

# Discrete-Time Interest Rate Modelling

Lane P. Hughston<sup>1</sup> and Andrea Macrina<sup>2,3</sup>

<sup>1</sup>Department of Mathematics, Imperial College, London SW7 2AZ, UK

<sup>2</sup>Department of Mathematics, King's College London, London, WC2R 2LS, UK

<sup>3</sup>Kyoto Institute of Economic Research, Kyoto University, Kyoto 606-8501, Japan

## Abstract

This paper presents an axiomatic scheme for interest rate models in discrete time. We take a pricing kernel approach, which builds in the arbitrage-free property and provides a link to equilibrium economics. We require that the pricing kernel be consistent with a pair of axioms, one giving the inter-temporal relations for dividend-paying assets, and the other ensuring the existence of a money-market asset. We show that the existence of a positive-return asset implies the existence of a previsible money-market account. A general expression for the price process of a limited-liability asset is derived. This expression includes two terms, one being the discounted risk-adjusted value of the dividend stream, the other characterising retained earnings. The vanishing of the latter is given by a transversality condition. We show (under the assumed axioms) that, in the case of a limited-liability asset with no permanently-retained earnings, the price process is given by the ratio of a pair of potentials. Explicit examples of discrete-time models are provided.

Keywords: Interest rates models; pricing kernels; financial time series; Flesaker-Hughston models; transversality condition; financial bubbles.

Email addresses: lane.hughston@ic.ac.uk, andrea.macrina@kcl.ac.uk

Presented at the 7th International ISAAC Congress, London, 13-17 July 2009.

## 1 Discrete-time asset pricing

Although discrete-time interest rate models are often introduced for computational purposes as a convenient approximation to the continuous-time situation, it is important to recognize that the theory can be developed in discrete time in an entirely satisfactory way in its own right, without reference to continuous time. Let  $\{t_i\}_{i=0,1,2,\dots}$  denote a time sequence where  $t_0$  is the present and  $t_{i+1} > t_i$  for all  $i \in \mathbb{N}_0$ . We assume that  $\{t_i\}$  is unbounded in the sense

that for any  $T$  there exists a value of  $i$  such that  $t_i > T$ . The economy is represented by a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  with a filtration  $\{\mathcal{F}_i\}_{i \geq 0}$  which we call the “market filtration”. Each asset is characterised by a pair  $\{S_{t_i}\}_{i \geq 0}$  and  $\{D_{t_i}\}_{i \geq 0}$  which we call the “value process” and the “dividend process”. We interpret  $D_{t_i}$  as the random cash flow paid out by the asset at time  $t_i$ . Then  $S_{t_i}$  denotes the “ex-dividend” value of the asset at  $t_i$ . For simplicity, we often write  $S_i = S_{t_i}$  and  $D_i = D_{t_i}$ . To ensure the absence of arbitrage, we assume the existence of a positive pricing kernel  $\{\pi_i\}_{i \geq 0}$ , and make the following assumptions:

**Axiom A.** For any asset with value process  $\{S_i\}_{i \geq 0}$  and dividend process  $\{D_i\}_{i \geq 0}$ , the process  $\{M_i\}_{i \geq 0}$  defined by

$$M_i = \pi_i S_i + \sum_{n=0}^i \pi_n D_n \quad (1.1)$$

is a martingale.

**Axiom B.** There exists a positive non-dividend-paying asset with value process  $\{\bar{B}_i\}_{i \geq 0}$  such that  $\bar{B}_{i+1} > \bar{B}_i$  for all  $i \in \mathbb{N}_0$ , and that for any  $b \in \mathbb{R}$  there exists a time  $t_i$  such that  $\bar{B}_i > b$ .

The notation  $\{\bar{B}_i\}$  distinguishes the positive return asset from the previsible money-market account  $\{B_i\}$  introduced later. Since  $\{\bar{B}_i\}$  is non-dividend paying, Axiom A implies that  $\{\pi_i \bar{B}_i\}$  is a martingale. Writing  $\bar{\rho}_i = \pi_i \bar{B}_i$ , we have  $\pi_i = \bar{\rho}_i / \bar{B}_i$ . Since  $\{\bar{B}_i\}$  is increasing,  $\{\pi_i\}$  is a supermartingale, and it follows from Axiom B that

$$\lim_{i \rightarrow \infty} \mathbb{E}[\pi_i] = 0. \quad (1.2)$$

We obtain the following result concerning limited-liability assets.

**Proposition 1.** Let  $S_i \geq 0$  and  $D_i \geq 0$  for all  $i \in \mathbb{N}$ . We have

$$S_i = \frac{m_i}{\pi_i} + \frac{1}{\pi_i} \mathbb{E}_i \left[ \sum_{n=i+1}^{\infty} \pi_n D_n \right], \quad (1.3)$$

where  $\{m_i\}$  is a non-negative martingale that vanishes if and only if:

$$\lim_{j \rightarrow \infty} \mathbb{E}[\pi_j S_j] = 0. \quad (1.4)$$

Thus  $\{m_i\}$  represents that part of the value of the asset that is “never paid out”. An idealised money-market account is of this nature, and so is a “permanent bubble” (cf. Tirole 1982, Armerin 2004). In the case of an asset for which the “transversality” condition (1.4) is satisfied, the price is directly related to the future dividend flow:

$$S_i = \frac{1}{\pi_i} \mathbb{E}_i \left[ \sum_{n=i+1}^{\infty} \pi_n D_n \right]. \quad (1.5)$$

This is the so-called “fundamental equation” often used as a basis for asset pricing (cf. Cochrane 2005). Alternatively we can write

$$S_i = \frac{1}{\pi_i}(\mathbb{E}_i[F_\infty] - F_i), \quad (1.6)$$

where  $F_i = \sum_{n=0}^i \pi_n D_n$ , and  $F_\infty = \lim_{i \rightarrow \infty} F_i$ . Hence the price of a pure dividend-paying asset can be expressed as a ratio of potentials, giving us a discrete-time analogue of a result of Rogers 1997.

## 2 Positive-return asset and pricing kernel

Let us introduce the notation

$$\bar{r}_i = \frac{\bar{B}_i - \bar{B}_{i-1}}{\bar{B}_{i-1}} \quad (2.1)$$

for the rate of return of the positive-return asset realised at time  $t_i$  on an investment made at  $t_{i-1}$ . The notation  $\bar{r}_i$  is used to distinguish the rate of return on  $\{\bar{B}_i\}$  from the rate of return  $r_i$  on the money market account  $\{B_i\}$  introduced later.

**Proposition 2.** *There exists an asset with constant value  $S_i = 1$  for all  $i \in \mathbb{N}_0$ , for which the associated cash flows are given by  $\{\bar{r}_i\}_{i \geq 1}$ .*

**Proposition 3.** *Let  $\{\bar{B}_i\}$  be a positive-return asset satisfying Axioms A and B, and let  $\{\bar{r}_i\}$  be its rate-of-return process. Then the pricing kernel can be expressed in the form  $\pi_i = \mathbb{E}_i[G_\infty] - G_i$ , where  $G_i = \sum_{n=1}^i \pi_n \bar{r}_n$  and  $G_\infty = \lim_{i \rightarrow \infty} G_i$ .*

There is a converse to this result that allows one to construct a system satisfying Axioms A and B from a strictly-increasing non-negative adapted process that converges and satisfies an integrability condition:

**Proposition 4.** *Let  $\{G_i\}_{i \geq 0}$  be a strictly-increasing adapted process with  $G_0 = 0$ , and  $\mathbb{E}[G_\infty] < \infty$ , where  $G_\infty = \lim_{i \rightarrow \infty} G_i$ . Let  $\{\pi_i\}$ ,  $\{\bar{r}_i\}$ , and  $\{\bar{B}_i\}$  be defined by  $\pi_i = \mathbb{E}_i[G_\infty] - G_i$  for  $i \geq 0$ ,  $\bar{r}_i = (G_i - G_{i-1})/\pi_i$  for  $i \geq 1$ , and  $\bar{B}_i = \prod_{n=1}^i (1 + \bar{r}_n)$  for  $i \geq 1$ , with  $\bar{B}_0 = 1$ . Let  $\{\bar{\rho}_i\}$  be defined by  $\bar{\rho}_i = \pi_i \bar{B}_i$  for  $i \geq 0$ . Then  $\{\bar{\rho}_i\}$  is a martingale, and  $\lim_{j \rightarrow \infty} \bar{B}_j = \infty$ , from which it follows that  $\{\pi_i\}$  and  $\{\bar{B}_i\}$  satisfy Axioms A and B.*

## 3 Discrete-time discount bond systems

The price  $P_{ij}$  at  $t_i$  of a discount bond that matures at  $t_j$  ( $i < j$ ) is

$$P_{ij} = \frac{1}{\pi_i} \mathbb{E}_i[\pi_j]. \quad (3.1)$$

Since  $\pi_i > 0$  for  $i \in \mathbb{N}$ , and  $\mathbb{E}_i[\pi_j] < \pi_i$  for  $i < j$ , it follows that  $0 < P_{ij} < 1$  for  $i < j$ . We observe that the “per-period” interest rate  $R_{ij}$  defined by  $P_{ij} = 1/(1 + R_{ij})$  is positive.

Since  $\{\pi_i\}$  is given, there is no need to model the volatility structure of the bonds. Thus, our scheme differs from the discrete-time models discussed in Heath *et al.* 1990, and Filipović & Zabczyk 2002. As an example of a class of discrete-time models set  $\pi_i = \alpha_i + \beta_i N_i$ , where  $\{\alpha_i\}$  and  $\{\beta_i\}$  are positive, strictly-decreasing deterministic processes with  $\lim_{i \rightarrow \infty} \alpha_i = 0$  and  $\lim_{i \rightarrow \infty} \beta_i = 0$ , and where  $\{N_i\}$  is a positive martingale. Then we have

$$P_{ij} = \frac{\alpha_j + \beta_j N_i}{\alpha_i + \beta_i N_i}, \quad (3.2)$$

giving a family of “rational” interest rate models. In a discrete-time setting we can produce models that do not necessarily have analogues in continuous time—for example, we can let  $\{N_i\}$  be the martingale associated with a branching process.

Any discount bond system consistent with our scheme admits a representation of the Flesaker-Hughston type (Rutkowski 1997, Jin & Glasserman 2001, Cairns 2004, Musiela & Rutkowski 2005, Björk 2009). More precisely, we have the following:

**Proposition 5.** *Let  $\{\pi_i\}$ ,  $\{\bar{B}_i\}$ ,  $\{P_{ij}\}$  satisfy Axioms A and B. Then there exists a family of positive martingales  $\{m_{in}\}_{0 \leq i \leq n}$ ,  $n \in \mathbb{N}$ , such that*

$$P_{ij} = \frac{\sum_{n=j+1}^{\infty} m_{in}}{\sum_{n=i+1}^{\infty} m_{in}}. \quad (3.3)$$

## 4 Construction of the money-market asset

Let us look now at the situation where the positive-return asset is previsible. Thus we assume that  $B_i$  is  $\mathcal{F}_{i-1}$ -measurable and we drop the “bar” over  $B_i$ . In that case we have

$$P_{i-1,i} = \frac{1}{\pi_{i-1}} \mathbb{E}_{i-1}[\pi_i] = \frac{B_{i-1}}{\rho_{i-1}} \mathbb{E}_{i-1} \left[ \frac{\rho_i}{B_i} \right] = \frac{B_{i-1}}{B_i}. \quad (4.1)$$

Hence, writing  $P_{i-1,i} = 1/(1 + r_i)$  where  $r_i = R_{i-1,i}$  we see that the rate of return on the money-market account is previsible, and is given by the one-period discount factor associated with the bond that matures at  $t_i$ .

Reverting to the general situation, it follows that if we are given a pricing kernel  $\{\pi_i\}$  on  $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_i\})$ , and a system of assets satisfying Axioms A and B, then we can construct a *candidate* for a previsible money market account by setting  $B_0 = 1$  and

$$B_i = (1 + r_i)(1 + r_{i-1}) \cdots (1 + r_1), \quad (4.2)$$

for  $i \geq 1$ , where  $r_i$  is defined by

$$r_i = \frac{\pi_{i-1}}{\mathbb{E}_{i-1}[\pi_i]} - 1. \quad (4.3)$$

We refer to  $\{B_i\}$  as the “natural” money-market account associated with  $\{\pi_i\}$ . To justify this terminology, we verify that  $\{B_i\}$ , so constructed, satisfies Axioms A and B. To this end, we note the following multiplicative decomposition. Let  $\{\pi_i\}$  be a positive supermartingale satisfying  $\mathbb{E}_i[\pi_j] < \pi_i$  for  $i < j$  and  $\lim_{j \rightarrow \infty} \pi_j = 0$ . Then we can write  $\pi_i = \rho_i/B_i$ , where

$$\rho_i = \frac{\pi_i}{\mathbb{E}_{i-1}[\pi_i]} \frac{\pi_{i-1}}{\mathbb{E}_{i-2}[\pi_{i-1}]} \cdots \frac{\pi_1}{\mathbb{E}_0[\pi_1]} \pi_0 \quad (4.4)$$

for  $i \geq 0$ , and

$$B_i = \frac{\pi_{i-1}}{\mathbb{E}_{i-1}[\pi_i]} \frac{\pi_{i-2}}{\mathbb{E}_{i-2}[\pi_{i-1}]} \cdots \frac{\pi_1}{\mathbb{E}_1[\pi_2]} \frac{\pi_0}{\mathbb{E}_0[\pi_1]} \quad (4.5)$$

for  $i \geq 1$ , with  $B_0 = 1$ . In this scheme we have

$$\rho_i = \frac{\pi_i}{\mathbb{E}_{i-1}[\pi_i]} \rho_{i-1}, \quad (4.6)$$

with  $\rho_0 = \pi_0$ ; and

$$B_i = \frac{\pi_{i-1}}{\mathbb{E}_{i-1}[\pi_i]} B_{i-1}, \quad (4.7)$$

with  $B_0 = 1$ . It is evident that  $\{\rho_i\}$  is  $\{\mathcal{F}_i\}$ -adapted, and that  $\{B_i\}$  is previsible and increasing. We establish the following:

**Proposition 6.** *Let  $\{\pi_i\}$  be a non-negative supermartingale such that  $\mathbb{E}_i[\pi_j] < \pi_i$  for all  $i < j \in \mathbb{N}_0$ , and  $\lim_{i \rightarrow \infty} \mathbb{E}[\pi_i] = 0$ . Let  $\{B_i\}$  be defined by  $B_0 = 1$  and  $B_i = \prod_{n=1}^i (1 + r_n)$  for  $i \geq 1$ , where  $1 + r_i = \pi_{i-1}/\mathbb{E}_{i-1}[\pi_i]$ , and set  $\rho_i = \pi_i B_i$  for  $i \geq 0$ . Then  $\{\rho_i\}$  is a martingale, and the interest rate system defined by  $\{\pi_i\}$ ,  $\{B_i\}$ ,  $\{P_{ij}\}$  satisfies Axioms A and B.*

A significant feature of Proposition 6 is that no integrability condition is required on  $\{\rho_i\}$ : the natural money market account defined above “automatically” satisfies Axiom A. Thus in place of Axiom B we can assume:

**Axiom B\*.** *There exists a positive non-dividend paying asset, the money-market account  $\{B_i\}_{i \geq 0}$ , having the properties that  $B_{i+1} > B_i$  for  $i \in \mathbb{N}_0$ , that  $B_i$  is  $\mathcal{F}_{i-1}$ -measurable for  $i \in \mathbb{N}$ , and that for any  $b \in \mathbb{R}$  there exists a  $t_i$  such that  $B_i > b$ .*

The content of Proposition 6 is that Axioms A and B together are equivalent to Axioms A and B\* together. Let us establish that the class of interest rate models satisfying Axioms A and B\* is non-vacuous. In particular, consider the “rational” model defined for some choice of  $\{N_i\}$ . It is an exercise to see that the previsible money market account is given for  $i = 0$  by  $B_0 = 1$  and for  $i \geq 1$  by

$$B_i = \prod_{n=1}^i \frac{\alpha_{n-1} + \beta_{n-1} N_{n-1}}{\alpha_n + \beta_n N_{n-1}}, \quad (4.8)$$

and that for  $\{\rho_i\}$  we have

$$\rho_i = \rho_0 \prod_{n=1}^i \frac{\alpha_n + \beta_n N_n}{\alpha_n + \beta_n N_{n-1}}, \quad (4.9)$$

where  $\rho_0 = \alpha_0 + \beta_0 N_0$ . One can check for each  $i \geq 0$  that  $\rho_i$  is bounded; therefore  $\{\rho_i\}$  is a martingale, and  $\{B_i\}$  satisfies Axioms A and B\*.

## 5 Doob decomposition

Consider now the Doob decomposition given by  $\pi_i = \mathbb{E}_i[A_\infty] - A_i$ , with

$$A_i = \sum_{n=0}^{i-1} (\pi_n - \mathbb{E}_n[\pi_{n+1}]) \quad (5.1)$$

as discussed, e.g., in Meyer 1966. It follows that

$$A_i = \sum_{n=0}^{i-1} \pi_n \left( 1 - \frac{\mathbb{E}_n[\pi_{n+1}]}{\pi_n} \right) = \sum_{n=0}^{i-1} \pi_n (1 - P_{n,n+1}) = \sum_{n=0}^{i-1} \pi_n r_{n+1} P_{n,n+1}, \quad (5.2)$$

where  $\{r_i\}$  is the previsible short rate process. The pricing kernel can therefore be put in the form

$$\pi_i = \mathbb{E}_i \left[ \sum_{n=i}^{\infty} \pi_n r_{n+1} P_{n,n+1} \right]. \quad (5.3)$$

Comparing (5.3) with the decomposition  $\pi_i = \mathbb{E}_i[G_\infty] - G_i$ ,  $G_i = \sum_{n=1}^i \pi_n \bar{r}_n$ , given in Proposition 3, we see that by setting

$$\bar{r}_i = \frac{r_i \pi_{i-1} P_{i-1,i}}{\pi_i} \quad (5.4)$$

we obtain a positive-return asset based on the Doob decomposition.

## 6 Foreign exchange processes

An extension of the material presented here to models for foreign exchange and inflation is pursued in Hughston & Macrina (2008). In particular, since the money-market account is a positive-return asset, by Proposition 3 we can write:

$$\pi_i = \mathbb{E}_i \left[ \sum_{n=i+1}^{\infty} \pi_n r_n \right]. \quad (6.1)$$

As a consequence, we see that the price process of a pure dividend-paying asset can be written in the following symmetrical form:

$$S_i = \frac{\mathbb{E}_i \left[ \sum_{n=i+1}^{\infty} \pi_n D_n \right]}{\mathbb{E}_i \left[ \sum_{n=i+1}^{\infty} \pi_n r_n \right]}. \quad (6.2)$$

In the case where  $\{S_i\}$  represents a foreign currency, the dividend process is the foreign interest rate, and both  $\{D_i\}$  and  $\{r_i\}$  are previsible.

## References

- [1] F. Armerin (2004) Aspects of cash flow valuation, *KTH Royal Institute of Technology*. Stockholm, Sweden.
- [2] T. Björk (2009) *Arbitrage Pricing in Continuous Time*, Oxford Finance.
- [3] A. J. G. Cairns (2004) *Interest Rate Models: An Introduction*, Princeton University Press.
- [4] J. H. Cochrane (2005) *Asset Pricing*, Princeton University Press.
- [5] D. Filipovic & J. Zabczyk (2002) Markovian term structure models in discrete time, *Annals of Applied Probability* **12**, 710-729.
- [6] D. Heath, R. Jarrow & A. Morton (1990) Bond pricing and the term structure of interest rates: a discrete time approximation, *Journal of Financial and Quantitative Analysis* **25**, 419-440.
- [7] L. P. Hughston, & A. Macrina (2008) Information, inflation, and interest. In *Advances in Mathematics of finance* (ed. L. Stettner). Banach Centre Publications **83** 117 (Warsaw, Poland: Polish Academy of Sciences 2008).
- [8] Y. Jin & P. Glasserman (2001) Equilibrium positive interest rates: a unified view, *Review of Financial Studies* **14**, 187-214.
- [9] P. A. Meyer (1966) *Probability and Potentials*, Blaisdell.
- [10] M. Musiela & M. Rutkowski (2005) *Martingale Methods in Financial Modelling*, Springer.
- [11] L. C. G. Rogers (1997) The potential approach to the term structure of interest rate and foreign exchange rates, *Mathematical Finance* **7**, 157-176.
- [12] M. Rutkowski (1997) A note on the Flesaker-Hughston model of the term structure of interest rates, *Applied Mathematical Finance* **4**, 151-163.
- [13] J. M. Tirole (1982) On the probability of speculation under rational expectation, *Econometrica* **50**, 1163-1181.