Robust utility maximization for diffusion market model with misspecified coefficients

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The paper studies the robust maximization of utility of terminal wealth in the diffusion financial market model. The underlying model consists with risky tradable asset, whose price is described by diffusion process with misspecified trend and volatility coefficients, and non-tradable asset with a known parameter. The robust utility functional is defined in terms of a HARA utility function. We give explicit characterization of the solution of the problem by means of a solution of the HJBI equation.

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1 Introduction

The purpose of the present paper is to study the robust maximization of utility of terminal wealth in the diffusion financial market model, where appreciate rate and volatility of the asset price are not known exactly.

The utility maximization problem was first studied by Merton (1971) in a classical Black-Scholes model. Using the Markov structure of the model he derived the Bellman equation for the value function of the problem and produced the closed-form solution of this equation in cases of power, logarithmic and exponential utility functions.

For general complete market models, it was shown by Pliska (1986), Cox and Huang (1989) and Karatzas et al (1987) that the optimal portfolio of the utility maximization problem is (up to a constant) equal to the density of the martingale measure, which is unique for complete markets. As shown by He and Pearson (1991) and Karatzas et al (1991), for incomplete markets described by Ito-processes, this method gives a duality characterization of optimal portfolios provided by the set of martingale measures. Their idea was to solve the dual problem of finding the suitable optimal martingale measure and then to express the solution of the primal problem by convex duality. Extending the domain of the dual problem the approach has been generalized to semimartingale models and under weaker conditions on the utility functions by Kramkov and Schachermayer (1999). All these papers consider a model of utility which assumes that beliefs are represented by a probability measure. In 1999, Chen and Epstein have introduced a continuous-time intertemporal version of multiple-priors utility in the case of a Brownian filtration. In this case, beliefs are represented by a set $\mathcal{P}$ of probability measures and the utility is defined as the minimum of expected utilities over the set $\mathcal{P}$. Independently, Cvitanic (2000) and Cvitanic and Karatzas (1999) examine, for a given option, hedging strategies that minimize the expected "shortfall" that is, the difference between the payoff and the terminal wealth. They consider the problem of determining a "worst-case" model $\tilde{Q}$, that is a model that maximizes the minimal shortfall risk over all possible priors $Q \in \mathcal{P}$. They show that under some assumptions their sup-inf problem can be written as an inf-sup problem. In 2004 Quenez studied the problem of utility maximization in an incomplete multiple-priors model, where asset prices are semimartingales. This problem corresponds to a sup-inf problem where the supremum is taken over the set of feasible wealths X (or portfolios) and where the infimum is taken over the set of priors $\mathcal{P}$. The author showed that, under suitable conditions, there exists a saddle-point for this problem. Moreover, Quenez developed a dual approach which consists in solving a dual minimization problem over the set of priors and supermartingale measures and showed how a solution of the dual problem induces one for the primal problem.

These sup-inf problems also can be called robust optimization problems since optimization involves an entire class $\mathcal{P}$ of possible probabilistic models and thus takes into account model risk. Optimal investment problems for such robust utility functionals were considered, among others, by Talay and Zheng (2002), Korn and Wilmott (2002), Quenez (2004), Schied (2005),(2008), Korn and Menkens (2005), Gundel (2005),Bordigoni at al. (2007), Schied and Wu (2005), Föllmer and Gundel (2006), Hernández-Hernández and Schied (2006, 2007) .

The numerous of publications are concerned to the case when one of these parameters is known exactly. In the case of unknown drift coefficient the existence of saddle point of corresponding minimax problem has been established and characterization of the optimal strategy has been obtained (see [5], [8], [7]). For the case of unknown volatility coefficients the construction of hedging strategy were given in the works [1], [3], [2], [16].

The most difficult case is to characterize the optimal strategy of minimax (or maximin) problem under uncertainty of both drift and volatility terms. Talay and Zheng [22] applied the PDE-based approach to the max-min problem and characterized the value as a viscosity solution of corresponding Hamilton-Jacob-Bellman-Isaacs (HJBI) equation. In general such problem does not admit a saddle point.

We consider incomplete diffusion financial market model which resembles to the model considered by Schied (2008), Hernández-Hernández and Schied (2006, 2007). We suppose that the market consists with riskless asset, risky tradable asset whose trend and volatility are misspecified and non-tradable asset with a known parameters. Different from Quenez (2004) and Schied (2008) approach we solve the maximin problem using HJBI equation which corresponds to the primal problem. In case of unknown trend and volatility coefficient such maximin problem doesn't have a saddle point in general. We are extending the set of model coefficients i.e. doing some "randomization" and by this way we are getting a problem with the saddle point. This gives us the possibility to replace maximin problem by minimax problem which is convenient to study HJBI equation properties. Particularly, we have found such form of this equation which coincides to equation obtained by Hernández-Hernández and Schied (2006) in case of known volatility. The solvability in classical sense of obtained equation is established and in case of specific drift coefficient HJBI equation is explicitly solved and saddle point (optimal portfolio and optimal coefficients) of maximin problem found as well.

The paper is organized as follows. In section 2, we describe the model and consider the misspecified coefficients as a generalized controls. Further we show the existence of saddle point of generalized max-min problem and derive HJBI equation for value function. In section 3 we prove the solvability in the classical sense of obtained PDE in the case of power utility and give explicit PDE-characterization of a robust maximization problem.

2 The generalized coefficients and existence of saddle point

Suppose that the financial market consists in a riskless asset

$$dS_t^0 = r(Y_t)S_t^0 dt \tag{2.1}$$

with $r(y) \geq 0$ and risky financial assets whose prices defined through stochastic differential equation (SDE)

$$\frac{dS_t}{S_t} = (\tilde{b}(Y_t) + \mu_t)dt + \sigma_t dw_t. \tag{2.2}$$

Here w_t is a standard Brownian motion and Y_t denotes a return of non-traded asset modelled by SDE

$$dY_t = \beta(Y_t)dt + \left(\rho dw_t + \sqrt{1-\rho^2}dw_t^{\perp}\right), \tag{2.3}$$

for a some correlation factor $\rho \in [0,1]$ and standard Brownian motion $w^{\perp}$, which is independent of w. We note $b = \tilde{b} - r$ and assume that

A1) $b(y),\ \beta(y),\ r(y)$ belongs to $C_b^1(\mathbb{R})$,

A2) $b'(y)$ belongs to $C_0(\mathbb{R})$, where $C_b^1(\mathbb{R})$ denotes the class of bounded continuous functions with bounded derivatives and $C_0(\mathbb{R})$ denotes the class of continuous function with compact supports.

Introduce the set $\tilde{\mathcal{U}}_K$ of all measurable process (μ_t, σ_t) with value in the set $K = [\mu_-, \mu_+] \times [\sigma_-, \sigma_+]$, where $0 \le \mu_- \le \mu_+,\ 0 \le \sigma_- \le \sigma_+$ and denote by $\mathcal{U}_K$ the subset of predictable processes from $\tilde{\mathcal{U}}_K$. By Π_x we denote the set of predictable processes such that $\int_0^T \pi_t^2 dt < \infty,\ P - a.s.$ and corresponding wealth process , defined as a solution of SDE

$$dX_t = (1 - \pi_t) X_t \frac{dS_t^0}{S_t^0} + \pi_t X_t \frac{dS_t}{S_t}, \tag{2.4}$$

$$X_0 = x, \tag{2.5}$$

satisfies condition $X_t(\pi) \ge 0$.

The objective of economic agent is to find the optimal robust strategy of the problem

$$\max_{\pi \in \Pi_x} \min_{(\mu,\sigma) \in \mathcal{U}_K} EU(X_T^{\mu,\sigma}(\pi)), \tag{2.6}$$

with

$$\begin{aligned} dX_t &= r(Y_t) X_t dt + \pi_t (b(Y_t) + \mu_t) dt + \pi_t \sigma_t dw_t, \quad X_0 = x, \\ dY_t &= \beta(Y_t) dt + \rho dw_t + \sqrt{1 - \rho^2} dw_t^{\perp}, \quad \eta_0 = y, \end{aligned} \tag{2.7}$$

where $U(x)$ is HARA[1)] utility function.

If we denote by $\nu_t(d\mu d\sigma)$ the regular conditional distribution of the pair of processes $(\mu, \sigma) \in \tilde{\mathcal{U}}_K$, with respect to filtration $\mathcal{F}_t$ and by (f, ν_t) the integral $\int_K f(\mu, \sigma) \nu_t(d\mu d\sigma)$, where $f(\mu, \sigma)$ is an arbitrary continuous function , we can perform the following extension maximin problem

$$\max_{\pi \in \Pi_x} \min_{(\mu,\sigma) \in \tilde{\mathcal{U}}_K} EU(X_T^{\mu,\sigma}(\pi)), \tag{2.8}$$

$$\begin{aligned} dX_t &= r(Y_t) X_t dt + \pi_t (b(Y_t) + (\mu, \nu_t)) dt + \pi_t \sqrt{(\sigma^2, \nu_t)} dw_t, \quad X_0 = x, \\ dY_t &= \beta(Y_t) dt + \rho \frac{(\sigma, \nu_t)}{\sqrt{(\sigma^2, \nu_t)}} dw_t + \sqrt{1 - \rho^2 \frac{(\sigma, \nu_t)^2}{(\sigma^2, \nu_t)}} dw_t^{\perp}, \quad \eta_0 = y. \end{aligned} \tag{2.9}$$

Introduce the set $\mathcal{P}(K)$ of probability distributions with support on K ($\mathcal{P}(K)$ is a compact metric space in a week topology, see [19]). Denote by ν_t the $\mathcal{P}(K)$-valued predictable process. Such type process usually called the generalized control in control theory. From now on we identify $\tilde{\mathcal{U}}_K$ to the set of generalized controls.

Remark 2.1. Let ${}^pY$ be the predictable projection of a process Y (see [17]). Then for $(\mu_t, \sigma_t),\ (\mu, \sigma) \in \tilde{\mathcal{U}}_K$ we have the equalities ${}^p\mu_t = (\mu, \nu_t),\ {}^p\sigma_t = (\sigma, \nu_t)$ and we can write

$$\begin{aligned} dX_t &= r(Y_t) X_t dt + \pi_t (b(Y_t) + {}^p\mu_t) dt + \pi_t \sqrt{{}^p\sigma_t^2} dw_t, \quad X_0 = x \\ dY_t &= \beta(Y_t) dt + \rho \frac{{}^p\sigma_t}{\sqrt{{}^p\sigma_t^2}} dw_t + \sqrt{1 - \rho^2 \frac{({}^p\sigma_t)^2}{{}^p\sigma_t^2}} dw_t^{\perp}, \quad \eta_0 = y. \end{aligned} \tag{2.10}$$

[1)] The function $U(\cdot)$ is a HARA (Hyperbolic Absolute Risk Aversion) utility if $-u_2'(x)/u'(x) = \gamma/x$, $\gamma < 1, \gamma \ne 0$

$\square$

Since

$$\begin{pmatrix} \pi_t\sqrt{(\sigma^2,\nu_t)} & 0 \\ \rho\frac{(\sigma,\nu_t)}{\sqrt{(\sigma^2,\nu_t)}} & \sqrt{1-\rho^2\frac{(\sigma,\nu_t)^2}{(\sigma^2,\nu_t)}} \end{pmatrix} \begin{pmatrix} \pi_t\sqrt{(\sigma^2,\nu_t)} & \rho\frac{(\sigma,\nu_t)}{\sqrt{(\sigma^2,\nu_t)}} \\ 0 & \sqrt{1-\rho^2\frac{(\sigma,\nu_t)^2}{(\sigma^2,\nu_t)}} \end{pmatrix}$$
$$= \begin{pmatrix} (\sigma^2,\nu_t)\pi_t^2 & \rho(\sigma,\nu_t)\pi_t \\ \rho(\sigma,\nu_t)\pi_t & 1 \end{pmatrix} \tag{2.11}$$

the generator of the process (X_t, Y_t) can be given by the function

$$\mathcal{H}^{\pi,\mu,\sigma}(x,y,p,q)$$
$$= \frac{1}{2}\,\pi^2\sigma^2 q_{11} + \rho\pi\sigma q_{12} + \frac{1}{2}\,q_{22} + xr(y)p_1 + \pi b(y)p_1 + \pi\mu p_1 + \beta(y)p_2. \tag{2.12}$$

For all $\nu \in \mathcal{P}(K), \pi \in R$ and $(x,y,p,q) \in R_+ \times R \times R^2 \times R^3$ we set

$$\mathcal{H}^{\pi,\nu}(x,y,p,q) = (\mathcal{H}^{\pi,\mu,\sigma}(x,y,p,q),\nu) \tag{2.13}$$

and

$$\mathcal{H}(x,y,p,q) = \max_{\pi\in\mathbb{R}}\min_{\nu\in\mathcal{P}(K)} \mathcal{H}^{\pi,\nu}(x,y,p,q). \tag{2.14}$$

Proposition 2.1. *For each fixed $(x,y,p,q) \in R_+ \times R \times R^2 \times R^3$, with $q_{11} < 0$ the function $(\pi,\nu) \to \mathcal{H}^{\pi,\mu,\sigma}(x,y,p,q)$ admits a saddle point (π^*,ν^*), i.e.*

$$\mathcal{H}^{\pi^*,\nu^*}(x,y,p,q) = \max_{\pi\in\mathbb{R}}\min_{\nu\in\mathcal{P}(K)} \mathcal{H}^{\pi,\nu}(x,y,p,q) = \min_{\nu\in\mathcal{P}(K)}\max_{\pi\in\mathbb{R}} \mathcal{H}^{\pi,\nu}(x,y,p,q). \tag{2.15}$$

Moreover

$$\max_{\pi\in\mathbb{R}}\min_{\nu\in\mathcal{P}(K)} \mathcal{H}^{\pi,\nu}(x,y,p,q) = \max_{\pi\in\mathbb{R}}\min_{(\mu,\sigma)\in K} \mathcal{H}^{\pi,\mu,\sigma}(x,y,p,q). \tag{2.16}$$

Proof. By Theorem of Neumann at al. (see Theorem lX.4.1 of [24]) for each positive n and fixed point (x,y,p,q) the function of measures $\lambda \in \mathcal{P}([-n,n]),\ \nu \in \mathcal{P}(K)$

$$(\lambda,\nu) \to \mathcal{H}^{\lambda,\nu}(x,y,p,q) = \int_{-n}^{n}\int_K \mathcal{H}^{\pi,\mu,\sigma}(x,y,p,q)\lambda(d\pi)\nu(d\mu d\sigma),$$

admits a saddle point (λ_n^*,ν_n^*), i.e.

$$\mathcal{H}^{\lambda_n^*,\nu_n^*}(x,y,p,q) = \max_{\lambda\in\mathcal{P}([-n,n])}\min_{\nu\in\mathcal{P}(K)} \mathcal{H}^{\lambda,\nu}(x,y,p,q)$$
$$= \min_{\nu\in\mathcal{P}(K)}\max_{\lambda\in\mathcal{P}([-n,n])} \mathcal{H}^{\lambda,\nu}(x,y,p,q). \tag{2.17}$$

By concavity of $\mathcal{H}^{\pi,\nu}$ with respect to π the maximizer $\pi_n^* = \arg\min \mathcal{H}^{\pi,\nu}$ is unique and thus $\lambda_n^* = \delta_{\pi_n^*}$ [2]. Therefore we have

$$\pi_n^* = \begin{cases} -n, & \text{if } -\dfrac{b(y)p_1 + (\mu,\nu_n^*)p_1 + (\sigma,\nu_n^*)\rho q_{12}}{(\sigma^2,\nu_n^*)q_{11}} < -n, \\ -\dfrac{b(y)p_1 + (\mu,\nu_n^*)p_1 + (\sigma,\nu_n^*)\rho q_{12}}{(\sigma^2,\nu_n^*)q_{11}}, & \text{if } -\dfrac{(b(y)p_1 + \mu,\nu_n^*)p_1 + (\sigma,\nu_n^*)\rho q_{12}}{(\sigma^2,\nu_n^*)q_{11}} \in [-n,n] \\ n, & \text{if } -\dfrac{(b(y)p_1 + \mu,\nu_n^*)p_1 + (\sigma,\nu_n^*)\rho q_{12}}{(\sigma^2,\nu_n^*)q_{11}} > n \end{cases}$$

and

$$\begin{aligned} \mathcal{H}^{\pi_n^*,\nu_n^*}(x,y,p,q) &= \max_{\pi\in[-n,n]} \min_{\nu\in\mathcal{P}(K)} \mathcal{H}^{\pi,\nu}(x,y,p,q) \\ &\equiv \frac{1}{2}q_{22} + \beta(y)p_2 + xr(y)p_1 \\ &\quad + \max_{\pi\in[-n,n]} \min_{\nu\in\mathcal{P}(K)} \left[\frac{1}{2}(\sigma^2,\nu)q_{11}\pi^2 + (\sigma,\nu)\rho q_{12}\pi + (b(y) + (\mu,\nu))p_1\pi\right] \\ &= \frac{1}{2}q_{22} + \beta(y)p_2 + xr(y)p_1 \\ &\quad + \min_{\nu\in\mathcal{P}(K)} \max_{\pi\in[-n,n]} \left[\frac{1}{2}(\sigma^2,\nu)q_{11}\pi^2 + (\sigma,\nu)\rho q_{12}\pi + (b(y) + (\mu,\nu))p_1\pi\right] \\ &\equiv \min_{\nu\in\mathcal{P}(K)} \max_{\pi\in[-n,n]} \mathcal{H}^{\pi,\nu}(x,y,p,q). \end{aligned} \tag{2.18}$$

By compactness of $\mathcal{P}(K)$ we can assume without loss of generality that the sequence ν_n^* is convergent to some ν^*. Thus

$$\pi_n^* \to \pi^* \equiv -\frac{b(y)p_1 + (\mu,\nu^*)p_1 + (\sigma,\nu^*)\rho q_{12}}{(\sigma^2,\nu^*)q_{11}} \quad \text{as } n \to \infty.$$

It remains to use the equalities

$$\begin{aligned} &\max_{\pi\in\mathbb{R}} \min_{\nu\in\mathcal{P}(K)} \mathcal{H}^{\pi,\nu}(x,y,p,q) \\ &\qquad = \lim_{n\to\infty} \max_{\pi\in[-n,n]} \min_{\nu\in\mathcal{P}(K)} \mathcal{H}^{\pi,\nu}(x,y,p,q) = \mathcal{H}^{\pi^*,\nu^*}(x,y,p,q) \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} &\min_{(\mu,\sigma)\in K} \max_{\pi\in\mathbb{R}} \mathcal{H}^{\pi,\mu,\sigma}(x,y,p,q) \\ &\qquad = \lim_{n\to\infty} \min_{(\mu,\sigma)\in K} \max_{\pi\in[-n,n]} \mathcal{H}^{\pi,\mu,\sigma}(x,y,p,q) = \mathcal{H}^{\pi^*,\nu^*}(x,y,p,q) \end{aligned} \tag{2.20}$$

to conclude that (π^*,ν^*) is saddle point of the problem.

[2] δ_a denotes a measure with support in the point a

On the other hand for each continuous function f on K

$$\min_{\nu\in\mathcal{P}(K)}(f,\nu) = \min_{(\mu,\sigma)\in K} f(\mu,\sigma),$$

since for $\nu^* = \arg\min_\nu(f,\nu)$ we have $\mathrm{supp}\nu^* \subseteq \{(\mu^*,\sigma^*)|f(\mu^*,\sigma^*) = \min f(\mu,\sigma)\}$. Hence

$$\max_{\pi\in[-n,n]}\min_{\nu\in\mathcal{P}(K)}\mathcal{H}^{\pi,\nu}(x,y,p,q) = \max_{\pi\in[-n,n]}\min_{(\mu,\sigma)\in K}\mathcal{H}^{\pi,\mu,\sigma}(x,y,p,q). \tag{2.21}$$

This equality together to (2.19),(2.20) prove (2.16). □

Now we define the value functions

$$\begin{aligned} v^-(t,x,y) &= \max_{\pi\in\Pi_x}\min_{(\mu,\sigma)\in\mathcal{U}_K} EU(X_T^{t,x,y}) \\ v^+(t,x,y) &= \min_{(\mu,\sigma)\in\bar{\mathcal{U}}_K}\max_{\pi\in\Pi_x} EU(X_T^{t,x,y}). \end{aligned} \tag{2.22}$$

Since the Isaacs condition is satisfied (by Proposition 2.1) there exists value of differential game $v \equiv v^+ = v^-$, which would be solution of HJBI equation

$$\frac{\partial}{\partial t}v(t,x,y) + \mathcal{H}(t,y,v_x(t,x,y),v_y(t,x,y),v_{xx}(t,x,y),v_{xy}(t,x,y),v_{yy}(t,x,y)) = 0, \tag{2.23}$$

$$v(T,x,y) = U(x). \tag{2.24}$$

It can be rewritten as

$$\begin{aligned} &\frac{\partial}{\partial t}v(t,x,y) + \frac{1}{2}v_{yy}(t,x,y) + \beta(y)v_y(t,x,y) + xr(y)v_x(t,x,y) \\ &\quad + \min_{\nu\in\mathcal{P}(K)}\max_{\pi\in R}\Big[\frac{1}{2}(\sigma^2,\nu)v_{xx}(t,x,y)\pi^2 + (\sigma,\nu)\rho v_{xy}(t,x,y)\pi \\ &\quad + (b(y)+(\mu,\nu))v_x(t,x,y)\pi\Big] = 0, \qquad (2.25) \\ &v(T,x,y) = U(x). \qquad (2.26) \end{aligned}$$

Simplifying the expression we have

$$\begin{aligned} &\min_{\nu\in\mathcal{P}(K)}\max_{\pi\in\mathbb{R}}\left[\frac{1}{2}(\sigma^2,\nu)q_{11}\pi^2 + (\sigma,\nu)\rho q_{12}\pi + b(y)p_1\pi + (\mu,\nu)p_1\pi\right] \\ &= \min_{\nu\in\mathcal{P}(K)}\left[\frac{((\sigma,\nu)\rho q_{12} + (b(y)+(\mu,\nu))p_1)^2}{-2(\sigma^2,\nu)q_{11}}\right] \\ &= \begin{cases} -\dfrac{p_1^2}{2q_{11}}\min\limits_{\nu\in\mathcal{P}(K)}\left[\dfrac{((\sigma,\nu)\kappa + b(y) + (\mu,\nu))^2}{(\sigma^2,\nu)}\right], & \text{if } p_1 \neq 0 \\ -\dfrac{\rho^2 q_{12}^2}{2\sigma_M}, & \text{if } p_1 = 0, \end{cases} \end{aligned} \tag{2.27}$$

where we suppose that $q_{11} < 0$ and use the notation $\kappa = \frac{\rho q_{12}}{p_1}$.

For the sake of simplicity we assume in addition

A3) $b(y) + \mu_- \geq 0,$ for all $y \in \mathbb{R}$.

Proposition 2.2. *There exists $\nu^* \in \mathcal{P}(K)$ of the form $\nu^* = \alpha\delta_{\mu_\pm,\sigma_-} + (1-\alpha)\delta_{\mu_\pm,\sigma_+}$, $0 \le \alpha \le 1$, such that*

$$\min_{\nu\in\mathcal{P}(K)} \left[\frac{((b(y)+\mu,\nu)+\kappa(\sigma,\nu))^2}{(\sigma^2,\nu)}\right] = \frac{((b(y)+\mu,\nu^*)+\kappa(\sigma,\nu^*))^2}{(\sigma^2,\nu^*)} \tag{2.28}$$

and

$$((\mu,\nu^*),(\sigma,\nu^*)) = \begin{cases} (\mu_+, \frac{\mu_+}{\kappa} + \frac{\sigma_-\sigma_+}{\sigma_M}), & \text{if } \kappa \in \left(-\infty, \dfrac{\mu_+\sigma_M}{\sigma_M\sigma_- - \sigma_+\sigma_-}\right] \\ (\mu_+,\sigma_-), & \text{if } \kappa \in \left(\dfrac{\mu_+\sigma_M}{\sigma_M\sigma_- - \sigma_+\sigma_-}, -\dfrac{\mu_+}{\sigma_-}\right] \\ (\kappa,-1) \text{ constant}, & \text{if } \kappa \in \left(-\dfrac{\mu_+}{\sigma_-}, -\dfrac{\mu_-}{\sigma_+}\right] \\ (\mu_-,\sigma_+), & \text{if } \kappa \in \left(-\dfrac{\mu_-}{\sigma_+}, \dfrac{\mu_-\sigma_M}{\sigma_M\sigma_+ - \sigma_+\sigma_-}\right] \\ (\mu_-, \frac{\mu_-}{\kappa} + \frac{\sigma_-\sigma_+}{\sigma_M}), & \text{if } \kappa \in \left(\dfrac{\mu_-\sigma_M}{\sigma_M\sigma_+ - \sigma_+\sigma_-}, \infty\right) \end{cases}, \tag{2.29}$$

$$\frac{((b(y)+\mu,\nu^*)+\kappa(\sigma,\nu^*))^2}{(\sigma^2,\nu^*)}$$
$$= \begin{cases} \dfrac{\kappa(2(b(y)+\mu_+)\sigma_M + \kappa\sigma_-\sigma_+)}{\sigma_M^2}, & \text{if } \kappa \in \left(-\infty, \dfrac{\mu_+\sigma_M}{\sigma_M\sigma_- - \sigma_+\sigma_-}\right] \\ \dfrac{(b(y)+\mu_+ + \kappa\sigma_-)^2}{\sigma_-^2}, & \text{if } \kappa \in \left(\dfrac{\mu_+\sigma_M}{\sigma_M\sigma_- - \sigma_+\sigma_-}, -\dfrac{\mu_+}{\sigma_-}\right] \\ 0, & \text{if } \kappa \in \left(-\dfrac{\mu_+}{\sigma_-}, -\dfrac{\mu_-}{\sigma_+}\right] \\ \dfrac{(b(y)+\mu_- + \kappa\sigma_+)^2}{\sigma_+^2}, & \text{if } \kappa \in \left(-\dfrac{\mu_-}{\sigma_+}, \dfrac{\mu_-\sigma_M}{\sigma_M\sigma_+ - \sigma_+\sigma_-}\right] \\ \dfrac{\kappa(2(b(y)+\mu_-)\sigma_M + \kappa\sigma_-\sigma_+)}{\sigma_M^2}, & \text{if } \kappa \in \left(\dfrac{\mu_-\sigma_M}{\sigma_M\sigma_+ - \sigma_+\sigma_-}, \infty\right) \end{cases}. \tag{2.30}$$

The proof is given in Appendix.

Corollary 1.

$$\min_{\nu\in\mathcal{P}(K)}\left[\frac{((b(y)+\mu,\nu)p_1+(\sigma,\nu)\rho q_{12})^2}{2(\sigma^2,\nu)q_{11}}\right]=\min_{(\mu,\sigma)\in K}\left[\frac{(b(y)p_1+\mu p_1+\sigma\rho q_{12})^2}{2(2\sigma_M\sigma-\sigma_-\sigma_+)q_{11}}\right]$$

$$=\begin{cases}\dfrac{\rho q_{12}(2p_1(b(y)+\mu_+)\sigma_M+\rho q_{12}\sigma_-\sigma_+)}{2q_{11}\sigma_M^2}, & \text{if } \dfrac{\rho q_{12}}{p_1}\in\left(-\infty,\dfrac{\mu_+\sigma_M}{\sigma_M\sigma_--\sigma_+\sigma_-}\right]\\ \dfrac{(p_1(b(y)+\mu_+)+\rho q_{12}\sigma_-)^2}{2q_{11}\sigma_-^2}, & \text{if } \dfrac{\rho q_{12}}{p_1}\in\left(\dfrac{\mu_+\sigma_M}{\sigma_M\sigma_--\sigma_+\sigma_-},-\dfrac{\mu_+}{\sigma_-}\right]\\ 0, & \text{if } \dfrac{\rho q_{12}}{p_1}\in\left(-\dfrac{\mu_+}{\sigma_-},-\dfrac{\mu_-}{\sigma_+}\right]\\ \dfrac{(p_1(b(y)+\mu_-)+\rho q_{12}\sigma_+)^2}{2q_{11}\sigma_+^2}, & \text{if } \dfrac{\rho q_{12}}{p_1}\in\left(-\dfrac{\mu_-}{\sigma_+},\dfrac{\mu_-\sigma_M}{\sigma_M\sigma_+-\sigma_+\sigma_-}\right]\\ \dfrac{\rho q_{12}(2p_1(b(y)+\mu_-)\sigma_M+\rho q_{12}\sigma_-\sigma_+)}{2q_{11}\sigma_M^2}, & \text{if } \dfrac{\rho q_{12}}{p_1}\in\left(\dfrac{\mu_-\sigma_M}{\sigma_M\sigma_+-\sigma_+\sigma_-},\infty\right)\\ \frac{\rho^2q_{12}^2}{2\sigma_M}, & \text{if } p_1=0\end{cases}. \qquad (2.31)$$

Proof. It is sufficient to verify that for $\nu^*=\alpha\delta_{\mu_\pm,\sigma_-}+(1-\alpha)\delta_{\mu_\pm,\sigma_+},\ 0\le\alpha\le 1$ we get $(\sigma^2,\nu^*)=2\sigma_M(\sigma,\nu^*)-\sigma_-\sigma_+$.

From this Corollary we obtain that the HJBI equation has the form

$$\frac{\partial}{\partial t}v(t,x,y)+\frac{1}{2}v_{yy}(t,x,y)+\beta(y)v_y(t,x,y)+xr(y)v_x(t,x,y)$$
$$-\min_{(\mu,\sigma)\in K}\frac{(b(y)v_x(t,x,y)+\mu v_x(t,x,y)+\rho\sigma v_{xy}(t,x,y))^2}{2(2\sigma_M\sigma-\sigma_-\sigma_+)v_{xx}(t,x,y)}=0, \qquad (2.32)$$
$$v(T,x,y)=U(x). \qquad (2.33)$$

Following to the Theorem 6 of [18] we can prove

Theorem 1 (Verification Theorem)**.** *Let $v(t,x,y)$ be a classical solution of (2.25), (2.26) with $v_{xx}<0$. Then there exists ν^* defined by (2.29) with $\kappa=\rho\frac{v_{xy}}{v_x}$ and the optimal strategy is given by*

$$\pi^*(t,x,y)=-\frac{(b(y)+\mu,\nu^*(t,x,y))v_x(t,x,y)+(\sigma,\nu^*(t,x,y))\rho v_{xy}(t,x,y)}{(2\sigma_M(\sigma,\nu^*(t,x,y)))-\sigma_-\sigma_+)v_{xx}(t,x,y)}, \qquad (2.34)$$

where

$$
\begin{aligned}
&((\mu,\nu^*(t,x,y)),(\sigma,\nu^*(t,x,y)))\\
&=\begin{cases}
\left(\mu_+,\dfrac{\mu_+ v_x(t,x,y)}{\rho v_{xy}(t,x,y)}+\dfrac{\sigma_-\sigma_+}{\sigma_M}\right), & \text{if } \dfrac{\rho v_{xy}(t,x,y)}{v_x(t,x,y)}\in\left(-\infty,\dfrac{\mu_+\sigma_M}{\sigma_M\sigma_- -\sigma_+\sigma_-}\right]\\
(\mu_+,\sigma_-), & \text{if } \dfrac{\rho v_{xy}(t,x,y)}{v_x(t,x,y)}\in\left(\dfrac{\mu_+\sigma_M}{\sigma_M\sigma_- -\sigma_+\sigma_-},-\dfrac{\mu_+}{\sigma_-}\right]\\
\left(\dfrac{\rho v_{xy}(t,x,y)}{v_x(t,x,y)},-1\right)\text{constant}, & \text{if } \dfrac{\rho v_{xy}(t,x,y)}{v_x(t,x,y)}\in\left(-\dfrac{\mu_+}{\sigma_-},-\dfrac{\mu_-}{\sigma_+}\right]\\
(\mu_-,\sigma_+), & \text{if } \dfrac{\rho v_{xy}(t,x,y)}{v_x(t,x,y)}\in\left(-\dfrac{\mu_-}{\sigma_+},\dfrac{\mu_-\sigma_M}{\sigma_M\sigma_+ -\sigma_+\sigma_-}\right]\\
\left(\mu_-,\dfrac{\mu_+ v_x(t,x,y)}{\rho v_{xy}(t,x,y)}+\dfrac{\sigma_-\sigma_+}{\sigma_M}\right), & \text{if } \dfrac{\rho v_{xy}(t,x,y)}{v_x(t,x,y)}\in\left(\dfrac{\mu_-\sigma_M}{\sigma_M\sigma_+ -\sigma_+\sigma_-},\infty\right)
\end{cases}. \qquad (2.35)
\end{aligned}
$$

3 The power utility case

We now consider the robust utility maximization problem with power utility $U(x)=\frac{1}{q}x^q$, with $q<1,\ q\neq 0$. Hence we obtain the equation

$$
\begin{aligned}
&\frac{\partial}{\partial t}v(t,x,y)+\frac{1}{2}v_{yy}(t,x,y)+\beta(y)v_y(t,x,y)+xr(y)v_x(t,x,y)\\
&\qquad-\min_{(\mu,\sigma)\in K}\frac{((b(y)+\mu)v_x(t,x,y)+\rho\sigma v_{xy}(t,x,y))^2}{2(2\sigma_M\sigma-\sigma_-\sigma_+)v_{xx}(t,x,y)}=0, \qquad (3.1)\\
&\qquad v(T,x,y)=\frac{1}{q}x^q. \qquad (3.2)
\end{aligned}
$$

The solution of this equation is of the form $v(t,x,y)=\frac{1}{q}x^q e^{u(t,y)}$, where u satisfies

$$
\begin{aligned}
&\frac{\partial}{\partial t}u(t,y)+\frac{1}{2}u_{yy}(t,y)+\beta(y)u_y(t,y)+\frac{1}{2}u_y^2(t,y)+qr(y)\\
&\qquad-\frac{1}{2(q-1)}\min_{(\mu,\sigma)\in K}\frac{(b(y)+\mu+\rho\sigma u_y(t,y))^2}{2\sigma_M\sigma-\sigma_-\sigma_+}=0, \qquad (3.3)\\
&\qquad u(T,y)=0. \qquad (3.4)
\end{aligned}
$$

It is evident that $v_{xx}(t,x,y)=(q-1)x^{q-2}e^{u(t,y)}<0$.

The equations (2.35) take the form

$$((\mu,\nu^*(t,y)),(\sigma,\nu^*(t,y)))$$
$$=\begin{cases}\left(\mu_+,\dfrac{\mu_+}{\rho}u_y(t,y)+\dfrac{\sigma_-\sigma_+}{\sigma_M}\right), & \text{if } \rho u_y(t,y)\in\left(-\infty,\dfrac{\mu_+\sigma_M}{\sigma_M\sigma_- -\sigma_+\sigma_-}\right]\\ (\mu_+,\sigma_-), & \text{if } \rho u_y(t,y)\in\left(\dfrac{\mu_+\sigma_M}{\sigma_M\sigma_- -\sigma_+\sigma_-},-\dfrac{\mu_+}{\sigma_-}\right]\\ (\rho u_y(t,y),-1)\text{constant}, & \text{if } \rho u_y(t,y)\in\left(-\dfrac{\mu_+}{\sigma_-},-\dfrac{\mu_-}{\sigma_+}\right]\\ (\mu_-,\sigma_+), & \text{if } \rho u_y(t,y)\in\left(-\dfrac{\mu_-}{\sigma_+},\dfrac{\mu_-\sigma_M}{\sigma_M\sigma_+ -\sigma_+\sigma_-}\right]\\ (\mu_-,\dfrac{\mu_+}{\rho}u_y(t,y)+\dfrac{\sigma_-\sigma_+}{\sigma_M}), & \text{if } \rho u_y(t,y)\in\left(\dfrac{\mu_-\sigma_M}{\sigma_M\sigma_+ -\sigma_+\sigma_-},\infty\right)\end{cases}. \quad (3.5)$$

Remark 3.1. By corollary 1 and (2.35) the equation (3.3) can be written as

$$\begin{aligned}&\frac{\partial}{\partial t}u(t,y)+\frac{1}{2}u_{yy}(t,y)+\beta(y)u_y(t,y)+\frac{1}{2}u_y^2(t,y)+qr(y)\\ &-\frac{\rho u_y(t,y)}{2(q-1)\sigma_M^2}(2(b(y)+\mu_+)\sigma_M+\sigma_-\sigma_+\rho u_y(t,y))\chi\left(\rho u_y(t,y)\le\frac{\mu_+\sigma_M}{\sigma_M\sigma_- -\sigma_+\sigma_-}\right)\\ &-\frac{1}{2(q-1)\sigma_-^2}(b(y)+\mu_+ +\rho\sigma_- u_y(t,y))^2\chi\left(\frac{\mu_+\sigma_M}{\sigma_M\sigma_- -\sigma_+\sigma_-}<\rho u_y(t,y)\le-\frac{\mu_+}{\sigma_-}\right)\\ &-\frac{1}{2(q-1)\sigma_+^2}(b(y)+\mu_- +\rho\sigma_+ u_y(t,y))^2\chi\left(-\frac{\mu_-}{\sigma_+}<\rho u_y(t,y)\le\frac{\mu_-\sigma_M}{\sigma_M\sigma_+ -\sigma_+\sigma_-}\right)\\ &-\frac{\rho u_y(t,y)}{2(q-1)\sigma_M^2}(2(b(y)+\mu_-)\sigma_M+\sigma_-\sigma_+\rho u_y(t,y))\chi\left(\rho u_y(t,y)>\frac{\mu_-\sigma_M}{\sigma_M\sigma_+ -\sigma_+\sigma_-}\right)=0, \quad (3.6)\\ &u(T,y)=0, \quad (3.7)\end{aligned}$$

where χ is the characteristic function.

Theorem 2. *Under conditions A1)-A3) the problem (3.3), (3.4) admits a classical solution with bounded $u_y(t,y)$ and a saddle point $(\nu^*(t,y),\pi^*(t,x,y))$ of the problem (2.6), (2.10) is defined by (3.5) and by*

$$\pi^*(t,x,y)=\frac{x}{1-q}\left(\frac{b(y)+(\mu,\nu^*(t,y))}{(\sigma^2,\nu^*(t,y))}+\rho\frac{(\sigma,\nu^*(t,y))}{(\sigma^2,\nu^*(t,y))}u_y(t,y)\right). \quad (3.8)$$

Proof. By condition A2) there exists $N\ge 0$ such that $b'(y)=0$, if $|y|>N$. Thus $b(y)=b^+$, if $y\ge N$ and $b(y)=b^-$, if $y\le -N$ for some constants b^+,b^-. The solution of (3.3) on the intervals $(-\infty,-N]$ and $[N,\infty)$ are $u^-(t)=-\frac{1}{2(q-1)}\frac{(b^-+\mu_-)^2}{\sigma_+^2}(T-t)$ and $u^+(t)=-\frac{1}{2(q-1)}\frac{(b^++\mu_-)^2}{\sigma_+^2}(T-t)$ respectively. Now we consider the Cauchy-Dirichlet problem on the bounded domain $(0,T)\times(-N,N)$

$$\frac{\partial}{\partial t}u(t,y)+\frac{1}{2}u_{yy}(t,y)+\beta(y)u_y(t,y)+\frac{1}{2}u_y^2(t,y)$$
$$-\frac{1}{2(q-1)}\min_{(\mu,\sigma)\in K}\frac{(b(y)+\mu+\rho\sigma u_y(t,y))^2}{2\sigma_M\sigma-\sigma_-\sigma_+}=0, \tag{3.9}$$
$$u(T,y)=0,\ u(t,\pm N)=u^{\pm}(t). \tag{3.10}$$

Suppose

$$a_1(t,y,u,p)=\frac{1}{2}p,$$
$$a(t,y,u,p)=\beta(y)p+\frac{1}{2}p-\frac{1}{2(q-1)}\min_{(\mu,\sigma)\in K}\frac{(b(y)+\mu+\rho\sigma p)^2}{2\sigma_M\sigma-\sigma_-\sigma_+}.$$

It is easy to see that a is Lipschitz function on the each ball of its domain, $a(t,y,u,0)$ is bounded below and all conditions of Theorem 6.2 chapt.V of ([15]) are satisfied. Therefore there exists a classical solution of (3.3),(3.4) with bounded $u_y(t,y)$ (the existence of classical solution follows also from Example 3.6 of [10] if we consider mixed problem with boundary conditions $u(T,y)=0,\ u_y(t,\pm N)+u(t,\pm N)=u^{\pm}(t)$).

Now we can use the Theorem 1. From (2.34) follows that the strategy is of the form

$$\pi^*(t,x,y)=-\frac{1}{q-1}\frac{b(y)+(\mu,\nu^*(t,y))+(\sigma,\nu^*(t,y))\rho u_y(t,y)}{2(\sigma,\nu^*(t,y))\sigma_M-\sigma_-\sigma_+}x$$
$$=\frac{1}{1-q}\frac{b(y)+(\mu,\nu^*(t,y))+(\sigma,\nu^*(t,y))\rho u_y(t,y)}{(\sigma^2,\nu^*(t,y))}x, \tag{3.11}$$

where $\nu^*(t,y)$ is defined by (3.5).

Corollary 2. *If $b=0$ then*

$$u(t,y)=-\frac{1}{2(q-1)}(T-t)\min_{(\mu,\sigma)\in K}\frac{\mu^2}{2\sigma_M\sigma-\sigma_-\sigma_+}=-\frac{1}{2(q-1)}(T-t)\frac{\mu_-^2}{\sigma_+^2}$$

is a solution of (3.3) and a saddle point of the maximin problem can be given explicitly

$$(\mu_t^*,\sigma_t^*)=(\mu_-,\sigma_+),\quad \pi^*(t,x,y)=-\frac{\mu_-}{2(q-1)\sigma_+^2}x.$$

Remark 3.2. When $\sigma_-=\sigma_+=\sigma_M$ we obtain

$$\frac{\partial}{\partial t}u(t,y)+\frac{1}{2}u_{yy}(t,y)+\beta(y)u_y(t,y)+\frac{1}{2}u_y^2(t,y)$$
$$-\frac{1}{2(q-1)\sigma_M^2}\min_{\mu_-\le\mu\le\mu_+}(b(y)+\mu+\rho\sigma_M u_y(t,y))^2$$
$$\equiv\frac{\partial}{\partial t}u(t,y)+\frac{1}{2}u_{yy}(t,y)+(2\rho\sigma_M b(y)+\beta(y))u_y(t,y)+\frac{1}{2}\left(1-\frac{\rho^2\sigma_M}{q-1}\right)u_y^2(t,y)$$
$$-\frac{1}{2(q-1)\sigma_M^2}\min_{\mu_-\le\mu\le\mu_+}((b(y)+\mu)^2+2\mu\rho\sigma_M u_y(t,y))=0, \tag{3.12}$$
$$u(T,y)=0. \tag{3.13}$$

The existence of classical solution of such type equation has been obtained by D. Hernández-Hernández and A. Schied in [8].

Remark 3.3. Instead of PDE (3.3) we can use BSDE with quadratic growth

$$dV_t = -\Big(\beta(w_t)Z_t + \frac{1}{2}Z_t^2 + qr(w_t) \\ - \frac{1}{2(q-1)} \min_{(\mu,\sigma)\in K} \frac{(b(w_t)+\mu+\rho\sigma Z_t)^2}{2\sigma_M\sigma - \sigma_-\sigma_+}\Big)dt + Z_t dw_t + Z_t^{\perp} dw_t^{\perp}, \tag{3.14}$$

$$V_T = 0. \tag{3.15}$$

which solvability follows from the results of [12], [23]. The strategy now is a solution of forward SDE

$$\pi_t^* = \frac{1}{1-q}\left(\frac{b(w_t)+(\mu,\nu_t^*(Z))}{(\sigma^2,\nu_t^*(Z))} + \rho\frac{(\sigma,\nu_t^*(Z))}{(\sigma^2,\nu_t^*(Z))}Z_t\right)X_t(\pi^*). \tag{3.16}$$

A Appendix

Each measure ν may be realized as a distribution of the pair of random variables (ξ,η) with the value in D. Simplifying the notation we $b(y)+\mu$ denote again by μ. Our aim is to characterize the dependence of the minimizer of the problem

$$\min_{\nu\in\mathcal{P}(K)}\left[\frac{((\mu,\nu)+\kappa(\sigma,\nu))^2}{(\sigma^2,\nu)}\right] = \min_{(\xi,\eta)\in K}\left[\frac{(E\xi+\kappa E\eta)^2}{E\eta^2}\right] \tag{A.1}$$

on the parameter $\kappa\in\mathbb{R}$.

Proposition A.1. *The pair*

$$(\xi^*,\eta^*) = \arg\min_{(\xi,\eta)\in K}\left[\frac{(E\xi+\kappa E\eta)^2}{E\eta^2}\right]$$

is such that ξ^* is number, η^* is Bernoulli random variables with value $\{\sigma_-,\sigma_+\}$ and their expectations are given as

$$(\xi^*, E\eta^*) = \begin{cases} \left(\mu_+, \dfrac{\mu_+}{\kappa} + \dfrac{\sigma_-\sigma_+}{\sigma_M}\right), & \text{if } \kappa\in\left(-\infty, \dfrac{\mu_+\sigma_M}{\sigma_M\sigma_- - \sigma_+\sigma_-}\right] \\ (\mu_+,\sigma_-), & \text{if } \kappa\in\left(\dfrac{\mu_+\sigma_M}{\sigma_M\sigma_- - \sigma_+\sigma_-}, -\dfrac{\mu_+}{\sigma_-}\right] \\ (\kappa,-1)\text{constant}, & \text{if } \kappa\in\left(-\dfrac{\mu_+}{\sigma_-}, -\dfrac{\mu_-}{\sigma_+}\right] \\ (\mu_-,\sigma_+), & \text{if } \kappa\in\left(-\dfrac{\mu_-}{\sigma_+}, \dfrac{\mu_-\sigma_M}{\sigma_M\sigma_+ - \sigma_+\sigma_-}\right] \\ \left(\mu_-, \dfrac{\mu_-}{\kappa} + \dfrac{\sigma_-\sigma_+}{\sigma_M}\right), & \text{if } \kappa\in\left(\dfrac{\mu_-\sigma_M}{\sigma_M\sigma_+ - \sigma_+\sigma_-}, \infty\right) \end{cases}. \tag{A.2}$$

Moreover

$$\frac{(\xi^*+\kappa E\eta^*)^2}{E\eta^{*2}} = \begin{cases} \dfrac{\kappa(2\mu_+\sigma_M+\kappa\sigma_-\sigma_+)}{\sigma_M^2}, & \text{if } \kappa\in\left(-\infty, \dfrac{\mu_+\sigma_M}{\sigma_M\sigma_- - \sigma_+\sigma_-}\right] \\ \dfrac{(\mu_++\kappa\sigma_-)^2}{\sigma_-^2}, & \text{if } \kappa\in\left(\dfrac{\mu_+\sigma_M}{\sigma_M\sigma_- - \sigma_+\sigma_-}, -\dfrac{\mu_+}{\sigma_-}\right] \\ 0, & \text{if } \kappa\in\left(-\dfrac{\mu_+}{\sigma_-}, -\dfrac{\mu_-}{\sigma_+}\right] \\ \dfrac{(\mu_-+\kappa\sigma_+)^2}{\sigma_+^2}, & \text{if } \kappa\in\left(-\dfrac{\mu_-}{\sigma_+}, \dfrac{\mu_-\sigma_M}{\sigma_M\sigma_+ - \sigma_+\sigma_-}\right] \\ \dfrac{\kappa(2\mu_-\sigma_M+\kappa\sigma_-\sigma_+)}{\sigma_M^2}, & \text{if } \kappa\in\left(\dfrac{\mu_-\sigma_M}{\sigma_M\sigma_+ - \sigma_+\sigma_-}, \infty\right) \end{cases}. \tag{A.3}$$

Proof. Let $(\mu_++\kappa\sigma_-)(\mu_-+\kappa\sigma_+)\leq 0$. Then by continuity of function $\mu+\kappa\sigma, (\mu,\sigma)\in K$, there exists $(\hat\mu,\hat\sigma)$ such that $\hat\mu+\kappa\hat\sigma=0$. Thus $(\hat\mu,\hat\sigma)\propto(\kappa,-1)$ and $\left[\frac{(E\xi^*+\kappa E\eta^*)^2}{E\eta^{*2}}\right]=0$. If $(\mu_++\kappa\sigma_-)(\mu_-+\kappa\sigma_+)>0$ then either $\kappa>\frac{\mu_-}{\sigma_+}$ and $\xi^*=\mu_-$ or $\kappa<-\frac{\mu_+}{\sigma_-}$ and $\xi^*=\mu_+$. Thus it is sufficient to study the minimization problem

$$\min_{\eta\in[\sigma_-,\sigma_+]}\left[\frac{(\mu_a+\kappa E\eta)^2}{E\eta^2}\right] \text{ for } a=+,-.$$

We will show that η^* is of the form $\eta^*=\sigma_-\chi_B+\sigma_+\chi_{B^c}$ for some event B. Indeed, if $E\eta^*=y$ then $E\eta^{*2}=2\sigma_My-\sigma_-\sigma_+$ and η^* is maximizer of the problem

$$\max_{\eta, E\eta=y} E\eta^2,$$

since for any η, with $E\eta=y$ we get

$$\begin{aligned} E\eta^2 &= E(\eta-\sigma_M)^2+2\sigma_My-\sigma_M^2 \\ &\leq \left(\frac{\sigma_+-\sigma_-}{2}\right)^2+2\sigma_My-\sigma_M^2 \\ &= 2\sigma_My-\sigma_-\sigma_+ = E\eta^{*2}. \end{aligned}$$

Hence

$$\min_{\eta\in[\sigma_-,\sigma_+]}\left[\frac{(\mu_a+\kappa E\eta)^2}{E\eta^2}\right] = \min_{\sigma_-\leq y\leq\sigma_+}\psi_a(y),$$

where $\psi_a(y)=\frac{(\mu_a+\kappa y)^2}{2\sigma_My-\sigma_-\sigma_+}$. Since

$$\psi_a'(y)=\frac{\kappa^2}{2\sigma_M}-\frac{\kappa^2}{2\sigma_M}\frac{(2\sigma_M\frac{\mu_a}{\kappa}+\sigma_-\sigma_+)^2}{(2\sigma_My-\sigma_-\sigma_+)^2}$$

the equation $\psi_a'(y)=0$ has two roots;

$$y_1^a=-\frac{\mu_a}{\kappa},\ y_2^a=\frac{\mu_a}{\kappa}+\frac{\sigma_-\sigma_+}{\sigma_M}.$$

If $y_1^a = -\frac{\mu_a}{\kappa} \in [\sigma_-, \sigma_+]$ then $y_2^a = \frac{\mu_a}{\kappa} + \frac{\sigma_-\sigma_+}{\sigma_M} \in [-\sigma_+ + \frac{\sigma_-\sigma_+}{\sigma_M}, -\sigma_- + \frac{\sigma_-\sigma_+}{\sigma_M}]$ and vise versa. Moreover $[\sigma_-, \sigma_+] \cap [-\sigma_+ + \frac{\sigma_-\sigma_+}{\sigma_M}, -\sigma_- + \frac{\sigma_-\sigma_+}{\sigma_M}] = \emptyset$. Since $\lim_{y\to\pm\infty} \psi_a(y) = \pm\infty$ then the least root is the maximizer and highest root is the minimizer. The case of $y_1^a \in [\sigma_-, \sigma_+]$ is equivalent to

$$\kappa \in [-\frac{\sigma_+}{\mu_a}, -\frac{\sigma_-}{\mu_a}]$$

and gives $\min \psi_a(y) = \psi_a(y_1^a) = 0$. From the relation $y_2^a \in [\sigma_-, \sigma_+]$ follows $-\sigma_+ + \frac{\sigma_-\sigma_+}{\sigma_M} \leq -\frac{\mu_a}{\kappa} \leq -\sigma_+ - \frac{\sigma_-\sigma_+}{\sigma_M}$ which equivalent to

$$\kappa \in (-\infty, \frac{\mu_a}{\sigma_- - \frac{\sigma_-\sigma_+}{\sigma_M}}] \cup [\frac{\mu_a}{\sigma_+ - \frac{\sigma_-\sigma_+}{\sigma_M}}, \infty).$$

In this case $\min_{\sigma_-\leq y\leq\sigma_+} \psi_a(y) = \psi_a(y_2^a) = \kappa\frac{2\mu_a + \kappa\sigma_-\sigma_+}{\sigma_M^2}$.

Now we will consider step by step the all possibilities of displacement of κ in the intervals formulated in Proposition.

1) $\kappa \in (-\infty, \frac{\mu_a}{\sigma_- - \frac{\sigma_-\sigma_+}{\sigma_M}}]$. Since $\frac{\mu_a}{\sigma_- - \frac{\sigma_-\sigma_+}{\sigma_M}} \leq -\frac{\mu_+}{\sigma_-}$ then $\kappa \in (-\infty, -\frac{\mu_+}{\sigma_-}]$ and $\xi^* = \mu_+$. Moreover $\min \psi_+(y) = \psi_+(y_2^+) = \kappa\frac{2\mu_+ + \kappa\sigma_-\sigma_+}{\sigma_M^2}$.

2) $\kappa \in (\frac{\mu_+}{\sigma_- - \frac{\sigma_-\sigma_+}{\sigma_M}}, -\frac{\mu_+}{\sigma_-}]$. From $\kappa \leq -\frac{\mu_+}{\sigma_-}$ follows that $y_1^+ = -\frac{\mu_+}{\kappa} < \sigma_-$ and from $\kappa > \frac{\mu_+}{\sigma_- - \frac{\sigma_-\sigma_+}{\sigma_M}}$ follows $y_2^+ = \frac{\mu_+}{\kappa} + \frac{\sigma_-\sigma_+}{\sigma_M} < \sigma_-$. Hence $\psi_+(y)$ is increasing on $[\sigma_-, \sigma_+]$ and $\arg\min_{\sigma_-\leq y\leq\sigma_+} \psi_+(y) = \sigma_-$.

3) $\kappa \in (-\frac{\mu_+}{\sigma_-}, -\frac{\mu_-}{\sigma_+}]$. Then $y_1^+ = -\frac{\mu_+}{\kappa} \in [\sigma_-, \sigma_+]$ and $\min \psi_+(y) = 0$.

4) $\kappa \in (-\frac{\mu_-}{\sigma_+}, \frac{\mu_-}{\sigma_+ - \frac{\sigma_-\sigma_+}{\sigma_M}}]$. Then $\frac{\mu_-}{\kappa} > \sigma_+ - \frac{\sigma_-\sigma_+}{\sigma_M}$ and $y_1^- = -\frac{\mu_-}{\kappa} < -\sigma_+ + \frac{\sigma_-\sigma_+}{\sigma_M} < \sigma_-$, $y_2^- = \frac{\mu_-}{\kappa} + \frac{\sigma_-\sigma_+}{\sigma_M} > \sigma_+$. Hence $\psi_-(y)$ is decreasing on $[\sigma_-, \sigma_+]$ and $\arg\min \psi_+(y) = \sigma_+$.

5) $\kappa \in (\frac{\mu_-}{\sigma_+ - \frac{\sigma_-\sigma_+}{\sigma_M}}, \infty]$. Then $\kappa > \frac{\mu_-}{\sigma_+}$ and $\xi^* = \mu_-$. On the other hand from $\frac{\mu_-}{\kappa} < \sigma_+ - \frac{\sigma_-\sigma_+}{\sigma_M}$ follows $y_2^- \in [\sigma_-, \sigma_+]$. Hence $\min_{\sigma_-\leq y\leq\sigma_+} \psi_-(y) = \psi_-(y_2^-)$.

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