The Moyal product is the matrix product

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Abstract

This is a short comment on the Moyal formula for deformation quantization. It is shown that the Moyal algebra of functions on the plane is canonically isomorphic to an algebra of matrices of infinite size.

1. Deformation quantization. Classical mechanical systems are mathematically described by symplectic manifolds, (M,ω), and their quantization is usually understood as a functor

$$(M,\omega) \longrightarrow \text{a Hilbert space } H,$$

which comes together with an association

$$\begin{matrix} \text{a function } f \text{ on } M \\ \text{“classical observable”} \end{matrix} \quad \longrightarrow \quad \begin{matrix} \text{an operator } \hat{f} \in \mathrm{End}H[[\hbar]] \\ \text{“quantum observable”} \end{matrix}$$

satisfying the conditions

(i) $\widehat{1} = \mathrm{Id}$,

(ii) $\widehat{\{f,g\}} = \lim_{\hbar\to 0} \frac{i}{\hbar}\left(\widehat{f}\,\widehat{g} - \widehat{g}\,\widehat{f}\right)$,

(iii) complex conjugation $\overset{\widehat{}}{\longrightarrow}$ transition to the adjoint,

where $\{\ ,\ \}$ is the Poisson bracket,

$$\{f,g\} := \omega^{-1}(df,dg).$$

Most symplectic manifolds arising in classical mechanics are total spaces of the cotangent bundles, $(T^*P, \text{standard symplectic form})$, to some n-dimensional *configuration* manifolds P, and the above association is often plagued with the ordering choice ambiguity.

Deformation quantization [BFFLS] offers a very different scheme in which the classical data, $(C^\infty(M),\omega)$, is mapped upon quantization to itself as a set but *not* as a ring. The usual commutative product of functions on M gets replaced by a new *deformed* product,

$$f *_\hbar g = fg + \sum_{n=1}^{\infty} \hbar^n D^n(f,g), \quad f,g \in C^\infty(M)[[\hbar]],$$

which is supposed to satisfy the conditions,

(i) D_n are $\mathbb{R}[[\hbar]]$-linear bidifferential operators of finite total order,

(ii) $1 *_\hbar f = f *_\hbar 1 = f$,

(iii) $\{f, g\} = \lim_{\hbar\to 0} \frac{1}{\hbar}(f *_\hbar g - g *_\hbar f)$,

(iv) $(f *_\hbar g) *_\hbar h = f *_\hbar (g *_\hbar h)$,

for all f, g and h in $C^\infty(M)[[\hbar]]$. The deformed product is associative, but no more commutative. The above mentioned ordering choice ambiguity shows itself again, this time in the form of an equivalence relation among star products: $*_\hbar \sim \star_\hbar$ if there exist $\mathbb{R}[[\hbar]]$-linear differential operators $Q_n : C^\infty(M) \to C^\infty(M)$ such that

$$Q(f \star_\hbar g) = (Qf) *_\hbar (Qg), \qquad \forall f, g \in C^\infty(M)[[\hbar]],$$

where $Q = \mathrm{Id} + \sum_{n\geq 1} Q_n$.

The spectral theory of observables can be studied, within the deformation quantization framework, via the so-called star-exponentials [BFFLS].

2. The Moyal product. This is the simplest example of deformation quantization. The symplectic input is $\mathbb{R}^{2n}$ with its standard 2-form $\omega = \sum_{i=1}^n dx^i \wedge dp_i$ and the deformed product is given by the following formula[1],

$$f *_\hbar g := e^{\sum_{a=1}^n \frac{\hbar}{2}\left(\frac{\partial^2}{\partial x^a \partial \tilde{p}_a} - \frac{\partial^2}{\partial p_a \partial \tilde{x}^a}\right)} f(x^b, p_b) g(\tilde{x}^c, \tilde{p}_c)\Big|_{\substack{x^a = \tilde{x}^a \\ p_a = \tilde{p}_a}}.$$

On the plane, $\mathbb{R}^2$, the Moyal product simplifies,

$$f *_\hbar g = \sum_{n=0}^\infty \frac{\hbar^n}{2^n n!} \sum_{k=0}^n \binom{n}{k} (-1)^k \frac{\partial^n f}{\partial x^{n-k} \partial p^k} \frac{\partial^n g}{\partial x^k \partial p^{n-k}}.$$

Quantum mechanically, this product corresponds to the Weyl (symmetric) ordering in the ring of observables $C^\infty(\mathbb{R}^2)[[\hbar]]$. The equivalent product,

$$\begin{aligned} f \star_\hbar g \;&:=\; e^{\frac{\hbar}{2}\frac{\partial^2}{\partial x \partial p}} \left((e^{-\frac{\hbar}{2}\frac{\partial^2}{\partial x \partial p}} f) *_\hbar (e^{-\frac{\hbar}{2}\frac{\partial^2}{\partial x \partial p}} g) \right) \\ &= \sum_{n=0}^\infty \frac{\hbar^n}{n!} \frac{\partial^n f}{\partial x^n} \frac{\partial^n g}{\partial p^n}, \end{aligned}$$

corresponds to the standard (non-symmetric) ordering.

From now on we assume that the symbols $*_\hbar$ and $\star_\hbar$ stand for the products just defined. As the space of observables we take the formal ring, $k[[x, p, \hbar]]$, where k is a field containing rational numbers (for example $\mathbb{R}$ or $\mathbb{C}$).

[1]Strictly speaking, we have to replace $\hbar$ by $i\hbar$ everywhere in this text to make things consistent with the usual physics conventions.

3. Matrix algebra. Let $\mathbb{Z}^{\geq 0}$ be the ring of non-negative integers, R the algebra of formal power series $k[[\hbar]]$ and I the maximal ideal in R. Denote by $\mathsf{Mat}_\infty^\hbar$ the algebra of all matrices, $(a_{ij})_{i,j\in\mathbb{Z}^{\geq 0}}$, such that

$$a_{ij} \in \begin{cases} R, & \text{if } i \geq j, \\ I^{j-i}, & \text{if } i < j. \end{cases}$$

Pictorially,

$$\mathsf{Mat}_\infty^\hbar = \begin{pmatrix} R & I & I^2 & I^3 & I^4 & \dots \\ R & R & I & I^2 & I^3 & \dots \\ R & R & R & I & I^2 & \dots \\ R & R & R & R & I & \dots \\ R & R & R & R & R & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

This display makes it obvious that the usual matrix multiplication in $\mathsf{Mat}_\infty^\hbar$ is well-defined.

The algebra $\mathsf{Mat}_\infty^\hbar$ is freely generated as an R-module by the matrices, $E_{a,b}$, $a,b\in\mathbb{Z}^{\geq 0}$, whose ij-th entry is, by definition, given by

$$(E_{a,b})_{ij} = \begin{cases} \frac{(b+k)!}{k!}\hbar^b, & \text{if } i = a+k, j = b+k, k = 0,1,2,\dots, \\ 0, & \text{otherwise.} \end{cases}$$

One may check that their matrix product is given by

$$E_{a,b}E_{c,d} = \sum_{n=0}^{\min(b,c)} \frac{\hbar^n b!c!}{n!(b-n)!(c-n)!} E_{a+c-n,b+d-n}. \tag{1}$$

4. Proposition. *There is a canonical isomorphism of associative algebras,*

$$\phi : (k[[x,p,\hbar]], \star_\hbar) \longrightarrow \mathsf{Mat}_\infty^\hbar$$

given on the generators as follows,

$$\phi(p^a x^b) = E_{a,b}.$$

This result implies in turn the main claim of this paper.

5. Main Theorem. *The Moyal algebra $(k[[x,p,\hbar]], *_\hbar)$ is canonically isomorphic to the matrix algebra $\mathsf{Mat}_\infty^\hbar$. On the generators, the isomorphism ψ is given by*

$$\psi(p^a x^b) = \sum_{n=0}^{\min(a,b)} \frac{\hbar^n a!b!}{2^n n!(a-n)!(b-n)!} E_{a-n,b-n}.$$

6. Example. To see how it all works, let us consider two monomials, p and x^2, with

$$p *_\hbar x^2 = px^2 - \hbar x.$$

We have

$$\psi(p) = E_{1,0} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

and

$$\psi(x^2) = E_{0,2} = \begin{pmatrix} 0 & 0 & \frac{2!}{0!}\hbar^2 & 0 & 0 & \dots \\ 0 & 0 & 0 & \frac{3!}{1!}\hbar^2 & 0 & \dots \\ 0 & 0 & 0 & 0 & \frac{4!}{2!}\hbar^2 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

implying

$$\begin{aligned} \psi(p)\psi(x^2) &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & \frac{2!}{0!}\hbar^2 & 0 & 0 & \dots \\ 0 & 0 & 0 & \frac{3!}{1!}\hbar^2 & 0 & \dots \\ 0 & 0 & 0 & 0 & \frac{4!}{2!}\hbar^2 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \\ &= E_{1,2} \\ &= (E_{1,2} + \hbar E_{0,1}) - \hbar E_{0,1} \\ &= \psi(px^2) - \hbar\phi(x) \\ &= \psi(p *_\hbar x^2). \end{aligned}$$

Analogously,

$$\begin{aligned} \psi(x^2)\phi(p) &= \begin{pmatrix} 0 & \frac{2!}{0!}\hbar^2 & 0 & 0 & 0 & \dots \\ 0 & 0 & \frac{3!}{1!}\hbar^2 & 0 & 0 & \dots \\ 0 & 0 & 0 & \frac{4!}{2!}\hbar^2 & 0 & \dots \\ 0 & 0 & 0 & 0 & \frac{5!}{3!}\hbar^2 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \\ &= E_{1,2} + 2\hbar E_{0,1} \\ &= (E_{1,2} + \hbar E_{0,1}) + \hbar E_{0,1} \\ &= \psi(px^2) + \hbar\phi(x) \\ &= \psi(x^2 *_\hbar p), \end{aligned}$$

which completes the check.

7. Proof of the Proposition. Once the matrices $E_{a,b}$ with the properties (1) are explicitly written down, the proof becomes obvious. Thus we have only to motivate our definition of $(E_{a,b})$s and, probably, indicate how one might check the key properties (1) without serious calculations:

1) In the first place, we have read these matrices out of the quantum space of the n-tuple point, $x^n = 0$, [Me]; the existence of the isomorphisms ϕ and ψ can be deduced from the projective limit of that construction.

2) Alternatively, one may check that the $\star_\hbar$-product of the functions

$$g_{a,b} := \frac{p^a x^b}{b!\hbar^b} e^{-\frac{px}{\hbar}}, \quad a, b = 0, 1, 2, \ldots,$$

is well-defined and is given by

$$g_{a,b} \star_\hbar g_{c,d} = \delta_{bc} g_{a,d},$$

where

$$\delta_{bc} = \begin{cases} 0, & \text{if } b \neq c, \\ 1, & \text{if } b = c. \end{cases}$$

This fact is very easily established from the obvious differential equations satisfied by $g_{a,d}$ and $g_{a,b} \star_\hbar g_{c,d}$ as well as their initial values (modulo the factor $p^a x^d$) at $x = p = 0$.

Next one notices that

$$p^a x^b = \sum_{n=0}^{\infty} \hbar^b \frac{(b+n)!}{n!} g_{a+n,b+n},$$

explaining both the isomorphism ϕ in the Proposition, and the properties (1) of the matrices $E_{a,b}$.

□

8. Proof of the Main Theorem. One may deduce this statement directly from the Proposition. Indirectly, one notices that the functions,

$$\begin{aligned} h_{a,b} &= e^{-\frac{\hbar}{2}\frac{\partial^2}{\partial x \partial p}} (g_{a,b}) \\ &= \frac{2}{b!\hbar^b} e^{-\frac{2xp}{\hbar}} e^{-\frac{\hbar}{4}\frac{\partial^2}{\partial x \partial p}} \left(p^a x^b\right) \end{aligned}$$

satisfy

$$h_{a,b} *_\hbar h_{c,d} = \delta_{bc} h_{a,d},$$

and that one has

$$e^{-\frac{\hbar}{2}\frac{\partial^2}{\partial x \partial p}} \left(p^a x^b\right) = \sum_{n=0}^{\infty} \hbar^b \frac{(b+n)!}{n!} h_{a+n,b+n}.$$

This explains the structure of the isomorphism ψ. □

References

[BFFLS] Bayen, F, Flato, M, Fronsdal, C., Lichnerovich, A. and Sternheimer, D.: Deformation quantization, I & II, Ann. Phys. **111**, 61-110 & 111-151 (1978)

[Mo] Moyal, J.E.: Quantum mechanics as a statistical theory, Proc. Cambridge Phil. Soc. **45**, 99-124 (1949)

[Me] Merkulov, S.A.: Deformation quantization of the n-tuple point, Commun. Math. Phys. **205**, 369-375 (1999)