 1 + k$, as well as $g_k = -1$ for $k = 1, \ldots, 9$, $g_{10} = 9$, $\gamma_i = 1 - (2i)/(n+1)$ for $i = 1, \ldots, n$. This is a rather extreme case of Example 1, with $g = 1$. The overall maximum is $\theta_{1,1} = 0.5184$, and the overall minimum is $\theta_{1,10} = 0.00485$. The company "$i = 1$" stays at the first rank longer than any other companies, because of its relatively strong name-based drift; whereas the company "$i = 10$" stays at the first rank only for a tiny amount of time, because of its relatively poor name-based drift.

Figure 1 shows a gray scale heat map for the different values of $\{\theta_{k,i}\}$; of course we know from Example 1 that $\theta_{10,i} = i/55$, $i = 1, \cdots, 10$. $\square$

For a larger number of companies, say $n \sim 5000$, it seems rather hopeless for the current computational environment to perform direct computations of $\theta_{k,i}$ via the sum of (5.16) over $(n-1)!$ permutations in general.

5.3. **Capital Distribution Curve.** The capital distribution curve is the log-log plot of market weights in descending order, as in (1.1). The empirical capital distribution curves, for the U.S. stock market 1929-1999, are shown in [12] (Figure 5.1 on page 95). Our next result computes the capital distribution curves directly from Proposition 3, using change of variables.

Corollary 5. *Under the assumptions of Proposition 3, the ranked market weights $\mu_{(1)}(\cdot), \ldots, \mu_{(n)}(\cdot)$ in (1.1), (4.2) have invariant distribution with*

$$\wp(m_1, \ldots, m_{n-1}) = \sum_{\mathbf{p} \in \Sigma_n} \Big[\theta_{\mathbf{p}} \cdot \prod_{k=1}^{n-1} \lambda_{\mathbf{p},k} \cdot \Big(\prod_{j=1}^{n} m_j^{\lambda_{\mathbf{p},j} - \lambda_{\mathbf{p},j-1} + 1} \Big)^{-1} \Big] \tag{5.18}$$

as its density, for $0 < m_n \le m_{n-1} \le \cdots \le m_1 < 1$ and $m_n = 1 - m_1 - \cdots - m_{n-1}$. Here we set $\lambda_{\mathbf{p},0} = 0 = \lambda_{\mathbf{p},n}$, $\mathbf{p} \in \Sigma_n$ for notational simplicity.

Moreover, the log-ranked market weights $\mathfrak{c}_k(\cdot) := \log \mu_{(k)}(\cdot)$ *have invariant distribution with density*

$$\wp(c_1, \ldots, c_{n-1}) = \sum_{\mathbf{p}\in\Sigma_n} \Big[\theta_{\mathbf{p}} \cdot \prod_{j=1}^{n-1} \Big(\lambda_{\mathbf{p},j} \cdot e^{-(\lambda_{\mathbf{p},j}-\lambda_{\mathbf{p},j+1})c_j} \Big) \cdot e^{\lambda_{p,n-1} c_n} \Big] \tag{5.19}$$

for $-\infty < c_n \le \cdots \le c_2 \le c_1 < 0\,,\ c_n = \log \big(1 - \sum_{j=1}^{n-1} e^{c_j}\big)\,.$

From the invariant density functions given by (5.11), (5.18) and (5.19), the piecewise linear capital distribution curve (1.1) has the expected slope

$$\mathbb{E}^{\nu}\Big[\frac{\log \mu_{(k+1)} - \log \mu_{(k)}}{\log(k+1) - \log k}\Big] = - \frac{\mathbb{E}^{\nu}\big(\Xi_k\big)}{\log(1+k^{-1})} = - \frac{\sum_{\mathbf{p}\in\Sigma_n} \theta_{\mathbf{p}} \lambda_{\mathbf{p},k}^{-1}}{\log(1+k^{-1})} \tag{5.20}$$

between the k^{th} and the $(k+1)^{\text{st}}$ ranked stocks for $k = 1, \ldots, n-1$, and the initial value

$$\mathbb{E}^{\nu}(\log \mu_{(1)}) = \mathbb{E}^{\nu}(\mathfrak{c}_1) = \mathbb{E}^{\nu}\Big[-\log \Big(1+e^{-\Xi_1}+e^{-(\Xi_1+\Xi_2)}+\cdots+e^{-(\Xi_1+\cdots+\Xi_{n-1})}\Big)\Big]$$

for the first rank. From (5.9) this expected initial value may be obtained through a Monte Carlo simulation of generating $(n-1)$ independent exponential random variables with intensities $\lambda_{\mathbf{p},j}$ for $j = 1, \ldots, n-1$, $\mathbf{p} \in \Sigma_n$. From (5.20) we obtain the following simple criterion for convexity (or concavity) of the expected capital distribution curves.

Corollary 6. *Under the assumptions of Proposition 3, a sufficient condition for the expected capital distribution curve* $\log k \mapsto \mathbb{E}^{\nu}(\log \mu_{(k)})$ *under the invariant distribution* ν *to be* convex *(respectively,* concave*), is that*

$$\lambda_{\mathbf{p},k+1} \log\Big(1 + \frac{1}{k+1}\Big) - \lambda_{\mathbf{p},k} \log\Big(1+\frac{1}{k}\Big) \ge 0\,; \quad \forall\ \mathbf{p} \in \Sigma_n \tag{5.21}$$

(resp., $\le$*) hold on each interval* $[\log k, \log(k+2)]$ *for each* $k = 1, \ldots n-2$, *where* $\lambda_{\mathbf{p},k}$ *is given in (5.10).*

Example 5. Let us consider the first-order Atlas model which is a combination of the "Atlas configuration" in Examples 1 with the further restrictions of Example 2; to wit, $g_n = (n-1)g$, $g_1 = \cdots = g_{n-1} = -g < 0$ for some $g > 0$, as well as $\gamma_i = 0$, $\rho_{i,j} = 0$, $1 \le i,j \le n$ and $\sigma_1^2 = \cdots = \sigma_n^2 = \sigma^2 > 0$ for some $\sigma^2 > 0$. From Corollary 6, the expected capital distribution curve is *convex* but *almost linear* for larger k. Indeed, the quantity $\lambda_{\mathbf{p},k} \log(1+k^{-1}) = 2(gk\,/\,\sigma^2) \cdot \log(1+k^{-1})$ increases in $k \ge 1$, and converges to one, as $k \uparrow \infty$, for all $\mathbf{p} \in \Sigma_n$, and so the difference in (5.21) is positive for each $k = 1, \ldots, n-2$ but decreases to zero quite rapidly in the order of $O(k^{-2})$, as $k \uparrow \infty$. Another explanation of such linearity ("Pareto line") of the capital distribution curves from an application of Poisson point processes can be found in Example 5.1.1 on page 94 of [12]. □

Example 6. Suppose now that we change only the rank-based variances in Example 5; namely, we take linearly growing variances $\sigma_k^2 = k\sigma^2$ for some $\sigma^2 > 0$, $k = 1, \ldots, n$. Then $\lambda_{\mathbf{p},k} \log(1+k^{-1}) = (4kg\,/\,[(2k+1)\sigma^2]) \cdot \log(1+$

k^{-1}) is decreasing in $k \geq 1$ for every $\mathbf{p} \in \Sigma_n$, and so the difference in (5.21) is negative for each $k = 1, \ldots, n-2$. Thus, from Corollary 6, the expected capital distribution curve becomes *concave*. □

Example 7. Consider a "pure" hybrid market defined by

$$dY_i(t) = \begin{cases} -\gamma_i \, dt + \rho_i \, dW_i(t) & \text{if } Y_i(t) \neq Y_{(n)}(t), \\ (g - \gamma_i) \, dt + \rho_i \, dW_i(t) & \text{if } Y_i(t) = Y_{(n)}(t), \end{cases}$$

for $i = 1, \ldots, n$ and $t \geq 0$, where $\gamma_i > 0$, $\rho_i > 0$, and $g = \sum_{i=1}^n \gamma_i$. We claim that for large n and $g \gg \gamma_i$ the capital distribution curve for this market is convex.

For large n and $g \gg \gamma_i$, the process $\big(Y_i(t) - Y_{(n)}\big)(t)$ is approximately exponentially distributed for t outside the set where $Y_i(t) = Y_{(n)}(t)$, with

$$P\{Y_i(t) - Y_{(n)}(t) > x\} \cong e^{-\alpha_i x},$$

where $\alpha_i = \rho_i^2 / 2\gamma_i$. It would seem that for large enough n, the Atlas stock would perform a role similar to a local time process, reflecting the stocks away from the Atlas position.

Now, if we let Y represent a generic member of the market, the distribution of $Y(t) - Y_{(n)}(t)$ will be mixed exponential, with

$$P\{Y(t) - Y_{(n)}(t) > x\} \cong \frac{1}{n} \sum_{i=1}^n e^{-\alpha_i x}.$$

In particular, for $1 \leq k < n$,

$$\frac{k}{n} \cong P\{\big(Y(t) - Y_{(n)}(t)\big) > \big(Y_{(k)}(t) - Y_{(n)}(t)\big)\} \cong \frac{1}{n} \sum_{i=1}^n \exp\big(-\alpha_i \big(Y_{(k)}(t) - Y_{(n)}(t)\big)\big),$$

and we can use this equation to determine the shape of the capital distribution curve.

We wish to determine the shape of the graph of $\log k = \log \sum_{i=1}^n e^{-\alpha_i x}$, where $\log k$ is considered to be a function of x. For $\varphi(x) := \sum_{i=1}^n e^{-\alpha_i x}$, we have

$$\frac{d^2}{dx^2} \log k = \frac{d^2}{dx^2} \log \varphi(x) = \frac{\varphi''(x)\varphi(x) - (\varphi'(x))^2}{(\varphi(x))^2}.$$

For this derivative to be nonnegative, it suffices that the numerator be nonnegative, and we have

$$\begin{aligned} \varphi''(x)\varphi(x) - (\varphi'(x))^2 &= \sum_{i=1}^n \alpha_i^2 e^{-\alpha_i x} \sum_{j=1}^n e^{-\alpha_j x} - \sum_{i,j=1}^n \alpha_i \alpha_j e^{-(\alpha_i + \alpha_j)x} \\ &= \sum_{i,j=1}^n (\alpha_i^2 - \alpha_i \alpha_j) e^{-(\alpha_i + \alpha_j)x} \\ &= \frac{1}{2} \sum_{i,j=1}^n (\alpha_i^2 - 2\alpha_i \alpha_j + \alpha_j^2) e^{-(\alpha_i + \alpha_j)x} \geq 0, \end{aligned}$$

so the graph is convex. It follows that the capital distribution curve of the pure hybrid market is likely to be convex. □

Example 8. To see different shapes of the expected capital distribution curve under different parameters apart from Examples 5 & 6, let us consider a pure hybrid market whose drift and volatility coefficients do not depend on ranks, except for the smallest (Atlas) stock. For example, take $n = 5000$, $g_k = 0$, $1 \le k \le n-1$, $g_n = c_*(2n-1)$, $\gamma_1 = -c_*$, $\gamma_i = -2c_*$, $2 \le i \le n$, $\sigma_k^2 = 0.075$, $1 \le k \le n$, and $\rho_{i,j} = 0$ for $1 \le i, j \le n$ with a parameter $c_* = 0.02$. These parameters satisfy the assumptions of Proposition 3. We cannot apply Corollary 6, because the difference in (5.21) is positive on $\{\mathbf{p} \in \Sigma_n : \mathbf{p}(k+1) \neq 1\}$ but negative on its (smaller) complement. The resulting expected capital distribution curve is *convex*; it is depicted in Figure 2. □

Example 9. Let us consider now a variant of this pure hybrid model, with a variance structure that is observed in practice. The parameters are the same as in Example 8, except for the different choices of the parameter c_* and for the rank-based variances $\sigma_k^2 := 0.075 + 6k \times 10^{-5}$ which are obtained from the smoothed annualized values for 1990-1999 data as in Section 5.4, page 109 of [12] (see page 2319 of [3]). The criterion from Corollary 6 cannot apply directly to this case, because the inequalities (5.21) do not hold for all $\mathbf{p} \in \Sigma_n$. The expected capital distribution curves under these parameters with (i) $c_* = 0.02$, (ii) $c_* = 0.03$, (iii) $c_* = 0.04$ are shown in Figure 3. The curve (i) is convex from the top rank to about the 25th rank, then turns concave until the lowest rank. The other curves (ii) & (iii) behave similarly. □

Example 10. Adopting the same parameter specifications in Example 9 (i) $c_* = 0.02$, except the rank-based drift, i.e., (iv) the upwind first ranked stock $g_1 = -0.016$, $g_k = 0, 2 \le k \le n-1$, $g_n = (0.02)(2n-1) + 0.016$ and (v) the windward top 50 stocks $g_1 = g_2 = \cdots = g_{50} = -0.016$, $g_k = 0, 51 \le k \le n-1$, $g_n = (0.02)(2n-1) + 0.8$, we obtain concave curves as in Figure 4. The observed average curve and the estimated curve of the first-order Atlas model for 1990-1999 (Figure 3 of [3], page 2320) are concave. The statistical inference for the capital distribution curves is an interesting problem that we do not discuss here. □

6. Portfolio Analysis

Let us consider investing in the market of (2.1) according to a portfolio rule $\Pi(\cdot) = (\Pi_1(\cdot), \ldots, \Pi_n(\cdot))'$. This is an $\{\mathcal{F}_t\}-$adapted, locally square-integrable process with $\sum_{i=1}^n \Pi_i(\cdot) = 1$. Each $\Pi_i(t)$ represents the proportion of the portfolio's wealth $V^\Pi(t)$ invested in stock i at time t, so

$$\frac{d\,V^\Pi(t)}{V^\Pi(t)} = \sum_{i=1}^n \Pi_i(t) \cdot \frac{d\,X_i(t)}{X_i(t)}, \quad V^\Pi(0) = 1. \tag{6.1}$$

For example, we may choose for every $t \in [0,\infty)$ the vector of market weights $\mu_i(t)$, $i = 1,\dots,n$ as in (4.2). We shall call the resulting $\Pi(\cdot) \equiv \mu(\cdot)$ the *market portfolio.*

For a constant-proportion portfolio $\Pi(\cdot) \equiv \pi \in \Gamma^n := \{(\pi_1,\dots,\pi_n)' \in \mathbb{R}^n \mid \sum_{i=1}^n \pi_i = 1\}$ (which of course the market portfolio is not), the solution of (6.1) is given by

$$d\,\log V^\pi(t) = \gamma^*_\pi(t)\,dt + \sum_{i=1}^n \pi_i\, d\,\log X_i(t)\,; \quad 0 \le t < \infty\,. \tag{6.2}$$

Here we denote by $\big(a_{ij}(t)\big)_{1\le i,j\le n} = S(Y(t))S(Y(t))'$ the covariance process from (2.4), and introduce

$$\gamma^*_\pi(t) := \frac{1}{2}\Big(\sum_{i=1}^n \pi_i\, a_{ii}(t) - \sum_{i,j=1}^n \pi_i\, a_{ij}(t)\,\pi_j\Big)\,; \quad 0 \le t < \infty\,, \tag{6.3}$$

the *excess growth rate* of the constant-proportion $\Pi(\cdot) \equiv \pi \in \Gamma^n$. Thus, for a constant-proportion portfolio we can write the solution of (6.1), namely

$$V^\pi(t) = w \cdot \exp\left[\sum_{i=1}^n \pi_i\Big\{\frac{A_{ii}(t)}{2} + \log\Big(\frac{X_i(t)}{X_i(0)}\Big)\Big\} - \frac{1}{2}\sum_{i,j=1}^n \pi_i A_{ij}(t)\pi_j\Big\}\right], \tag{6.4}$$

as in (2.4) of [17], where $A_{ij}(\cdot) = \int_0^\cdot a_{ij}(t)\,dt$; we set $A(\cdot) := (A_{ij}(\cdot))_{1\le i,j\le n}$.

6.1. **Target Portfolio.** Let us assume that, for every $(t,\omega) \in [0,\infty) \times \Omega$, there exists a vector $\Pi^*(t,\omega) := (\Pi^*_1(t,\omega),\dots,\Pi^*_n(t,\omega))' \in \Gamma^n$ that attains the maximum of the wealth $V^\pi(t,\omega)$ over vectors $\pi \in \Gamma^n$; and that the resulting process $\Pi^*(\cdot)$ defines a portfolio. Along with Cover [8] & Jamshidian [17], we shall call this $\Pi^*(\cdot)$ *Target Portfolio*, and

$$V_*(t) := \max_{\pi\in\Gamma^n} V^\pi(t)\,, \qquad 0 \le t < \infty \tag{6.5}$$

the *Target Performance* for the model. (The quantity of (6.5) is not necessarily equal to, and will typically be very different from, the performance $V^{\Pi^*}(\cdot)$ of the portfolio Π^*.)

The Target Performance $V_*(\cdot)$ exceeds the performance of the leading stock, of the value-line index (the geometric mean), and of any arithmetic average (such as the DJIA): to wit, taking $X_1(0) = \cdots = X_n(0) = 1$, we have for every vector $(\alpha_1,\dots,\alpha_n)' \in \Gamma^n_+ := \{(\pi_1,\dots,\pi_n)' \in \Gamma^n | \pi_i \ge 0\,, i = 1,\dots,n\}$ the almost sure comparisons

$$V_*(\cdot) \ge \max\Big[\max_{1\le i\le n} X_i(\cdot)\,,\ \Big(\prod_{j=1}^n X_j(\cdot)\Big)^{1/n},\ \sum_{j=1}^n \alpha_j X_j(\cdot)\Big]\,. \tag{6.6}$$

Under the assumptions of Proposition 1, the limits $\theta_{\mathbf{p}}$ of the average occupation times in (3.8) exist almost surely, and so do the limits of the average covariance rate $\mathfrak{a}^{\infty}_{ij} := \lim_{T\to\infty} A_{ij}(T)\,/\,T$, i.e., $\mathfrak{a}^{\infty} := \big(\mathfrak{a}^{\infty}_{ij}\big)_{1\le i,j\le n}$ is

$$\begin{aligned}\mathfrak{a}^{\infty} &= \lim_{T\to\infty}\frac{1}{T}\int_0^T \big(a_{ij}(t)\big)_{1\le i,j\le n}\,d\,t\\ &= \lim_{T\to\infty}\frac{1}{T}\int_0^T \sum_{\mathbf{p}\in\Sigma_n} \mathbf{1}_{\mathcal{R}_{\mathbf{p}}}(Y(s))\cdot \mathfrak{s}_{\mathbf{p}}\mathfrak{s}'_{\mathbf{p}}\,d\,t = \sum_{\mathbf{p}\in\Sigma_n}\theta_{\mathbf{p}}\,\mathfrak{s}_{\mathbf{p}}\,\mathfrak{s}'_{\mathbf{p}}\,,\end{aligned}$$

with $\mathfrak{s}_{\mathbf{p}}$ defined in (2.3). It follows from (6.4) that the asymptotic long-term-average growth rate of a constant-proportion portfolio $\pi\in\Gamma^n$ is

$$\lim_{T\to\infty}\frac{1}{T}\log V^{\pi}(T) = \gamma + \frac{1}{2}\Big(\sum_{i=1}^n \pi_i\mathfrak{a}^{\infty}_{ii} - \sum_{i,j=1}^n \pi_i\mathfrak{a}^{\infty}_{ij}\pi_j\Big) =: \gamma + \gamma^{\infty}_{\pi}\,. \tag{6.7}$$

Maximizing this expression over $\pi\in\Gamma^n$ amounts to maximizing over constant-proportion portfolios $\gamma^{\infty}_{\pi} = (1/2)\big(\sum_{i=1}^n \pi_i\mathfrak{a}^{\infty}_{ii} - \sum_{i,j=1}^n \pi_i\mathfrak{a}^{\infty}_{ij}\pi_j\big)$, the excess growth rate that corresponds to the asymptotic covariance structure.

We shall call *Asymptotic Target Portfolio* a vector $\bar\pi = (\bar\pi_1,\ldots,\bar\pi_n)'\in\Gamma^n$ that attains $\max_{\pi\in\Gamma^n}\gamma^{\infty}_{\pi}$. We can regard this as an *asymptotic growth-optimal* portfolio over all constant-proportion portfolios, in the sense that $\lim_{T\to\infty}(1/T)\,\log(V^{\pi}(T)/V^{\bar\pi}(T))\le 0$ holds a.s. for every $\pi\in\Gamma^n$.

Example 11. When there is no covariance structure by name, i.e., $\rho_{i,j}\equiv 0$ for every $1\le i,j\le n$, we have $A_{ij}(\cdot)\equiv 0$ for $i\ne j$. In this case, we compute a target portfolio $\Pi^*(\cdot)$ as

$$\begin{aligned}\Pi^*_i(t) \;=\; &\Big(2A_{ii}(t)\sum_{j=1}^n\frac{1}{A_{jj}(t)}\Big)^{-1}\Big[\,2-n-2\sum_{j=1}^n\frac{1}{A_{jj}(t)}\log\Big(\frac{X_j(t)}{X_j(0)}\Big)\Big]\\ &+\frac{1}{2}+\frac{1}{A_{ii}(t)}\log\Big(\frac{X_i(t)}{X_i(0)}\Big)\,;\qquad i=1,\ldots,n\,,\end{aligned}\tag{6.8}$$

and an asymptotic target portfolio by

$$\bar\pi_i = \frac{1}{2}\Big[\,1-\frac{n-2}{\mathfrak{a}^{\infty}_{ii}}\Big(\sum_{j=1}^n\frac{1}{\mathfrak{a}^{\infty}_{jj}}\Big)^{-1}\Big] = \lim_{t\to\infty}\Pi^*_i(t)\,;\quad i=1,\ldots,n\,,\ \text{a.s.} \tag{6.9}$$

The portfolio $\bar\pi$ has exactly the same long-term growth rate as the target performance in (6.5), namely

$$\lim_{T\to\infty}\frac{1}{T}\,\log\Big(\frac{V^{\bar\pi}(T)}{V_*(T)}\Big) = 0\,,\qquad \text{a.s.;} \tag{6.10}$$

on the other hand, it outperforms the overall market rather significantly over long time horizons, namely

$$\begin{aligned}(6.11)\qquad &\lim_{T\to\infty}\frac{1}{T}\log\Big(\frac{V^{\bar\pi}(T)}{V^{\mu}(T)}\Big)=\frac{1}{2}\sum_{i=1}^{n}\bar\pi_i(1-\bar\pi_i)\mathfrak{a}_{ii}^{\infty}\\ &=\frac{1}{8}\Big[\sum_{i=1}^{n}\mathfrak{a}_{ii}^{\infty}-(n-2)^2\Big(\sum_{j=1}^{n}\frac{1}{\mathfrak{a}_{jj}^{\infty}}\Big)^{-1}\Big]\ge\frac{n-1}{2}\Big(\sum_{i=1}^{n}\frac{1}{\mathfrak{a}_{ii}^{\infty}}\Big)^{-1}\end{aligned}$$

a.s., from the arithmetic mean - harmonic mean inequality. □

With Cover [8] and Jamshidian [17], we shall say that stock i is *asymptotically active*, if for the expression of (6.9) we have $\bar\pi_i>0$; and that the entire *market is asymptotically active,* if all its stocks are asymptotically active, that is, if $\bar\pi\in\Gamma^n_{++}:=\{(\pi_1,\dots,\pi_n)'\in\Gamma^n\,|\,\pi_i>0\,,\ i=1,\dots,n\}$.

Example 12. A sufficient condition for asymptotic activity of the model with $n\ge3$ under the condition of Proposition 3, is obtained from (6.9) as

$$(6.12)\qquad \frac{1}{\mathfrak{a}_{ii}^{\infty}}<\frac{1}{n-2}\Big(\sum_{\ell=1}^{n}\frac{1}{\mathfrak{a}_{\ell\ell}^{\infty}}\Big),\qquad\text{or equivalently}$$

$$(6.13)\quad \Big(\sum_{\mathbf{p}\in\Sigma_n}\sigma^2_{\mathbf{p}^{-1}(i)}\prod_{j=1}^{n-1}\lambda_{\mathbf{p},j}^{-1}\Big)^{-1}<\frac{1}{n-2}\Big[\sum_{\ell=1}^{n}\Big(\sum_{\mathbf{p}\in\Sigma_n}\sigma^2_{\mathbf{p}^{-1}(\ell)}\prod_{j=1}^{n-1}\lambda_{\mathbf{p},j}^{-1}\Big)^{-1}\Big],$$

for every $i=1,\cdots,n$, with $\lambda_{\mathbf{p},j}$ defined in (5.10). This is the case in the constant variance model $\sigma_1^2=\cdots=\sigma_n^2$. In general, it seems that the drift and volatility coefficients have non-trivial effects on the condition (6.13). □

6.2. **Universal Portfolio.** The *Universal Portfolio* of Cover [8] and Jamshidian [17] is defined as

$$\widehat\Pi_i(t):=\frac{\int_{\Gamma^n_+}\pi\,V^{\pi}(t)\,d\pi}{\int_{\Gamma^n_+}V^{\pi}(t)\,d\pi}\,;\qquad 0\le t<\infty\,,\ 1\le i\le n\,.$$

The wealth process of this portfolio is given by the "performance-weighting"

$$V^{\widehat\Pi}(t)=\frac{\int_{\Gamma^n_+}V^{\pi}(t)\,d\pi}{\int_{\Gamma^n_+}d\pi}\,;\qquad 0\le t<\infty\,,$$

as can be checked easily. It follows from Theorem 2.4 of Jamshidian [17] that the Universal Portfolio does not lag significantly behind the Target Portfolio: its performance lag is only polynomial in time under an asymptotically active model. To wit, there exists then a positive constant C, such that

$$\frac{V^{\widehat\Pi}(T)}{V_*(T)}\sim CT^{-(n-1)/2}\quad\text{as}\quad T\to\infty$$

holds almost surely, thus also

$$\lim_{T\to\infty}\frac{1}{T}\log\Big(\frac{V^{\widehat{\Pi}}(T)}{V^{\bar{\pi}}(T)}\Big)=\lim_{T\to\infty}\frac{1}{T}\log\Big(\frac{V^{\widehat{\Pi}}(T)}{V_*(T)}\Big)=0\,.$$

In the context of the hybrid model, under the assumptions of Proposition 3 and of Example 12, the Universal Portfolio attains the long-term growth rate of the asymptotic target portfolio $\bar{\pi}$ (that is, the maximal long-term growth rate achievable through constant-proportion portfolios), *as if the structure of the market were known.*

6.3. **Growth-Optimal Portfolio.** We shall call *growth-optimal* a portfolio $\varpi(\cdot)$ that satisfies the inequality $\lim_{T\to\infty}(1/T)\,\log(V^{\Pi}(T)/V^{\varpi}(T))\le 0$ almost surely, for any portfolio $\Pi(\cdot)$.

In order to find such a growth-optimal portfolio under no-name based correlation $\rho_{i,j}\equiv 0$ for $1\le i,j\le n$, we need to maximize over $\pi\in\Gamma^n$ the quantity

$$\sum_{i=1}^{n}\Big(\widetilde{\gamma}_i(t)+\frac{1}{2}a_{ii}(t)\Big)\pi_i-\frac{1}{2}\sum_{i=1}^{n}a_{ii}(t)^2\pi_i^2\,,$$

where $\widetilde{\gamma}_i(t)\;=\;\sum_{\mathbf{p}\in\Sigma_n}\mathbf{1}_{\mathcal{R}_\mathbf{p}}(Y(t))\,g_{\mathbf{p}^{-1}(i)}+\gamma_i+\gamma$ is the i^{th} element of $G(Y(t))$ of (2.4). By the Lagrange multiplier method, we obtain

$$\varpi_i(t)=\frac{1}{2}+\frac{\widetilde{\gamma}_i(t)+\overline{\gamma}(t)}{a_{ii}(t)}\,;\qquad i=1,\dots,n\,,\quad 0\le t<\infty$$

where the constraint $\sum_{i=1}^{n}\varpi_i(t)=1$ is enforced by the multiplier

$$\overline{\gamma}(t)\;=\;\Big(\sum_{i=1}^{n}\frac{1}{a_{ii}(t)}\Big)^{-1}\Big(1-\frac{n}{2}-\sum_{j=1}^{n}\frac{\widetilde{\gamma}_j(t)}{a_{jj}(t)}\Big)\,.$$

The growth rate of this portfolio's performance $V^{\varpi}(\cdot)$ is

$$\frac{n\gamma}{2}+\frac{1}{2}\sum_{i=1}^{n}\frac{\widetilde{\gamma}_i^2(t)}{a_{ii}(t)}-\frac{\overline{\gamma}^2(t)}{2}\sum_{i=1}^{n}\frac{1}{a_{ii}(t)}+\frac{1}{8}\sum_{i=1}^{n}a_{ii}(t)\,.$$

• In order to make some comparisons, let us specialize to the equal-variance case, i.e., $\sigma_1^2=\cdots=\sigma_n^2=\sigma^2$ with no name-based correlations $\rho_{i,j}\equiv 0$; we obtain

$$\lim_{T\to\infty}\frac{1}{T}\log V^{\varpi}(T)=\gamma+\frac{\sigma^2}{2}\Big(1-\frac{1}{n}\Big)+\frac{1}{2\sigma^2}\Big(\sum_{k=1}^{n}g_k^2-\sum_{i=1}^{n}\gamma_i^2\Big)\,.\tag{6.14}$$

On the other hand, the Universal Portfolio and the Asymptotic Target Portfolio have the same long-term growth rate, namely

$$\lim_{T\to\infty}\frac{1}{T}\log V^{\bar{\pi}}(T)=\lim_{T\to\infty}\frac{1}{T}\log V^{\widehat{\Pi}}(T)=\gamma+\frac{\sigma^2}{2}\Big(1-\frac{1}{n}\Big)\,.\tag{6.15}$$

Under the conditions of (2.2) and (3.2), we can verify

$$\sum_{k=1}^{n} g_k^2 > \sum_{i=1}^{n} \gamma_i^2 . \tag{6.16}$$

To show (6.16), we may assume without loss of generality $\gamma_1 \geq \cdots \geq \gamma_n$ and hence that there exists $(\delta_1, \ldots, \delta_{n-1})' \in (\mathbb{R}_+)^{n-1} \setminus \{0\}$ such that $g_k = -(\gamma_k + \delta_k)$ for $k = 1, \ldots, n-1$, and $g_n = -\gamma_n + (\delta_1 + \cdots + \delta_{n-1})$ for (2.2) and (3.2). Then we obtain

$$\begin{aligned}\sum_{k=1}^{n} g_k^2 &= \sum_{i=1}^{n-1} (\gamma_i + \delta_i)^2 + \big(- \gamma_n + (\delta_1 + \cdots + \delta_{n-1})\big)^2 \\ &= \sum_{i=1}^{n} \gamma_i^2 + \sum_{i=1}^{n-1} \big(\delta_i^2 + 2\delta_i(\gamma_i - \gamma_n)\big) + \Big(\sum_{i=1}^{n-1} \delta_i\Big)^2 > \sum_{i=1}^{n} \gamma_i^2 .\end{aligned}$$

Thus we observe from (6.14), (6.15), (6.16) that the growth-optimal portfolio $\varpi(\cdot)$ dominates in the long run both the universal portfolio $\widehat{\Pi}(\cdot)$ and the asymptotic target portfolio $\bar{\pi}$, a.s. The advantage of the universal portfolio is that it can be constructed with total oblivion as to what the actual values of the parameters of the model might be; some of these may be quite hard to estimate in practice. By contrast, constructing the growth-optimal portfolio $\varpi(\cdot)$ requires knowledge of all the model parameters, and keeping track of the positions of all stocks in all ranks at all times.

7. Appendix

7.1. **Proof of Lemma 1.** The stochastic exponential

$$\zeta(t) = \exp\Big[- \int_0^t \langle \xi(u) , d\, W(u) \rangle - \frac{1}{2} \int_0^t \| \xi(u) \|^2 \, d\, u \Big] ; \quad 0 \leq t < \infty$$

is a continuous martingale, where $\xi(t) := S^{-1}(Y(t)) G(Y(t))$ for $0 \leq t < \infty$ and $\|x\|^2 := \sum_{j=1}^{n} x_j^2$, $x \in \mathbb{R}^n$ and $\langle x, y \rangle = \sum_{j=1}^{n} x_j y_j$, $x, y \in \mathbb{R}^n$. Recall that $S(\cdot)$, $S^{-1}(\cdot)$ and $G(\cdot)$ in (2.4) are bounded. By Girsanov's theorem

$$\widetilde{W}(t) := W(t) + \int_0^t \sigma^{-1}(Y(u)) \mu\,(Y(u))\, d\,u\,, \; 0 \leq t < \infty$$

is an $n-$dimensional Brownian motion under the new probability measure $\mathbb{Q}$, locally equivalent to $\mathbb{P}$, that satisfies

$$\mathbb{Q}(C) = \mathbb{E}^{\mathbb{P}}(\zeta(T)\, \mathbf{1}_C) ; \; C \in \mathcal{F}_T \, , 0 \leq T < \infty . \tag{7.1}$$

Thus, equation (2.4) under $\mathbb{P}$ is reduced to

$$d\, Y(t) = S(Y(t))\, d\, \widetilde{W}(t) ; \quad 0 \leq t < T , \quad \text{under } \mathbb{Q} . \tag{7.2}$$

7.1.1. *Local time of Bessel Process.* Let us denote by $Z(\cdot)$ the square of the $\delta-$dimensional Bessel process with $0 < \delta < 2$. This process reaches the origin with probability one, but is subject to instantaneous reflection there, i.e., the local time $\Lambda_Z(\cdot)$ of $Z(\cdot)$ at the origin is zero:

$$\Lambda_Z(t) = \delta \int_0^t \mathbf{1}_{\{Z(s)=0\}} \, ds = 0 \qquad \text{and so } \mathrm{Leb}(\{t : Z(t) = 0\}) = 0$$

holds for all $t \in [0, \infty)$ almost surely. (Proposition XI.1.5. of [21]) The same is true for the $\delta-$dimensional Bessel process $\mathfrak{r}^\delta(\cdot) = \sqrt{|Z(\cdot)|}$ with $1 < \delta < 2$, i.e., $\mathrm{Leb}(\{t : \mathfrak{r}^\delta(t) = 0\}) = 0$ almost surely, and hence

$$\Lambda_{\mathfrak{r}^\delta}(t) = \frac{\delta - 1}{2} \int_0^t \mathbf{1}_{\{\mathfrak{r}^\delta(s)=0\}} \frac{ds}{\mathfrak{r}^\delta(s)} = 0; \quad 0 \le t < \infty, \quad 1 < \delta < 2. \tag{7.3}$$

On the other hand, for $\delta \ge 2$ the origin is never reached at all. We conclude that the local time of $\delta-$dimensional Bessel process $\mathfrak{r}^\delta(\cdot)$ is *identically equal to zero*, i.e., $\Lambda_{\mathfrak{r}^\delta}(\cdot) \equiv 0$, for any $\delta \in (1, \infty)$.

7.1.2. *Comparisons with Bessel Processes.* Now let us fix integers $1 < i < j < k \le n$. Under $\mathbb{Q}$ in (7.1) we shall compare the rank gap process

$$\eta(t) := \max_{\ell=i,j,k} Y_\ell(t) - \min_{m=i,j,k} Y_m(t)$$

with a Bessel process of dimension $\delta > 1$, using Lemmata 5 and 6 below.

We introduce the function $g(y) := [(y_i - y_j)^2 + (y_j - y_k)^2 + (y_k - y_i)^2]^{1/2}$ for $y \in \mathbb{R}^n$, and note the comparison $\sqrt{3}\,\eta(\cdot) \ge g(Y(\cdot))$. An application of Itô's rule to $g(Y(\cdot))$ yields the semimartingale decomposition $d\,g(Y(t)) = h(Y(t))\,dt + d\,\Theta(t)$, where we introduce the $(n \times 3)$ matrix $D_{ijk} := (d_i, d_j, d_k)$ with $(n \times 1)$ vectors $d_i := \mathfrak{e}_i - \mathfrak{e}_j$, $d_j := \mathfrak{e}_j - \mathfrak{e}_k$, $d_k := \mathfrak{e}_k - \mathfrak{e}_i$, we denote by $\mathfrak{e}_i$, $i = 1, \ldots, n$ the i^{th} unit vector in $\mathbb{R}^n$, and

$$\begin{aligned}
h(y) &:= \frac{(R(y) - 1)Q(y)}{2\,g(y)}, \quad R(y) := \frac{\mathrm{Tr}(D'_{ijk} S(y) S'(y) D_{ijk})}{Q(y)}, \\
Q(y) &:= \frac{y' D_{ijk} D'_{ijk} S(y) S(y)' D_{ijk} D'_{ijk} y}{y' D_{ijk} D'_{ijk} y}; \quad y \in \mathbb{R}^n \setminus \mathcal{Z}, \\
\mathcal{Z} &:= \{y \in \mathbb{R}^n \,|\, g(y) = \big(y' D_{ijk} D'_{ijk} y\big) = 0\}, \\
\Theta(t) &:= \int_0^t \Big(\sum_{\ell=i,j,k} \frac{S'(y) d_\ell d'_\ell y}{g(y)} \Big|_{y=Y(s)} \Big) \, d\widetilde{W}(s), \\
\langle \Theta \rangle(t) &= \int_0^t Q(Y(s))\, ds; \quad 0 \le t < \infty.
\end{aligned} \tag{7.4}$$

Here note that under the assumption on (2.3), and because $3D_{ijk}D'_{ijk} = D_{ijk}D'_{ijk}D_{ijk}D'_{ijk}$, we have

$$Q(\cdot) = \frac{3\,y' D_{ijk} D'_{ijk} S(\cdot) S(\cdot)' D_{ijk} D'_{ijk} y}{y' D_{ijk} D'_{ijk} D_{ijk} D'_{ijk} y} \ge 3 \min_{\mathbf{p} \in \Sigma_n} \min_{\ell=1,\ldots,n} \tilde{\lambda}_{\ell,\mathbf{p}} > 0 \tag{7.5}$$

in $\mathbb{R}^n \setminus \mathcal{Z}$, where $\tilde{\lambda}_{\ell,\mathbf{p}}, \ell = 1, \ldots, n$ are the eigenvalues of the positive-definite matrices $\mathfrak{s}_{\mathbf{p}}\mathfrak{s}'_{\mathbf{p}}$ for $\mathbf{p} \in \Sigma_n$, and so $\langle \Theta \rangle(\cdot)$ is strictly increasing when $Y(\cdot) \in \mathbb{R}^n \setminus \mathcal{Z}$. Now define the stopping time $\tau_u := \inf\{t \geq 0 \,|\, \langle \Theta \rangle(t) \geq u\}$, and note

$$\mathfrak{G}(u) := g(Y(\tau_u)) = g(Y(0)) + \int_0^{\tau_u} h(Y(t))\, dt + \widetilde{B}(u)\,; \quad 0 \leq u < \infty\,,$$

where $\widetilde{B}(u) := \Theta(\tau_u)$, $0 \leq u < \infty$ is a standard Brownian motion, by the Dambis-Dubins-Schwartz theorem of time-change for martingales. Thus, with $\mathfrak{d}(u) := R(Y(\tau_u))$ we can write

$$d\,\mathfrak{G}(u) = \frac{\mathfrak{d}(u) - 1}{2\mathfrak{G}(u)}\, d\,u + d\,\widetilde{B}(u)\,; \quad 0 \leq u < \infty\,, \quad \mathfrak{G}(0) = g(Y(0))\,.$$

The dynamics of the process $\mathfrak{G}(\cdot)$ are comparable to those of a Bessel process $\mathfrak{r}^{\delta}(\cdot)$ with dimension δ, generated by the same $\widetilde{B}(\cdot)$ and started at the same initial point $g(Y(0))$. Since $S(\cdot)S(\cdot)'$ is positive definite under (2.3) and rank $(D_{ijk}) = 2$, the (3×3) matrix $D'_{ijk}S(\cdot)S(\cdot)'D_{ijk}$ is non-negative definite and the number of its non-zero eigenvalues is equal to $\text{rank}(D'_{ijk}S(\cdot)S(\cdot)'D_{ijk}) = 2$. Let us denote by $\bar{\lambda}_{\ell,\mathbf{p}}$, $\ell = 1, 2, 3$ the eigenvalues of $D'_{ijk}\mathfrak{s}_{\mathbf{p}}\mathfrak{s}_{\mathbf{p}'}D_{ijk}$ for $\mathbf{p} \in \Sigma_n$. Then for $R(\cdot)$ in (7.4) we obtain

$$R(\cdot) \geq \delta_0 := \min_{\mathbf{p} \in \Sigma_n} \left(\frac{\sum_{\ell=1}^{3} \bar{\lambda}_{\ell,\mathbf{p}}}{\max_{1 \leq \ell \leq 3} \bar{\lambda}_{\ell,\mathbf{p}}} \right) > 1 \text{ in } \mathbb{R}^n \setminus \mathcal{Z} \tag{7.6}$$

and so $\mathfrak{d}(\cdot) \geq \delta_0 > 1$ when $Y(\tau_\cdot) \in \mathbb{R}^n \setminus \mathcal{Z}$. By a comparison argument similar to that in the proof of Lemma 2.1 of [16], we may show that $\mathfrak{G}(t) \geq \mathfrak{r}^{\delta_0}(t)$ for $0 \leq t < \infty$ a.s. Thus $\sqrt{3}\eta(t) \geq g(Y(t)) = \mathfrak{G}(\langle \Theta \rangle(t))$ implies $\sqrt{3}\eta(t) \geq \mathfrak{r}^{\delta_0}(\langle \Theta \rangle(t))$ for $0 \leq t < \infty$, a.s., and so we obtain the following result.

Lemma 5. *For the process $Y(\cdot)$ of (7.2) with (2.3), the multiple $\sqrt{3}\eta(\cdot)$ of the rank-gap process dominates, a.s. under $\mathbb{Q}$, a time-changed Bessel process $\tilde{\mathfrak{r}}(\cdot) := \mathfrak{r}^{\delta_0}(\langle \Theta \rangle(\cdot))$ with dimension δ_0 as in (7.6):*

$$\mathbb{Q}\big(\sqrt{3}\eta(t) \geq \tilde{\mathfrak{r}}(t)\,;\ 0 \leq t < \infty\big) = 1\,.$$

Lemma 6. *Under $\mathbb{Q}$, the rank-gap process $\eta(\cdot)$ satisfies $\langle \eta \rangle(t) \leq c_1 t$, $0 \leq t < \infty$ a.s. for some constant $c_1 > 0$ and the local time $\Lambda_\eta(\cdot)$ of $\eta(\cdot)$ at the origin is identically equal to zero.*

In fact, since the diffusion coefficient matrix $S(\cdot)$ of $Y(\cdot)$ in (7.2) is bounded and positive definite under (2.3), there exist such constant c_1 that $\langle \eta \rangle(t) \leq c_1 t$ for $0 \leq t < \infty$ a.s. Moreover, from (7.5) and Lemma 5, there exists a constant $c_2 := \min_{\mathbf{p} \in \Sigma_n, \ell = 1, \ldots, n} \tilde{\lambda}_{\ell,\mathbf{p}} > 0$, such that $\langle \Theta \rangle(t) \geq c_2 t$ for $0 \leq t < \infty$ a.s. It follows from the representation of local times (Theorem

VI. 1.7 of [21]) and (7.3) with Lemma 5 that

$$
\begin{aligned}
\Lambda_\eta(t) &= \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t \mathbf{1}_{\{0 \le \eta(s) < \varepsilon\}} \, d\langle \eta \rangle(s) \le \lim_{\varepsilon \downarrow 0} \frac{\sqrt{3}c_1}{2\,\varepsilon} \int_0^t \mathbf{1}_{\{0 \le \sqrt{3}\eta(s) < \varepsilon\}} \, d\, s \\
&\le \lim_{\varepsilon \downarrow 0} \frac{\sqrt{3}c_1}{2\,\varepsilon} \int_0^t \mathbf{1}_{\{0 \le \tilde{\mathfrak{r}}(s) < \varepsilon\}} \, d\, s \le \lim_{\varepsilon \downarrow 0} \frac{\sqrt{3}\, c_1}{2\, c_2\, \varepsilon} \int_0^{\langle \Theta \rangle (t)} \mathbf{1}_{\{0 \le \mathfrak{r}^\delta(u) < \varepsilon\}} \, d\, u \\
&\le \sqrt{3}\, c_1\, c_2^{-1}\, \Lambda_{\mathfrak{r}^\delta}(\langle \Theta \rangle(t)) \equiv 0\,; \quad 0 \le t < \infty\,.
\end{aligned} \tag{7.7}
$$

Define an increasing family of events $C_T := \{\Lambda_\eta(t) > 0 \text{ for some } t \in [0, T]\}$, $T \ge 0$. By Lemma 6 we obtain $\mathbb{Q}(C_\infty) = 0$ and $0 = \mathbb{Q}(C_\ell) = \mathbb{P}(C_\ell)$ for $\ell \ge 1$. Then $\mathbb{P}(\Lambda_\eta(t) > 0 \text{ for some } t \ge 0) = \mathbb{P}(\cup_{\ell=1}^\infty C_\ell) = \lim_{\ell=\infty} \mathbb{P}(C_\ell) = 0$. Thus the local time $\Lambda_\eta(t)$ of rank gap process $\eta(\cdot)$ for $(Y_i(\cdot)\,, Y_j(\cdot)\,, Y_k(\cdot)\,)$ is zero for $0 \le t < \infty$ a.s. under $\mathbb{P}$.

Since the choice of i, j, k is arbitrary, there is no local time generated by the rank gap process of any three coordinates. The rank gap process of more than three coordinates (e.g. $\max_{\ell=h,i,j,k} Y_\ell(\cdot) - \min_{m=h,i,j,k} Y_m(\cdot)$) dominates that of any three sub-coordinates. Therefore, by a similar argument as (7.7) and its consequence, any local time of rank gap process of more than three coordinates is zero for $0 \le t < \infty$ a.s. *under* $\mathbb{P}$.

To establish (4.4) from this and (4.3), and thus complete the proof of Lemma 1, consider any integers (ranks) $1 \le a \le \ell < m \le b \le n$ with $b - a \ge 2$, and observe that we have almost surely:

$$
0 \equiv \Lambda^{a,b}(t) = \int_0^t 1_{\{Z_a(s) = Z_b(s)\}} \, d\big(Z_a(s) - Z_b(s)\big)
$$

$$
= \int_0^t 1_{\{Z_a(s) = Z_b(s)\}} \, d\big(Z_a(s) - Z_\ell(s)\big) + \int_0^t 1_{\{Z_a(s) = Z_b(s)\}} \, d\big(Z_\ell(s) - Z_m(s)\big)
$$

$$
+ \int_0^t 1_{\{Z_a(s) = Z_b(s)\}} \, d\big(Z_m(s) - Z_b(s)\big)
$$

$$
= \int_0^t 1_{\{Z_a(s) = Z_b(s)\}} \, d\big(\Lambda^{a,\ell}(s) + \Lambda^{\ell,m}(s) + \Lambda^{m,b}(s)\big) \ge \int_0^t 1_{\{Z_a(s) = Z_b(s)\}} \, d\Lambda^{\ell,m}(s) \ge 0\,.
$$

The a.s. equality $\int_0^t 1_{\{Z_a(s) = Z_b(s)\}} \, d\Lambda^{\ell,m}(s) = 0$ follows readily from this, as does

$$
\int_0^t 1_{\{N_k(t) \ge 3\}} \Big(\sum_{\ell=k+1}^n d\Lambda^{k,\ell}(s) - \sum_{\ell=1}^{k-1} d\Lambda^{\ell,k}(s) \Big) \;=\; 0
$$

and thus (4.4) as well. □

7.2. **Proof of Lemma 2.** For each $k = 1, \ldots, n-1$ the local time $\Lambda^{k,k+1}(\cdot)$ is a continuous additive functional of $(\Xi(\cdot), \mathfrak{P}_\cdot)$ with support in $\mathfrak{F}_k$, and the expectation of $\Lambda^{k,k+1}(t)$ with respect to the invariant distribution $\nu(\cdot, \cdot)$ is finite for $t \ge 0$.

It follows from the theory of additive functionals [2] that there is a finite measure $\nu_k(\cdot,\cdot)$ on $\mathfrak{F}_k \times \Sigma_n$ such that

$$\frac{1}{T}\,\mathbb{E}_\nu\Big[\int_0^T g(\Xi(s),\mathfrak{P}_s)\,d\,\Lambda^{k,k+1}(s)\Big] \;=\; \frac{1}{2}\int_{\mathfrak{F}_k\times\Sigma_n} g(z,\mathbf{p})\,d\,\nu_k(z,\mathbf{p}) \tag{7.8}$$

for every bounded measurable function $g : \mathfrak{F}_k \times \Sigma_n \mapsto \mathbb{R}$. Let us denote by $\nu_{0k}(\cdot) = \nu_k(\cdot,\Sigma_n)$ the marginal distribution on $\mathfrak{F}_k$. The absolute continuity of $\nu_{0k}(\cdot)$ with respect to $(n-1)-$dimensional Lebesgue measure is argued by localization and the properties of Reflected Brownian motion as in Theorem 7.1, Lemmata 7.7 and 7.9 of [14].

Now, by an application of Itô's rule, for $f \in C_b^2((\mathbb{R}_+)^{n-1})$ we obtain

$$f(\Xi(T)) = f(\Xi(0)) + \int_0^T \langle \nabla f(\Xi(s)), d\,\zeta^{\mathrm{mart}}(s)\rangle$$
$$+\sum_{k=1}^{n-1}\int_0^T \big[\mathcal{D}_k f\big](\Xi(s))\,d\,\Lambda^{k,k+1}(s) + \int_0^T \big[\mathcal{A} f\big](\Xi(s),\mathfrak{P}_s)\,d\,s\,; \quad T\ge 0\,,$$

where $\zeta^{\mathrm{mart}}(\cdot)$ is the martingale part of $\zeta(\cdot)$ and $\mathcal{D}_k$ and $\mathcal{A}$ are differential operators defined in (5.3). Taking expectations with respect to $\mathbb{P}$ and then integrating for the initial values with respect to the stationary distribution $\nu(\cdot,\cdot)$ with Fubini's theorem and (7.8), we obtain

$$0 = \frac{T}{2}\sum_{k=1}^{n-1}\int_{\mathfrak{F}_k}\big[\mathcal{D}_k f\big](z)\,d\,\nu_{0k}(z) + T\int_{(\mathbb{R}_+)^{n-1}\times\Sigma_n}\big[\mathcal{A} f\big](z,\mathbf{p})d\,\nu(z,\mathbf{p})\,.$$

Dividing by $T>0$, we obtain the basic adjoint relationship (5.6).

7.3. **A Sanity Check of Corollary 4.** In this section we verify that $(\theta_{k,i})_{1\le i,k\le n}$ in (5.16) satisfy (4.10). Since $\theta_{k,i}$ is homogeneous in the product $\prod_{j=1}^{n-1}[-4(\sigma_j+\sigma_{j+1}^2)^{-1}]$, it suffices to show $\sum_{k=1}^n \widetilde{\theta}_{k,i}\big(g_k+\gamma_i\big)=0$ where we use the modifications $\widetilde{\theta}_{k,i} := \sum_{\{\mathbf{p}(k)=i\}}\widetilde{\theta}_{\mathbf{p}}$,

$$\widetilde{\theta}_{\mathbf{p}} := \Big(\sum_{\mathbf{q}\in\Sigma_n}\prod_{j=1}^{n-1}\widetilde{\lambda}_{\mathbf{q},j}\Big)^{-1}\prod_{j=1}^{n-1}\widetilde{\lambda}_{\mathbf{p},j}^{-1}\,, \quad \widetilde{\lambda}_{\mathbf{p},j} := \sum_{\ell=1}^{j}\big(g_\ell+\gamma_{\mathbf{p}(\ell)}\big)$$

of $(\theta_{k,i}\,,\theta_{\mathbf{p}},\,\lambda_{\mathbf{p},j})$, $1\le i,j,k\le n$, $\mathbf{p}\in\Sigma_n$ for notational simplicity. Note that $\widetilde{\lambda}_{\mathbf{p},n}=0$ from (2.2) for $\mathbf{p}\in\Sigma_n$.

First, observe for $\ell=2,\dots,n$ and $i=1,\dots,n$,

$$\sum_{\{\mathbf{p}:\,\mathbf{p}(\ell-1)=i\}}\widetilde{\lambda}_{\mathbf{p},\ell-1}\widetilde{\theta}_{\mathbf{p}} + \sum_{\{\mathbf{p}:\,\mathbf{p}(\ell)=i\}}(g_\ell+\gamma_i)\widetilde{\theta}_{\mathbf{p}} = \sum_{\{\mathbf{p}:\,\mathbf{p}(\ell)=i\}}\widetilde{\lambda}_{\mathbf{p},\ell}\widetilde{\theta}_{\mathbf{p}}\,. \tag{7.9}$$

In fact, for every i,ℓ define another permutation $\widetilde{\mathbf{p}}$ from a (fixed) permutation $\mathbf{p}\in\{\mathbf{q}\in\Sigma_n\,:\,\mathbf{q}(\ell-1)=i\}$ by

$$\widetilde{\mathbf{p}}(k) := \widetilde{\mathbf{p}}(k\,;\mathbf{p}) = \begin{cases}\mathbf{p}(k)\,, & k=1,\dots,\ell-2,\,\ell+1,\dots,n\,,\\ \mathbf{p}(\ell)\,, & k=\ell-1\,,\\ i\,, & k=\ell\,,\end{cases}$$

which is obtained by exchanging $(\ell-1)$ st and ℓ th elements of $\mathbf{p} \in \{\mathbf{q} \in \Sigma_n : \mathbf{q}(\ell-1) = i\}$, and also define $M := (\sum_{\mathbf{q}\in\Sigma_n} \prod_{j=1}^{n-1} \widetilde{\lambda}_{\mathbf{q},j})^{-1}$ here. Then $\widetilde{\lambda}_{\mathbf{p},j} = \widetilde{\lambda}_{\widetilde{\mathbf{p}},j}$ for $j \neq \ell-1$ and hence the left-hand of (7.9) is

$$\sum_{\{\mathbf{p}:\,\mathbf{p}(\ell-1)=i\}} \widetilde{\lambda}_{\mathbf{p},\ell-1} \cdot M \prod_{j=1}^{n-1} \widetilde{\lambda}_{\mathbf{p},j}^{-1} + \sum_{\{\mathbf{p}:\,\mathbf{p}(\ell)=i\}} (g_\ell + \gamma_i) M \prod_{j=1}^{n-1} \widetilde{\lambda}_{\mathbf{p},j}^{-1}$$
$$= \sum_{\{\widetilde{\mathbf{p}}:\,\widetilde{\mathbf{p}}(\ell)=i\}} M \prod_{j\neq\ell-1}^{n-1} \widetilde{\lambda}_{\widetilde{\mathbf{p}},j}^{-1} + \sum_{\{\widetilde{\mathbf{p}}:\,\widetilde{\mathbf{p}}(\ell)=i\}} (g_\ell + \gamma_{\widetilde{p}(\ell)}) M \prod_{j=1}^{n-1} \widetilde{\lambda}_{\widetilde{\mathbf{p}},j}^{-1}$$
$$= \sum_{\{\widetilde{\mathbf{p}}:\,\widetilde{\mathbf{p}}(\ell)=i\}} [\widetilde{\lambda}_{\widetilde{\mathbf{p}},\ell-1} + g_\ell + \gamma_{\widetilde{\mathbf{p}}(\ell)}] \cdot M \prod_{j=1}^{n-1} \widetilde{\lambda}_{\widetilde{\mathbf{p}},j}^{-1} = \sum_{\{\mathbf{p}:\,\mathbf{p}(\ell)=i\}} \widetilde{\lambda}_{\mathbf{p},\ell} \widetilde{\theta}_{\mathbf{p}} \,,$$

which is the right-hand of (7.9). Now applying (7.9) for $\ell = 2, \ldots, n$, we obtain

$$\sum_{k=1}^{n} (g_k + \gamma_i)\widetilde{\theta}_{k,i} = (g_1 + \gamma_i)\widetilde{\theta}_{1,i} + (g_2 + \gamma_i)\widetilde{\theta}_{2,i} + \sum_{k=3}^{n} (g_k + \gamma_i)\widetilde{\theta}_{k,i}$$
$$= \sum_{\{\mathbf{p}:\,\mathbf{p}(2)=i\}} \widetilde{\lambda}_{\mathbf{p},2}\widetilde{\theta}_{\mathbf{p}} + \sum_{k=3}^{n} (g_k + \gamma_i)\widetilde{\theta}_{k,i} = \cdots = \sum_{\{\mathbf{p}:\,\mathbf{p}(n)=i\}} \widetilde{\lambda}_{\mathbf{p},n}\widetilde{\theta}_{\mathbf{p}} = 0 \,,$$

for $i = 1, \ldots, n$, because $\widetilde{\lambda}_{\mathbf{p},n} = 0$ for $\mathbf{p} \in \Sigma_n$. Therefore, (4.10) is satisfied.

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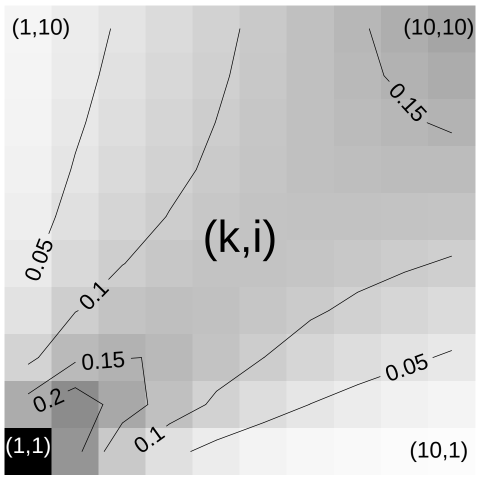

FIGURE 1. Different values of $\{\theta_{k,i}\}$ for (k,i), when the parameters are specified for an extreme case in Example 4.

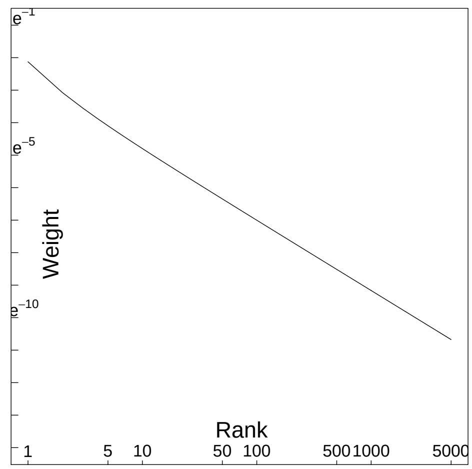

FIGURE 2. Expected capital distribution curve for the pure hybrid model in Example 8.

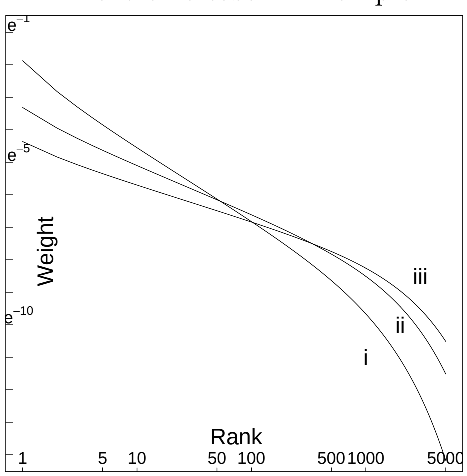

FIGURE 3. Expected capital distribution curves for the hybrid model in Example 9.

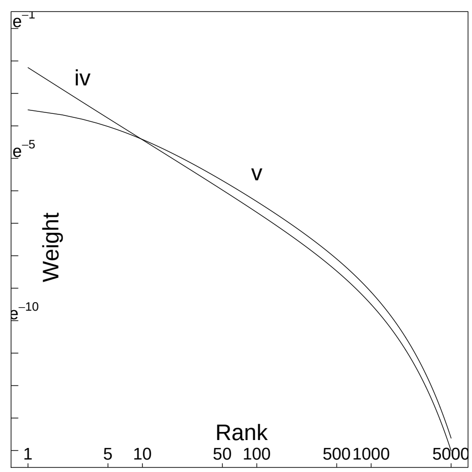

FIGURE 4. Expected capital distribution curves for the hybrid model in Example 10.