LPTHE-00-04

# Dual Baxter equations and quantization of Affine Jacobian.

F.A. Smirnov [0]

Laboratoire de Physique Théorique et Hautes Energies [1]
Université Pierre et Marie Curie, Tour 16, $1^{er}$ étage, 4 place Jussieu
75252 Paris Cedex 05, France

**Abstract.** A quantum integrable model is considered which describes a quantization of affine hyper-elliptic Jacobian. This model is shown to possess the property of duality: a dual model with inverse Planck constant exists such that the eigen-functions of its Hamiltonians coincide with the eigen-functions of Hamiltonians of the original model. We explain that this duality can be considered as duality between homologies and cohomologies of quantized affine hyper-elliptic Jacobian.

[0] Membre du CNRS
[1] *Laboratoire associé au CNRS.*

# 1 Introduction.

The relation between the affine Jacobian and integrable models is well known (*cf.* [1]). In the paper [2] we have shown that the algebra of functions on the affine Jacobian is generated by action of hamiltonian vector fields from finite number of functions. The latter functions are coefficients of highest non-vanishing cohomologies of the affine Jacobian. Actually, the idea that such description of the algebra of functions is possible appeared from the paper [3] which considers the structure of the algebra of observables for quantum and classical Toda chain.

In the present paper we give quantum version of [2]. A quantum mechanical model is formulated which gives a quantization of the affine Jacobian. As usual in Quantum Mechanics we can describe not the variety itself but the algebra of functions on it (observables). We need to show that the quantum algebra of observables possesses essential property of corresponding classical algebra of functions. In our case this property is the possibility of creating every observable from a finite number of observables (cohomologies) by action of Hamiltonians.

In the process of realization of this program we find the Baxter equations which describe the spectrum of the model. It happens that these equations possess the property of duality: there is dual model with inverse Planck constant for which the eigen-vectors are the same. The algebras of observables of two dual models commute. The next ingredient of our study is the method of separation of variables developed by Sklyanin [4]. Using this method we present the matrix elements of any observable in terms of certain integrals.

We show that the integrals in question are expresses in terms of deformed Abelian integrals (*cf.*[3, 5]). The observables for both dual models are defined in terms of cohomologies. The most beautiful feature of our construction is that in these cohomologies enter the integrals for matrix elements in such a way that the cohomologies of dual model play role of homologies for original one and vise a versa. We consider this relation between week-strong duality in quantum theory with duality between homologies and cohomologies as the most important conclusion of this paper.

# 2 Affine Jacobian.

In this section we briefly summarize necessary facts concerning relation between integrable models and algebraic geometry following the paper [2]. The reason for repeating certain facts from [2] is that we shall need them in slightly different situation.

Consider $2 \times 2$ matrix which depends polynomially on the parameter $z$:

$$m(z) = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix}$$

where the matrix elements are polynomials of the form:

$$a(z) = z^{g+1} + a_1 z^g + \cdots + a_{g+1}, \tag{1}$$
$$b(z) = z^g + b_1 z^{g-1} + \cdots + b_g,$$
$$c(z) = c_2 z^g + c_3 z^{g-1} \cdots + c_{g+2},$$
$$d(z) = d_2 z^{g-1} + d_3 z^{g-2} \cdots + d_{g+1}$$

In the the affine space $\mathbb{C}^{4g+2}$ with coordinates $a_1, \cdots, a_{g+1}$, $b_1, \cdots, b_g$, $c_2, \cdots, c_{g+2}$, $d_2, \cdots, d_{g+1}$ consider the (2g+1)-dimensional affine variety $\mathcal{M}$ defined as quadric

$$f(z) \equiv a(z)d(z) - b(z)c(z) = 1 \tag{2}$$

We consider this simplest situation, but in principle it is possible to put arbitrary polynomial of degree $2g$ in RHS.

On the quadric $\mathcal{M}$ let us consider the sections $J_{\text{aff}}(t)$ defined by the equations:

$$a(z) + d(z) = t(z) \tag{3}$$

where $t(z)$ is given polynomial of the form:

$$t(z) = z^{g+1} + z^g t_1 + \cdots + t_{g+1}, \tag{4}$$

The notation $J_{\text{aff}}(t)$ stands for affine Jacobi variety. The definition of affine Jacobi variety and its equivalence to $J_{\text{aff}}(t)$ described above are given in the Appendix A. We include Appendix A because there is minor difference with the situation considered in [1] and [2]. The variety $\mathcal{M}$ is foliated into the affine Jacobians $J_{\text{aff}}(t)$. Mechanical model described below provides a clever way of describing this foliation.

We would like to understand the geometrical meaning of quantum integrable models. The general philosophy teaches that in order to describe the quantization of a manifold one has to deform the algebra of functions on this manifold preserving certain essential properties of this algebra. The classical algebra must allow the Poisson structure in order that quantization is possible.

Certain Poisson brackets for the coefficients of matrix $m(z)$ can be introduced. We do not write them down explicitly, if needed they can be obtained taking classical limit of the commutation relations (14). The algebra

$$\widehat{\mathcal{A}} = \mathbb{C}\,[\, a_1, \cdots, a_{g+1}, b_1, \cdots, b_g, c_2, \cdots, c_{g+2}, d_2, \cdots, d_{g+1}]$$

becomes a Poisson algebra. The most important properties of this Poisson structure are the following. First, the coefficients of the determinant $f(z)$ belong to the center of the Poisson algebra, so, the equation (2) is consistent with Poisson structure. Second, the trace $t(z)$ generates commutative sub-algebra:

$$\{t(z), t(z')\} = 0$$

It can be shown, actually, that the coefficient $t_{g+1}$ of the trace belongs to the center, it is convenient to put $t_{g+1} = (-1)^{g+1}2$. The subject of our study is the algebra of functions on the $\mathcal{M}$:

$$\mathcal{A} = \frac{\widehat{\mathcal{A}}}{\{f(z) = 1,\ t_{g+1} = (-1)^{g+1}2\}}$$

on which the Poisson structure is well defined.

The Poisson commutative algebra generated by the coefficients $t_1, \cdots, t_g$ is called the algebra of integrals of motion. Introduce the commuting vector-fields

$$\partial_i g = \{t_i, g\}, \qquad i = 1, \cdots g$$

The vector-fields $\partial_j$ describe motion along the sub-varieties $J_{\text{aff}}(t)$.

One can think of these vector-fields as

$$\partial_j = \frac{\partial}{\partial \tau_j}$$

where $\tau_j$ are "times" corresponding to the integrals of motion $t_j$. Define the ring of integrals of motion

$$\mathcal{T} = \mathbb{C}\left[t_1, \cdots, t_g\right] \tag{5}$$

Introduce the space of differential forms $C^k$ with basis

$$x\, d\tau_{i_1} \wedge \cdots \wedge d\tau_{i_k}, \qquad x \in \mathcal{A}$$

and the differential

$$d = \partial_j d\tau_j$$

Consider corresponding cohomologies $H^k$. In the paper [2] the arguments are given in favour of the following

**Conjecture 1.** *The cohomologies $H^k$ are finite-dimensional over the ring $\mathcal{T}$, they are isomorphic to the cohomologies of the affine variety $J_{\text{aff}}(t)$ with $t$ in generic position.*

On the algebra $\mathcal{A}$ and on the spaces $C^k$ one can introduce degree [2]. Take the basis of $H^g$ considered as a vector space over $\mathcal{T}$ which is composed of homogeneous representatives

$$\Omega_\alpha = g_\alpha\, d\tau_1 \wedge \cdots \wedge d\tau_g$$

where $\alpha$ takes finite number of values. The fact of foliation of $\mathcal{M}$ into varieties $J_{\text{aff}}(t)$ corresponds to the following statement concerning the algebra $\mathcal{A}$ [2]

**Proposition 1.** *Every element $x$ of $\mathcal{A}$ can be presented in the form*

$$x = \sum_\alpha p_\alpha(\partial_1, \cdots, \partial_g) g_\alpha \tag{6}$$

*where $p_\alpha(\partial_1, \cdots, \partial_g)$ are polynomials of $\partial_1, \cdots, \partial_g$ with coefficients in $\mathcal{T}$.*

The representation (6) is not unique, the equations

$$\sum_\alpha p_\alpha(\partial_1, \cdots, \partial_g) g_\alpha = 0 \tag{7}$$

are counted by $H^{g-1}$ [2].

The formula (6) can be useful only if we are able to control the cohomologies. Concerning these cohomologies we adopt several conjectures following [2].

# 3 Conjectured form of cohomologies.

The affine variety $J_{\text{aff}}(t)$ allows the following description. Consider the hyper-elliptic curve $X$ of genus $g$:

$$w^2 - t(z)w + 1 = 0 \tag{8}$$

This curve has two points over the point $z = \infty$ which we denote by $\infty^{\pm}$.

Consider a matrix $m(z)$ satisfying (2). Take the zeros of $b(z)$:

$$b(z) = \prod_{j=1}^{g} (z - z_j)$$

and

$$w_j = d(z_j)$$

Obviously $z_j, w_j$ satisfy the equation of the curve $X$ (8). Thus $m(z)$ defines a point $\mathcal{P}$ (divisor) on the symmetrized $g$-th power $X[g]$ of the curve $X$. The divisor $\mathcal{P}$ consists of the points $p_j = (z_j, w_j) \in X$.

Oppositely one can reconstruct $m(z)$ starting form the divisor $\mathcal{P}$. Corresponding map is singular, the singularities being located on

$$D = \{\mathcal{P} \mid p_i = \sigma(p_j) \text{ for some } i, j \text{ or } p_i = \infty^{\pm} \text{ for some } i\} \tag{9}$$

Where $\sigma$ is hyper-elliptic involution. Thus the alternative description of $J_{\text{aff}}(t)$ is

$$J_{\text{aff}}(t) = X[g] - D$$

Consider the meromorphic differentials on $X$ with singularities at $\infty^{\pm}$. We chose the following basis of these differentials:

$$\begin{aligned} \mu_k(p) &= z^{g+k}\,\frac{dz}{y}, \quad -g \le k \le 0 \\ \mu_k(p) &= \big[y\frac{d}{dz}(z^{k-g-1}y)\big]_{\ge}\,\frac{dz}{y}, \quad k \ge 1 \end{aligned} \tag{10}$$

where $p = (z, w)$, $y = 2w - t(z)$, $[\ ]_{\ge}$ means that only non-negative degrees of Laurent series in the brackets are taken.

The form

$$\widetilde{\mu}_k = \sum_i \mu_k(p_i)$$

is viewed as a form on $J_{\text{aff}}(t)$. It is easy to see that the forms $\mu_k$ (hence $\widetilde{\mu}_k$) with $k \ge g + 1$ are exact. Consider the space $W^m$ with the basis:

$$\Omega_{k_1,\cdots,k_m} = \widetilde{\mu}_{k_1} \wedge \cdots \wedge \widetilde{\mu}_{k_m}$$

where $-g \le k_j \le g$. Following [2] we adopt the

**Conjecture 2.** *We have*

$$H^m = \frac{W^m}{\sigma \wedge W^{m-2}} \tag{11}$$

*where*

$$\sigma = \sum_{j=1}^{g} \widetilde{\mu}_j \wedge \widetilde{\mu}_{-j}$$

According to (9) the singularities of differential forms occur either at $p_i = \sigma(p_j)$ or at $p_i = \infty^{\pm}$. The non-trivial essence of Conjecture 2 is that the first kind of singularities can be eliminated by adding exact forms. There are $(g-1)$-forms singular at $p_i = \sigma(p_j)$ such that these singularities disappear after applying $d$. This is the origin of the space $\sigma \wedge W^{k-2}$ [2].

Consider briefly the dual picture. On the affine curve with punctures at $\infty^{\pm}$ there are $2g+1$ non-trivial cycles $\delta_k$ with $k = -g, \cdots, g$. The cycles $\delta_k$ , $k < 0$ are a-cycles, the cycles $\delta_k$ , $k > 0$ are b-cycles and $\delta_0$ is the cycle around $\infty^+$. One defines the cycles $\widetilde{\delta}_k$ on the symmetrical power of the affine curve. The $\wedge$-operation is introduced for these cycles by duality with cohomologies. The non-trivial consequence of Conjecture 2 is that every cycle on $J_{\text{aff}}(t)$ can be constructed by wedging $\widetilde{\delta}_k$. The formula dual to (11) is

$$H_m = \frac{W_m}{\sigma' \wedge W_{m-2}} \tag{12}$$

where $W_m$ is spanned by

$$\Delta_{k_1,\cdots,k_m} = \widetilde{\delta}_{k_1} \wedge \cdots \wedge \widetilde{\delta}_{k_m}$$

and

$$\sigma' = \sum_{j=1}^{g} \widetilde{\delta}_j \wedge \widetilde{\delta}_{-j}$$

We need to factorize over $\sigma' \wedge W_{m-2}$ because the 2-cycle $\sigma'$ intersects with $D$.

Let us return to the relation of $H^g$ to the algebra $\mathcal{A}$. Notice that

$$d\tau_1 \wedge \cdots \wedge d\tau_g \simeq \widetilde{\mu}_1 \wedge \cdots \wedge \widetilde{\mu}_g \equiv \Omega$$

The functions

$$x_{k_1,\cdots,k_g} = \Omega^{-1}\, \Omega_{k_1,\cdots,k_g}$$

are symmetric polynomials of $z_1, \cdots, z_g$. Recall that $b_1, \cdots, b_g$ are nothing but elementary symmetric polynomials of $z_1, \cdots, z_g$. Hence the coefficients of cohomologies have the form:

$$g_\alpha = g_\alpha(b_1, \cdots, b_g)$$

The dimension of $H^g$ is determined by Conjecture 2:

$$\alpha = 1, \cdots, \tbinom{2g+1}{g} - \tbinom{2g+1}{g-2},$$

The equations (7) are consequences of the following ones

$$\sum_{k=1}^{g} \partial_k \big(\, \Omega^{-1}\, (\mu_{-k} \wedge \Omega_{k_1,\cdots,k_{g-1}})\big) = 0, \quad \forall\, \Omega_{k_1,\cdots,k_{g-1}} \in W^{g-1} \tag{13}$$

# 4 Quantization of affine Jacobian.

Let us consider a quanization of algebra $\mathcal{A}$. The parameter of deformation (Planck constant) is denoted by $\gamma$, we shall also use

$$q = e^{i\gamma}$$

Consider the $2\times$ 2 matrix $\boldsymbol{m}(z)$ with noncommuting entries. Suppose that the dependence on the spectral parameter $z$ is exactly the same as in classical case (1). The variables $\boldsymbol{a}_j$, $\boldsymbol{b}_j$, $\boldsymbol{c}_j$, $\boldsymbol{d}_j$ are subject to commutation relations which are summarized as follows:

$$\begin{aligned} & r_{21}(z_1,z_2)\,\boldsymbol{m}_1(z_1)\,k_{12}(z_1)\,s_{12}\,\boldsymbol{m}_2(z_2)\,k_{21}(z_2) = \\ & = \boldsymbol{m}_2(z_2)\,k_{21}(z_2)\,s_{21}\,\boldsymbol{m}_1(z_1)\,k_{12}(z_1)\,r_{12}(z_1,z_2) \end{aligned} \tag{14}$$

where usual conventions are used: the equation (14) is written in the tensor product $\mathbb{C}^2\otimes\mathbb{C}^2$, $a_1 = a\otimes I$, $a_2 = I\otimes 2$, $a_{21} = Pa_{12}P$ where $P$ is the operation of permuattions. The $\mathbb{C}$-number matrices $r$, $k$, $s$ are:

$$\begin{aligned} & r_{12}(z_1,z_2) = \frac{z_1-qz_2}{1-q}\,(I\otimes I) + \frac{z_1+qz_2}{1+q}\,(\sigma^3\otimes\sigma^3) + \\ & + 2\,(z_1\sigma^-\otimes\sigma^+ + z_2\sigma^+\otimes\sigma^-), \\ & k_{12}(z) = I\otimes(I-\sigma^3) + \left(q^{-\sigma^3} + z(q^2-1)\sigma^-\right)\otimes(I+\sigma^3), \\ & s_{12} = I\otimes I - (q-q^{-1})\sigma^-\otimes\sigma^+ \end{aligned} \tag{15}$$

These commutation relations are important because they respect the form of matrix $\boldsymbol{m}(z)$ prescribed by (1), we shall explain how they are related to more usual r-matrix relations in the next section.

Define the polynomials:

$$\begin{aligned} & \boldsymbol{t}(z) = q\boldsymbol{a}(z) + q^2\boldsymbol{d}(z) - z(q^2-1)\boldsymbol{b}(z) \\ & \boldsymbol{f}(z) = q\boldsymbol{d}(z)\boldsymbol{t}(zq^{-2}) - q^2\boldsymbol{d}(z)\boldsymbol{d}(zq^{-2}) - q\boldsymbol{b}(z)\boldsymbol{c}(zq^{-2}) \end{aligned} \tag{16}$$

The algebra $\widehat{\mathcal{A}}(q)$ is generated by $\boldsymbol{a}_1,\cdots,\boldsymbol{a}_{g+1}$, $\boldsymbol{b}_1,\cdots,\boldsymbol{b}_g$, $\boldsymbol{c}_2,\cdots,\boldsymbol{c}_{g+2}$, $\boldsymbol{d}_2,\cdots,\boldsymbol{d}_{g+1}$, The polynomial $\boldsymbol{f}(z)$ belongs to the center of $\widehat{\mathcal{A}}(q)$. The coefficients of $\boldsymbol{t}(z)$ are commuting, actually, $\boldsymbol{t}_{g+1}$ belongs to the center of $\widehat{\mathcal{A}}(q)$. We define:

$$\mathcal{A}(q) = \frac{\widehat{\mathcal{A}}(q)}{\{\boldsymbol{f}(z)=1,\ \boldsymbol{t}_{g+1} = (-1)^{g+1}2\}}$$

The non-commutative algebra $\mathcal{A}(q)$ defines a quantization of the algebra of function on the quadric $\mathcal{M}$. However, we cannot define directly the quantization of the algebra of functions on the affine Jacobian because the coefficients of $\boldsymbol{t}(z)$ are not in the center of $\mathcal{A}(q)$. What we can do is to describe the quantum version of Proposition 1 and of description of cohomologies. The exposition will be more detailed than in the classical case.

Like in the paper [3] we accept the following

**Conjecture 3.** *The algebra $\mathcal{A}(q)$ is spanned as linear space by elements of the form:*

$$x = p_L(\boldsymbol{t}_1, \cdots, \boldsymbol{t}_g) g(\boldsymbol{b}_1, \cdots, \boldsymbol{b}_g) p_R(\boldsymbol{t}_1, \cdots, \boldsymbol{t}_g) \tag{17}$$

*where $p_L(\boldsymbol{t}_1, \cdots, \boldsymbol{t}_g)$, $g(\boldsymbol{b}_1, \cdots, \boldsymbol{b}_g)$, $p_R(\boldsymbol{t}_1, \cdots, \boldsymbol{t}_g)$ are polynomials.*

We were not able to prove this statement, however, since the algebra $\mathcal{A}(q)$ is graded we can check it degree by degree. This has been done up to degree 8. Notice the similarity between the representation (17) and the representation for spin operators proved in [6]. Conjecture 3 implies that certain generalization of the results of [6] is possible. In fact the formula (17) is similar to the formula (6): we can either symmetrize or anti-symmetrize $\boldsymbol{t}_j$ in (17) which corresponds in classics to multipilation by $t_j$ or to applying $\partial_j$. In order to have complete agreement with clasical case we have to show that only finitely many different polynomials $g(\boldsymbol{b}_1, \cdots \boldsymbol{b}_g)$ (cohomologies) create entier algebra $\mathcal{A}(q)$.

Notice that the commutation relations (14) imply in particular that

$$[\boldsymbol{b}(z), \boldsymbol{b}(z')] = 0$$

which means that we have the commutative family of operators $\boldsymbol{z}_j$ defined by

$$\boldsymbol{b}(z) = \prod (z - \boldsymbol{z}_j)$$

So, every polynomials $g(\boldsymbol{b}_1, \cdots, \boldsymbol{b}_g)$ can be considered as symmetric polynomial of $\boldsymbol{z}_j$ and vice versa.

It is very convenient to use the following formal definitions. Consider the ring $\mathcal{T}$ defined in (5). By $\mathcal{V}^k$ we denote the space of anti-symmetric polynomials of $k$ variables such that their degrees with respect to every variable is not less than 1 with coefficients in $\mathcal{T} \otimes \mathcal{T}$. In other words $\mathcal{V}^k$ is the space spanned by the polynomials:

$$p_L \cdot h \cdot p_R \equiv p_L(t_1, \cdots, t_g) h(z_1, \cdots, z_k) p_R(t'_1, \cdots, t'_g)$$

where $h$ is anti-symmetric, vanishing when one of $z_j$ vanishes. The following operations can be defined.

1. Multiplication by $t_j$ and $t'_j$.

2. Operation $\wedge \ : \ \mathcal{V}^k \otimes \mathcal{V}^l \to \mathcal{V}^{k+l}$ which is defined as follows:

$$(p_L \cdot h \cdot p_R) \wedge (p'_L \cdot h' \cdot p'_R) = p_L p'_L \cdot (h \wedge h') \cdot p_R p'_R$$

where

$$(h \wedge h')(z_1, \cdots, z_{k+l}) =$$
$$= \frac{1}{k!\, l!} \sum_{\pi \in S_{k+l}} (-1)^{\pi} \, h(z_{\pi(1)}, \cdots, z_{\pi(k)}) h'(z_{\pi(k+1)}, \cdots, z_{\pi(k+l)})$$

We have a map

$$\mathcal{V}^g \xrightarrow{\chi} \mathcal{A}(q)$$

defined on the basis elements as

$$\chi(p_L \cdot h \cdot p_R) = p_L(\boldsymbol{t}_1, \cdots, \boldsymbol{t}_g)\, \frac{h(\boldsymbol{z}_1, \cdots, \boldsymbol{z}_g)}{\prod \boldsymbol{z}_i \prod_{i<j}(\boldsymbol{z}_i - \boldsymbol{z}_j)}\, p_R(\boldsymbol{t}_1, \cdots, \boldsymbol{t}_g)$$

and continued linearly. The Conjecture 3 states that this map is surjective. We want to describe the kernel of the map $\chi$.

First, consider the space $\mathcal{V}^1$. The elements of this space are polynomials of one variable $z$ with coefficients in $\mathcal{T} \otimes \mathcal{T}$. In Appendix B we describe certain basis in $\mathcal{V}^1$ considered as a linear space over $\mathcal{T} \otimes \mathcal{T}$. The basis in question consists of the polynomials: $s_k$ with $k \geq -g$ such that the degree of $s_k$ with respect to $z$ equals $g + k + 1$. The kernel of $\chi$ is the joint of three sub-spaces, let us describe them.

1. For $k \geq g + 1$ we have:

$$\chi\big(s_k \wedge \mathcal{V}^{g-1}\big) = 0 \tag{18}$$

2. Consider $c \in \mathcal{V}^2$ defined as

$$c = \sum_{j=1}^{g} s_j \wedge s_{-j}$$

we have

$$\chi\big(c \wedge \mathcal{V}^{g-2}\big) = 0 \tag{19}$$

3. Consider $d \in \mathcal{V}^1$ defined as

$$d = (t_j - t'_j)s_{-j}$$

we have

$$\chi\big(d \wedge \mathcal{V}^{k-1}\big) = 0 \tag{20}$$

The construction of the space

$$\frac{\mathcal{V}^g}{\mathrm{Ker}(\chi)} \simeq \mathcal{A}(q)$$

is in complete correspondence with the classiccal case. In classics we start with all the 1-forms $\widetilde{\mu}_k$. Imposing (18) corresponds to throwing away the exact forms and working with $\widetilde{\mu}_k$ for $k = -g \cdots, g$ only. Imposing (19) corresponds to factorizing over $\sigma \wedge W^{k-2}$ in classics. Finally, (20) corresponds to the equation (13).

The origin of the equations (18), (19), (20) will be explained in the Section 9. There should be purely algebraic method of prooving these equations, but we do not know it. It is important to mention that accepting Conjecture 3 we are forced to conclude that the kernel of $\chi$ is completely described by the equations (18), (19), (20). This is proved by calculation of characters similarly to that of [3].

# 5 The realization of $\mathbf{A(q)}$.

We want to describe a realization of the algebra $\mathcal{A}(q)$ in a space of functions. Consider the quantum mechanical system described by the operators $x_j$ with $j = 1, \cdots, 2g+2$ and $y$ (zero mode). The operators $x_j$ and $y$ are self-adjoint, they satisfy the commutation relations:

$$\begin{aligned} x_k x_l &= q^2 x_l x_k \qquad k < l, \\ y x_k &= q^2 x_k y \qquad \forall k \end{aligned}$$

The hamiltonian of the system is

$$\boldsymbol{h} = q^{-1} \sum_{k=1}^{2g+2} x_k x_{k-1}^{-1}$$

where

$$x_{2g+3} \equiv qyx_1$$

Physically this model defines the simplest lattice regularization of the chiral Bose field with modified energy-momentum tensor.

It is useful to double the number of degrees of freedom. Consider the algebra $A$ generated by two operators $u$ and $v$ satisfying the commutation relations:

$$uv = qvu$$

Take the algebra $A^{\otimes(2g+2)}$ the operators $u_j$, $v_j$ ($j = 1, \cdots, 2g+2$) are defined as $u$ and $v$ acting in $j$-th tensor component. The original operators $x_i$ are expressed in terms of $u_i, v_i$ as follows:

$$x_k = v_k \prod_{j=1}^{k-1} u_j^{-2}, \quad y = \prod_{j=1}^{2g+2} u_j$$

Consider the "monodromy matrix"

$$\widetilde{\boldsymbol{m}}(z) = \begin{pmatrix} \widetilde{\boldsymbol{a}}(z) & \widetilde{\boldsymbol{b}}(z) \\ \widetilde{\boldsymbol{c}}(z) & \widetilde{\boldsymbol{d}}(z) \end{pmatrix} = l_{2g+2}(z) \, \cdots \, l_1(z) \tag{21}$$

where the l-operators are

$$l(z) = \frac{1}{\sqrt{z}} \begin{pmatrix} zu & -qvu \\ zv^{-1}u^{-1} & 0 \end{pmatrix} \tag{22}$$

This is a particular case of more genaral l-operator $l(z, \kappa)$ in which the last matrix element is not 0 but $\kappa z u$, the model corresponding to the latter l-operator is a subject of study in a series of papers [7].

The matrix elements of the matrix $\boldsymbol{m}(z)$ satisfy the commutation relations:

$$r_{12}(z_1, z_2) \widetilde{\boldsymbol{m}}_1(z_1) \widetilde{\boldsymbol{m}}_2(z_2) = \widetilde{\boldsymbol{m}}_2(z_2) \widetilde{\boldsymbol{m}}_1(z_1) r_{12}(z_1, z_2) \tag{23}$$

where the r-matrix $r_{12}(z_1, z_2)$ is defined ealier (15). These are canonical r-matrix commutation relations. The quantum determinant of the matrix $\widetilde{\boldsymbol{m}}(z)$ is defined by

$$\boldsymbol{f}(z) = \widetilde{\boldsymbol{d}}(z)\widetilde{\boldsymbol{a}}(zq^{-2}) - \widetilde{\boldsymbol{b}}(z)\widetilde{\boldsymbol{c}}(zq^{-2}),$$

it belongs to the center, in our realization of $\widetilde{\boldsymbol{m}}(z)$ one has $\boldsymbol{f}(z) = 1$. The trace of $\widetilde{\boldsymbol{m}}(z)$ generates commuting quantities, we denote this trace as follows:

$$\widetilde{\boldsymbol{a}}(z) + \widetilde{\boldsymbol{d}}(z) = y\boldsymbol{t}(z)$$

The matrix elements of the matrix $\widetilde{m}(z)$ are of the form:

$$\begin{aligned}
\widetilde{\boldsymbol{a}}(z) &= \widetilde{\boldsymbol{a}}_0 z^{g+1} + \widetilde{\boldsymbol{a}}_1 z^g + \cdots + \widetilde{\boldsymbol{a}}_{g+1}, \\
\widetilde{\boldsymbol{b}}(z) &= \widetilde{\boldsymbol{b}}_0 z^g + \widetilde{\boldsymbol{b}}_1 z^{g-1} + \cdots + \widetilde{\boldsymbol{b}}_g, \\
\widetilde{\boldsymbol{c}}(z) &= \widetilde{\boldsymbol{c}}_1 z^{g+1} + \widetilde{\boldsymbol{c}}_2 z^g + \cdots + \widetilde{\boldsymbol{c}}_{g+1} z, \\
\widetilde{\boldsymbol{d}}(z) &= \widetilde{\boldsymbol{d}}_1 z^g + \widetilde{\boldsymbol{d}}_2 z^{g-1} + \cdots + \widetilde{\boldsymbol{d}}_g z
\end{aligned} \tag{24}$$

where, in particular, $\widetilde{\boldsymbol{a}}_0 = y$. This form of polynomials $\widetilde{\boldsymbol{a}}(z)$, $\widetilde{\boldsymbol{b}}(z)$, $\widetilde{\boldsymbol{c}}(z)$, $\widetilde{\boldsymbol{d}}(z)$ does not corresponds to what we have in the classical model of affine Jacobian. This is the reason for modifying the matrix $\widetilde{m}(z)$ as follows:

$$\boldsymbol{m}(z) = \begin{pmatrix} \widetilde{\boldsymbol{a}}_0\widetilde{\boldsymbol{b}}_0^{-1} & 0 \\ -\widetilde{\boldsymbol{d}}_1\widetilde{\boldsymbol{b}}_0^{-1} & 1 \end{pmatrix} \widetilde{\boldsymbol{m}}(z) \begin{pmatrix} \widetilde{\boldsymbol{b}}_0\widetilde{\boldsymbol{a}}_0^{-1} & 0 \\ q\widetilde{\boldsymbol{d}}_1\widetilde{\boldsymbol{a}}_0^{-1} & 1 \end{pmatrix}$$

The matrix elements of this matrix have structure (1), they satisfy closed commutation relations (14), the operators $\boldsymbol{f}(z)$ and $\boldsymbol{t}(z)$ defined for these two matrices coincide, in particular we have

$$\boldsymbol{t}_1 = \boldsymbol{h}$$

Thus the modification of matrix $\widetilde{\boldsymbol{m}}(z)$ which is necessary for relation to the affine Jacobian is responcible for appearing of strangely looking commutation relations (14).

# 6 Q-operator.

Our first goal is to define Baxter's Q-operator. Let us realize the operators $v, u$ in $L_2(\mathbb{R})$ as follows

$$v = e^{\varphi}, \qquad u = e^{i\gamma\frac{d}{d\varphi}}$$

We shall work in the $\varphi$-representation, i.e. in the space $\mathfrak{H} = (L_2(\mathbb{R}))^{\otimes(2g+2)}$. Following the standart procedure (*cf.* [8]) one introduces the vectors $Q(\zeta \mid \psi_1, \cdots, \psi_{2g+2})$ which depend on

$$\zeta = \tfrac{1}{2}\log z$$

and $2g+2$ additional parameters, $\psi_j$, and satisfy the equation:

$$\begin{aligned}
(-1)^{g+1}\boldsymbol{t}(z)Q(\zeta \mid \psi_1, \cdots, \psi_{2g+2}) = \\
= Q(\zeta + i\gamma \mid \psi_1, \cdots, \psi_{2g+2}) + Q(\zeta - i\gamma \mid \psi_1, \cdots, \psi_{2g+2})
\end{aligned}$$

In $\varphi$-representation the "components" of these vectors are given by

$$Q(\varphi_1,\cdots,\varphi_{2g+2}\mid\zeta\mid\psi_1,\cdots,\psi_{2g+2}) = \\ = e^{\frac{1}{2}(1+\frac{\pi}{\gamma})\zeta+\frac{1}{4i\gamma}\zeta^2}\prod_{k=1}^{2g+2}\lambda(\zeta\mid\varphi_k-\psi_k\,)\langle\,\varphi_k\mid\psi_{k-1}\,\rangle \tag{25}$$

where $\psi_0\equiv\psi_{2g+2}$,

$$\langle\,\varphi\mid\psi\,\rangle = e^{\frac{1}{4i\gamma}(2\varphi\psi-\varphi^2)}, \\ \lambda(\zeta\mid\psi) = e^{-\frac{1}{2i\gamma}\zeta\psi}\,\Phi(\psi-\zeta)\,e^{\frac{\pi+\gamma}{\gamma}(\psi-\zeta)}, \tag{26}$$

and the function $\Phi(\varphi)$ satisfies the functional equation:

$$\frac{\Phi(\varphi+i\gamma)}{\Phi(\varphi-i\gamma)} = \frac{1}{1+e^{\varphi}} \tag{27}$$

The solution to this equation is

$$\Phi(\varphi) = \exp\left(\int\limits_{\mathbb{R}+i0}\frac{e^{ik\varphi}}{4\sinh\gamma k\,\sinh\pi k}\frac{dk}{k}\right)$$

This wonderful function and its applications can be found in [9].

As usual we want to consider $Q(\varphi_1,\cdots,\varphi_{2g+2}\mid\zeta\mid\psi_1,\cdots,\psi_{2g+2})$ as the kernel of an operator:

$$Q(\varphi_1,\cdots,\varphi_{2g+2}\mid\zeta\mid\psi_1,\cdots,\psi_{2g+2}) = \langle\,\varphi_1,\cdots,\varphi_{2g+2}\mid\boldsymbol{\mathcal{Q}}(\zeta)\mid\psi_1,\cdots,\psi_{2g+2}\,\rangle$$

The subtle point is that we have to use mixed representations: the vectors $\mid\psi\,\rangle$ are the eigenvectors of the operators

$$w\equiv e^{\psi} = uvu$$

Notice that this justifies the notation $\langle\,\varphi\mid\psi\,\rangle$ in (26), and that

$$[\,\psi,\varphi\,] = 2i\gamma$$

The operators $\boldsymbol{\mathcal{Q}}(\zeta)$ satisfy the equations

$$(-1)^{g+1}\boldsymbol{t}(z)\boldsymbol{\mathcal{Q}}(\zeta) = \boldsymbol{\mathcal{Q}}(\zeta+i\gamma)+\boldsymbol{\mathcal{Q}}(\zeta-i\gamma) \tag{28}$$

This is famous Baxter's equation.

Before going further let us discuss the properties of the operator $\boldsymbol{\mathcal{Q}}(\zeta)$. We have

$$\overline{\Phi(\varphi)} = \Phi(\bar{\varphi}); \\ \Phi(\varphi)\sim\exp\left(\frac{1}{4i\gamma}\varphi^2\right), \qquad \text{as} \quad \varphi\to\infty \tag{29}$$

so, the kernel of $\boldsymbol{\mathcal{Q}}(\zeta)$ for $\zeta \in \mathbb{R}$ is an oscilating function, and it is rather clear that our operator is well defined on the functions of $\psi_j$ of Schwartz class ($\mathrm{Sch}_\psi$) sending them to functions of $\varphi_j$ which are also of Schwartz class ($\mathrm{Sch}_\varphi$). Using the equations (29) one easily finds the kernel $\langle\, \psi \mid \boldsymbol{\mathcal{Q}}^*(\zeta) \mid \varphi \,\rangle$ of the adjoint operator $\boldsymbol{\mathcal{Q}}^*(\zeta)$ (we consider the case of real $\zeta$). Further, notice that the l-operator can be rewritten as

$$l(z) = \frac{1}{\sqrt{z}} \begin{pmatrix} zu & -qu^{-1}w \\ zuw^{-1} & 0 \end{pmatrix} \tag{30}$$

Applying to this l-operator the same procedure as before one finds that $\boldsymbol{\mathcal{Q}}^*(\zeta)$ also solves the Baxter equation (28):

$$(-1)^{g+1}\boldsymbol{t}(z)\boldsymbol{\mathcal{Q}}^*(\zeta) = \boldsymbol{\mathcal{Q}}^*(\zeta + i\gamma) + \boldsymbol{\mathcal{Q}}^*(\zeta - i\gamma)$$

It can be shown that actually

$$\boldsymbol{\mathcal{Q}}(\zeta) = \boldsymbol{\mathcal{Q}}^*(\zeta) \qquad \text{for} \qquad \zeta \in \mathbb{R}$$

Considering the kernel of the operator $\boldsymbol{\mathcal{Q}}^*(\zeta)$ one finds that this operator acts from $\mathrm{Sch}_\varphi$ to $\mathrm{Sch}_\psi$. So, the products $\boldsymbol{\mathcal{Q}}(\zeta)\boldsymbol{\mathcal{Q}}(\zeta')$ are well defined at least for $\zeta, \zeta' \in \mathbb{R}$.

We want to show that the operators $\boldsymbol{\mathcal{Q}}(\zeta)$ constitute a commutative family:

$$[\, \boldsymbol{\mathcal{Q}}(\zeta), \boldsymbol{\mathcal{Q}}(\zeta') \,] = 0 \tag{31}$$

To this end we want to show that the operator $\boldsymbol{\mathcal{Q}}(\zeta)$ can be rewritten as

$$\boldsymbol{\mathcal{Q}}(\zeta) = \mathrm{tr}_a \left( \mathcal{L}_{a2q+2}(\zeta) \,\cdots\, \mathcal{L}_{a1}(\zeta) \right) \tag{32}$$

where the operators $\mathcal{L}_{aj}(\zeta)$ act in the tensor product of the "auxilary space" labeled by $a$ and of the "quantum space" where $\varphi_j$, $\psi_j$ act. Actually in our case the "auxilary space" will be isomorphic to the "quantum space", i.e. we shall have a universal l-operator. If the operators $\mathcal{L}_{aj}(\zeta)$ satisfy Yang-Baxter equations with some R-matrix then the commutativity (31) follows from the standart argument.

To find the representation (32) rewrite (25) as

$$\boldsymbol{\mathcal{Q}}(\zeta) = e^{\frac{1}{2}(1+\frac{\pi}{\gamma})\zeta + \frac{1}{4i\gamma}\zeta^2} \int \prod_{j=1}^{2g+2} d\varphi'_j d\psi'_j \langle\, \psi'_j \mid \mathcal{L}_{aj}(\zeta) \mid \varphi'_j \,\rangle \langle\, \varphi'_j \mid \psi'_{j-1} \,\rangle$$

where $\varphi'_j$, $\psi'_j$ are operators acting in the "auxiliary space", $\psi'_0 = \psi'_{2g+2}$. So, (32) indeed takes place if the kernel of the "universal" l-operator is given by

$$\langle\, \varphi' \mid \otimes \langle\, \psi \mid \; \mathcal{L}(\zeta) \; \mid \psi' \,\rangle \otimes \mid \varphi \,\rangle = \delta(\varphi - \varphi')\delta(\psi - \psi')\lambda(\zeta \mid \varphi - \psi)$$

Hence the formula (32) holds for the operators $\mathcal{L}_{aj}(\zeta)$ of the form:

$$\mathcal{L}_{12}(\zeta) = \mathcal{P}_{12}\widehat{\mathcal{L}}_{12}(\zeta)$$

where $\mathcal{P}_{12}$ is the operator of permutation, and the operator $\widehat{\mathcal{L}}_{12}(\zeta)$ acts in the tensor product as follows:

$$\widehat{\mathcal{L}}_{12}(\zeta) = \lambda(\zeta \mid \varphi \otimes I - I \otimes \psi)$$

Thus the operator $\boldsymbol{\mathcal{Q}}(\zeta)$ can be considered as trace of "universal" monodromy matrix and the commutativity (31) follows from the Yang-Baxter equation:

$$\widehat{\mathcal{R}}_{12}(\zeta_1-\zeta_2)\,\widehat{\mathcal{L}}_{23}(\zeta_1)\,\widehat{\mathcal{L}}_{12}(\zeta_2)=\widehat{\mathcal{L}}_{23}(\zeta_2)\,\widehat{\mathcal{L}}_{12}(\zeta_1)\,\widehat{\mathcal{R}}_{23}(\zeta_1-\zeta_2) \tag{33}$$

with the simple r-matrix:

$$\widehat{\mathcal{R}}_{12}(\zeta)=\exp\left(\frac{(I\otimes\psi-\varphi\otimes I)\zeta}{2i\gamma}\right)$$

The Yang-Baxter equation (33) in our case is almost trivial. In the case of the more general l-operator $\widehat{l}(z,\kappa)$ mentioned above we would need to use a more complicated r-matrix and the proof of Yang-Baxter equations needs some non-trivial identities [10].

The self-ajoint (for real $\zeta_1,\zeta_2$) operators $\boldsymbol{\mathcal{Q}}(\zeta_1)$, $\boldsymbol{\mathcal{Q}}(\zeta_2)$ commute, hence the eigenvectors of $\boldsymbol{\mathcal{Q}}(\zeta)$ do not depend on $\zeta$. Actually, the operator $\boldsymbol{\mathcal{Q}}(\zeta)$ is an entire function of $\zeta$. The kernel of $\boldsymbol{\mathcal{Q}}(\zeta)$ has poles, but in the process of analytical continuation the poles never pinch the contour of integration. The Baxter's equation (28) implies that $\boldsymbol{\mathcal{Q}}(\zeta_1)$ and $\boldsymbol{t}(z_2)$ also commute. Suppose that $\mathcal{Q}(\zeta)$ ant $t(z)$ are eigen-values of these operators, due to the equation (28) they satisfy

$$(-1)^{g+1}t(z)\mathcal{Q}(\zeta)=\mathcal{Q}(\zeta+i\gamma)+\mathcal{Q}(\zeta-i\gamma) \tag{34}$$

Let us discuss further analytical properties of $\mathcal{Q}(\zeta)$. Since the operator $\boldsymbol{\mathcal{Q}}(\zeta)$ is an entire functions of $\zeta$ the eigen-value $\mathcal{Q}(\zeta)$ is an entire function as well. As it has been said $\boldsymbol{t}(0)=\boldsymbol{t}_{g+1}$ belongs to the center of the algebra defined by the commutation relations (23), so, we can fix it. It is convenient to put $\boldsymbol{t}_{g+1}=(-1)^{g+1}2$ which allows to reqiure that

$$\mathcal{Q}(\zeta)\to 1,\qquad \zeta\to-\infty \tag{35}$$

From quasi-classical consideration which are completely parallel to those from [8, 3] it is naturally to conjecture that the eigenvalues of $\mathcal{Q}(\zeta)$ have zeros only on the real axis and that asymptotically for $\zeta\to\infty$ one has:

$$\mathcal{Q}(\zeta)\sim e^{-(g+1)(1+\frac{\pi}{\gamma})\zeta}\cos\left(\frac{(g+1)\zeta^2}{\gamma}+\frac{\pi}{4}\right) \tag{36}$$

The important question is whether the equations (34) together with the analytical properties described above are sufficient to find the spectrum of commuting Hamiltonians. In our opinion it is impossible, the additional information is needed which is provided in the following section.

# 7 Duality.

Consider the function $\Phi(\zeta)$. The most interesting property of this function is its duality: together with the equation (27) it satisfies the equation

$$\frac{\Phi(\varphi+i\pi)}{\Phi(\varphi-i\pi)}=\frac{1}{1+e^{\frac{\pi}{\gamma}\varphi}}$$

Using this property and the definition of the operator $\boldsymbol{\mathcal{Q}}(\zeta)$ one finds that there is dual equation for $\boldsymbol{\mathcal{Q}}(\zeta)$:

$$(-1)^{g+1}\boldsymbol{T}(Z)\boldsymbol{\mathcal{Q}}(\zeta) = \boldsymbol{\mathcal{Q}}(\zeta+\pi i) + \boldsymbol{\mathcal{Q}}(\zeta-\pi i) \tag{37}$$

where

$$Z = e^{\frac{2\pi}{\gamma}\zeta},$$

and $\boldsymbol{T}(Z)$ is the trace of the monodromy matrix

$$\widetilde{\boldsymbol{M}}(Z) = L_{2g+2}(Z)\ \cdots\ L_1(Z)$$

with

$$L(Z) = \frac{1}{\sqrt{Z}}\begin{pmatrix} ZU^{-1} & -QVU \\ ZV^{-1}U^{-1} & 0 \end{pmatrix}$$

The dual operators

$$U = e^{\frac{\pi}{\gamma}\varphi}, \qquad V = e^{\pi i\frac{d}{d\varphi}}$$

satisfy the commutation relations

$$UV = QVU$$

with dual

$$Q = e^{i\frac{\pi^2}{\gamma}}$$

The only non-trivial commutation relations of $u, v$ with $U, V$ are

$$uV = -Vu, \qquad vU = -Uv$$

which means that

$$S\ (l(z)\otimes I)(I\otimes L(Z)) = (I\otimes L(Z))(l(z)\otimes I)\ S$$

with $S = \sigma^3\otimes\sigma^3$. From here it is obvious that

$$[\ \boldsymbol{t}(z), \boldsymbol{T}(Z)\ ] = 0$$

All that is the result of manifest duality of the kernel of $\boldsymbol{\mathcal{Q}}(\zeta)$ with respect to change:

$$\gamma\to\frac{\pi^2}{\gamma}, \qquad \zeta\to\frac{\pi}{\gamma}\zeta, \qquad \varphi_j\to\frac{\pi}{\gamma}\varphi_j, \qquad \psi_j\to\frac{\pi}{\gamma}\psi_j$$

It is clear that $\boldsymbol{T}(Z_1)$ and $\boldsymbol{\mathcal{Q}}(\zeta_2)$ commute, so, the equation (37) implies the equation for eigen-values:

$$(-1)^{g+1}T(Z)\mathcal{Q}(\zeta) = \mathcal{Q}(\zeta+\pi i) + \mathcal{Q}(\zeta-\pi i) \tag{38}$$

The function $\mathcal{Q}(\zeta)$ is not an entire function of $z$ as it is the case in other situations (for example [11]), that is why the equation (28) alone does not look strong enough to

define it. However, the equation (37) controlling the behaviour of $\mathcal{Q}(\zeta)$ under the $2\pi i$ -rotation in $z$-plane must provide the missing information.

So, our main conjecture is the following

**Conjecture 4.** *The spectrum on $\boldsymbol{t}(z)$ (and, simultaneously, of $\boldsymbol{T}(Z)$) is described by all solutions of the equations (34) and (38) such that*
1. *$t(z)$ and $T(Z)$ are polynomials of degree $g+1$.*
2. *$\mathcal{Q}(\zeta)$ is an entire function of $\zeta$.*
3. *$\mathcal{Q}(\zeta)$ satisfies (35) and (36).*
4. *All the zeros of $\mathcal{Q}(\zeta)$ in the strip $-(\pi+\gamma) < \mathrm{Im}(\zeta) < (\pi+\gamma)$ are real.*

# 8 Separation of variables.

The relation of integrable models to the algebraic geometry can be completely understood in the framework of separation of variables.

We have already mentioned that

$$[\, \boldsymbol{b}(z), \boldsymbol{b}(z') \,] = 0$$

which implies commutativity of the operators $\boldsymbol{z}_j$ defined as roots of $\boldsymbol{b}(z)$. Consider the operators

$$\boldsymbol{w}_j = (-1)^{g+1} q\, \boldsymbol{d}(\overleftarrow{\boldsymbol{z}}_j)$$

where $d(\overleftarrow{\boldsymbol{z}_j})$ means that $\boldsymbol{z}_j$ which does not commute with coefficients of $\boldsymbol{d}(z)$ is substituted to this polynomial form the left. Following Sklyanin [4] one shows that

$$\boldsymbol{z}_j \boldsymbol{w}_k = \boldsymbol{w}_k \boldsymbol{z}_j, \quad j \neq k; \qquad \boldsymbol{z}_j \boldsymbol{w}_j = q^2 \boldsymbol{w}_j \boldsymbol{z}_j$$

and

$$\boldsymbol{w}_j^2 - \boldsymbol{w}_j \boldsymbol{t}(\overleftarrow{\boldsymbol{z}}_j) + 1 = 0 \tag{39}$$

Introduce the operators

$$\boldsymbol{\zeta}_j = \tfrac{1}{2} \log(\boldsymbol{z}_j)$$

and consider the wave-function corresponding to given set of eigen-values of integral of motion $\boldsymbol{t}_1, \cdots, \boldsymbol{t}_g$ in $\boldsymbol{\zeta}$-representation. The equation (39) implies [4] that we can look for this wave function in the form:

$$\langle\, \zeta_1, \cdots, \zeta_g \mid t_1, \cdots, t_g \,\rangle = \mathcal{Q}(\zeta_1) \cdots \mathcal{Q}(\zeta_g)$$

where $\mathcal{Q}(\zeta)$ satisfies the equation:

$$\mathcal{Q}(\zeta + i\gamma) + \mathcal{Q}(\zeta - i\gamma) = (-1)^{g+1} t(z) \mathcal{Q}(\zeta)$$

where $t(z)$ is constructed from the eigen-values $t$. This equation coincides with the equation (28) written for particular eigen-values. So, following [8] we claim that the wave function in separated variables is defined by eigen-value of the operator $\boldsymbol{\mathcal{Q}}$ which connects two different approaches to integrable models.

Notice that the vector $\mid t_1, \cdots, t_g \rangle$ is eigen-vector for the operators $\boldsymbol{T}_1, \cdots, \boldsymbol{T}_g$ since the function $\mathcal{Q}(\zeta)$ satisfies the equation (37). In order to identify explicitly the eigen-vales of $\boldsymbol{T}_1, \cdots, \boldsymbol{T}_g$ we shall write $\mid t_1, \cdots, t_g; T_1, \cdots, T_g \rangle$.

We have the algebra of operators $\mathcal{A}(q)$ and the dual algebra $\mathcal{A}(Q)$ which act in the same space $\mathfrak{H}$. All the operators from $\mathcal{A}(q)$ commute with the operators from $\mathcal{A}(Q)$. The fundamental property of $\mathcal{A}(q)$ is that it is spanned as linear space by the elements of the form (17) according to Conjecture 3. Similar fact must be true for $\mathcal{A}(Q)$. Taking these facts together one realizes the algebra $\mathcal{A}(q) \cdot \mathcal{A}(Q)$ is spanned by the elements of the form:

$$\begin{aligned} \mathcal{X} =& xX = p_L(\boldsymbol{t}_1, \cdots, \boldsymbol{t}_g) P_L(\boldsymbol{T}_1, \cdots, \boldsymbol{T}_g) \\ & \times g(\boldsymbol{b}_1, \cdots, \boldsymbol{b}_g) G(\boldsymbol{B}_1, \cdots, \boldsymbol{B}_g)\ P_R(\boldsymbol{T}_1, \cdots, \boldsymbol{T}_g) p_R(\boldsymbol{t}_1, \cdots, \boldsymbol{t}_g) \end{aligned} \tag{40}$$

We denote by $h(\boldsymbol{z}_1, \cdots, \boldsymbol{z}_g)$ and $H(\boldsymbol{Z}_1, \cdots, \boldsymbol{Z}_g)$ anti-symmetric polynomials obtained from $g(\boldsymbol{b}_1, \cdots, \boldsymbol{b}_g)$ and $G(\boldsymbol{B}_1, \cdots, \boldsymbol{B}_g)$, for example:

$$h(\boldsymbol{z}_1, \cdots, \boldsymbol{z}_g) = \prod \boldsymbol{z}_i \prod_{i<j} (\boldsymbol{z}_i - \boldsymbol{z}_j) g(\boldsymbol{b}_1(\boldsymbol{z}_1, \cdots, \boldsymbol{z}_g), \cdots, \boldsymbol{b}_g(\boldsymbol{z}_1, \cdots, \boldsymbol{z}_g))$$

Let us consider the matrix element of the operator $\mathcal{X}$ between two eigen-vectors of Hamiltonians. The wave functions are real for real $\zeta$. By requirement of self-ajointness of $\boldsymbol{t}(z)$ and $\boldsymbol{T}(Z)$ one defines the scalar product (*cf.* [4]). The matrix element in question is

$$\begin{aligned} & \langle\ t_1, \cdots t_g\ ;\ T_1, \cdots, T_g \mid \mathcal{X} \mid t'_1, \cdots t'_g\ ;\ T'_1, \cdots, T'_g\ \rangle = \\ & = p_L(t_1, \cdots, t_g) p_R(t'_1, \cdots, t'_g) P_L(T_1, \cdots, T_g) P_R(T'_1, \cdots, T'_g) \\ & \times \int_{-\infty}^{\infty} d\zeta_1 \cdots \int_{-\infty}^{\infty} d\zeta_g\ h(z_1, \cdots, z_g) H(Z_1, \cdots, Z_g) \prod_{j=1}^{g} \mathcal{Q}(\zeta_j) \mathcal{Q}'(\zeta_j) \end{aligned} \tag{41}$$

When does the integral for the matrix element (41) converge? Suppose that

$$h \sim z_j^{g+k+1},\ H \sim Z_j^{g+l+1} \qquad \text{when} \quad \zeta_j \to \infty$$

then the integrand in the matrix element behaves when $\zeta_j \to \infty$ as

$$\exp 2\zeta_j \big( (k-1) + \frac{\pi}{\gamma}(l-1) \big)$$

hence for generic $\gamma$ the integral converges only if $k = 1, l = 0$ or $k = 0, l = 1$. When $\gamma$ is small we can allow the operators with $l = 0$ and $k < \frac{\pi}{\gamma}$, oppositely, when $\gamma$ is big the operators with $k = 0$ and $l < \frac{\gamma}{\pi}$ are allowed. The limits $\gamma \to 0$ and $\gamma \to \infty$ are two dual quasi-classical limits. For these limits the operators $l = 0, \forall k$ and $k = 0, \forall l$ respectively define the classical observables. At least these operators must be defined in the quantum case: if the quantization procedure makes sense the principle of correspondence must hold. Hence, the fact that in general only two operators with $k = 1, l = 0$ or $k = 0, l = 1$ lead to convergent integrals means that some

regularization of these integrals is needed. The regularized integrals in question must allow to define the matrix element (41) for arbitrary $k, l$, they have to coincide with usual integrals whenever the latter are applicable, they must satisfy some additional requirements which will be discussed in the Section 9. The origin of these additional requirements is in the cohomological construction explained in Section 4.

Notice that any anti-symmetric with respect to $z_1, \cdots, z_g$ and $Z_1, \cdots, Z_g$ polynomial

$$h(z_1, \cdots, z_g)H(Z_1, \cdots, Z_g)$$

can be presented as linear combination of products of Schur-type determinants

$$\det|z_i^{k_j}|\,\det|Z_i^{l_j}|$$

where $\{k_1, \cdots, k_g\}$ and $\{l_1, \cdots, l_g\}$ are arbitrary sets of positive integers. So, the integrals (41) can be expressed in terms of 1-fold integrals

$$\langle\, l \mid L \,\rangle \simeq \int\limits_{-\infty}^{\infty} \mathcal{Q}(\zeta)\mathcal{Q}'(\zeta)l(z)L(Z)d\zeta \tag{42}$$

where $l$ and $L$ are polynomials such that $l(0) = 0$ and $L(0) = 0$. The symbol $\simeq$ means that the integrals in RHS are not always defined, the regularization is defined in th Appendix B, in the next section we describe results of this regularization..

# 9 Deformed Abelian differentials.

In Appendix B we define the polynomials $s_k(z)$. These polynomials are of the form

$$\begin{aligned} s_k(z) &= z^{g+1+k}, \qquad -g \le k \le 0 \\ s_k(z) &= \frac{1}{i\gamma}\left(\frac{q^k-1}{q^k+1}\right) z^{g+1+k} + \cdots, \quad k \ge 1 \end{aligned} \tag{43}$$

where $\cdots$ stands for terms of lower degree (containing $t_j$, $t'_j$ in coefficients) explicitly given in Appendix B.

In the classical case every polynomial defines an Abelian differentials on the affine curve $X - \infty^{\pm}$. Similarly we consider the polynomials $s_k$ as corresponding to "deformed Abelian differentials." Let us be more precise. The regularized integrals are defined in Appendix B in such a way that they satisfy several conditions. First of them is

$$\langle\, s_k \mid S_l \,\rangle = 0 \qquad k \ge g+1, \quad \forall l \tag{44}$$

$$\langle\, s_k \mid S_l \,\rangle = 0 \qquad \forall k, \quad l \ge g+1 \tag{45}$$

Due to (44) we consider the polynomials $s_k$, $k \ge g+1$ as corresponding to exact forms. The polynomials $s_k$ with $k = -1, \cdots, -g$ correspond to first kind differentials, $s_o$ corresponds to the third kind one, $s_k$ with $k = 1, \cdots, g$ correspond to second kind differentials.

Explicitly the relation with classical case is as follows. Consider the case $t(z) = t'(z)$ and take the limit:

$$r_k = z^{-1} \lim_{\gamma \to 0} s_k(z)$$

Then the classical Abelian differential related to $s_k$ is

$$\mu_k = \frac{r_k(z)}{y} dz$$

Similar interpretation can be given to $S_k$ which correspond to Abelian differentials in dual classical limit $\gamma \to \infty$. However, the most interesting feature of our construction is that together with this cohomological interpretation an alternative "homological" one is possible. The polynomials $S_k$, $k \geq g+1$ correspond to retractable cycles according to (45). The polynomials $S_k$ for $k = \pm 1, \cdots, \pm g$ are interpreted as analogues $a$ and $b$ cycles $\delta_k$ on the "deformed affine curve", $S_0$ corresponds to cycle $\delta_0$ around $\infty^+$ which is non-trivial on the affine curve.The pairing $\langle\, l \mid L \,\rangle$ defines the integral of differential defined by $l$ over cycle defined by $L$. The asymptotic of the integrals $\langle\, l \mid L \,\rangle$ in the classical limit $\gamma \to 0$ are, indeed, described by Abelian integrals. Certainly, the opposite interpretation ($l$-cycle, $L$-differential ) is possible which corresponds to dual classical limit. It is not the first time that this kind of objects appears [5], but it is the first time that we observe real duality between two classical limits.

Let us define the pairing between two polynomials $l_1$ and $l_2$

$$l_1 \circ l_2 = \lim_{\Lambda \to \infty} \int_{\Lambda}^{\Lambda + i\pi} \big[ \mathcal{Q}(\zeta)\mathcal{Q}'(\zeta) l_1(z) \delta_\gamma^{-1}(\mathcal{Q}\mathcal{Q}' l_2)(\zeta - i\pi) + \\ + \mathcal{Q}(\zeta - \pi i)\mathcal{Q}'(\zeta - i\pi) l_1(z) \delta_\gamma^{-1}(\mathcal{Q}\mathcal{Q}' l_2)(\zeta - i\gamma) \big] d\zeta \tag{46}$$

One can show that these formulae give well-defined anti-symmetric pairings which correspond classically to natural pairing between meromorphic differentials

$$\omega_1 \circ \omega_2 = \mathrm{res}_{p=\infty^+} \left( \omega_1(p) \int^p \omega_2 \right)$$

The polynomials $s_{\pm j}$ and for $j = 1, \cdots, g$ constitute canonical basis:

$$s_k \circ s_l = \mathrm{sgn}(k-l) \delta_{k,-l}$$

Similarly, to introduce the definition of $L_1 \circ L_2$ it is sufficient to do necessary replacements in (46): $l_i \leftrightarrow L_i$, $z \leftrightarrow Z$, $\gamma \leftrightarrow \pi$. The polynomials $S_{\pm j}$ are canonically conjugated.

The following anti-symmetric polynomials play role of 2-forms $\sigma$ and $\sigma'$ used in classics:

$$c(z_1, z_2) = \sum_{j=1}^{g} \left( s_{-j}(z_1) s_j(z_2) - s_j(z_1) s_{-j}(z_2) \right), \tag{47}$$

$$C(Z_1, Z_2) = \sum_{j=1}^{g} \left( S_{-j}(Z_1) S_j(Z_2) - S_j(Z_1) S_{-j}(Z_2) \right)$$

As usual [5] the most important property of deformed Abelian integrals is that the Riemann bilinear relations remain valid after the deformation. Namely, consider the the following $2g \times 2g$ period matrix: $P$ with the matrix elements

$$P_{kl} = \langle\, s_k \mid S_l \,\rangle \qquad\qquad k, l = -g, \cdots, -1, 1, \cdots, g$$

The deformed Riemann bilinear identity is formulated as

**Proposition 2.** *The matrix $P$ belongs to the symplectic group:*

$$P \in Sp(2g) \tag{48}$$

This Proposition 2 is equivalent to a number of bilinear relations between the deformed Abelian integrals. Proving them it is convenient to consider the domain of small $\gamma$ ($\gamma < \pi/n$) when the regularization of integrals simplifies, and then continue analytically with respect to $\gamma$. Still the proof is rather complicated technically: it is based on non-trivial properties of the regularized integrals. We do not give this bulky proof here.

There is one more relation for deformed Abelian integrals. One can check that

$$\langle\, d \mid S_k \,\rangle = 0, \quad \forall k, \qquad d = \sum_{j=1}^{g} (t_j - t'_j) s_{-j} \tag{49}$$

$$\langle\, s_k \mid D \,\rangle = 0, \quad \forall k, \qquad D = \sum_{j=1}^{g} (T_j - T'_j) S_{-j} \tag{50}$$

The relations (49) do not have direct analogue in terms of Abelian integrals, recall that we put $t_j = t'_j$ taking classical limit which turns the relation into triviality. However, there is another way of taking the classical limit where this equation is important [3].

# 10 Return to quantization of affine Jacobian.

Let us return to the main subject of this paper: quantization of affine Jacobian. Consider any $x \in \mathcal{A}(q)$. Such $x$ is identified with $x \cdot I \in \mathcal{A}(q) \cdot \mathcal{A}(Q)$, so, due to Conjecture 3 the matrix element of $x$ can be presented as

$$\langle\, t_1, \cdots t_g \mid x \mid t'_1, \cdots t'_g \,\rangle = p_L(t_1, \cdots, t_g) p_R(t'_1, \cdots, t'_g) \tag{51}$$
$$\times \int_{-\infty}^{\infty} d\zeta_1 \cdots \int_{-\infty}^{\infty} d\zeta_g \; h(z_1, \cdots, z_g) H_I(Z_1, \cdots, Z_g) \prod_{j=1}^{g} \mathcal{Q}(\zeta_j) \mathcal{Q}'(\zeta_j)$$

where the eigen-values of $\boldsymbol{T}_j$ are the same at the left and at the right, so, we do not write them explicitly, the polynomial $H_I$ which corresponds to $X = I$ is given by

$$H_I(Z_1, \cdots, Z_g) = \prod_{j=1}^{g} Z_j \prod_{i<j} (Z_i - Z_j)$$

notice that

$$H_I = S_{-1} \wedge \cdots \wedge S_{-g} \tag{52}$$

The formula for the matrix elements (51) for small $\gamma$ (when no regularization of integrals is needed) can be deduced rigorously starting from realization of $\mathcal{A}(q)$ defined in Section 4. The following equations follow respectively from (44), (49), (48) (recall notations of Section 4):

$$s_k \wedge \mathcal{V}^{g-1} \simeq 0, \quad k \geq g+1 \tag{53}$$

$$c \wedge \mathcal{V}^{g-2} \simeq 0, \tag{54}$$

$$d \wedge \mathcal{V}^{g-1} \simeq 0, \tag{55}$$

where $\simeq$ means that these expression vanish being substitute into the integral (51). The equation (54) needs explanation. To prove this equation one has to take into account the Riemann bilinear identity (48) and the formula (52); notice that

$$S_{-i} \circ S_{-j} = 0, \quad 1 \leq i, j \leq g$$

The formula for the matrix elements (51) can be rigorously deduced for small $\gamma$. Hence the equations (53), (54), (55) lead to equation for operators. The latter equations are obtained applying the operation $\chi$ (Section 4):

$$\chi\big(s_k \wedge \mathcal{V}^{g-1}\big) = 0, \quad k \geq g+1 \tag{56}$$

$$\chi\big(c \wedge \mathcal{V}^{g-2}\big) = 0, \tag{57}$$

$$\chi\big(d \wedge \mathcal{V}^{k-1}\big) = 0 \tag{58}$$

We conclude that the formulae for the polynomials $s_k$ needed in Section 4 are exactly the same as given in (43). Thus we put together the algebraic part of this work with the analytical one.

On the other hand the equations (56, 57, 58) are of purely algebraic character, so, if they are valid for small $\gamma$ they must be valid always. That is why we regularized the integrals for matrix element in order that the equations (53), (54), (55) hold for any $\gamma$.

Moreover, there is a dual model and we can consider the operators $\mathcal{X} = xX$ from $\mathcal{A}(q) \cdot \mathcal{A}(Q)$. The equations (56), (57), (58) and dual equations still have to be valid. The regularized integrals are defined in such a way that it is the case. The equations (56), (58) and their duals clearly follow from (44), (45) and (49), (50). The most interesting is the equation (57). Due to the Riemann bilinear relation this equation follows from

$$c \wedge \mathcal{V}^{g-2} \simeq 0 \tag{59}$$

which is true if the sub-space $c \wedge \mathcal{V}^{g-2}$ of the space $\mathcal{V}^g$ is convoluted with the sub-space

$$\frac{\mathcal{V}_g}{C \wedge \mathcal{V}_{g-2}}$$

where $\mathcal{V}_k$ is the same as $\mathcal{V}^k$ for dual model (this notation is not occasional: the space $\mathcal{V}_k$ plays role of $k$-cycles for $k$-forms from $\mathcal{V}^k$). In other words we impose the equation

$$C \wedge \mathcal{V}_{g-2} = 0$$

and the dual equation (59) is imposed automatically due to the Riemann bilinear relation.

Let us discuss the classical limit in some more details. Consider the hyper-elliptic curve $X$. If we realize this curve as characteristic equation of classical analogue of the monodromy matrix $\widetilde{\boldsymbol{m}}(z)$ (21) the branch points of the curve can be shown real non-negative. Actually requiring $t_{g+1} = (-1)^{g+1}2$ we put one of branch points at $z = 0$. Thus the branch points are $0 = q_1 < \cdots < q_{2g+2}$. The Riemann surface is realized as two-sheet covering of the plane of $z$ with cuts $I_k = [q_{2k-1}, q_{2k}]$, $k = 1, \cdots, g+1$. The canonical a-cycles $\delta_{-j}$ and b-cycles $\delta_j$ are shown on the ***fig. 1*** follows:

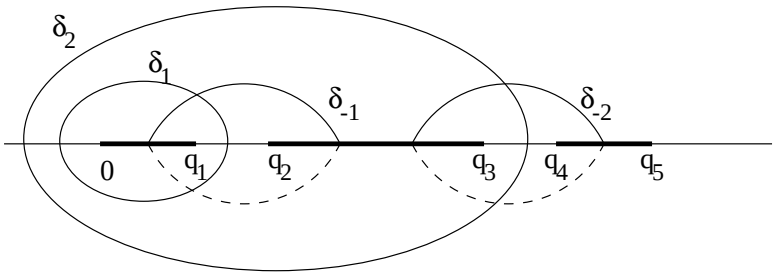


*fig. 1*

Under the classical dynamics every of the separated variables $z_j$ oscillates in the interval: $q_{2j-1} \leq z_j \leq q_{2j}$, topologically it corresponds to motion along the a-cycle $\delta_{-j}$. One can show that the integral $\langle\ s_k \mid S_{-j}\ \rangle$ is described in the classical limit $\gamma \to 0$ by $\delta_{-j}$ of differential $\mu_k$. Thus the $g$-cycle (52) corresponds to classical trajectory $\delta_{-1} \wedge \cdots \wedge \delta_{-g}$. Recall that the cycle (52) corresponds to insertion of unit operator of the dual model. Introducing other dual operators one gets integrals with respect to both a-cycles and b-cycles. Classically the coresponding trajectories are not real, but the factorization by $\sigma' \wedge W_{m-2}$ in (12) guarantees that the classical non-real trajectories are not singular. We would like to finish this paper with this topological interpretation of the dual model.

# 11 Appendix A.

In this Appendix we shall give the canonical definition of affine Jacobi variety $J_{\text{aff}}(t)$. Consider hyper-elliptic curve $X$ of genus $g$:

$$w^2 - t(z)w + 1 = 0$$

We have the canonical basis with a-cycles $\delta_k$, $-g \le k < 0$ and b-cycles $\delta_k$, $0 < k \le g$. Associate with this basis the basis of normalized holomorphic differentials $\omega_j$:

$$\int\limits_{\delta_{-i}} \omega_j = \delta_{ij}, \quad B_{ij} = \int\limits_{\delta_i} \omega_j$$

The Jacobi variety of this curve is the $g$-dimensional complex torus:

$$J(t) = \frac{\mathbb{C}^g}{\mathbb{Z}^g \times B\mathbb{Z}^g}$$

With every point $p \in X$ we identify the point $\alpha(p) \in J(t)$ with coordinates:

$$\alpha_j(p) = \int\limits_b^p \omega_j$$

for the reference point $b$ it is convenient to take one of the branch points. The curve $X$ has two points over the point $z = \infty$, denote them by $\infty^{\pm}$ and consider the $(g-1)$-dimensional subvariety of $J(t)$ defined by

$$\Theta^{\pm} = \{\zeta \in J(t) \mid \theta(\zeta + \alpha(\infty^-))\theta(\zeta + \alpha(\infty^+)) = 0\}$$

where $\theta$ is Riemann theta-function. It can be shown that there exist an isomorphism:

$$J_{\text{aff}}(t) \simeq J(t) - \Theta^{\pm} \tag{60}$$

The equivalence of this description with the description in terms of divisors (Section 1) is due to the Abel map $X[g] \to J(t)$ explicitely given by

$$\zeta = \alpha(\mathcal{P}) + \Delta, \quad \alpha(\mathcal{P}) = \sum \alpha(p_j)$$

where $\Delta$ is the Riemann characteristic.

# 12 Appendix B.

In this Appendix we describe the regularization of integrals which has been used in the paper.

Define:

$$\delta_\xi(f(\zeta)) = f(\zeta + i\xi) - f(\zeta),$$
$$\Delta_\xi(f(\zeta)) = f(\zeta + i\xi) - f(\zeta - i\xi)$$

Introduce the polynomials

$$\begin{aligned}
s_k(z) = \frac{1}{2i\gamma}\big\{ & t(z)\Delta_\gamma^{-1}[z^{k-g-1}t(z)]_> + t'(z)\Delta_\gamma^{-1}[z^{k-g-1}t'(z)]_> - \\
& - t(z)\Delta_\gamma^{-1}[z^{k-g-1}q^{2(g+1-k)}t'(zq^{-2})]_> - \\
& - t'(z)\Delta_\gamma^{-1}[z^{k-g-1}q^{2(g+1-k)}t(zq^{-2})]_> - \\
& - \tfrac{1}{2}\left(t'(z)[z^{k-g-1}t(z)]_> + t(z)[z^{k-g-1}t'(z)]_>\right) + \\
& + (q^{2(g+1-k)k} - q^{2(k-g-1)})[z^{k-g-1}]_>\big\}, \qquad k \geq 0; \\
s_k(z) = z^{g+1+k}, & \qquad -g \leq k \leq 0;
\end{aligned}$$

where the notation $[\ \ ]_>$ means that only the positive degrees of Laurent series in brackets are taken. Obviously $\deg(s_k) = g+1+k$. Further, with every function $f(\zeta)$ associate the functions:

$$\begin{aligned}
u[f](\zeta) = \frac{1}{2i\gamma}\big\{ & t(z)\Delta_\gamma^{-1}\left(f(\zeta)t(z)\right) + t'(z)\Delta_\gamma^{-1}\left(f(\zeta)t'(z)\right) - \\
& - t(z)\Delta_\gamma^{-1}\left(f(\zeta - i\gamma)t'(zq^{-2})\right) - t'(z)\Delta_\gamma^{-1}\left(f(\zeta - i\gamma)t(zq^{-2})\right) - \\
& - f(\zeta)t(z)t'(z) + f(\zeta + i\gamma) - f(\zeta - i\gamma)\big\}, \\
v[f](\zeta) = \frac{1}{2i\gamma}\big\{ & (-1)^{g+1}\big(\Delta_\gamma^{-1}\left(f(\zeta - i\gamma)t(zq^{-2})\right)\mathcal{Q}(\zeta)\mathcal{Q}'(\zeta - i\gamma) + \\
& + \Delta_\gamma^{-1}\left(f(\zeta - i\gamma)t'(zq^{-2})\right)\mathcal{Q}(\zeta - i\gamma)\mathcal{Q}'(\zeta) - \\
& - \Delta_\gamma^{-1}\left(f(\zeta)t(z)\right)\mathcal{Q}(\zeta - i\gamma)\mathcal{Q}'(\zeta) \\
& - \Delta_\gamma^{-1}\left(f(\zeta)t'(z)\right)\mathcal{Q}(\zeta)\mathcal{Q}'(\zeta - i\gamma)\big) + \\
& + f(\zeta)\mathcal{Q}(\zeta - i\gamma)\mathcal{Q}'(\zeta - i\gamma) + f(\zeta - i\gamma)\mathcal{Q}(\zeta)\mathcal{Q}'(\zeta)\big\}
\end{aligned}$$

Define

$$s_k^-(\zeta) = \begin{cases} -s_k(z) + u[f](\zeta), & f = z^{k-g-1}, \quad k \geq 1; \\ -s_0(z) + u[f](\zeta) + (-1)^{g+1}2, & f = \zeta z^{-g-1},\ k = 0; \\ -z^{g+1+k}, & -g \leq k \leq -1 \end{cases}$$

and

$$p_k(\zeta) = \begin{cases} v[f](\zeta), & f = z^{k-g-1},\ k \geq 1; \\ v[f](\zeta), & f = \zeta z^{-g-1},\ k = 0; \\ 0, & -g \leq k \leq -1 \end{cases}$$

These definitions imply that

$$\left(s_k(z) + s_k^-(z)\right)\mathcal{Q}(\zeta)\mathcal{Q}'(\zeta) = \delta_\gamma(p_k(\zeta)) \tag{61}$$

Similarly one introduces the functions $S_k(Z)$, $S_k^-(\zeta)$, $P_k(\zeta)$ changing everywhere $z$ by $Z$, $q$ by $Q$ and $i\gamma$-shift of $\zeta$ by $i\pi$-shift of $\zeta$. One has

$$(S_k(z) + S_k^-(z))\ \mathcal{Q}(\zeta)\mathcal{Q}'(\zeta) = \delta_\pi(P_k(\zeta)) \tag{62}$$

Our goal is to define a pairing $\langle\ l \mid L\ \rangle$ between two arbitrary polynomials $l(z)$ and $L(Z)$ such that $l(0) = 0$, $L(0) = 0$. Notice that every such polynomial $l$ ($L$) can be presented as linear combination of polynomials $s_k$ ($S_k$).

Consider the following two pictures:

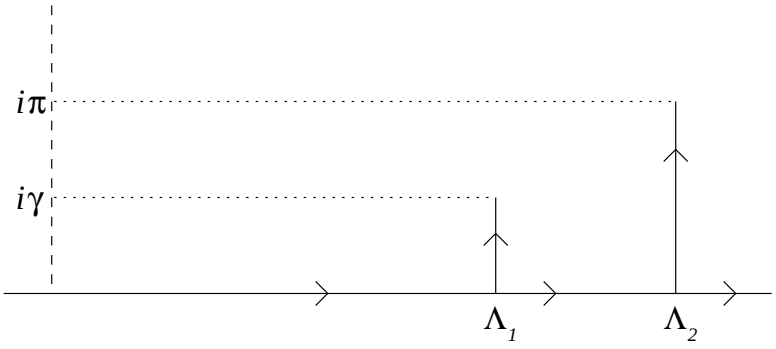


*fig. 2a*

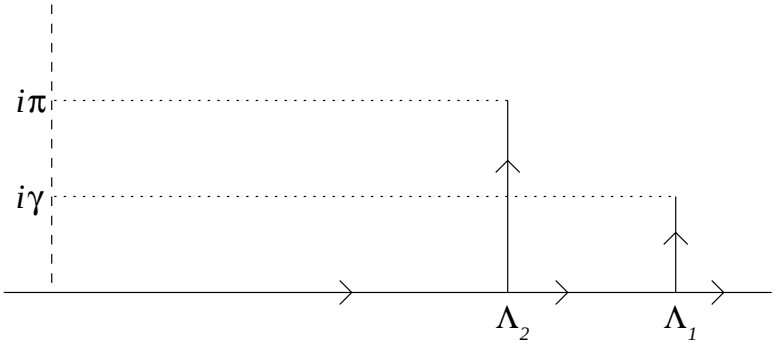


*fig. 2b*

We define

$$\langle s_k \mid S_l\rangle \equiv \int_{-\infty}^{\Lambda_1} \mathcal{Q}(\zeta)\mathcal{Q}'(\zeta)s_k(z)S_l(Z)d\zeta +$$
$$+ \int_{\Lambda_1}^{\Lambda_2} \mathcal{Q}(\zeta)\mathcal{Q}'(\zeta)s_k^-(\zeta)S_l(Z)d\zeta + \int_{\Lambda_2}^{\infty} \mathcal{Q}(\zeta)\mathcal{Q}'(\zeta)s_k^-(\zeta)S_l^-(\zeta)d\zeta -$$
$$- \int_{\Lambda_1}^{\Lambda_1+i\gamma} S_l(Z)p_k(\zeta)d\zeta - \int_{\Lambda_2}^{\Lambda_2+i\pi} s_k^-(\zeta)P_l(\zeta)d\zeta, \tag{63}$$

see (*fig. 2a*). The first integral in RHS converges at $-\infty$ because $l(0) = L(0) = 0$. The equations (61,62) guarantee that the regularization (63) does not depend on $\Lambda_1$, $\Lambda_2$ if they remain ordered: $\Lambda_1 < \Lambda_2$. Moreover, let us transform (*fig. 2a*) into (*fig. 2b*). The alternative definition of regularized integral referring to (*fig. 2b*) is

$$\langle s_k \mid S_l \rangle \equiv \int\limits_{-\infty}^{\Lambda_2} \mathcal{Q}(\zeta)\mathcal{Q}'(\zeta)s_k(z)S_l(Z)d\zeta +$$
$$+ \int\limits_{\Lambda_2}^{\Lambda_1} \mathcal{Q}(\zeta)\mathcal{Q}'(\zeta)s_k(z)S_l^-(\zeta)d\zeta + \int\limits_{\Lambda_1}^{\infty} \mathcal{Q}(\zeta)\mathcal{Q}'(\zeta)s_k^-(\zeta)S_l^-(\zeta)d\zeta -$$
$$- \int\limits_{\Lambda_1}^{\Lambda_1+i\gamma} s_k(z)P_l(\zeta)d\zeta - \int\limits_{\Lambda_2}^{\Lambda_2+i\pi} S_l^-(\zeta)p_k(\zeta)d\zeta \qquad (64)$$

The equivalence of the regularizations (63) and (64) is based on the following fact. It is easy to realize that for any $l$ and $L$ there exist a function $X_{kl}(\zeta)$ such that

$$(S_l(Z) - S_l^-(\zeta))p_k(\zeta) = \delta_\pi(X_{kl}(\zeta)),$$
$$(s_k(z) - s_k^-(\zeta))P_l(\zeta) = \delta_\gamma(X_{kl}(\zeta))$$

The equivalence in question follows from the equality:

$$\int\limits_{\Lambda}^{\Lambda+i\gamma} (S_l(Z) - S_l^-(\zeta))\, g(\zeta)d\zeta = \left( \int\limits_{\Lambda+i\pi}^{\Lambda+i\pi+i\gamma} - \int\limits_{\Lambda}^{\Lambda+i\gamma} \right) X_{kl}(\zeta)d\zeta =$$
$$= \left( \int\limits_{\Lambda+i\gamma}^{\Lambda+i\pi+i\gamma} - \int\limits_{\Lambda}^{\Lambda+i\pi} \right) X_{kl}(\zeta)d\zeta = \int\limits_{\Lambda}^{\Lambda+i\pi} (s_k(z) - s_k^-(\zeta))\, G(\zeta)d\zeta$$

# References


[1] D. Mumford, *Tata Lectures on Theta*, vol. I and II, Birkhäuser, Boston (1983)

[2] A. Nakayashiki, F.A. Smirnov, *Cohomologies of Affine Jacobi Varieties and Integrable Systems.* math-ph/0001017

[3] F.A. Smirnov, J. Phys. A: Math. Gen. 31 (1988) 8953

[4] E.K. Sklyanin, *Separation of Variables*, Progress in Theoretical Physics Supplement 118 (1995) 35

[5] F.A. Smirnov, Lett.Math.Phys. 36 (1996) 267

[6] J.M. Maillet, V. Terras, *On the quantum inverse scattering problem*, hep-th/9911030

[7] L.D. Faddeev, A.Yu. Volkov, *Algebraic Quantization of Integrable Models in Discrete Space-time*, hep-th/9710039, and references therein

[8] V. Pasquier and M. Gaudin, J.Phys. A (1992) 5243

[9] L.D. Faddeev, Lett. Math. Phys. 34 (1995) 249

[10] L.D. Faddeev, R.M. Kashaev, A.Yu. Volkov, *to be published*

[11] V.V. Bazhanov, S.L. Lukyanov, A.B. Zamolodchikov, Comm. Math. Phys. 177 (1996) 381; Comm. Math. Phys. 190 (1997) 247