Inf-convolution of G-expectations

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Abstract

In this paper we will discuss the optimal risk transfer problems when risk measures are generated by G-expectations, and we present the relationship between inf-convolution of G-expectations and the inf-convolution of drivers G.

Keywords: inf-convolution, G-expectation, G-normal distribution, G-Brownian motion

1 Introduction

Coherent risk measures were introduced by Artzner et al. [1] in finite probability spaces and lately by Delbaen [8,9] in general probability spaces. The family of coherent risk measures was extended later by Föllmer and Schied [10,11] and, independently, by Frittelli and Rosazza Gianin [12,13] to the class of convex risk measures.

The notion of g-expectations was introduced by Peng [15] as solutions to a class of nonlinear Backward Stochastic Differential Equations (BSDE in short) which were first studied by Pardoux and Peng [14]. Financial applications were discussed in detail by El Karoui et al. [6].

Let us introduce the optimal risk transfer model we are concerned with. This model can be briefly described as follows:

Two economic agents A and B are considered, who assess the risk associated with their respective positions by risk measures ρ_A and ρ_B. The issuer, agent A, with the total risk capital X, wants to issue a financial product F and sell it to agent B for the price π in order to reduce his risk

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exposure. His objective is to minimize $\rho_A(X - F + \pi)$ with respect to F and π, while the interest of buyer B is not to be exposed to a greater risk after the transaction:

$$\rho_B(F - \pi) \leq \rho_B(0).$$

Using the cash translation invariance property, this optimization problem can be rewritten in the simpler form

$$\inf_F \{\rho_A(X - F) + \rho_B(F)\}.$$

This problem was first studied by El Karoui and Pauline Barrieu [2,3,4] for convex risk measures, in particular those described by g-expectation.

Related with the pioneering paper [1] on coherent risk measures, sublinear expectations (or, more generally, convex expectations, see [10,11,13]) have become more and more popular for modeling such risk measures. Indeed, in any sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ a coherent risk measure ρ can be defined in a simple way by putting $\rho(X) := \hat{\mathbb{E}}[-X]$, for $X \in \mathcal{H}$.

The notion of a sublinear expectation named G-expectation was first introduced by Peng [17,18] in 2006. Compared with g-expectations, the theory of G-expectation is intrinsic in the sense that it is not based on a given (linear) probability space. A G-expectation is a fully nonlinear expectation. It characterizes the variance uncertainty of a random variable. We recall that the problem of mean uncertainty has been studied by Chen-Epstein through g-expectation in [5]. Under this fully nonlinear G-expectation, a new type of Itô's formula has been obtained, and the existence and uniqueness for stochastic differential equation driven by a G-Brownian motion have been shown. For a more detailed description the reader is referred to Peng's recent papers [17,18,19].

This paper focuses on the mentioned optimization problem where the g-risk measures are replaced by one dimensional G-expectations, i.e., the problem:

$$\hat{\mathbb{E}}_{G_1} \Box\, \hat{\mathbb{E}}_{G_2}[X] := \inf_F \{\hat{\mathbb{E}}_{G_1}[X - F] + \hat{\mathbb{E}}_{G_2}[F]\}.$$

The main aim of this paper is to present the relationship between the above introduced operator $\hat{\mathbb{E}}_{G_1} \Box\, \hat{\mathbb{E}}_{G_2}[\cdot]$ and the G-expectation $\hat{\mathbb{E}}_{G_1 \Box G_2}[\cdot]$. More precisely, we show that both operators coincide if $G_1 \Box G_2 \neq -\infty$.

In this paper we constrain ourselves to one dimensional G-expectation, the multi-dimensional case is much more complicated and we hope to study this case in a forthcoming publication.

Our approach is mainly based on the recent results by Peng [19] which allow to show that $\hat{\mathbb{E}}_{G_1} \Box\, \hat{\mathbb{E}}_{G_2}[\cdot]$ constructed by inf-convolution of $\hat{\mathbb{E}}_{G_1}[\cdot]$ and $\hat{\mathbb{E}}_{G_2}[\cdot]$ satisfies the properties of G-expectation. To our best knowledge, this is the first paper that uses the results of Theorem 4.1.3 of [19] to prove that a given nonlinear expectation is a G-expectation.

This paper is organized as follows: while basic definitions and properties of G-expectation and G-Brownian Motion are recalled in Section 2, Section 3 states and proves the main result of this paper: If $G_1\Box G_2 \neq -\infty$, then $\hat{\mathbb{E}}_{G_1}\Box\, \hat{\mathbb{E}}_{G_2}[\cdot]$ also is a G-expectation and

$$\hat{\mathbb{E}}_{G_1}\Box\, \hat{\mathbb{E}}_{G_2}[\cdot] = \hat{\mathbb{E}}_{G_1\Box G_2}[\cdot].$$

2 Notation and Preliminaries

The aim of this section is to recall some basic definitions and properties of G-expectations and G-Brownian motions, which will be needed in the sequel. The reader interested in a more detailed description of these notions is referred to Peng's recent papers [17,18,19].
Adapting Peng's approach in [19], we let Ω be a given nonempty fundamental space and $\mathcal{H}$ be a linear space of real functions defined on Ω such that :

i) $1 \in \mathcal{H}$.
ii) $\mathcal{H}$ is stable with respect to local Lipschitz functions, i.e. for all $n \geq 1$, and for all $X_1, ..., X_n \in \mathcal{H}$, $\varphi \in C_{l,lip}(\mathbb{R}^n)$, it holds also $\varphi(X_1, ..., X_n) \in \mathcal{H}$.

Recall that $C_{l,lip}(\mathbb{R}^n)$ denotes the space of all local Lipschitz functions φ over $\mathbb{R}^n$ satisfying

$$|\varphi(x) - \varphi(y)| \leq C(1 + |x|^m + |y|^m)|x - y|, x, y \in \mathbb{R}^n,$$

for some $C > 0, m \in \mathbb{N}$ depending on φ. The set $\mathcal{H}$ is interpreted as the space of random variables defined on Ω.

Definition 2.1 A sublinear expectation $\hat{\mathbb{E}}$ on $\mathcal{H}$ is a functional $\mathcal{H} \to \mathbb{R}$ with the following properties : for all $X, Y \in \mathcal{H}$, we have
(a) Monotonicity: if $X \geq Y$ then $\hat{\mathbb{E}}[X] \geq \hat{\mathbb{E}}[Y]$.
(b) Preservation of constants: $\hat{\mathbb{E}}[c] = c$, for all reals c.
(c) Sub-additivity (or property of self-dominacy):

$$\hat{\mathbb{E}}[X] - \hat{\mathbb{E}}[Y] \leq \hat{\mathbb{E}}[X - Y].$$

(d) Positive homogeneity: $\hat{\mathbb{E}}[\lambda X] = \lambda\hat{\mathbb{E}}[X], \forall \lambda \geq 0$.

The triple $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a sublinear expectation space. It generalizes the classical case of the linear expectation $E[X] = \int_\Omega X dP$, $X \in L^1(\Omega, \mathcal{F}, \mathcal{P})$, over a probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Moreover, $\rho(X) = \hat{\mathbb{E}}[-X]$ defines a coherent risk measure on $\mathcal{H}$.

Definition 2.2 For arbitrary $n, m \geq 1$, a random vector $Y = (Y_1, Y_2, ..., Y_n) \in \mathcal{H}^n$ $(= \mathcal{H} \times \mathcal{H} \times ... \times \mathcal{H})$ is said to be independent of $X \in \mathcal{H}^m$ under $\hat{\mathbb{E}}[\cdot]$ if for each test function $\varphi \in C_{l,lip}(\mathbb{R}^{n+m})$ we have

$$\hat{\mathbb{E}}[\varphi(X, Y)] = \hat{\mathbb{E}}[\hat{\mathbb{E}}[\varphi(x, Y)]_{x=X}].$$

Remark: In the case of linear expectation, this notion of independence is just the classical one. It is important to note that under sublinear expectations the condition Y is independent to X does not imply automatically that X is independent to Y.

Let $X = (X_1, ..., X_n) \in \mathcal{H}^n$ be a given random vector. We define a functional on $C_{l,lip}(\mathbb{R}^n)$ by

$$\hat{\mathbb{F}}_X[\varphi] := \hat{\mathbb{E}}[\varphi(X)], \varphi \in C_{l,lip}(\mathbb{R}^n).$$

It's easy to check that $\hat{\mathbb{F}}_X[\cdot]$ is a sublinear expectation defined on $(\mathbb{R}^n, C_{l,lip}(\mathbb{R}^n))$.

Definition 2.3 Given two sublinear expectation spaces $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ and $(\widetilde{\Omega}, \widetilde{\mathcal{H}}, \widetilde{\mathbb{E}})$, two random vectors $X \in \mathcal{H}^n$ and $Y \in \widetilde{\mathcal{H}}^n$ are said to be identically distributed if for each test function $\varphi \in C_{l,lip}(\mathbb{R}^n)$

$$\hat{\mathbb{F}}_X[\varphi] = \widetilde{\mathbb{F}}_Y[\varphi].$$

We now introduce the important notion of G-normal distribution. For this, let $0 \leq \underline{\sigma} \leq \overline{\sigma} \in \mathbb{R}$, and let G be the sublinear function:

$$G(\alpha) = \frac{1}{2}(\overline{\sigma}^2\alpha^+ - \underline{\sigma}^2\alpha^-), \alpha \in \mathbb{R}.$$

As usual $\alpha^+ = max\{0, \alpha\}$ and $\alpha^- = (-\alpha)^+$. Given an arbitrary initial condition $\varphi \in C_{l,lip}(\mathbb{R})$, we denote by u_φ the unique viscosity solution of the following parabolic partial differential equation (PDE):

$$\begin{aligned} &\partial_t u_\varphi(t, x) = G(\partial^2_{xx} u_\varphi(t, x)), \qquad &(t, x) \in (0, \infty) \times \mathbb{R}, \\ &u_\varphi(0, x) = \varphi(x), &x \in \mathbb{R}. \end{aligned}$$

Definition 2.4 : A random variable X in a sub-expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called $G_{\underline{\sigma},\overline{\sigma}}$-normal distributed, and we write $X \sim \mathcal{N}(0; [\underline{\sigma}^2, \overline{\sigma}^2])$, if for all $\varphi \in C_{l,lip}(\mathbb{R})$,

$$\hat{\mathbb{E}}[\varphi(x + \sqrt{t}X)] := u_\varphi(t, x), \qquad (t, x) \in [0, \infty) \times \mathbb{R}.$$

Remark: From [18], we have the following Kolmogrov-Chapman chain rule:

$$u_\varphi(t+s,x) = \hat{\mathbb{E}}[u_\varphi(t, x+\sqrt{s}X)], \quad s \geq 0.$$

In what follows we will take as fundamental space Ω the space $C_0(\mathbb{R}^+)$ of all real-valued continuous functions $(\omega_t)_{t\in\mathbb{R}^+}$ with $\omega_0 = 0$, equipped with the topology generated by the uniform convergence on compacts.
For each fixed $T \geq 0$, we consider the following space of local Lipschitz functionals :

$$\begin{aligned}\mathcal{H}_T &= Lip(\mathcal{F}_T) : \\ &= \{X(\omega) = \varphi(\omega_{t_1}, ..., \omega_{t_m}), t_1, ..., t_m \in [0,T], \varphi \in C_{l,lip}(\mathbb{R}^m), m \geq 1\}.\end{aligned}$$

Furthermore, for $0 \leq s \leq t$, we define

$$\begin{aligned}\mathcal{H}_t^s &= Lip(\mathcal{F}_t^s) : \\ &= \{X(\omega) = \varphi(\omega_{t_2} - \omega_{t_1}, ..., \omega_{t_{m+1}} - \omega_{t_m}), t_1, ..., t_{m+1} \in [s,t], \\ &\quad \varphi \in C_{l,lip}(\mathbb{R}^m), m \geq 1\}.\end{aligned}$$

It is clear that $\mathcal{H}_t^s \subseteq \mathcal{H}_t \subseteq Lip(\mathcal{F}_T)$, for $s \leq t \leq T$. We also introduce the space

$$\mathcal{H} = Lip(\mathcal{F}) := \bigcup_{n=1}^{\infty} Lip(\mathcal{F}_n).$$

Obviously, $Lip(\mathcal{F}_t^s)$, $Lip(\mathcal{F}_T)$ and $Lip(\mathcal{F})$ are vector lattices.

We will consider the canonical space and set

$$B_t(\omega) = \omega_t, t \in [0,\infty), \text{ for } \omega \in \Omega.$$

Obviously, for each $t \in [0,\infty), B_t \in Lip(\mathcal{F}_t)$. Let $G(a) = G_{\underline{\sigma},\overline{\sigma}}(a) = \frac{1}{2}(\overline{\sigma}^2 a^+ - \underline{\sigma}^2 a^-), a \in \mathbb{R}$. We now introduce a sublinear expectation $\hat{\mathbb{E}}$ defined on $\mathcal{H}_T = Lip(\mathcal{F}_T)$, as well as on $\mathcal{H} = Lip(\mathcal{F})$, via the following procedure: For each $X \in \mathcal{H}_T$ with

$$X = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, ..., B_{t_m} - B_{t_{m-1}}),$$

and for all $\varphi \in C_{l,lip}(\mathbb{R}^m)$ and $0 = t_0 \leq t_1 < ... < t_m \leq T,\ m \geq 1$, we set

$$\begin{aligned}&\hat{\mathbb{E}}[\varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, ..., B_{t_m} - B_{t_{m-1}})] \\ &= \widetilde{\mathbb{E}}[\varphi(\sqrt{t_1 - t_0}\xi_1, ..., \sqrt{t_m - t_{m-1}}\xi_m)],\end{aligned}$$

where $(\xi_1, ..., \xi_m)$ is an m-dimensional random vector in some sublinear expectation space $(\widetilde{\Omega}, \widetilde{\mathcal{H}}, \widetilde{\mathbb{E}})$, such that $\xi_i \sim \mathcal{N}(0; [\underline{\sigma}^2, \overline{\sigma}^2])$ and ξ_{i+1} is independent of $(\xi_1, ..., \xi_i)$, for all $i = 1, ..., m-1, m \in \mathbb{N}$. The related conditional expectation of $X = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, ..., B_{t_m} - B_{t_{m-1}})$ under $\mathcal{H}_{t_j}$ is defined by

$$\begin{aligned}\hat{\mathbb{E}}[X|\mathcal{H}_{t_j}] &= \hat{\mathbb{E}}[\varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, ..., B_{t_m} - B_{t_{m-1}})|\mathcal{H}_{t_j}] \\ &= \psi(B_{t_1} - B_{t_0}, ..., B_{t_j} - B_{t_{j-1}})\end{aligned}$$

where

$$\psi(x_1, ..., x_j) = \widetilde{\mathbb{E}}[\varphi(x_1, ..., x_j, \sqrt{t_{j+1} - t_j}\xi_{j+1}, ..., \sqrt{t_m - t_{m-1}}\xi_m)].$$

We know from [18,19] that $\hat{\mathbb{E}}[\cdot]$ defines consistently a sublinear expectation on $Lip(\mathcal{F})$, satisfying (a)-(d) in Definition 2.1. The reader interested in a more detailed discussion is referred to [18,19].

Definition 2.5 The expectation $\hat{\mathbb{E}}[\cdot] : Lip(\mathcal{F}) \to \mathbb{R}$ defined through the above procedure is called $G_{\underline{\sigma},\overline{\sigma}}$-expectation. The corresponding canonical process $(B_t)_{t\geq 0}$ in the sublinear expectation is called a $G_{\underline{\sigma},\overline{\sigma}}$-Brownian motion on $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$.
At the end of this section we list some useful properties that we will need in Section 3.

Proposition 2.6 ([18,19]) The following properties of $\hat{\mathbb{E}}[\cdot|\mathcal{H}_t]$ hold for all $X, Y \in \mathcal{H} = Lip(\mathcal{F})$:
(a')If $X \geq Y$, then $\hat{\mathbb{E}}[X|\mathcal{H}_t] \geq \hat{\mathbb{E}}[Y|\mathcal{H}_t]$.
(b')$\hat{\mathbb{E}}[\eta|\mathcal{H}_t] = \eta$, for each $t \in [0, \infty)$ and $\eta \in \mathcal{H}_t$.
(c')$\hat{\mathbb{E}}[X|\mathcal{H}_t] - \hat{\mathbb{E}}[Y|\mathcal{H}_t] \leq \hat{\mathbb{E}}[X - Y|\mathcal{H}_t]$.
(d')$\hat{\mathbb{E}}[\eta X|\mathcal{H}_t] = \eta^+\hat{\mathbb{E}}[X|\mathcal{H}_t] + \eta^-\hat{\mathbb{E}}[-X|\mathcal{H}_t]$, for each $\eta \in \mathcal{H}_t$.
We also have

$$\hat{\mathbb{E}}[\hat{\mathbb{E}}[X|\mathcal{H}_t]|\mathcal{H}_s] = \hat{\mathbb{E}}[X|\mathcal{H}_{t\wedge s}], \text{ and in particular, } \hat{\mathbb{E}}[\hat{\mathbb{E}}[X|\mathcal{H}_t]] = \hat{\mathbb{E}}[X].$$

For each $X \in Lip(\mathcal{F}_T^t)$, $\hat{\mathbb{E}}[X|\mathcal{H}_t] = \hat{\mathbb{E}}[X]$, moreover, the properties (b') and (c') imply: $\hat{\mathbb{E}}[X + \eta|\mathcal{H}_t] = \hat{\mathbb{E}}[X|\mathcal{H}_t] + \eta$, whenever $\eta \in \mathcal{H}_t$.
We will need also the following two propositions, and for proofs the reader is referred to [18,19].

Proposition 2.7 For each convex function φ and each concave function ψ with $\varphi(B_t)$ and $\psi(B_t) \in \mathcal{H}_t$, we have $\hat{\mathbb{E}}[\varphi(B_t)] = \mathbb{E}[\varphi(\overline{\sigma}W_t)]$ and $\hat{\mathbb{E}}[\psi(B_t)] = \mathbb{E}[\psi(\underline{\sigma}W_t)]$, where $(W_t)_{t\geq 0}$ is a Brownian motion under the linear expectation $\mathbb{E}$.

Proposition 2.8 Let $\hat{\mathbb{E}}_1[\cdot]$ and $\hat{\mathbb{E}}_2[\cdot]$ be a $G_{\underline{\sigma}_1,\overline{\sigma}_1}$ and a $G_{\underline{\sigma}_2,\overline{\sigma}_2}$ expectation on the space $(\Omega, \mathcal{H})$, respectively. Then, if $[\underline{\sigma}_1, \overline{\sigma}_1] \subseteq [\underline{\sigma}_2, \overline{\sigma}_2]$, we have $\hat{\mathbb{E}}_1[X] \leq \hat{\mathbb{E}}_2[X]$ and $\hat{\mathbb{E}}_1[X|\mathcal{H}_t] \leq \hat{\mathbb{E}}_2[X|\mathcal{H}_t]$, for all $X \in \mathcal{H}$ and all $t \geq 0$.

3 Inf-convolution of G-expectations

The aim of this section is to state the main result of this paper, that is the relationship between the inf-convolution $\hat{\mathbb{E}}_{G_1}\Box\, \hat{\mathbb{E}}_{G_2}[\cdot]$ and the G-expectation $\hat{\mathbb{E}}_{G_1\Box G_2}[\cdot]$. We begin with the definitions necessary for the understanding of these both expressions.
For given $0 \le \underline{\sigma}_i \le \overline{\sigma}_i \in \mathbb{R}$, i=1,2, let $G_i = G_{\underline{\sigma}_i,\overline{\sigma}_i}$ and we denote by $\hat{\mathbb{E}}_i[\cdot]$ the G_i-expectation $\hat{\mathbb{E}}_{G_i}[\cdot]$ on $(\Omega, \mathcal{H})$ $(= (C_0(\mathbb{R}^+), Lip(\mathcal{F})))$. The inf-convolution of $\hat{\mathbb{E}}_1[\cdot]$ with $\hat{\mathbb{E}}_2[\cdot]$, denoted by $\hat{\mathbb{E}}_1\Box\, \hat{\mathbb{E}}_2[\cdot]$ is defined as :

$$\hat{\mathbb{E}}_1\Box\, \hat{\mathbb{E}}_2[X] = \inf_{F\in\mathcal{H}}\{\hat{\mathbb{E}}_1[X-F] + \hat{\mathbb{E}}_2[F]\}, \quad X \in \mathcal{H}.$$

Notice that $\hat{\mathbb{E}}_1\Box\, \hat{\mathbb{E}}_2[\cdot]: \ \mathcal{H} \to \mathbb{R}\cup\{-\infty\}$.
In the same way we define

$$G_1\Box G_2(x) = \inf_{y\in\mathbb{R}}\{G_1(x-y) + G_2(y)\}, \ \ x \in \mathbb{R}.$$

Observe also that $G_1\Box G_2(\cdot): \ \mathbb{R} \to \ \mathbb{R}\cup\{-\infty\}$. It is easy to check that $G_1\Box G_2(\cdot)$ has the following form:

$$G_1\Box G_2(x) = \begin{cases} -\infty, & [\underline{\sigma}_1,\overline{\sigma}_1]\cap[\underline{\sigma}_2,\overline{\sigma}_2] = \emptyset; \\ \frac{1}{2}(\overline{\sigma}^2x^+ - \underline{\sigma}^2x^-), & [\underline{\sigma}_1,\overline{\sigma}_1]\cap[\underline{\sigma}_2,\overline{\sigma}_2] = [\underline{\sigma},\overline{\sigma}] \neq \emptyset. \end{cases}$$

If $G_1\Box G_2(\cdot) = -\infty$, then also $\hat{\mathbb{E}}_1\Box\, \hat{\mathbb{E}}_2[\cdot] = -\infty$. More precisely, we have the following proposition:

Proposition 3.1 If $[\underline{\sigma}_1,\overline{\sigma}_1]\cap[\underline{\sigma}_2,\overline{\sigma}_2] = \emptyset$, then $\hat{\mathbb{E}}_1\Box\hat{\mathbb{E}}_2[X] = -\infty$, for all $X \in \mathcal{H}$.
Proof: Without loss of generality we may suppose $\overline{\sigma}_1 < \underline{\sigma}_2$. Choosing $F = -\lambda B_t^2, \lambda > 0, t > 0$, we then have due to Proposition 2.7 that for all $X \in \mathcal{H}$,

$$\begin{aligned} &\hat{\mathbb{E}}_1[X-F] + \hat{\mathbb{E}}_2[F] \\ &= \hat{\mathbb{E}}_1[X+\lambda B_t^2] + \hat{\mathbb{E}}_2[-\lambda B_t^2] \\ &\le \hat{\mathbb{E}}_1[X] + \hat{\mathbb{E}}_1[\lambda B_t^2] + \hat{\mathbb{E}}_2[-\lambda B_t^2] \\ &\le \hat{\mathbb{E}}_1[X] + \lambda\overline{\sigma}_1^2 t - \lambda\underline{\sigma}_2^2 t. \end{aligned}$$

Letting $\lambda \to \infty$, we obtain $\hat{\mathbb{E}}_1\Box\, \hat{\mathbb{E}}_2[X] = -\infty$. ■

If $[\underline{\sigma}_1,\overline{\sigma}_1]\cap[\underline{\sigma}_2,\overline{\sigma}_2]$ is not empty we have the following theorem, which is the main result of this paper.

Theorem 3.2 Let $\hat{\mathbb{E}}_1[\cdot]$ and $\hat{\mathbb{E}}_2[\cdot]$ be the two G-expectations on the space $(\Omega, \mathcal{H})$, which have been defined above. If $G_1\Box G_2(\cdot) \neq -\infty$, then

$\hat{\mathbb{E}}_1\Box\ \hat{\mathbb{E}}_2[\cdot]$ is a G-expectation on $(\Omega,\mathcal{H})$ and has the driver $G_1\Box G_2$, i.e., $\hat{\mathbb{E}}_1\Box\ \hat{\mathbb{E}}_2[\cdot]=\hat{\mathbb{E}}_{G_1\Box G_2}[\cdot]$.

Let us first discuss Theorem 3.2 in the special case.

Lemma 3.3 Let $[\underline{\sigma}_1,\overline{\sigma}_1]\subseteq[\underline{\sigma}_2,\overline{\sigma}_2]$. Then $G_1\Box G_2(\cdot)=G_1(\cdot)$, as well as $\hat{\mathbb{E}}_1\Box\ \hat{\mathbb{E}}_2[\cdot]=\hat{\mathbb{E}}_1[\cdot]$.

Proof: We already know that $G_1\Box G_2(\cdot)=G_1(\cdot)$, so it remains only to prove that $\hat{\mathbb{E}}_1\Box\ \hat{\mathbb{E}}_2[\cdot]=\hat{\mathbb{E}}_1[\cdot]$. For this we note that, firstly, by choosing $F=0$ in the definition of $\hat{\mathbb{E}}_1\Box\ \hat{\mathbb{E}}_2$, we get $\hat{\mathbb{E}}_1\Box\ \hat{\mathbb{E}}_2\leq\hat{\mathbb{E}}_i,\ i=1,2$.
On the other hand, due to Proposition 2.8 we know that $\hat{\mathbb{E}}_1\leq\hat{\mathbb{E}}_2$. Thus, from the subadditivity of $\hat{\mathbb{E}}_1[\cdot]$,

$$\hat{\mathbb{E}}_1[X-F]+\hat{\mathbb{E}}_2[F]\geq\hat{\mathbb{E}}_1[X-F]+\hat{\mathbb{E}}_1[F]\geq\hat{\mathbb{E}}_1[X],\ \ F\in\mathcal{H}.$$

Consequently, $\hat{\mathbb{E}}_1\Box\hat{\mathbb{E}}_2[\cdot]=\hat{\mathbb{E}}_1[\cdot]$. Thus, Theorem 3.2 holds true in this special case.
The case $[\underline{\sigma}_1,\overline{\sigma}_1]\supseteq[\underline{\sigma}_2,\overline{\sigma}_2]$ can be treated analogously. ■

The situation becomes more complicate if neither $[\underline{\sigma}_1,\overline{\sigma}_1]\subseteq[\underline{\sigma}_2,\overline{\sigma}_2]$ nor $[\underline{\sigma}_2,\overline{\sigma}_2]\subseteq[\underline{\sigma}_1,\overline{\sigma}_1]$. Without loss of generality, we suppose that $[\underline{\sigma}_1,\overline{\sigma}_1]\bigcap[\underline{\sigma}_2,\overline{\sigma}_2]=[\underline{\sigma}_2,\overline{\sigma}_1]$. In this case

$$G_1\Box G_2(x)=\frac{1}{2}(\overline{\sigma}_1^2x^+-\underline{\sigma}_2^2x^-)=G_3(x),\ \ x\in\mathbb{R},$$

where $G_3=G_{\underline{\sigma}_2,\overline{\sigma}_1}$. By $\hat{\mathbb{E}}_3[\cdot]$ we denote the G-expectation on $(\Omega,\mathcal{H})$ with driver $G_3(\cdot)$. The above notations will be kept for the rest of the paper. Our aim is to prove that $\hat{\mathbb{E}}_1\Box\ \hat{\mathbb{E}}_2[\cdot]=\hat{\mathbb{E}}_3[\cdot]$.

The proof is based on Theorem 4.1.3 in Peng's paper [19]; this theorem characterizes the intrinsic properties of G-Brownian motions and G-expectations.

Lemma 3.4 (see Theorem 4.1.3, Peng [19]) Let $(\widetilde{B}_t)_{t\geq0}$ be a process defined in the sub-expectation space $(\widetilde{\Omega},\widetilde{\mathcal{H}},\widetilde{\mathbb{E}})$ such that
(i) $\widetilde{B}_0=0$;
(ii) For each $t,s\geq0$, the increment $\widetilde{B}_{t+s}-\widetilde{B}_t$ has the same distribution as $\widetilde{B}_s$ and is independent of $(\widetilde{B}_{t_1},\widetilde{B}_{t_2},...,\widetilde{B}_{t_n})$, for all $0\leq t_1,...,t_n\leq t,n\geq1$.
(iii) $\widetilde{\mathbb{E}}[\widetilde{B}_t]=\widetilde{\mathbb{E}}[-\widetilde{B}_t]=0$, and $\lim_{t\downarrow0}\widetilde{\mathbb{E}}[|\widetilde{B}_t|^3]t^{-1}=0$.
Then $(\widetilde{B}_t)_{t\geq0}$ is a $G_{\underline{\sigma},\overline{\sigma}}$-Brownian motion with $\overline{\sigma}^2=\widetilde{\mathbb{E}}[\widetilde{B}_1^2]$ and $\underline{\sigma}^2=-\widetilde{\mathbb{E}}[-\widetilde{B}_1^2]$.

In the sequel, in order to prove Theorem 3.2 we will show that the inf-convolution $\hat{\mathbb{E}}_1\Box\ \hat{\mathbb{E}}_2[\cdot]$ is a sublinear expectation on $(\Omega,\mathcal{H})$. This will make

Lemma 3.4 applicable. More precisely, we will show that under the sublinear expectation $\hat{\mathbb{E}}_1\Box\,\hat{\mathbb{E}}_2[\cdot]$ the canonical process $(B_t)_{t\geq 0}$ satisfies the assumptions of Lemma 3.4 for $\overline{\sigma} = \overline{\sigma}_1, \underline{\sigma} = \underline{\sigma}_2$. This has as consequence that $(B_t)_{t\geq 0}$ is a $G_{\underline{\sigma}_2,\overline{\sigma}_1}$-Brownian motion under $\hat{\mathbb{E}}_1\Box\,\hat{\mathbb{E}}_2[\cdot]$, and implies that $\hat{\mathbb{E}}_1\Box\,\hat{\mathbb{E}}_2[\cdot] = \hat{\mathbb{E}}_3[\cdot]$.

Proposition 3.5 Under the assumption $[\underline{\sigma}_1, \overline{\sigma}_1]\bigcap[\underline{\sigma}_2, \overline{\sigma}_2] = [\underline{\sigma}_2, \overline{\sigma}_1]$, the inf-convolution $\hat{\mathbb{E}}_1\Box\,\hat{\mathbb{E}}_2[\cdot]$ is a sublinear expectation on $(\Omega, \mathcal{H})$.
Proof: (a) Monotonicity: The monotonicity is an immediate consequence of that of the G-expectation $\hat{\mathbb{E}}_1[\cdot]$.

(b) Preservation of constants: From the preservation of constants property and the subadditivity of $\hat{\mathbb{E}}_1$, we have

$$
\begin{aligned}
&\hat{\mathbb{E}}_1\Box\,\hat{\mathbb{E}}_2[c] \\
&= \inf_{F\in\mathcal{H}}\{\hat{\mathbb{E}}_1[c-F] + \hat{\mathbb{E}}_2[F]\} \\
&= c + \inf_{F\in\mathcal{H}}\{\hat{\mathbb{E}}_1[-F] + \hat{\mathbb{E}}_2[F]\} \\
&\geq c + \inf_{F\in\mathcal{H}}\{\hat{\mathbb{E}}_3[-F] + \hat{\mathbb{E}}_3[F]\} \\
&\geq c.
\end{aligned}
$$

The latter lines follow from the fact that $\hat{\mathbb{E}}_3 \leq \hat{\mathbb{E}}_i, i = 1, 2$, and the subadditivity of $\hat{\mathbb{E}}_3$. Moreover, by taking F=0 in the definition of $\hat{\mathbb{E}}_1\Box\,\hat{\mathbb{E}}_2[c]$ we get the converse inequality.

(c) Sub-additivity: Given arbitrary fixed $X, Y \in \mathcal{H}$, in virtue of the subadditivity of $\hat{\mathbb{E}}_1[\cdot]$ and $\hat{\mathbb{E}}_2[\cdot]$, we have for all $F_1, F_2 \in \mathcal{H}$

$$
\begin{aligned}
&\hat{\mathbb{E}}_1[X - Y - F_1] + \hat{\mathbb{E}}_2[F_1] + \hat{\mathbb{E}}_1[Y - F_2] + \hat{\mathbb{E}}_2[F_2] \\
&\geq \hat{\mathbb{E}}_1[X - (F_1 + F_2)] + \hat{\mathbb{E}}_2[F_1 + F_2].
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&\hat{\mathbb{E}}_1\Box\,\hat{\mathbb{E}}_2[X - Y] + \hat{\mathbb{E}}_1\Box\,\hat{\mathbb{E}}_2[Y] \\
&= \inf_{F_1,F_2\in\mathcal{H}}\{\hat{\mathbb{E}}_1[X - Y - F_1] + \hat{\mathbb{E}}_2[F_1] + \hat{\mathbb{E}}_1[Y - F_2] + \hat{\mathbb{E}}_2[F_2]\} \\
&\geq \inf_{F_1,F_2\in\mathcal{H}}\{\hat{\mathbb{E}}_1[X - F_1 - F_2] + \hat{\mathbb{E}}_2[F_1 + F_2]\} \\
&= \hat{\mathbb{E}}_1\Box\,\hat{\mathbb{E}}_2[X].
\end{aligned}
$$

(d)Finally, the positive homogeneity is an easy consequence of that of $\hat{\mathbb{E}}_1[\cdot]$ and $\hat{\mathbb{E}}_2[\cdot]$. ■

The following series of statements has as objective to prove that the canonical process $(B_t)_{t\geq 0}$ satisfies under the sublinear expectation $\hat{\mathbb{E}}_1\Box\,\hat{\mathbb{E}}_2[\cdot]$ the assumptions of Lemma 3.4.

Lemma 3.6: Let φ be a convex or concave function such that $\varphi(B_t) \in \mathcal{H}$, then $\hat{\mathbb{E}}_1\Box\, \hat{\mathbb{E}}_2[\varphi(B_t)] = \hat{\mathbb{E}}_3[\varphi(B_t)]$.
Proof: We only prove the convex case, the proof for concave φ is analogous. If φ is convex we have according to Proposition 2.7 ,

$$\hat{\mathbb{E}}_3[\varphi(B_t)] = \mathbb{E}[\varphi(\overline{\sigma}_1 W_t)] = \hat{\mathbb{E}}_1[\varphi(B_t)].$$

By Proposition 2.8 we know that $\hat{\mathbb{E}}_i[\cdot] \geq \hat{\mathbb{E}}_3[\cdot], i = 1, 2$, and consequently, also $\hat{\mathbb{E}}_1\Box\, \hat{\mathbb{E}}_2[\cdot] \geq \hat{\mathbb{E}}_3[\cdot]$.
On the other hand, since obviously, $\hat{\mathbb{E}}_1\Box\, \hat{\mathbb{E}}_2[\cdot] \leq \hat{\mathbb{E}}_1[\cdot]$, we get, for convex functions φ, $\hat{\mathbb{E}}_1\Box\, \hat{\mathbb{E}}_2[\varphi(B_t)] = \hat{\mathbb{E}}_3[\varphi(B_t)]$. Similarly we can prove the concave case. ■

Remark: From Proposition 3.5 we know already that $\hat{\mathbb{E}}_1\Box\, \hat{\mathbb{E}}_2[\cdot]$ is a sub-linear expectation. This implies $\hat{\mathbb{E}}_1\Box\, \hat{\mathbb{E}}_2[0] = 0$. From Lemma 3.6, we have that $F^* = 0$ is an optimal control when φ is convex, while the optimal control is $F^* = \varphi(B_t)$ when φ is concave. Moreover,

$$\hat{\mathbb{E}}_1\Box\hat{\mathbb{E}}_2[-B_t] = \hat{\mathbb{E}}_1\Box\hat{\mathbb{E}}_2[B_t] = 0$$

$$\hat{\mathbb{E}}_1\Box\, \hat{\mathbb{E}}_2[B_t^2] = \overline{\sigma}_1^2 t,\ \hat{\mathbb{E}}_1\Box\, \hat{\mathbb{E}}_2[-B_t^2] = -\underline{\sigma}_2^2 t.$$

Lemma 3.7: We have $\frac{\hat{\mathbb{E}}_1\Box\, \hat{\mathbb{E}}_2[|B_t|^3]}{t} \rightarrow\ 0$, as $t \rightarrow\ 0$.
Proof: Since $\varphi(x) = |x|^3$ is convex, we obtain due to Lemma 3.6 that:

$$\hat{\mathbb{E}}_1\Box\, \hat{\mathbb{E}}_2[|B_t|^3] = \hat{\mathbb{E}}_3[|B_t|^3] = \overline{\sigma}_1^3 \mathbb{E}[|W_1|^3] t^{3/2},$$

where $(W_t)_{t\geq 0}$ is Brownian motion under the linear expectation $\mathbb{E}$. The statement follows now easily.

Proposition 3.8: We have

$$\hat{\mathbb{E}}_1\Box\, \hat{\mathbb{E}}_2[\varphi(B_t - B_s)] = \hat{\mathbb{E}}_1\Box\, \hat{\mathbb{E}}_2[\varphi(B_{t-s})],\quad t \geq s \geq 0, \varphi \in C_{l,lip}(\mathbb{R}).$$

The proof of Proposition 3.8 is rather technical. To improve the readability of the paper, the proof is postponed to the annex.

Lemma 3.9: For each $t \geq s$, $B_t - B_s$ is independent of $(B_{t_1}, B_{t_2}, ..., B_{t_n})$ under the sub-linear expectation $\hat{\mathbb{E}}_1\Box\, \hat{\mathbb{E}}_2[\cdot]$, for each $n \in \mathbb{N}, 0 \leq t_1, ..., t_n \leq s$, that is, for all $\varphi \in C_{l,lip}(\mathbb{R}^{n+1})$

$$\begin{aligned}&\hat{\mathbb{E}}_1\Box\, \hat{\mathbb{E}}_2[\varphi(B_{t_1}, B_{t_2}, ..., B_{t_n}, B_t - B_s)]\\ =\ &\hat{\mathbb{E}}_1\Box\, \hat{\mathbb{E}}_2[\hat{\mathbb{E}}_1\Box\, \hat{\mathbb{E}}_2[\varphi(x_1, ..., x_n, B_t - B_s)]|_{(x_1,...,x_n)=(B_{t_1},...,B_{t_n})}].\end{aligned}$$

We shift also the proof of Lemma 3.9 to the annex.

We are now able to give the proof of Theorem 3.2:

Proof (of Theorem 3.2): It is sufficient to apply Lemma 3.4. Due to the above statements, we know that the canonical process $(B_t)_{t\geq 0}$ is a G-Brownian motion under the sublinear expectation $\hat{\mathbb{E}}_1 \square \hat{\mathbb{E}}_2[\cdot]$. Consequently $\hat{\mathbb{E}}_1 \square \hat{\mathbb{E}}_2[\cdot]$ is a G-expectation on the space $(\Omega, \mathcal{H})$ and has the driver $G_1 \square G_2 = G_{\underline{\sigma}_2, \overline{\sigma}_1}$. ■

Given n sublinear expectations $\hat{\mathbb{E}}_1, ..., \hat{\mathbb{E}}_n$ we define iteratively

$$\hat{\mathbb{E}}_1 \square \hat{\mathbb{E}}_2 \square \hat{\mathbb{E}}_3 := (\hat{\mathbb{E}}_1 \square \hat{\mathbb{E}}_2) \square \hat{\mathbb{E}}_3,$$

and

$$\hat{\mathbb{E}}_1 \square \hat{\mathbb{E}}_2 \square ... \square \hat{\mathbb{E}}_k := (\hat{\mathbb{E}}_1 \square \hat{\mathbb{E}}_2 \square ... \square \hat{\mathbb{E}}_{k-1}) \square \hat{\mathbb{E}}_k, \ 3 \leq k \leq n.$$

Then from Theorem 3.2 it follows:

Corollary 3.10: Let $0 \leq \underline{\sigma}_i \leq \overline{\sigma}_i, \ 1 \leq i \leq n$, and denote by $\hat{\mathbb{E}}_i[\cdot]$ the $G_{\underline{\sigma}_i, \ \overline{\sigma}_i}$-expectation on the space $(\Omega, \mathcal{H})$. Then under the assumption $\bigcap_{i=1}^n [\underline{\sigma}_i, \overline{\sigma}_i] \neq \emptyset$, $\hat{\mathbb{E}}_1 \square \hat{\mathbb{E}}_2 \square ... \square \hat{\mathbb{E}}_n[\cdot]$ also is a G-expectation and has the driver $G_{\underline{\sigma}_1, \overline{\sigma}_1} \square G_{\underline{\sigma}_2, \overline{\sigma}_2} \square ... \square G_{\underline{\sigma}_n, \overline{\sigma}_n}$. Moreover, for any permutation $i_1, ..., i_n$ of the natural numbers 1,...,n it holds:

$$\hat{\mathbb{E}}_1 \square \hat{\mathbb{E}}_2 \square ... \square \hat{\mathbb{E}}_n[\cdot] = \hat{\mathbb{E}}_{i_1} \square \hat{\mathbb{E}}_{i_2} \square ... \square \hat{\mathbb{E}}_{i_n}[\cdot].$$

Remark: If $\bigcap_{i=1}^n [\underline{\sigma}_i, \overline{\sigma}_i]$ is empty, then $\hat{\mathbb{E}}_1 \square \hat{\mathbb{E}}_2 \square ... \square \hat{\mathbb{E}}_n[\cdot] = -\infty$, otherwise $\hat{\mathbb{E}}_1 \square \hat{\mathbb{E}}_2 \square ... \square \hat{\mathbb{E}}_n[\cdot]$ is a $G_{\underline{\sigma}, \overline{\sigma}}$-expectation, where $[\underline{\sigma}, \overline{\sigma}] = \bigcap_{i=1}^n [\underline{\sigma}_i, \overline{\sigma}_i]$.

4 Annex

4.1 Proof of Proposition 3.8

We begin with the proof of Proposition 3.8. For this we need the following two lemmas.

Lemma 4.1: For all $T > 0$ and all $X \in \mathcal{H}_T$, we have

$$\inf_{F \in \mathcal{H}_T} \{\hat{\mathbb{E}}_1[X - F] + \hat{\mathbb{E}}_2[F]\} = \inf_{F \in \mathcal{H}} \{\hat{\mathbb{E}}_1[X - F] + \hat{\mathbb{E}}_2[F]\}.$$

Proof: From $\mathcal{H}_T \subseteq \mathcal{H}$ we see that

$$\inf_{F\in\mathcal{H}_T}\{\hat{\mathbb{E}}_1[X-F]+\hat{\mathbb{E}}_2[F]\} \geq \inf_{F\in\mathcal{H}}\{\hat{\mathbb{E}}_1[X-F]+\hat{\mathbb{E}}_2[F]\}.$$

Thus it remains to prove the converse inequality.
First we notice that, due to Proposition 2.8 and the subadditivity of $\hat{\mathbb{E}}_3$, for any $F \in \mathcal{H}$,

$$\hat{\mathbb{E}}_2[F|\mathcal{H}_T]+\hat{\mathbb{E}}_1[-F|\mathcal{H}_T] \geq \hat{\mathbb{E}}_3[F|\mathcal{H}_T]+\hat{\mathbb{E}}_3[-F|\mathcal{H}_T] \geq 0.$$

Consequently, for all $X \in \mathcal{H}_T$ and all $F \in \mathcal{H}$,

$$\begin{aligned}
&\hat{\mathbb{E}}_1[X-F]+\hat{\mathbb{E}}_2[F] \\
&= \hat{\mathbb{E}}_1[\hat{\mathbb{E}}_1[X-F|\mathcal{H}_T]]+\hat{\mathbb{E}}_2[F] \\
&= \hat{\mathbb{E}}_1[X+\hat{\mathbb{E}}_1[-F|\mathcal{H}_T]]+\hat{\mathbb{E}}_2[F] \\
&= \hat{\mathbb{E}}_1[X-(-\hat{\mathbb{E}}_1[-F|\mathcal{H}_T])]+\hat{\mathbb{E}}_2[-\hat{\mathbb{E}}_1[-F|\mathcal{H}_T]] \\
&\quad -\hat{\mathbb{E}}_2[-\hat{\mathbb{E}}_1[-F|\mathcal{H}_T]]+\hat{\mathbb{E}}_2[\hat{\mathbb{E}}_2[F|\mathcal{H}_T]] \\
&\geq \hat{\mathbb{E}}_1[X-(-\hat{\mathbb{E}}_1[-F|\mathcal{H}_T])]+\hat{\mathbb{E}}_2[-\hat{\mathbb{E}}_1[-F|\mathcal{H}_T]] \\
&\geq \inf_{F\in\mathcal{H}_T}\{\hat{\mathbb{E}}_1[X-F]+\hat{\mathbb{E}}_2[F]\}.
\end{aligned}$$

The statement now follows easily. ■

Lemma 4.2: For all $X \in \mathcal{H}_t^s, 0 \leq s \leq t$, the following holds true:

$$\inf_{F\in\mathcal{H}_t}\{\hat{\mathbb{E}}_1[X-F]+\hat{\mathbb{E}}_2[F]\} = \inf_{F\in\mathcal{H}_t^s}\{\hat{\mathbb{E}}_1[X-F]+\hat{\mathbb{E}}_2[F]\}.$$

Proof: Firstly, from $\mathcal{H}_t^s \subseteq \mathcal{H}_t$, we have, obviously, for all $X \in \mathcal{H}_t^s$,

$$\inf_{F\in\mathcal{H}_t}\{\hat{\mathbb{E}}_1[X-F]+\hat{\mathbb{E}}_2[F]\} \leq \inf_{F\in\mathcal{H}_t^s}\{\hat{\mathbb{E}}_1[X-F]+\hat{\mathbb{E}}_2[F]\}.$$

Secondly, for any $X \in \mathcal{H}_t^s$ and $F \in \mathcal{H}_t$, we can suppose without loss of generality that $X = \varphi(B_{t_1}-B_s,...,B_{t_n}-B_s)$ and $F = \psi(B_{t_1'},B_{t_2'},...,B_{t_k'},B_{t_1}-B_s,...,B_{t_n}-B_s)$, where $t_1',...,t_k' \in [0,s]$, $t_1,...,t_n \in [s,t], n,k \in \mathbb{N}, \varphi \in C_{l,lip}(\mathbb{R}^n)$ and $\psi \in C_{l,lip}(\mathbb{R}^{n+k})$.
To simplify the notation we put:

$$Y_1 = (B_{t_1'},B_{t_2'},...,B_{t_k'}), Y_2 = (B_{t_1}-B_s,...,B_{t_n}-B_s), \mathbf{x} = (x_1,x_2,...,x_k).$$

Then,

$$
\begin{aligned}
&\hat{\mathbb{E}}_1[X-F]+\hat{\mathbb{E}}_2[F] \\
&=\hat{\mathbb{E}}_1[\hat{\mathbb{E}}_1[\varphi(Y_2)-\psi(Y_1,Y_2)|\mathcal{H}_s]]+\hat{\mathbb{E}}_2[F] \\
&=\hat{\mathbb{E}}_1[\hat{\mathbb{E}}_1[\varphi(Y_2)-\psi(\mathbf{x},Y_2)]|_{\mathbf{x}=Y_1}]+\hat{\mathbb{E}}_2[F] \\
&=\hat{\mathbb{E}}_1[(\hat{\mathbb{E}}_1[\varphi(Y_2)-\psi(\mathbf{x},Y_2)]+\hat{\mathbb{E}}_2[\psi(\mathbf{x},Y_2)]-\hat{\mathbb{E}}_2[\psi(\mathbf{x},Y_2)])|_{\mathbf{x}=Y_1}]+\hat{\mathbb{E}}_2[F] \\
&\geq \hat{\mathbb{E}}_1[\inf_{F\in\mathcal{H}_t^s}\{\hat{\mathbb{E}}_1[X-F]+\hat{\mathbb{E}}_2[F]\}-\hat{\mathbb{E}}_2[\psi(\mathbf{x},Y_2)]|_{\mathbf{x}=Y_1}]+\hat{\mathbb{E}}_2[F] \\
&=\inf_{F\in\mathcal{H}_t^s}\{\hat{\mathbb{E}}_1[X-F]+\hat{\mathbb{E}}_2[F]\}+\hat{\mathbb{E}}_1[-\hat{\mathbb{E}}_2[\psi(\mathbf{x},Y_2)]|_{\mathbf{x}=Y_1}] \\
&\quad +\hat{\mathbb{E}}_2[\hat{\mathbb{E}}_2[\psi(\mathbf{x},Y_2)]|_{\mathbf{x}=Y_1}] \\
&\geq \inf_{F\in\mathcal{H}_t^s}\{\hat{\mathbb{E}}_1[X-F]+\hat{\mathbb{E}}_2[F]\}.
\end{aligned}
$$

Thus the proof is complete now. ■

Now we are able to prove Proposition 3.8.

Proof (of Proposition 3.8): For arbitrarily fixed $s \geq 0$, we put $\widetilde{B}_t = B_{t+s} - B_s,\ t \geq 0$. Then, obviously, $\mathcal{H}_{t+s}^s = \widetilde{\mathcal{H}}_t,\ t \geq 0$, where $\widetilde{\mathcal{H}}_t$ is generated by $\widetilde{B}_t$. Moreover, $\widetilde{B}_t$ is a G-Brownian Motion under $\hat{\mathbb{E}}_1$ and $\hat{\mathbb{E}}_2$.
According to the Lemmas 4.1 and 4.2, we have the following:

$$
\begin{aligned}
&\hat{\mathbb{E}}_1\Box\,\hat{\mathbb{E}}_2[\varphi(B_t-B_s)] \\
&=\inf_{F\in\mathcal{H}_t^s}\{\hat{\mathbb{E}}_1[\varphi(B_t-B_s)-F]+\hat{\mathbb{E}}_2[F]\} \\
&=\inf_{F\in\widetilde{\mathcal{H}}_{t-s}}\{\hat{\mathbb{E}}_1[\varphi(\tilde{B}_{t-s})-F]+\hat{\mathbb{E}}_2[F]\} \\
&=\inf_{F\in\mathcal{H}_{t-s}}\{\hat{\mathbb{E}}_1[\varphi(B_{t-s})-F]+\hat{\mathbb{E}}_2[F]\} \\
&=\hat{\mathbb{E}}_1\Box\,\hat{\mathbb{E}}_2[\varphi(B_{t-s})].
\end{aligned}
$$

Thus the proof of Proposition 3.8 is complete now. ■

4.2 Proof of Lemma 3.9

Let us come now to the proof of Lemma 3.9, which we split into a sequel of lemmas.

Lemma 4.3: For all $\varphi \in C_{l,lip}(\mathbb{R}^{n+1}), n \in \mathbb{N}$ and $0 \leq t_1, ..., t_n \leq s \leq t$, it holds:

$$
\begin{aligned}
&\hat{\mathbb{E}}_1\Box\,\hat{\mathbb{E}}_2[\varphi(B_{t_1},B_{t_2},...,B_{t_n},B_t-B_s)] \\
&\geq \hat{\mathbb{E}}_1\Box\,\hat{\mathbb{E}}_2[\hat{\mathbb{E}}_1\Box\,\hat{\mathbb{E}}_2[\varphi(x_1,...,x_n,B_t-B_s)]|_{(x_1,...,x_n)=(B_{t_1},...,B_{t_n})}].
\end{aligned}
$$

Proof: Let $X = \varphi(B_{t_1}, B_{t_2}, ..., B_{t_n}, B_t - B_s)$. Without loss of generality we can suppose that $F \in \mathcal{H}$ has the form $\psi(B_{t'_1}, B_{t'_2}, ..., B_{t'_k}, B_{t'_{k+1}} - B_s, ..., B_{t'_m} - B_s)$, where $0 \leq t_1, ..., t_n, t'_1, ..., t'_k \leq s,\ t'_{k+1}, ..., t'_m \geq s, m \geq k, m, k \in \mathbb{N}$, and $\varphi \in C_{l,lip}(\mathbb{R}^{n+1}), \psi \in C_{l,lip}(\mathbb{R}^m)$.

For simplifying the notation we put:

$$\mathbf{x}_1 = (x_1, ..., x_n), \mathbf{x}_2 = (x'_1, ..., x'_k), Y_1 = (B_{t_1}, B_{t_2}, ..., B_{t_n}),$$
$$Y_2 = (B_{t'_1}, ..., B_{t'_k}), Y_3 = (B_{t'_{k+1}} - B_s, ..., B_{t'_m} - B_s).$$

Then

$$\begin{aligned}
&\hat{\mathbb{E}}_1[X - F] + \hat{\mathbb{E}}_2[F] = \hat{\mathbb{E}}_1[\hat{\mathbb{E}}_1[X - F|\mathcal{H}_s]] + \hat{\mathbb{E}}_2[\hat{\mathbb{E}}_2[F|\mathcal{H}_s]] \\
&= \hat{\mathbb{E}}_1[\hat{\mathbb{E}}_1[\varphi(\mathbf{x}_1, B_t - B_s) - \psi(\mathbf{x}_2, Y_3)]|_{\mathbf{x}_1 = Y_1, \mathbf{x}_2 = Y_2}] \\
&\quad + \hat{\mathbb{E}}_2[\hat{\mathbb{E}}_2[\psi(\mathbf{x}_2, Y_3)]|_{\mathbf{x}_2 = Y_2}] \\
&= \hat{\mathbb{E}}_1[(\hat{\mathbb{E}}_1[\varphi(\mathbf{x}_1, B_t - B_s) - \psi(\mathbf{x}_2, Y_3)] + \hat{\mathbb{E}}_2[\psi(\mathbf{x}_2, Y_3)] \\
&\quad - \hat{\mathbb{E}}_2[\psi(\mathbf{x}_2, Y_3)])|_{\mathbf{x}_1 = Y_1, \mathbf{x}_2 = Y_2}] + \hat{\mathbb{E}}_2[\hat{\mathbb{E}}_2[\psi(\mathbf{x}_2, Y_3)]|_{\mathbf{x}_2 = Y_2}] \\
&\geq \hat{\mathbb{E}}_1[(\hat{\mathbb{E}}_1 \Box\, \hat{\mathbb{E}}_2[\varphi(\mathbf{x}_1, B_t - B_s)] - \hat{\mathbb{E}}_2[\psi(\mathbf{x}_2, Y_3)])|_{\mathbf{x}_1 = Y_1, \mathbf{x}_2 = Y_2}] \\
&\quad + \hat{\mathbb{E}}_2[\hat{\mathbb{E}}_2[\psi(\mathbf{x}_2, Y_3)]|_{\mathbf{x}_2 = Y_2}] \\
&\geq \hat{\mathbb{E}}_1 \Box\, \hat{\mathbb{E}}_2[\hat{\mathbb{E}}_1 \Box\, \hat{\mathbb{E}}_2[\varphi(\mathbf{x}_1, B_t - B_s)]|_{\mathbf{x}_1 = Y_1}] \\
&= \hat{\mathbb{E}}_1 \Box\, \hat{\mathbb{E}}_2[\hat{\mathbb{E}}_1 \Box\, \hat{\mathbb{E}}_2[\varphi(x_1, ..., x_n, B_t - B_s)]|_{(x_1, ..., x_n) = (B_{t_1}, ..., B_{t_n})}].
\end{aligned}$$

Hence, we get

$$\begin{aligned}
&\hat{\mathbb{E}}_1 \Box\, \hat{\mathbb{E}}_2[\varphi(B_{t_1}, B_{t_2}, ..., B_{t_n}, B_t - B_s)] \\
&\geq \hat{\mathbb{E}}_1 \Box\, \hat{\mathbb{E}}_2[\hat{\mathbb{E}}_1 \Box\, \hat{\mathbb{E}}_2[\varphi(x_1, ..., x_n, B_t - B_s)]|_{(x_1, ..., x_n) = (B_{t_1}, ..., B_{t_n})}].
\end{aligned}$$

The proof of the Lemma 4.3 is complete now. ■

Let $Lip(\mathbb{R}^n), n \in \mathbb{N}$, denote the space of bounded Lipschitz functions $\varphi \in Lip(\mathbb{R}^n)$ satisfying:

$$|\varphi(x) - \varphi(y)| \leq C|x - y| \qquad x, y \in \mathbb{R}^n,$$

where C is a constant only depending on φ.

The proof that

$$\begin{aligned}
&\hat{\mathbb{E}}_1 \Box\, \hat{\mathbb{E}}_2[\varphi(B_{t_1}, B_{t_2}, ..., B_{t_n}, B_t - B_s)] \\
&\leq \hat{\mathbb{E}}_1 \Box\, \hat{\mathbb{E}}_2[\hat{\mathbb{E}}_1 \Box\, \hat{\mathbb{E}}_2[\varphi(x_1, ..., x_n, B_t - B_s)]|_{(x_1, ..., x_n) = (B_{t_1}, ..., B_{t_n})}]
\end{aligned}$$

is much more difficult than that of the converse inequality. For the proof we need the following statements.

Lemma 4.4: We assume that the random variable $\varphi(B_{t_1}, B_{t_2}-B_{t_1}, ..., B_{t_n}-B_{t_{n-1}})$, with $t_i \leq t_{i+1},\ i=1,...,n-1,\ n\in\mathbb{N}$ and $\varphi \in Lip(\mathbb{R}^n)$, satisfies the following assumption: there exist $L, M \geq 0$ s.t. $|\varphi| \leq L$, and $\varphi(x,y)=0$, for all $(x,y)\in[-M,M]^c\times\mathbb{R}^{n-1}$.
We define

$$\begin{aligned}\phi(x) :&= \hat{\mathbb{E}}_1 \square \hat{\mathbb{E}}_2[\varphi(x, B_{t_2}-B_{t_1}, ..., B_{t_n}-B_{t_{n-1}})] \\ &= \inf_{F\in\mathcal{H}^{t_1}_{t_n}} \{\hat{\mathbb{E}}_1[\varphi(x, B_{t_2}-B_{t_1}, ..., B_{t_n}-B_{t_{n-1}})-F]+\hat{\mathbb{E}}_2[F]\}.\end{aligned}$$

Then we have the existence of an ε-optimal $\widetilde{\psi}(x)$ of the form $\psi(x, B_{t'_2}-B_{t_1}, ..., B_{t'_{l+1}}-B_{t_1})$, i.e., for any $\varepsilon>0$ we can find a finite dimensional function $\psi(x,\cdot)\in C_{l,lip}(\mathbb{R}^l), l\geq 1$, such that, for suitable $t'_2, ..., t'_{l+1}\geq t_1$,

$$\begin{aligned}\widetilde{\psi}(x) := \hat{\mathbb{E}}_1[&\varphi(x, B_{t_2}-B_{t_1}, ..., B_{t_n}-B_{t_{n-1}}) \\ &-\psi(x, B_{t'_2}-B_{t_1}, ..., B_{t'_{l+1}}-B_{t_1})] \\ &+\hat{\mathbb{E}}_2[\psi(x, B_{t'_2}-B_{t_1}, ..., B_{t'_{l+1}}-B_{t_1})]\end{aligned}$$

satisfies

$$|\widetilde{\psi}(x)-\phi(x)|\leq\varepsilon.$$

Proof: Since $\varphi\in Lip(\mathbb{R}^n)$, we find for any $\varepsilon>0$ some sufficiently large $J\geq 1$ s.t. for all $x,\tilde{x}\in\mathbb{R}$ with $|x-\widetilde{x}|\leq\frac{2M}{J}$ it holds $|\varphi(x,y)-\varphi(\widetilde{x},y)|\leq\varepsilon/6$. We then let $-M=x_0\leq x_1\leq ....\leq x_J=M$, be such that $|x_{j+1}-x_j|=\frac{2M}{J},\ 0\leq j\leq J-1$.
On the other hand, for every fixed j there are some $m_j\geq 1,\ t_{i,j}\geq t_1$ $(2\leq i\leq m_j)$ and $\psi^{x_j}\in C_{l,lip}(\mathbb{R}^{m_j-1})$, such that

$$\begin{aligned}\phi(x_j)\leq \hat{\mathbb{E}}_1[&\varphi(x_j, B_{t_2}-B_{t_1}, ..., B_{t_n}-B_{t_{n-1}}) \\ &-\psi^{x_j}(B_{t_{2,j}}-B_{t_1}, ..., B_{t_{m_j,j}}-B_{t_1})] \\ &+\hat{\mathbb{E}}_2[\psi^{x_j}(B_{t_{2,j}}-B_{t_1}, ..., B_{t_{m_j,j}}-B_{t_1})] \\ &\leq \phi(x_j)+\varepsilon/6.\end{aligned}$$

Since there are only a finite number of j we can find a finite dimensional function denoted by $\psi(x_j,y),\ y\in\mathbb{R}^l$, s.t. for each fixed j, $\psi(x_j,\cdot)\in C_{l,lip}(\mathbb{R}^l)$ and

$$\psi(x_j, B_{t'_2}-B_{t_1}, ..., B_{t'_{l+1}}-B_{t_1}) = \psi^{x_j}(B_{t_{2,j}}-B_{t_1}, ..., B_{t_{m_j,j}}-B_{t_1}),$$

where $\{t'_2, ..., t'_{l+1}\}=\bigcup_{j=1}^J\{t_{2,j}, ..., t_{m_j,j}\}$.

With the convention $\psi(x_0, y) = \psi(x_J, y) = 0, y \in \mathbb{R}^l$, we define

$$\psi(x,y) := \begin{cases} \frac{x_{j+1}-x}{x_{j+1}-x_j}\psi(x_j,y) + \frac{x-x_j}{x_{j+1}-x_j}\psi(x_{j+1},y), & x \in [x_j, x_{j+1}], \\ & 0 \le j \le J-1, \\ 0, & \text{otherwise.} \end{cases}$$

Obviously, $\psi(x,y) \in C_{l,lip}(\mathbb{R}^{l+1})$.

We now introduce $\widetilde{\psi}(x)$:

$$\begin{aligned} &= \hat{\mathbb{E}}_1[\varphi(x, B_{t_2} - B_{t_1}, ..., B_{t_n} - B_{t_{n-1}}) - \psi(x, B_{t'_2} - B_{t_1}, ..., B_{t'_{l+1}} - B_{t_1})] \\ &\qquad + \hat{\mathbb{E}}_2[\psi(x, B_{t'_2} - B_{t_1}, ..., B_{t'_{l+1}} - B_{t_1})]. \end{aligned}$$

If $x \notin [-M, M]$, $\varphi(x,\cdot) = 0$ and $\psi(x,\cdot) = 0$. Consequently, $\widetilde{\psi}(x) = 0$. Moreover, from Proposition 3.5 we have that for $x \notin [-M,M]$ also $\phi(x) = 0$. Then $\widetilde{\psi}(x) = \phi(x) = 0$ when $x \notin [-M,M]$, and we have also $|\widetilde{\psi}(x_j) - \phi(x_j)| \le \varepsilon/6$ for each j. We also recall that, for all $0 \le j \le J-1$ and all $x \in [x_j, x_{j+1}]$,

$$|\varphi(x,y) - \varphi(x_j,y)| \le \varepsilon/6, \text{ for all } y \in \mathbb{R}^{n-1}.$$

Our objective is to estimate

$$|\widetilde{\psi}(x) - \phi(x)| \le |\widetilde{\psi}(x) - \phi(x_j)| + |\phi(x_j) - \phi(x)|.$$

For this end we notice that, with the notation:

$$Y_1 = (B_{t_2} - B_{t_1}, ..., B_{t_n} - B_{t_{n-1}}), Y_2 = (B_{t'_2} - B_{t_1}, ..., B_{t'_{l+1}} - B_{t_1}),$$

we have from the definition of $\phi(x)$ and $\phi(x_j)$ and from the properties of $\hat{\mathbb{E}}_1 \Box \hat{\mathbb{E}}_2$ as sublinear expectation:

$$|\phi(x) - \phi(x_j)| \le \hat{\mathbb{E}}_1 \Box \hat{\mathbb{E}}_2[|\varphi(x,Y_1) - \varphi(x_j,Y_1)|] \le \varepsilon/6.$$

On the other hand, since $|\varphi(x,Y_1) - \varphi(x_j,Y_1)| \le \varepsilon/6$,

$$\begin{aligned} &|\widetilde{\psi}(x) - \phi(x_j)| \\ &= |\hat{\mathbb{E}}_1[\varphi(x,Y_1) - \psi(x,Y_2)] + \hat{\mathbb{E}}_2[\psi(x,Y_2)] - \phi(x_j)| \\ &\le | \hat{\mathbb{E}}_1[\varphi(x_j,Y_1) - \psi(x,Y_2)] + \hat{\mathbb{E}}_2[\psi(x,Y_2)] - \phi(x_j) \mid + \varepsilon/6. \end{aligned}$$

Due to the definition of $\phi(x_j)$, the latter expression without module is non-negative. Thus,

$$
\begin{aligned}
&|\widetilde{\psi}(x)-\phi(x_j)| \\
&\leq \hat{\mathbb{E}}_1[\varphi(x_j,Y_1)-\psi(x,Y_2)]+\hat{\mathbb{E}}_2[\psi(x,Y_2)]-\phi(x_j)+\varepsilon/6 \\
&\leq \hat{\mathbb{E}}_1[\frac{x_{j+1}-x}{x_{j+1}-x_j}(\varphi(x_j,Y_1)-\psi(x_j,Y_2))+\frac{x-x_j}{x_{j+1}-x_j}(\varphi(x_{j+1},Y_1) \\
&-\psi(x_{j+1},Y_2))]+\hat{\mathbb{E}}_2[\frac{x_{j+1}-x}{x_{j+1}-x_j}\psi(x_j,Y_2)+\frac{x-x_j}{x_{j+1}-x_j}\psi(x_{j+1},Y_2)] \\
&-\phi(x_j)+2\varepsilon/6 \\
&\leq \frac{x_{j+1}-x}{x_{j+1}-x_j}\{\hat{\mathbb{E}}_1[\varphi(x_j,Y_1)-\psi(x_j,Y_2)]+\hat{\mathbb{E}}_2[\psi(x_j,Y_2)]-\phi(x_j)\} \\
&+\frac{x-x_j}{x_{j+1}-x_j}\{\hat{\mathbb{E}}_1[\varphi(x_{j+1},Y_1)-\psi(x_{j+1},Y_2)]+\hat{\mathbb{E}}_2[\psi(x_{j+1},Y_2)]-\phi(x_j)\} \\
&+2\varepsilon/6.
\end{aligned}
$$

Hence, due to the choice of ψ^{x_j} and $\psi^{x_{j+1}}$,

$$|\widetilde{\psi}(x)-\phi(x_j)|\leq 5\varepsilon/6.$$

This latter estimate combined with the fact that for $|\phi(x)-\phi(x_j)|\leq\varepsilon/6$ then yields

$$|\widetilde{\psi}(x)-\phi(x)|\leq\varepsilon.$$

The proof of Lemma 4.4 is complete now. ■

Lemma 4.4 allows to prove the following:

Lemma 4.5: Let $\varphi\in\ Lip(\mathbb{R}^n)$ be bounded and such that, for some real $M>0$, $\text{supp}(\varphi)\subset[-M,M]\times\mathbb{R}^{n-1}$. Then, for all $0\leq t_1\leq t_2...\leq t_n$,

$$
\begin{aligned}
&\hat{\mathbb{E}}_1\square\hat{\mathbb{E}}_2[\varphi(B_{t_1},B_{t_2}-B_{t_1},...,B_{t_n}-B_{t_{n-1}})] \\
&=\hat{\mathbb{E}}_1\square\hat{\mathbb{E}}_2[\hat{\mathbb{E}}_1\square\hat{\mathbb{E}}_2[\varphi(x,B_{t_2}-B_{t_1},...,B_{t_n}-B_{t_{n-1}})]|_{x=B_{t_1}}].
\end{aligned}
$$

Proof: Firstly, it follows directly from Lemma 4.3 that:

$$
\begin{aligned}
&\hat{\mathbb{E}}_1\square\hat{\mathbb{E}}_2[\varphi(B_{t_1},B_{t_2}-B_{t_1},...,B_{t_n}-B_{t_{n-1}})] \\
&\geq\hat{\mathbb{E}}_1\square\hat{\mathbb{E}}_2[\hat{\mathbb{E}}_1\square\hat{\mathbb{E}}_2[\varphi(x,B_{t_2}-B_{t_1},...,B_{t_n}-B_{t_{n-1}})]|_{x=B_{t_1}}]. \quad (1)
\end{aligned}
$$

Secondly, from Lemma 4.4 we know that for any $\varepsilon>0$ there is some $\psi\in C_{l,lip}(\mathbb{R}^{l+1})$ such that $|\widetilde{\psi}(x)-\phi(x)|\leq\varepsilon$, for all $x\in\mathbb{R}$, where $\widetilde{\psi}(x)$ and $\phi(x)$ have been introduced in Lemma 4.4 .
Due to Lemma 4.1, there is $\widetilde{\phi}(B_{t''_1},....,B_{t''_k})\in\mathcal{H}_{t_1},0\leq t''_1,...,t''_k\leq t_1,k\in\mathbb{N}$, such that

$$|\hat{\mathbb{E}}_1[\phi(B_{t_1})-\widetilde{\phi}(B_{t''_1},....,B_{t''_k})]+\hat{\mathbb{E}}_2[\widetilde{\phi}(B_{t''_1},....,B_{t''_k})]-\hat{\mathbb{E}}_1\square\hat{\mathbb{E}}_2[\phi(B_{t_1})]|\leq\varepsilon.$$

For $t_2', ..., t_{l+1}' \geq t_1$ from the definition of $\widetilde{\psi}(x)$ in Lemma 4.4 we put

$$\psi'(x) = \hat{\mathbb{E}}_2[\psi(x, B_{t_2'} - B_{t_1}, ..., B_{t_{l+1}'} - B_{t_1})]$$

and

$$F = \psi(B_{t_1}, B_{t_2'} - B_{t_1}, ..., B_{t_{l+1}'} - B_{t_1}) + \widetilde{\phi}(B_{t_1''}, ..., B_{t_k''}) - \psi'(B_{t_1}).$$

Notice that

$$\hat{\mathbb{E}}_2[F|\mathcal{H}_{t_1}] = \widetilde{\phi}(B_{t_1''}, ..., B_{t_k''})$$

and

$$\hat{\mathbb{E}}_1[\varphi(B_{t_1}, B_{t_2} - B_{t_1}, ..., B_{t_n} - B_{t_{n-1}}) - F|\mathcal{H}_{t_1}] = \widetilde{\psi}(B_{t_1}) - \widetilde{\phi}(B_{t_1''}, ..., B_{t_k''}).$$

Then, due to the choice of $\widetilde{\phi}(B_{t_1''}, ..., B_{t_k''})$,

$$\begin{aligned}
&\hat{\mathbb{E}}_1 \Box \hat{\mathbb{E}}_2[\varphi(B_{t_1}, B_{t_2} - B_{t_1}, ..., B_{t_n} - B_{t_{n-1}})] - \hat{\mathbb{E}}_1 \Box \hat{\mathbb{E}}_2[\phi(B_{t_1})] \\
&\leq \hat{\mathbb{E}}_1[\varphi(B_{t_1}, B_{t_2} - B_{t_1}, ..., B_{t_n} - B_{t_{n-1}}) - F] + \hat{\mathbb{E}}_2[\hat{\mathbb{E}}_2[F|\mathcal{H}_{t_1}]] \\
&\quad -(\hat{\mathbb{E}}_1[\phi(B_{t_1}) - \widetilde{\phi}(B_{t_1''}, ..., B_{t_k''})] + \hat{\mathbb{E}}_2[\widetilde{\phi}(B_{t_1''}, ..., B_{t_k''})]) + \varepsilon \\
&= \hat{\mathbb{E}}_1[\varphi(B_{t_1}, B_{t_2} - B_{t_1}, ..., B_{t_n} - B_{t_{n-1}}) - F] \\
&\quad -\hat{\mathbb{E}}_1[\phi(B_{t_1}) - \widetilde{\phi}(B_{t_1''}, ..., B_{t_k''})] + \varepsilon \\
&= \hat{\mathbb{E}}_1[\hat{\mathbb{E}}_1[\varphi(B_{t_1}, B_{t_2} - B_{t_1}, ..., B_{t_n} - B_{t_{n-1}}) - F|\mathcal{H}_{t_1}]] \\
&\quad -\hat{\mathbb{E}}_1[\phi(B_{t_1}) - \widetilde{\phi}(B_{t_1''}, ..., B_{t_k''})] + \varepsilon \\
&= \hat{\mathbb{E}}_1[\widetilde{\psi}(B_{t_1}) - \widetilde{\phi}(B_{t_1''}, ..., B_{t_k''})] - \hat{\mathbb{E}}_1[\phi(B_{t_1}) - \widetilde{\phi}(B_{t_1''}, ..., B_{t_k''})] + \varepsilon \\
&\leq \hat{\mathbb{E}}_1[|\phi(B_{t_1}) - \widetilde{\psi}(B_{t_1})|] + \varepsilon \\
&\leq 2\varepsilon.
\end{aligned}$$

From the definition of ϕ in Lemma 4.4 and the arbitrariness of $\varepsilon > 0$ it follows then that

$$\begin{aligned}
&\hat{\mathbb{E}}_1 \Box \hat{\mathbb{E}}_2[\varphi(B_{t_1}, B_{t_2} - B_{t_1}, ..., B_{t_n} - B_{t_{n-1}})] \\
&\leq \hat{\mathbb{E}}_1 \Box \hat{\mathbb{E}}_2[\hat{\mathbb{E}}_1 \Box \hat{\mathbb{E}}_2[\varphi(x, B_{t_2} - B_{t_1}, ..., B_{t_n} - B_{t_{n-1}})]|_{x=B_{t_1}}].
\end{aligned}$$

This together with (1) yields the wished statement. The proof of Lemma 4.5 is complete now. ■

In the next statement we extend Lemma 4.5 to general functions of $Lip(\mathbb{R}^n)$.

Lemma 4.6: Let $\varphi \in Lip(\mathbb{R}^n)$, $n \geq 1$, and $t_n \geq t_{n-1} \geq ... \geq t_1 \geq 0$. Then

$$\begin{aligned}
&\hat{\mathbb{E}}_1 \Box \hat{\mathbb{E}}_2[\varphi(B_{t_1}, B_{t_2} - B_{t_1}, ..., B_{t_n} - B_{t_{n-1}})] \\
&= \hat{\mathbb{E}}_1 \Box \hat{\mathbb{E}}_2[\hat{\mathbb{E}}_1 \Box \hat{\mathbb{E}}_2[\varphi(x, B_{t_2} - B_{t_1}, ..., B_{t_n} - B_{t_{n-1}})]|_{x=B_{t_1}}].
\end{aligned}$$

Proof: Let $L > 0$ be such that $|\varphi| \leq L$. Given an arbitrarily large $M > 0$ we define, for all $y \in \mathbb{R}^{n-1}$,

$$\widetilde{\varphi}(x, y) := \begin{cases} \varphi(x, y), & x \in [-M, M] \\ \varphi(-M, y)(M + 1 + x), & x \in [-M - 1, -M] \\ \varphi(M, y)(M + 1 - x), & x \in [M, M + 1] \\ 0, & \text{otherwise.} \end{cases}$$

Obviously, $\widetilde{\varphi}$ satisfies the assumptions of Lemma 4.5.
Letting

$$\widetilde{\varphi}'(x) = \hat{\mathbb{E}}_1 \square \hat{\mathbb{E}}_2[\widetilde{\varphi}(x, B_{t_2} - B_{t_1}, ..., B_{t_n} - B_{t_{n-1}})]$$

and

$$\phi(x) = \hat{\mathbb{E}}_1 \square \hat{\mathbb{E}}_2[\varphi(x, B_{t_2} - B_{t_1}, ..., B_{t_n} - B_{t_{n-1}})],$$

we have

$$\begin{aligned} & |\phi(x) - \widetilde{\varphi}'(x)| \\ = & |\hat{\mathbb{E}}_1 \square \hat{\mathbb{E}}_2[\varphi(x, B_{t_2} - B_{t_1}, ..., B_{t_n} - B_{t_{n-1}})] \\ & -\hat{\mathbb{E}}_1 \square \hat{\mathbb{E}}_2[\widetilde{\varphi}(x, B_{t_2} - B_{t_1}, ..., B_{t_n} - B_{t_{n-1}})]| \\ \leq & \hat{\mathbb{E}}_1 \square \hat{\mathbb{E}}_2[|\varphi(x, B_{t_2} - B_{t_1}, ..., B_{t_n} - B_{t_{n-1}}) \\ & -\widetilde{\varphi}(x, B_{t_2} - B_{t_1}, ..., B_{t_n} - B_{t_{n-1}})|] \\ \leq & \frac{2L}{M}|x|. \end{aligned}$$

Consequently,

$$\begin{aligned} & |\hat{\mathbb{E}}_1 \square \hat{\mathbb{E}}_2[\phi(B_{t_1})] - \hat{\mathbb{E}}_1 \square \hat{\mathbb{E}}_2[\widetilde{\varphi}'(B_{t_1})]| \leq \hat{\mathbb{E}}_1 \square \hat{\mathbb{E}}_2[|\phi(B_{t_1}) - \widetilde{\varphi}'(B_{t_1})|] \\ & \leq \hat{\mathbb{E}}_1 \square \hat{\mathbb{E}}_2[\frac{2L}{M}|B_{t_1}|] = \frac{2L}{M}\hat{\mathbb{E}}_1 \square \hat{\mathbb{E}}_2[|B_{t_1}|]. \end{aligned}$$

On the other hand, from the definition of $\widetilde{\varphi}$ we also obtain

$$\begin{aligned} & |\hat{\mathbb{E}}_1 \square \hat{\mathbb{E}}_2[\varphi(B_{t_1}, B_{t_2} - B_{t_1}, ..., B_{t_n} - B_{t_{n-1}})] \\ & \quad -\hat{\mathbb{E}}_1 \square \hat{\mathbb{E}}_2[\widetilde{\varphi}(B_{t_1}, B_{t_2} - B_{t_1}, ..., B_{t_n} - B_{t_{n-1}})]| \\ \leq & \hat{\mathbb{E}}_1 \square \hat{\mathbb{E}}_2[|\varphi(B_{t_1}, B_{t_2} - B_{t_1}, ..., B_{t_n} - B_{t_{n-1}}) \\ & \quad -\widetilde{\varphi}(B_{t_1}, B_{t_2} - B_{t_1}, ..., B_{t_n} - B_{t_{n-1}})|] \\ \leq & \frac{2L}{M}\hat{\mathbb{E}}_1 \square \hat{\mathbb{E}}_2[|B_{t_1}|]. \end{aligned}$$

Thus, since due to Lemma 4.5

$$\hat{\mathbb{E}}_1 \square \hat{\mathbb{E}}_2[\widetilde{\varphi}(B_{t_1}, B_{t_2} - B_{t_1}, ..., B_{t_n} - B_{t_{n-1}})] = \hat{\mathbb{E}}_1 \square \hat{\mathbb{E}}_2[\widetilde{\varphi}'(B_{t_1})],$$

we get by letting $M \mapsto +\infty$ the relation

$$\begin{aligned}
&\hat{\mathbb{E}}_1\Box\hat{\mathbb{E}}_2[\varphi(B_{t_1}, B_{t_2}-B_{t_1}, ..., B_{t_n}-B_{t_{n-1}})] \\
&= \hat{\mathbb{E}}_1\Box\hat{\mathbb{E}}_2[\hat{\mathbb{E}}_1\Box\hat{\mathbb{E}}_2[\varphi(x, B_{t_2}-B_{t_1}, ..., B_{t_n}-B_{t_{n-1}})]|_{x=B_{t_1}}]].
\end{aligned}$$

The proof of Lemma 4.6 is complete. ■

Lemma 4.7: For all $\varphi \in Lip(\mathbb{R}^{n-1})$, $n \geq 1$, and $0 \leq t_1 \leq t_2 \leq ... \leq t_n$, we have

$$\begin{aligned}
&\hat{\mathbb{E}}_1\Box\hat{\mathbb{E}}_2[\varphi(B_{t_2}-B_{t_1}, ..., B_{t_n}-B_{t_{n-1}})] \\
&= \hat{\mathbb{E}}_1\Box\hat{\mathbb{E}}_2[\hat{\mathbb{E}}_1\Box\hat{\mathbb{E}}_2[\varphi(y, B_{t_3}-B_{t_2}, ..., B_{t_n}-B_{t_{n-1}})]|_{y=B_{t_2}-B_{t_1}}]
\end{aligned}$$

Proof: Lemma 4.2 allows to repeat the arguments of the Lemmas 4.3 to 4.6 in $\mathcal{H}_{t_1}^{t_n}$. The result of Lemma 4.7 then follows. ■

Finally, we have:

Lemma 4.8: Let $\varphi \in Lip(\mathbb{R}^{n+1})$, $n \geq 1$ and $0 \leq t_1, ..., t_n \leq s$. Then

$$\begin{aligned}
&\hat{\mathbb{E}}_1\Box\hat{\mathbb{E}}_2[\varphi(B_{t_1}, B_{t_2}, ..., B_{t_n}, B_t - B_s)] \\
&= \hat{\mathbb{E}}_1\Box\hat{\mathbb{E}}_2[\hat{\mathbb{E}}_1\Box\hat{\mathbb{E}}_2[\varphi(x_1, ..., x_n, B_t - B_s)]|_{(x_1,...,x_n)=(B_{t_1}, B_{t_2}, ..., B_{t_n})}]].
\end{aligned}$$

Proof: Without any loss of generality we can suppose $0 \leq t_1 \leq t_2 \leq ... \leq t_n$. Then there is some $\widetilde{\varphi} \in Lip(\mathbb{R}^{n+1})$ such that $\varphi(B_{t_1}, B_{t_2}, ..., B_{t_n}, B_t - B_s) = \widetilde{\varphi}(B_{t_1}, B_{t_2}-B_{t_1}, ..., B_{t_n}-B_{t_{n-1}}, B_t - B_s) \in \mathcal{H}_t$. With the notation $\mathbf{x} = (x_1, ..., x_n)$, and due to the Lemmas 4.1 to 4.7 we have

$$\begin{aligned}
&\hat{\mathbb{E}}_1\Box\hat{\mathbb{E}}_2[\varphi(B_{t_1}, B_{t_2}, ..., B_{t_n}, B_t - B_s)] \\
&= \hat{\mathbb{E}}_1\Box\hat{\mathbb{E}}_2[\widetilde{\varphi}(B_{t_1}, B_{t_2}-B_{t_1}, ..., B_{t_n}-B_{t_{n-1}}, B_t - B_s)] \\
&\qquad . \quad . \quad . \quad . \\
&= \hat{\mathbb{E}}_1\Box\hat{\mathbb{E}}_2[\hat{\mathbb{E}}_1\Box\hat{\mathbb{E}}_2[\widetilde{\varphi}(\mathbf{x}, B_t - B_s)]|_{\mathbf{x}=(B_{t_1}, B_{t_2}-B_{t_1}, ..., B_{t_n}-B_{t_{n-1}})}]] \\
&= \hat{\mathbb{E}}_1\Box\hat{\mathbb{E}}_2[\hat{\mathbb{E}}_1\Box\hat{\mathbb{E}}_2[\varphi(\mathbf{x}, B_t - B_s)]|_{\mathbf{x}=(B_{t_1}, B_{t_2}, ..., B_{t_n})}]] \\
&= \hat{\mathbb{E}}_1\Box\hat{\mathbb{E}}_2[\hat{\mathbb{E}}_1\Box\hat{\mathbb{E}}_2[\varphi(x_1, ..., x_n, B_t - B_s)]|_{(x_1,...,x_n)=(B_{t_1}, B_{t_2}, ..., B_{t_n})}]].
\end{aligned}$$

The proof of Lemma 4.8 is complete now. ■

Let us now come to the proof of Lemma 3.9.
Proof (of Lemma 3.9) : In a first step, we will prove that for each $\varphi \in C_{l,lip}(\mathbb{R}^{n+1})$ there exists a sequence of bounded Lipschitz functions $(\varphi_N)_{N\geq 1}$ such that

$$\begin{aligned}
&\hat{\mathbb{E}}_1[|\varphi_N(B_{t_1}, B_{t_2}, ..., B_{t_n}, B_t - B_s) - \varphi(B_{t_1}, B_{t_2}, ..., B_{t_n}, B_t - B_s)|] \\
&\longrightarrow 0, \quad \text{as} \quad N \longrightarrow \infty.
\end{aligned}$$

For this end we put

$$l_N(x) = (x \wedge N) \vee (-N), N \geq 1, \ x \in \mathbb{R},$$

and

$$\varphi_N(x_1, ..., x_{n+1}) = \varphi(l_N(x_1), ..., l_N(x_{n+1})),$$

and we notice that

$$|x - l_N(x)| \leq \frac{|x|^2}{N}, \text{ for all } x \in \mathbb{R}.$$

Obviously, the functions φ_N are bounded and Lipschitz, and, moreover,

$$\begin{aligned}
&|\varphi_N(x_1, ..., x_{n+1}) - \varphi(x_1, ..., x_{n+1})| \\
&= |\varphi(l_N(x_1), ..., l_N(x_{n+1})) - \varphi(x_1, ..., x_{n+1})| \\
&\leq C(1 + |x_1|^m + ... + |x_{n+1}|^m)\sqrt{\sum_{i=1}^{n+1} \frac{|x_i|^4}{N^2}} \\
&= \frac{C(1 + |x_1|^m + ... + |x_{n+1}|^m)\sqrt{\sum_{i=1}^{n+1} |x_i|^4}}{N},
\end{aligned}$$

where C and $m \geq 0$ are constants only depending on φ. Then, in virtue of the finiteness of $\hat{\mathbb{E}}_1[(1 + |B_{t_1}|^m + ... + |B_{t_n}|^m + |B_t - B_s|^m)(\sum_{i=1}^n |B_{t_i}|^4 + |B_t - B_s|^4)^{\frac{1}{2}}]$, we get

$\hat{\mathbb{E}}_1[|\varphi_N(B_{t_1}, B_{t_2}, ..., B_{t_n}, B_t - B_s) - \varphi(B_{t_1}, B_{t_2}, ..., B_{t_n}, B_t - B_s)|] \longrightarrow 0,$
as $N \longrightarrow \infty$.

Let $\mathbf{x}_1 = (x_1, ..., x_n)$ and $Y_1 = (B_{t_1}, B_{t_2}, ..., B_{t_n})$. Then, due to our above convergence result,

$$\begin{aligned}
&|\hat{\mathbb{E}}_1 \Box \hat{\mathbb{E}}_2[\varphi_N(Y_1, B_t - B_s)] - \hat{\mathbb{E}}_1 \Box \hat{\mathbb{E}}_2[\varphi(Y_1, B_t - B_s)]| \\
&\leq \hat{\mathbb{E}}_1 \Box \hat{\mathbb{E}}_2[|\varphi_N(Y_1, B_t - B_s) - \varphi(Y_1, B_t - B_s)|] \\
&\leq \hat{\mathbb{E}}_1[|\varphi_N(Y_1, B_t - B_s) - \varphi(Y_1, B_t - B_s)|] \\
&\longrightarrow 0, \quad \text{as} \quad N \longrightarrow \infty,
\end{aligned}$$

and, from Lemma 4.3,

$$\begin{aligned}
&|\hat{\mathbb{E}}_1 \Box \hat{\mathbb{E}}_2[\hat{\mathbb{E}}_1 \Box \hat{\mathbb{E}}_2[\varphi_N(\mathbf{x}_1, B_t - B_s)]|_{\mathbf{x}_1 = Y_1}] \\
&-\hat{\mathbb{E}}_1 \Box \hat{\mathbb{E}}_2[\hat{\mathbb{E}}_1 \Box \hat{\mathbb{E}}_2[\varphi(\mathbf{x}_1, B_t - B_s)]|_{\mathbf{x}_1 = Y_1}]| \\
&\leq \hat{\mathbb{E}}_1 \Box \hat{\mathbb{E}}_2[\hat{\mathbb{E}}_1 \Box \hat{\mathbb{E}}_2[|\varphi_N(\mathbf{x}_1, B_t - B_s) - \varphi(\mathbf{x}_1, B_t - B_s)|]|_{\mathbf{x}_1 = Y_1}] \\
&\leq \hat{\mathbb{E}}_1[|\varphi_N(Y_1, B_t - B_s) - \varphi(Y_1, B_t - B_s)|] \\
&\longrightarrow 0, \quad \text{as} \quad N \longrightarrow \infty.
\end{aligned}$$

On the other hand, from Lemma 4.8 we have

$$\begin{aligned}&\hat{\mathbb{E}}_1\Box\hat{\mathbb{E}}_2[\varphi_N(B_{t_1}, B_{t_2}, ..., B_{t_n}, B_t - B_s)]\\ &= \hat{\mathbb{E}}_1\Box\hat{\mathbb{E}}_2[\hat{\mathbb{E}}_1\Box\hat{\mathbb{E}}_2[\varphi_N(x_1, ..., x_n, B_t - B_s)]|_{(x_1,...,x_n)=(B_{t_1}, B_{t_2}, ..., B_{t_n})}]].\end{aligned}$$

Combining the above results we can conclude that

$$\begin{aligned}&\hat{\mathbb{E}}_1\Box\hat{\mathbb{E}}_2[\varphi(B_{t_1}, B_{t_2}, ..., B_{t_n}, B_t - B_s)]\\ &= \hat{\mathbb{E}}_1\Box\hat{\mathbb{E}}_2[\hat{\mathbb{E}}_1\Box\hat{\mathbb{E}}_2[\varphi(x_1, ..., x_n, B_t - B_s)]|_{(x_1,...,x_n)=(B_{t_1}, B_{t_2}, ..., B_{t_n})}]].\end{aligned}$$

The proof is complete now. ■

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