A RIEMANN-ROCH THEOREM FOR ONE-DIMENSIONAL COMPLEX GROUPOIDS

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Abstract

We consider a smooth groupoid of the form $\Sigma \rtimes \Gamma$ where Σ is a Riemann surface and Γ a discrete pseudogroup acting on Σ by local conformal diffeomorphisms. After defining a K-cycle on the crossed product $C_0(\Sigma) \rtimes \Gamma$ generalising the classical Dolbeault complex, we compute its Chern character in cyclic cohomology, using the index theorem of Connes and Moscovici. This involves in particular a generalisation of the Euler class constructed from the modular automorphism group of the von Neumann algebra $L^\infty(\Sigma) \rtimes \Gamma$.

I. Introduction

In a series of papers [4, 5], Connes and Moscovici proved a general index theorem for transversally (hypo)elliptic operators on foliations. After constructing K-cycles on the algebra crossed product $C_0(M) \rtimes \Gamma$, where Γ is a discrete pseudogroup acting on the manifold M by local diffeomorphisms [4], they developed a theory of characteristic classes for actions of Hopf algebras that generalise the usual Chern-Weil construction to the non-commutative case [5, 6]. The Chern character of the concerned K-cycles is then captured in the periodic cyclic cohomology of a particular Hopf algebra encoding the action of the diffeomorphisms on M. The nice thing is that this cyclic cohomology can be completely exhausted as Gelfand-Fuchs cohomology and renders the index computable.
We shall illustrate these methods with a specific example, namely the crossed product of a Riemann surface Σ by a discrete pseudogroup Γ of local conformal mappings. We find that the relevant characteristic classes are the fundamental class $[\Sigma]$ and a cyclic 2-cocycle on $C_c^\infty(\Sigma) \rtimes \Gamma$ generalising the (Poincaré dual of the) usual Euler class. When applied to the K-cycle represented by the Dolbeault operator of $\Sigma \rtimes \Gamma$, this yields a non-commutative version of the Riemann-Roch theorem. Throughout the text we also stress the crucial role played by the modular automorphism group of the von Neumann algebra $L^\infty(\Sigma) \rtimes \Gamma$.

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II. The Dolbeault K-cycle

Let Σ be a Riemann surface without boundary and Γ a pseudogroup of local conformal mappings of Σ into itself. We want to define a K-cycle on the algebra $C_0(\Sigma) \rtimes \Gamma$ generalising the classical Dolbeault complex. Following [4], the first step consists in lifting the action of Γ to the bundle P over Σ, whose fiber at point x is the set of Kähler metrics corresponding to the complex structure of Σ at x. By the obvious correspondence metric $\leftrightarrow$ volume form, P is the $\mathbb{R}_+^*$-principal bundle of densities on Σ. The pseudogroup Γ acts canonically on P and we consider the crossed procuct $C_0(P) \rtimes \Gamma$.
Let ν be a smooth volume form on Σ. As in [2], this gives a weight on the von Neumann algebra $L^\infty(\Sigma) \rtimes \Gamma$ together with a representative σ of its modular automorphism group. Moreover σ leaves $C_0(\Sigma) \rtimes \Gamma$ globally invariant and one has

$$C_0(P) \rtimes \Gamma = (C_0(\Sigma) \rtimes \Gamma) \rtimes_\sigma \mathbb{R} \;, \tag{1}$$

where the space P is identified with $\Sigma \times \mathbb{R}$ thanks to the choice of the global section ν. Therefore one has a Thom-Connes isomorphism [1]

$$K_i(C_0(\Sigma) \rtimes \Gamma) \to K_{i+1}(C_0(P) \rtimes \Gamma) \;, \quad i = 0, 1 \;, \tag{2}$$

and we shall obtain the desired K-homology class on $C_0(P) \rtimes \Gamma$. The reason for working on P rather than Σ is that P carries quasi Γ-invariant metric structures, allowing the construction of K-cycles represented by differential hypoelliptic operators [4].

More precisely, consider the product $P \times \mathbb{R}$, viewed as a bundle over Σ with 2-dimensional fiber. The action of Γ extends to $P \times \mathbb{R}$ by making $\mathbb{R}$ invariant. Up to another Thom isomorphism, the K-cycle may be defined on $C_0(P \times \mathbb{R}) \rtimes \Gamma = (C_0(P) \rtimes \Gamma) \otimes C_0(\mathbb{R})$. By a choice of horizontal subspaces on the bundle $P \times \mathbb{R}$, one can lift the Dolbeault operator $\overline{\partial}$ of Σ. This yields the horizontal operator $Q_H = \overline{\partial} + \overline{\partial}^*$, where the adjoint $\overline{\partial}^*$ is taken relative to the L^2-norm given by the canonical invariant measure on $P \times \mathbb{R}$ (see [4] for details). Finally, consider the signature operator of the fibers, $Q_V = d_V d_V^* - d_V^* d_V$, where d_V is the vertical differential. Then the sum $Q = Q_H + Q_V$ is a hypoelliptic operator representing our Dolbeault K-cycle.
This construction ensures that the principal symbol of Q is completely canonical, because related only to the fibration of $P \times \mathbb{R}$ over Σ, and hence is invariant under Γ. Another choice of horizontal subspaces does not change the leading term of the symbol of Q. This is basically the reason why Q allows to construct a spectral triple (of even parity) for the algebra $C_c^\infty(P \times \mathbb{R}) \rtimes \Gamma$.
If $\Gamma = \mathrm{Id}$, then $C_0(P \times \mathbb{R}) \rtimes \Gamma = C_0(\Sigma) \otimes C_0(\mathbb{R}^2)$ and the addition of Q_V to Q_H is nothing else but a Thom isomorphism in K-homology

$$K^*(C_0(\Sigma)) \to K^*(C_0(P \times \mathbb{R})) \tag{3}$$

sending the classical Dolbeault elliptic operator $\overline{\partial} + \overline{\partial}^*$ to Q.

Now we want to compute the Chern character of Q in the periodic cyclic cohomology $H^*(C_c^\infty(P \times \mathbb{R}) \rtimes \Gamma)$ using the index theorem of [5]. We need first to construct an *odd* cycle by tensoring the Dolbeault complex with the spectral triple of the real line $(C_c^\infty(\mathbb{R}), L^2(\mathbb{R}), i\frac{\partial}{\partial x})$. In this way we get a differential operator $Q' = Q + i\frac{\partial}{\partial x}$ whose Chern character lives in the cyclic cohomology of $(C_c^\infty(P) \rtimes \Gamma) \otimes C_c^\infty(\mathbb{R}^2)$. By Bott periodicity it is just the cup product

$$\mathrm{ch}_*(Q') = \varphi \# [\mathbb{R}^2] \tag{4}$$

of a cyclic cocycle $\varphi \in HC^*(C_c^\infty(P) \rtimes \Gamma)$ by the fundamental class of $\mathbb{R}^2$. The main theorem of [5] states that φ can be computed from Gelfand-Fuchs cohomology, after transiting through the cyclic cohomology of a particular Hopf algebra. We perform the explicit computation in the remaining of the paper.

III. The Hopf algebra and its cyclic cohomology

First we reduce to the case of a flat Riemann surface, since for any groupoid $\Sigma \rtimes \Gamma$ one can find a flat surface Σ' and a pseudogroup Γ' acting by conformal transformations on Σ' such that $C_0(\Sigma') \rtimes \Gamma'$ is Morita equivalent to $C_0(\Sigma) \rtimes \Gamma$ (see [5] and section V below).

Let then Σ be a flat Riemann surface and $(z, \overline{z})$ a complex coordinate system corresponding to the complex structure of Σ. Let F be the $Gl(1, \mathbb{C})$-principal bundle over Σ of frames corresponding to the conformal structure. F is gifted with the coordinate system $(z, \overline{z}, y, \overline{y})$, $y, \overline{y} \in \mathbb{C}^*$. A point of F is the frame

$$(y\partial_z, \overline{y}\partial_{\overline{z}}) \quad \text{at } (z, \overline{z}) \ . \tag{5}$$

The action of a discrete pseudogroup Γ of conformal transformations on Σ can be lifted to an action on F by pushforward on frames. More precisely, a holomorphic transformation $\psi \in \Gamma$ acts on the coordinates by

$$z \ \rightarrow \ \psi(z) \qquad \mathrm{Dom}\psi \subset F \tag{6}$$

$$y \ \rightarrow \ \psi'(z) y \ , \qquad \psi'(z) = \partial_z \psi(z) \ . \tag{7}$$

Let $C_c^\infty(F)$ be the algebra of smooth complex-valued functions with compact support on F, and consider the crossed product $\mathcal{A} = C_c^\infty(F) \rtimes \Gamma$. $\mathcal{A}$ is the associative algebra linearly generated by elements of the form fU_ψ^* with $\psi \in \Gamma$, $f \in C_c^\infty(F)$, $\mathrm{supp} f \subset \mathrm{Dom}\psi$. We adopt the notation $U_\psi \equiv U_{\psi^{-1}}^*$ for the inverse of U_ψ^*. The multiplication rule

$$f_1 U_{\psi_1}^* f_2 U_{\psi_2}^* = f_1 \, (f_2 \circ \psi_1) U_{\psi_2\psi_1}^* \tag{8}$$

makes good sense thanks to the condition $\mathrm{supp} f_i \subset \mathrm{Dom}\psi_i$. We introduce now the differential operators

$$X = y\partial_z \qquad Y = y\partial_y \qquad \overline{X} = \overline{y}\partial_{\overline{z}} \qquad \overline{Y} = \overline{y}\partial_{\overline{y}} \tag{9}$$

forming a basis of the set of smooth vector fields viewed as a module over $C^\infty(F)$. These operators act on $\mathcal{A}$ in a natural way:

$$X.(fU_\psi^*) = (X.f)U_\psi^* \ , \qquad Y.(fU_\psi^*) = (Y.f)U_\psi^* \tag{10}$$

and similarly for $\overline{X}, \overline{Y}$. Remark that the system $(z, \overline{z})$ determines a smooth volume form $\frac{dz\wedge d\overline{z}}{2i}$ on Σ. This in turn gives a representative σ of the modular automorphism group of $L^\infty(\Sigma) \rtimes \Gamma$, whose action on $C_c^\infty(\Sigma) \rtimes \Gamma$reads (cf. [3] chap. III)

$$\sigma_t(fU_\psi^*) = |\psi'|^{2it} fU_\psi^* \ , \quad t \in \mathbb{R} \ . \tag{11}$$

We let D be the derivation corresponding to the infinitesimal action of σ:

$$D = -i\frac{d}{dt}\sigma_t|_{t=0} \qquad D(fU_\psi^*) = \ln|\psi'|^2 fU_\psi^* \ . \tag{12}$$

The operators $\delta_n, \overline{\delta}_n$, $n \geq 1$ are defined recursively

$$\delta_n = \underbrace{[X, ...[X}_{n}, D]...] \qquad \overline{\delta}_n = \underbrace{[\overline{X}, ...[\overline{X}}_{n}, D]...] \ . \tag{13}$$

Their action on $\mathcal{A}$ are explicitly given by

$$\delta_n(fU_\psi^*) = y^n \partial_z^n(\ln\psi')fU_\psi^* \ , \qquad \overline{\delta}_n(fU_\psi^*) = y^n \partial_z^n(\ln\overline{\psi'})fU_\psi^* \ . \tag{14}$$

Thus $\delta_n, \overline{\delta}_n$ represent in some sense the Taylor expansion of D. All these operators fulfill the commutation relations

$$\begin{aligned} [Y, X] &= X \qquad [Y, \delta_n] \ = \ n\delta_n \\ [X, \delta_n] &= \delta_{n+1} \qquad [\delta_n, \delta_m] \ = \ 0 \end{aligned} \tag{15}$$

and similarly for the conjugates $\overline{X}, \overline{Y}, \overline{\delta}_n$. Thus $\{X, Y, \delta_n, \overline{X}, \overline{Y}, \overline{\delta}_n\}_{n\geq 1}$ form a basis of a (complex) Lie algebra. Let $\mathcal{H}$ be its enveloping algebra. The remarkable fact is that $\mathcal{H}$ is a Hopf algebra. First, the coproduct $\Delta : \mathcal{H} \to \mathcal{H}\otimes\mathcal{H}$ is determined by the action of $\mathcal{H}$ on $\mathcal{A}$:

$$\Delta h(a_1 \otimes a_2) = h(a_1 a_2) \quad \forall h \in \mathcal{H}, a_i \in \mathcal{A} \ . \tag{16}$$

One has

$$\begin{aligned} \Delta X &= 1 \otimes X + X \otimes 1 + \delta_1 \otimes Y \\ \Delta Y &= 1 \otimes Y + Y \otimes 1 \qquad \Delta\delta_1 \ = \ 1 \otimes \delta_1 + \delta_1 \otimes 1 \ . \end{aligned} \tag{17}$$

$\Delta\delta_n$ for $n > 1$ is obtained recursively from (13) using the fact that Δ is an algebra homomorphism, $\Delta(h_1 h_2) = \Delta h_1 \Delta h_2$. Similarly for the conjugate elements. The counit $\varepsilon : \mathcal{H} \to \mathbb{C}$ satisfies simply $\varepsilon(1) = 1$, $\varepsilon(h) = 0 \ \forall h \neq 1$.
Finally, $\mathcal{H}$ has an antipode $S : \mathcal{H} \to \mathcal{H}$, determined uniquely by the condition $m \circ S \otimes \mathrm{Id} \circ \Delta = m \circ \mathrm{Id} \otimes S \circ \Delta = \eta\varepsilon$, where $m : \mathcal{H}\otimes\mathcal{H} \to \mathcal{H}$ is the multiplication and $\eta : \mathbb{C} \to \mathcal{H}$ the unit of $\mathcal{H}$. One finds

$$S(X) = -X + \delta_1 Y \quad S(Y) = -Y \quad S(\delta_1) = -\delta_1 \ . \tag{18}$$

Since S is an antiautomorphism: $S(h_1h_2) = S(h_2)S(h_1)$, the values of $S(\delta_n)$, $n > 1$ follow.

We are interested now in the cyclic cohomology of $\mathcal{H}$ [5, 6]. As a space, the cochain complex $C^*(\mathcal{H})$ is the tensor algebra over $\mathcal{H}$:

$$C^*(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes n} \ . \tag{19}$$

The crucial step is the construction of a characteristic map

$$\gamma : \mathcal{H}^{\otimes n} \to C^n(\mathcal{A}, \mathcal{A}^*) \tag{20}$$

from the cochain complex of $\mathcal{H}$ to the Hochschild complex of $\mathcal{A}$ with coefficients in $\mathcal{A}^*$ [3]. First F has a canonical Γ-invariant measure $dv = dzd\overline{z}\frac{dyd\overline{y}}{(y\overline{y})^2}$. This yields a trace τ on $\mathcal{A}$:

$$\begin{aligned} \tau(f) &= \int_F f\, dv \qquad f \in C_c^\infty(F) \ , \\ \tau(fU_\psi^*) &= 0 \qquad \text{if } \psi \neq 1 \ . \end{aligned} \tag{21}$$

Then the characteristic map sends the n-cochain $h_1 \otimes ... \otimes h_n \in \mathcal{H}^{\otimes n}$ to the Hochschild cochain $\gamma(h_1 \otimes ... \otimes h_n) \in C^n(\mathcal{A}, \mathcal{A}^*)$ given by

$$\gamma(h_1 \otimes ... \otimes h_n)(a_0, ..., a_n) = \tau(a_0 h_1(a_1)...h_n(a_n)) \ , \qquad a_i \in \mathcal{A} \ . \tag{22}$$

The cyclic cohomology of $\mathcal{H}$ is defined such that γ is a morphism of cyclic complexes. One introduces the face operators $\delta^i : \mathcal{H}^{\otimes(n-1)} \to \mathcal{H}^{\otimes n}$ for $0 \leq i \leq n$:

$$\begin{aligned} \delta^0(h_1 \otimes ... \otimes h_{n-1}) &= 1 \otimes h_1 \otimes ... \otimes h_{n-1} \\ \delta^i(h_1 \otimes ... \otimes h_{n-1}) &= h_1 \otimes ... \otimes \Delta h_i \otimes ... \otimes h_{n-1} \qquad 1 \leq i \leq n-1 \\ \delta^n(h_1 \otimes ... \otimes h_{n-1}) &= h_1 \otimes ... \otimes h_{n-1} \otimes 1 \end{aligned} \tag{23}$$

as well as the degeneracy operators $\sigma_i : \mathcal{H}^{\otimes(n+1)} \to \mathcal{H}^{\otimes n}$

$$\sigma_i(h_1 \otimes ... \otimes h_{n+1}) = h_1 \otimes ... \varepsilon(h_{i+1}) ... \otimes h_{n+1} \qquad 0 \leq i \leq n \ . \tag{24}$$

Next, the cyclic structure is provided by the antipode S and the multiplication of $\mathcal{H}$. Consider the twisted antipode $\tilde{S} = (\delta \otimes S) \circ \Delta$, where $\delta : \mathcal{H} \to \mathbb{C}$ is a character such that

$$\tau(h(a)b) = \tau(a\tilde{S}(h)(b)) \qquad \forall a, b \in \mathcal{A} \ . \tag{25}$$

This last formula plays the role of ordinary integration by parts. One finds:

$$\begin{aligned} \delta(1) &= 1 \ , \qquad \delta(Y) = \delta(\overline{Y}) = 1 \\ \delta(X) &= \delta(\overline{X}) = \delta(\delta_n) = \delta(\overline{\delta}_n) = 0 \quad \forall n \geq 1 \ . \end{aligned} \tag{26}$$

The definition implies $\tilde{S}^2 = 1$. Connes and Moscovici proved in [6] that the latter identity is sufficient to ensure the existence of a cyclicity operator $\tau_n : \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes n}$

$$\tau_n(h_1 \otimes ... \otimes h_n) = (\Delta^{n-1}\tilde{S}(h_1)) \cdot h_2 \otimes ... \otimes h_n \otimes 1 \ , \tag{27}$$

with $(\tau_n)^{n+1} = 1$. Now $C^*(\mathcal{H})$ endowed with $\delta^i, \sigma_i, \tau_n$ defines a cyclic complex. The Hochschild coboundary operator $b : \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes(n+1)}$ is

$$b = \sum_{i=0}^{n+1} (-)^i \delta^i \tag{28}$$

and Connes' operator $B : \mathcal{H}^{\otimes(n+1)} \to \mathcal{H}^{\otimes n}$ is

$$B = \sum_{i=0}^{n} (-)^{ni} (\tau_n)^i B_0 \qquad B_0 = \sigma_n \tau_{n+1} + (-)^n \sigma_n \ . \tag{29}$$

They fulfill the usual relations $B^2 = b^2 = bB + Bb = 0$, so that $C^*(\mathcal{H}, b, B)$ is a bicomplex. We define the cyclic cohomology $HC^*(\mathcal{H})$ as the b-cohomology of the subcomplex of cyclic cochains. The corresponding *periodic* cyclic cohomology $H^*(\mathcal{H})$ is isomorphic to the cohomology of the bicomplex $C^*(\mathcal{H}, b, B)$ [3]. Furthermore, the definitions of $\delta^i, \sigma_i, \tau_n$ imply that γ is a morphism of cyclic complexes. Consequently, γ passes to cyclic cohomology

$$\gamma : HC^*(\mathcal{H}) \to HC^*(\mathcal{A}) \ , \tag{30}$$

as well as to periodic cyclic cohomology

$$\gamma : H^*(\mathcal{H}) \to H^*(\mathcal{A}) \ . \tag{31}$$

In fact we are not interested in the frame bundle F but rather in the bundle of metrics $P = F/SO(2)$, where $SO(2) \subset Gl(1, \mathbb{C})$ is the group of rotations of frames. P is gifted with the coordinate chart $(z, \overline{z}, r)$ where the radial coordinate r is obtained from the decomposition

$$y = e^{-r+i\theta} \qquad r \in \mathbb{R} \ , \theta \in [0, 2\pi) \ . \tag{32}$$

The pseudogroup Γ still acts on P by

$$\begin{aligned} z &\to \psi(z) \qquad \overline{z} \to \overline{\psi(z)} \\ r &\to r - \frac{1}{2} \ln |\psi'(z)|^2 \ . \end{aligned} \tag{33}$$

Define $\mathcal{A}_1 = \mathcal{A}^{SO(2)} \subset \mathcal{A}$ the subalgebra of elements of $\mathcal{A}$ invariant under the (right) action of $SO(2)$ on F. $\mathcal{A}_1$ is canonically isomorphic to the crossed product $C_c^\infty(P) \rtimes \Gamma$. P carries a Γ-invariant measure $dv_1 = e^{2r} dz d\overline{z} dr$, so that there is a trace on $\mathcal{A}_1$, namely

$$\begin{aligned} \tau_1(f) &= \int_P f \, dv_1 \qquad f \in C_c^\infty(P) \\ \tau_1(fU_\psi^*) &= 0 \qquad \text{if } \psi \neq 1 \ . \end{aligned} \tag{34}$$

Thus passing to $SO(2)$-invariants yields an induced characteristic map

$$\gamma_1 : HC^*(\mathcal{H}, SO(2)) \to HC^*(\mathcal{A}_1) \tag{35}$$

from the relative cyclic cohomology of $\mathcal{H}$, with $\gamma_1(h_1 \otimes ... \otimes h_n)(a_0, ..., a_n) = \tau_1(a_0 h_1(a_1)...h_1(a_n))$, $a_i \in \mathcal{A}_1$, where $h_1 \otimes ... \otimes h_n$ represents an element of $HC^*(\mathcal{H}, SO(2))$. The map γ_1 generalises the classical Chern-Weil construction of characteristic classes from connexions and curvatures. In the crossed product case $\Sigma \rtimes \Gamma$, these classes are captured by the periodic cyclic cohomology of $\mathcal{H}$. The authors of [5] computed the latter as Gelfand-Fuchs cohomology. This is the subject of the next section.

IV. Gelfand-Fuchs cohomology

Let G be the group of complex analytic transformations of $\mathbb{C}$. G has a unique decomposition $G = G_1 G_2$, where G_1 is the group of affine transformations

$$x \to ax + b\ , \qquad x \in \mathbb{C}\ ,\ a, b \in \mathbb{C} \tag{36}$$

and G_2 is the group of transformations of the form

$$x \to x + o(x)\ . \tag{37}$$

Any element of G is then the composition $k \circ \psi$ for $k \in G_1$, $\psi \in G_2$. Since G_2 is the left quotient of G by G_1, G_1 acts on G_2 from the right: for $k \in G_1$, $\psi \in G_2$, one has $\psi \triangleleft k \in G_2$. Similarly, G_2 acts on G_1 from the left: $\psi \triangleright k \in G_1$.

Remark that G_1 is the crossed product $\mathbb{C} \rtimes Gl(1, \mathbb{C})$. The space $\mathbb{C} \times Gl(1, \mathbb{C})$ is a prototype for the frame bundle F of a flat Riemann surface. This motivates the notation $a = y$, $b = z$ for the coordinates on G_1. Under this identification, the left action of G_2 on G_1 corresponds to the action of G_2 on F: for a holomorphic transformation $\psi \in G_2$, one has

$$z \to \psi(z)\ , \qquad y \to \psi'(z) y\ , \tag{38}$$

with $\psi(0) = 0$, $\psi'(0) = 1$. Furthermore, the vector fields $X, \overline{X}, Y, \overline{Y}$ form a basis of invariant vector fields for the left action of G_1 on itself, i.e. a basis of the (complexified) Lie algebra of G_1. Its dual basis is given by the left-invariant 1-forms (Maurer-Cartan form)

$$\begin{aligned} \omega_{-1} &= y^{-1} dz \qquad & \overline{\omega}_{-1} &= \overline{y}^{-1} d\overline{z} \\ \omega_0 &= y^{-1} dy \qquad & \overline{\omega}_0 &= \overline{y}^{-1} d\overline{y}\ . \end{aligned} \tag{39}$$

The left action $G_2 \triangleright G_1$ implies a right action of G_2 on forms by pullback. One has in particular, for $\psi \in G_2$,

$$\omega_{-1} \circ \psi = \omega_{-1} \qquad \omega_0 \circ \psi = \omega_0 + y \partial_z \ln \psi' \omega_{-1} \qquad \text{and c.c.} \tag{40}$$

Consider now the discrete crossed product $\mathcal{H}_* = C_c^\infty(G_1) \rtimes G_2$ where G_2 acts on $C_c^\infty(G_1)$ by pullback. As a coalgebra, $\mathcal{H}$ is dual to the algebra $\mathcal{H}_*$. One has a natural action of $\mathcal{H}$ on $\mathcal{H}_*$:

$$\begin{aligned} X.(fU_\psi^*) &= X.fU_\psi^* \qquad f \in C_c^\infty(G_1), \psi \in G_2 \ , \\ \delta_n(fU_\psi^*) &= y^n \partial_z^n \ln \psi' \, fU_\psi^* \ , \end{aligned} \tag{41}$$

and so on with $Y, \overline{X}$... The operators $\delta_n, \overline{\delta_n}$ have in fact an interpretation in terms of coordinates on the group G_2: for $\psi \in G_2$, $\delta_n(\psi)$ is by definition the value of the function $\delta_n(U_\psi^*)U_\psi$ at $1 \in G_1$. For any $k \in G_1$, one has

$$[\delta_n(U_\psi^*)U_\psi](k) = \delta_n(\psi \triangleleft k) \ . \tag{42}$$

Note that (40) rewrites

$$\omega_0 \circ \psi = \omega_0 + \delta_1(\psi \triangleleft k)\omega_{-1} \qquad \text{at } k \in G_1 \ . \tag{43}$$

The Hopf subalgebra of $\mathcal{H}$ generated by $\delta_n, \overline{\delta_n}$, $n \geq 1$, corresponds to the commutative Hopf algebra of functions on G_2 which are *polynomial* in these coordinates.

Let A be the complexification of the formal Lie algebra of G. It coincides with the jets of holomorphic and antiholomorphic vector fields of any order on $\mathbb{C}$:

$$\begin{array}{lllll} \partial_x & , & x\partial_x \ ,..., \ x^n\partial_x \ ,... & & x \in \mathbb{C} \\ \partial_{\overline{x}} & , & \overline{x}\partial_{\overline{x}} \ ,..., \ \overline{x}^n\partial_{\overline{x}} \ ,... & & \end{array} \tag{44}$$

The Lie bracket between the elements of the above basis is thus

$$\begin{aligned} [x^n\partial_x, x^m\partial_x] &= (m-n)x^{n+m-1}\partial_x \qquad \text{and c.c.} \\ [x^n\partial_x, \overline{x}^m\partial_{\overline{x}}] &= 0 \ . \end{aligned} \tag{45}$$

Define the generator of dilatations $H = x\partial_x + \overline{x}\partial_{\overline{x}}$ and of rotations $J = x\partial_x - \overline{x}\partial_{\overline{x}}$. They fulfill the properties

$$\begin{aligned} [H, x^n\partial_x] &= (n-1)x^n\partial_x \qquad [H, \overline{x}^n\partial_{\overline{x}}] = (n-1)\overline{x}^n\partial_{\overline{x}} \\ [J, x^n\partial_x] &= (n-1)x^n\partial_x \qquad [J, \overline{x}^n\partial_{\overline{x}}] = -(n-1)\overline{x}^n\partial_{\overline{x}} \ . \end{aligned} \tag{46}$$

We are interested in the Lie algebra cohomology of A (see [7]). The complex $C^*(A)$ of cochains is the exterior algebra generated by the dual basis $\{\omega^n, \overline{\omega}^n\}_{n\geq -1}$:

$$\begin{aligned} \omega^n(x^m\partial_x) &= \delta^m_{n+1} \qquad \omega^n(\overline{x}^m\partial_{\overline{x}}) = 0 \\ \overline{\omega}^n(x^m\partial_x) &= 0 \qquad \overline{\omega}^n(\overline{x}^m\partial_{\overline{x}}) = \delta^m_{n+1} \qquad \forall n \geq -1, m \geq 0 \ , \end{aligned} \tag{47}$$

and the coboundary operator is uniquely defined by its action on 1-cochains

$$d\omega(X,Y) = -\omega([X,Y]) \qquad \forall X, Y \in A \ . \tag{48}$$

¿From [5] we know that the *periodic* cyclic cohomology $H^*(\mathcal{H}, SO(2))$ is isomorphic to the relative Lie algebra cohomology $H^*(A, SO(2))$, i.e. the cohomology of the basic subcomplex of cochains on A relative to the Cartan operation (L, i) of J:

$$L_J\omega = (i_J d + d i_J)\omega \qquad \forall \omega \in C^*(A) \ . \tag{49}$$

We say that a cochain $\omega \in C^*(A)$ is of weight r if $L_H\omega = -r\omega$. Remark that

$$L_H\omega^n = -n\omega^n \ , \qquad L_H\overline{\omega}^n = -n\overline{\omega}^n \qquad \forall n \geq -1 \ , \tag{50}$$

so that $C^*(A)$ is the direct sum, for $r \geq -2$, of the spaces $C^*_r(A)$ of weight r. Since $[H, J] = 0$, $C^*_r(A)$ is stable under the Cartan operation of J and we note $C^*_r(A, SO(2))$ the complex of basic cochains of weight r. Then we have

$$C^*(A, SO(2)) = \bigoplus_{r=-2}^{\infty} C^*_r(A, SO(2)) \ . \tag{51}$$

For any cocycle $\omega \in C^*_r(A, SO(2))$,

$$L_H\omega = d i_H \omega = -r\omega \tag{52}$$

so that $C^*_r(A, SO(2))$ is acyclic whenever $r \neq 0$. Hence $H^*(A, SO(2))$ is equal to the cohomology of the finite-dimensional subcomplex $C^*_0(A, SO(2))$. The direct computation gives

$$\begin{array}{lll} H^0(A, SO(2)) = \mathbb{C} & \text{with representative} & 1 \\ H^2(A, SO(2)) = \mathbb{C} & \quad '' & \omega^{-1}\omega^1 \\ H^3(A, SO(2)) = \mathbb{C} & \quad '' & (\omega^{-1}\omega^1 - \overline{\omega}^{-1}\overline{\omega}^1)(\omega^0 + \overline{\omega}^0) \\ H^5(A, SO(2)) = \mathbb{C} & \quad '' & \omega^1\omega^{-1}\overline{\omega}^1\overline{\omega}^{-1}(\omega^0 + \overline{\omega}^0) \end{array} \tag{53}$$

The other cohomology groups vanish.

Next we construct a map C from $C^*(A)$ to the bicomplex $(C^{n,m}, d_1, d_2)_{n,m\in\mathbb{Z}}$ of [3] chap. III.2.δ. Let $\Omega^m(G_1)$ be the space m-forms on G_1. $C^{n,m}$ is the space of totally antisymmetric maps $\gamma : {G_2}^{n+1} \to \Omega^m(G_1)$ such that

$$\gamma(g_0 g, ..., g_n g) = \gamma(g_0, ..., g_n) \circ g \qquad g_i \in G_2, g \in G \ , \tag{54}$$

where $g_i g$ is given by the right action of G on G_2, and G acts on $\Omega^*(G_1)$ by pullback (left action of G on G_1).
The first differential $d_1 : C^{n,m} \to C^{n+1,m}$ is

$$(d_1\gamma)(g_0, ..., g_{n+1}) = (-)^m \sum_{i=0}^{n+1} (-)^i \gamma(g_0, ..., \overset{\vee}{g_i}, ..., g_{n+1}) \ , \tag{55}$$

and $d_2 : C^{n,m} \to C^{n,m+1}$ is just the de Rham coboundary on $\Omega^*(G_1)$:

$$(d_2\gamma)(g_0, ..., g_n) = d(\gamma(g_o, ..., g_n)) \ . \tag{56}$$

Of course ${d_1}^2 = {d_2}^2 = d_1 d_2 + d_2 d_1 = 0$. Remark that for $\gamma \in C^{n,m}$, the invariance property (54) implies

$$\gamma(g_0, ..., g_n) \circ k = \gamma(g_0 \triangleleft k, ..., g_n \triangleleft k) \qquad \forall k \in G_1 \ , \tag{57}$$

in other words the value of $\gamma(g_0, ..., g_n) \in \Omega^m(G_1)$ at k is deduced from its value at 1.

Let us describe now the construction of C. As a vector space, the Lie algebra A is just the direct sum $\mathbf{G}_1 \oplus \mathbf{G}_2$, $\mathbf{G}_i$ being the (complexified) Lie algebra of G_i. The cochain complex $C^*(A)$ is then the exterior product $\Lambda A^* = \Lambda \mathbf{G}_1{}^* \otimes \Lambda \mathbf{G}_2{}^*$. One identifies $\mathbf{G}_1{}^*$ with the cotangent space $T_1^*(G_1)$ of G_1 at the identity. Since G_2 fixes $1 \in G_1$, there is a right action of G_2 on $\Lambda \mathbf{G}_1{}^*$ by pullback. The basis $\{\omega^{-1}, \omega^0, \overline{\omega}^{-1}, \overline{\omega}^0\}$ of $\mathbf{G}_1{}^*$ is represented by left-invariant one-forms on G_1 through the identification

$$\begin{array}{rclrcl} \omega^{-1} & \rightarrow & -\omega_{-1} = -y^{-1}dz & \overline{\omega}^{-1} & \rightarrow & -\overline{\omega}_{-1} = -\overline{y}^{-1}d\overline{z} \\ \omega^0 & \rightarrow & -\omega^0 = -y^{-1}dy & \overline{\omega}^0 & \rightarrow & -\overline{\omega}^0 = -\overline{y}^{-1}d\overline{y} \ , \end{array} \tag{58}$$

and the right action of $\psi \in G_2$ reads (cf. (40))

$$\omega^{-1} \cdot \psi = \omega^{-1} \ , \qquad \omega^0 \cdot \psi = \omega^0 + \delta_1(\psi)\omega^{-1} \ . \tag{59}$$

Next, we view a cochain $\omega \in C^*(A)$ as a cochain of the Lie algebra of G_2 with coefficients in the right G_2-module $\Lambda \mathbf{G}_1{}^*$. It is represented by a $\Lambda \mathbf{G}_1{}^*$-valued right-invariant form μ on G_2. Then $C(\omega) \in C^{*,*}$ evaluated on $(g_0, ..., g_n) \in {G_2}^{n+1}$ is a differential form on G_1 whose value at $1 \in G_1$ is

$$C(\omega)(g_0, ..., g_n) = \int_{\Delta(g_0, ..., g_n)} \mu \quad \in \Lambda T_1^*(G_1) \ , \tag{60}$$

where $\Delta(g_0, ..., g_n)$ is the affine simplex in the coordinates $\delta_i, \overline{\delta_i}$, with vertices $(g_0, ..., g_n)$. Let $\{\rho_j\}$ be a basis of left-invariant forms on G_1. Then

$$C(\omega)(g_0, ..., g_n) = \sum_j p_j(g_0, ..., g_n)\rho_j \qquad \text{at } 1 \in G_1 \ , \tag{61}$$

where $p_j(g_0, ..., g_n)$ are polynomials in the coordinates $\delta_i, \overline{\delta_i}$. The invariance property (54) enables us to compute the value of $C(\omega)(g_0, ..., g_n)$ at any $k \in G_1$,

$$C(\omega)(g_0, ..., g_n)(k) = \sum_j p_j(g_0 \triangleleft k, ..., g_n \triangleleft k)\rho_j \tag{62}$$

because $\rho_j \circ k = \rho_j$.

Connes and Moscovici showed in [5] that C is a morphism from $C^*(A, d)$ to the bicomplex $(C^{n,m}, d_1, d_2)_{n,m \in \mathbb{Z}}$. In the relative case, it restricts to a morphism from $C^*(A, SO(2), d)$ to the subcomplex $(C^{n,m}_{bas.}, d_1, d_2)$ of antisymmetric cochains on G_2 with values in the *basic* de Rham cohomology $\Omega^*(P) = \Omega^*(G_1/SO(2))$.

It remains to compute the image of $H^*(A, SO(2))$ by C. We restrict ourselves to even cocycles, i.e. the unit $1 \in H^0(A, SO(2))$ and the first Chern class $c_1 \in H^2(A, SO(2))$, defined as the class

$$c_1 = [2\omega^{-1}\omega^1] \ . \tag{63}$$

One has $C(1) \in C^{0,0}_{bas.}$. The immediate result is

$$C(1)(g_0) = 1 \ , \qquad g_0 \in G_2 \ . \tag{64}$$

For the first Chern class, we must transform c_1 into a right-invariant form on G_2 with values in $\Lambda T^*_1(G_1)$. We already know that ω^{-1} is represented by $-\omega_{-1} = -y^{-1}dz$, which satisfies $\omega_{-1} \circ \psi = \omega_{-1}$, $\forall \psi \in G_2$. Next, the Taylor expansion of an element $\psi \in G_2$ can be expressed in the coordinates δ_n thanks to the obvious formula

$$\ln \psi'(x) = \sum_{n=1}^{\infty} \frac{1}{n!} \delta_n(\psi) x^n \ , \qquad \forall x \in \mathbb{C} \ . \tag{65}$$

One finds:

$$\psi(x) = x + \frac{1}{2}\delta_1(\psi)x^2 + \frac{1}{3!}(\delta_2(\psi) + \delta_1(\psi)^2)x^3 + O(x^4) \ . \tag{66}$$

It shows that the cochain $\omega^1 \in C^*(A)$ is represented by the right-invariant 1-form $\frac{1}{2}d\delta_1$ on G_2. Thus at $1 \in G_1$, $C(c_1) \in C^{1,1}_{bas.}$ is given by

$$\begin{aligned} C(c_1)(g_0, g_1) &= \int_{\Delta(g_0,g_1)} -\omega_{-1} d\delta_1 \\ &= -\omega_{-1}(\delta_1(g_1) - \delta_1(g_0)) \qquad g_i \in G_2 \ , \end{aligned} \tag{67}$$

and at $k \in G_1$, the 1-form $C(c_1)(g_0, g_1)$ is

$$C(c_1)(g_0, g_1) = -\omega_{-1}(\delta_1(g_1 \triangleleft k) - \delta_1(g_0 \triangleleft k)) \ . \tag{68}$$

Since $\omega_{-1} = y^{-1}dz$ and $\delta_1(g \triangleleft k) = y\partial_z \ln g'(z)$, z and y being the coordinates of k, one has explicitly

$$C(c_1)(g_0, g_1) = -dz(\partial_z \ln {g_1}'(z) - \partial_z \ln {g_0}'(z)) \ . \tag{69}$$

It is a basic form on G_1 relative to $SO(2)$, then descends to a form on $P = G_1/SO(2)$ as expected.

The last step is to use the map Φ of [3] theorem 14 p.220 from $(C^{n,m}, d_1, d_2)$ to the (b, B) bicomplex of the discrete crossed product $C^\infty_c(P) \rtimes G_2$. Define the algebra

$$\mathcal{B} = \Omega^*(P) \hat{\otimes} \Lambda\mathbb{C}(G_2') \ , \tag{70}$$

where $\Lambda\mathbb{C}(G_2')$ is the exterior algebra generated by the elements δ_ψ, $\psi \in G_2$, with $\delta_e = 0$ for the identity e of G_2. With the de Rham coboundary d of $\Omega^*(P)$, $\mathcal{B}$ is a differential algebra. Now form the crossed product $\mathcal{B} \rtimes G_2$, with multiplication rules

$$\begin{aligned} U^*_\psi \alpha U_\psi &= \alpha \circ \psi \ , \qquad \alpha \in \Omega^*(P), \psi \in G_2 \\ U^*_{\psi_1} \delta_{\psi_2} U_{\psi_1} &= \delta_{\psi_2 \circ \psi_1} - \delta_{\psi_1} \ , \qquad \psi_i \in G_2 \ . \end{aligned} \tag{71}$$

Endow $\mathcal{B} \rtimes G_2$ with the differential $\tilde{d}$ acting on an element bU^*_ψ as

$$\tilde{d}(bU^*_\psi) = dbU^*_\psi - (-)^{\partial b} b\delta_\psi U^*_\psi \ , \tag{72}$$

where db comes from the de Rham coboundary of $\Omega^*(P)$. The map

$$\Phi : (C^{*,*}, d_1, d_2) \to (C_c^\infty(P) \rtimes G_2, b, B) \tag{73}$$

is constructed as follows. Let $\gamma \in C^{n,m}_{bas.}$. It yields a linear form $\tilde{\gamma}$ on $\mathcal{B} \rtimes G_2$:

$$\begin{aligned} \tilde{\gamma}(\alpha \otimes \delta_{g_1} ... \delta_{g_n}) &= \int_P \alpha \wedge \gamma(1, g_1, ..., g_n) \ , \qquad \alpha \in \Omega^*(P), g_i \in G_2 \\ \tilde{\gamma}(bU^*_\psi) &= 0 \qquad \text{if} \ \ \psi \neq 1 \ . \end{aligned} \tag{74}$$

Then $\Phi(\gamma)$ is the following l-cochain on $C_c^\infty(P) \rtimes G_2$, $l = \dim P - m + n$

$$\begin{aligned} \Phi(\gamma)(x_0, ..., x_l) &= \frac{n!}{(l+1)!} \sum_{j=0}^{l} (-)^{j(l-j)} \tilde{\gamma}(\tilde{d}x_{j+1} ... \tilde{d}x_l x_0 \tilde{d}x_1 ... \tilde{d}x_j) \ , \\ & \quad x_i \in C_c^\infty(P) \rtimes G_2 \subset \mathcal{B} \rtimes G_2 \ . \end{aligned} \tag{75}$$

The essential tool is that Φ is a morphism of bicomplexes:

$$\Phi(d_1\gamma) = b\Phi(\gamma) \ , \qquad \Phi(d_2\gamma) = B\Phi(\gamma) \ . \tag{76}$$

Moreover, if $d_1\gamma = d_2\gamma = 0$, $\Phi(\gamma)$ is a cyclic cocycle. This happens in our case. Since P is a 3-dimensional manifold, the image of $C(1)$ under Φ is the cyclic 3-cocycle

$$\Phi(C(1))(x_0, ..., x_3) = \int_P x_0 dx_1 ... dx_3 \ , \qquad x_i \in C_c^\infty(P) \rtimes G_2 \ , \tag{77}$$

where $d(fU^*_\psi) = df U^*_\psi$ for $f \in C_c^\infty(P)$, $\psi \in G_2$, and the integration is extended over $\Omega^*(P) \rtimes G_2$ by setting

$$\int_P \alpha U^*_\psi = 0 \qquad \text{if} \ \psi \neq 1, \ \alpha \in \Omega^*(P) \ . \tag{78}$$

The image of $\gamma = C(c_1)$ is more complicated to compute. One has

$$\tilde{\gamma}(\alpha \otimes \delta_g) = - \int_P \alpha \wedge y^{-1} dz \delta_1(g \lhd k) \ , \qquad \alpha \in \Omega^2(P), g \in G_2 \tag{79}$$

where $y^{-1} dz \delta_1(g \lhd k) = dz \partial_z \ln g'(z)$ is, of course, a 1-form on P. $\Phi(\gamma)$ is the cyclic 3-cocycle

$$\begin{aligned} \Phi(\gamma)(f_0 U^*_{\psi_0}, ..., f_3 U^*_{\psi_3}) &= -\tilde{\gamma}(f_0 U^*_{\psi_0} df_1 U^*_{\psi_1} df_2 U^*_{\psi_2} f_3 \delta_{\psi_3} U^*_{\psi_3} \\ & \quad + f_0 U^*_{\psi_0} df_1 U^*_{\psi_1} f_2 \delta_{\psi_2} U^*_{\psi_2} df_3 U^*_{\psi_3} \\ & \quad + f_0 U^*_{\psi_0} f_1 \delta_{\psi_1} U^*_{\psi_1} df_2 U^*_{\psi_2} df_3 U^*_{\psi_3}) \end{aligned}$$

$$\begin{aligned} = \ & \tilde{\gamma}(f_0 \, (df_1 \circ \psi_0) \, (df_2 \circ \psi_1\psi_0) \, (f_3 \circ \psi_2\psi_1\psi_0) \delta_{\psi_2\psi_1\psi_0} \\ & + f_0 \, (df_1 \circ \psi_0) \, (f_2 \circ \psi_1\psi_0) \, (df_3 \circ \psi_2\psi_1\psi_0)(\delta_{\psi_2\psi_1\psi_0} - \delta_{\psi_1\psi_0}) \\ & - f_0 \, (f_1 \circ \psi_0) \, (df_2 \circ \psi_1\psi_0) \, (df_3 \circ \psi_2\psi_1\psi_0)(\delta_{\psi_1\psi_0} - \delta_{\psi_0})) \ , \end{aligned} \tag{80}$$

upon assuming that $\psi_3\psi_2\psi_1\psi_0 = \mathrm{Id}$. Using the relation

$$\delta_1(\psi \triangleleft k) = [\delta_1(U_\psi^*)U_\psi](k) \ , \qquad \forall k \in G_1, \psi \in G_2 \tag{81}$$

the computation gives

$$\Phi(\gamma)(x_0, ..., x_3) = \int_P x_0(dx_1 dx_2 \delta_1(x_3) + dx_1 \delta_1(x_2) dx_3 + \delta_1(x_1) dx_2 dx_3) y^{-1} dz \ . \tag{82}$$

Now recall that P has an invariant volume form $dv_1 = e^{2r} dz d\overline{z} dr$. The differential df of a function on P makes use of the horizontal $X = y\partial_z$, $\overline{X} = \overline{y}\partial_{\overline{z}}$ and vertical $Y + \overline{Y} = -\partial_r$ vector fields:

$$df = y^{-1} dz X.f + \overline{y}^{-1} d\overline{z}\overline{X}.f - dr(Y + \overline{Y}).f \ . \tag{83}$$

Then using the relations (40) one sees that $\Phi(C(c_1))$ is a sum of terms involving the Hopf algebra

$$\Phi(C(c_1))(x_0, ..., x_3) = \sum_i \int_P x_0 h_1^i(x_1)...h_3^i(x_3) dv_1 \ , \tag{84}$$

where the sum $\sum_i h_1^i \otimes h_2^i \otimes h_3^i$ is a cyclic 3-cocycle of $\mathcal{H}$ relative to $SO(2)$. This follows from the existence of the characteristic map (35)

$$HC^*(\mathcal{H}, SO(2)) \to HC^*(C_c^\infty(P) \rtimes G_2) \tag{85}$$

and the duality between $\mathcal{H}$ and $\mathcal{H}_* = C_c^\infty(G_1) \rtimes G_2$ (cf. [5]).

Returning to the initial situation, where F is the frame bundle of a flat Riemann surface Σ, and $P = F/SO(2)$ the bundle of metrics, the above computation shows that the cyclic 3-cocycle on $\mathcal{A}_1 = C_c^\infty(P) \rtimes \Gamma$

$$[c_1](a_0, ..., a_3) = \sum_i \int_P a_0 h_1^i(a_1)...h_3^i(a_3) dv_1 \ , \qquad a_i \in \mathcal{A}_1 \ , \tag{86}$$

is the image of $C(c_1)$ by the characteristic map $HC^*(\mathcal{H}, SO(2)) \to HC^*(\mathcal{A}_1)$. Also the fundamental class

$$[P](a_0, ..., a_3) = \int_P a_0 da_1 da_2 da_3 \tag{87}$$

is in the range of the characteristic map.

Since Connes and Moscovici showed that the Gelfand-Fuchs cohomology $H^*(A, SO(2))$ is isomorphic to the periodic cyclic cohomology of $\mathcal{H}$, we have completely determined the odd part of the range of the characteristic map. We can summarize the result in the following

Proposition 1 *Under the characteristic map*

$$H^*(A, SO(2)) \simeq H^*(\mathcal{H}, SO(2)) \to H^*(\mathcal{A}_1) \ , \tag{88}$$

the unit $1 \in H^0(A, SO(2))$ *maps to the fundamental class* $[P]$ *represented by the cyclic 3-cocycle*

$$[P](a_0, ..., a_3) = \int_P a_0 da_1 da_2 da_3 \ , \qquad a_i \in \mathcal{A}_1 \ , \tag{89}$$

and the first Chern class $c_1 \in H^2(A, SO(2))$ *gives the cocycle* $[c_1] \in HC^3(\mathcal{A}_1)$*:*

$$[c_1](a_0, ..., a_3) = \int_P a_0(da_1 da_2 \delta_1(a_3) + da_1 \delta_1(a_2) da_3 + \delta_1(a_1) da_2 da_3) y^{-1} dz \ . \tag{90}$$

In section II we considered an odd K-cycle on $C_0(P \times \mathbb{R}^2) \rtimes \Gamma$ represented by a differential operator Q', which is equivalent, up to Bott periodicity, to an odd K-cycle on $C_0(P) \rtimes \Gamma$. Q' is a matrix-valued polynomial in the vector fields $X, \overline{X}, Y + \overline{Y}$ and the partial derivatives along the two directions of $\mathbb{R}^2$. Its Chern character is the cup product

$$\mathrm{ch}_*(Q') = \varphi \# [\mathbb{R}^2] \tag{91}$$

of a cyclic cocycle $\varphi \in HC^{odd}(C_c^\infty(P) \rtimes \Gamma)$ by the fundamental class of $\mathbb{R}^2$. The index theorem of Connes and Moscovici states that φ is in the range of the characteristic map (we have to assume that the action of Γ on Σ has no fixed point). Hence it is a linear combination of the characteristic classes $[P]$ and $[c_1]$. We shall determine the coefficients by using the classical Riemann-Roch theorem.

V. A Riemann-Roch theorem for crossed products

We shall first use the Thom isomorphism in K-theory [1]

$$K_i(C_0(\Sigma) \rtimes \Gamma) \to K_{i+1}(C_0(P) \rtimes \Gamma) \tag{92}$$

to descend the characteristic classes $[P]$ and $[c_1]$ down to the cyclic cohomology of $C_c^\infty(\Sigma) \rtimes \Gamma$. Recall that $C_0(P) \rtimes \Gamma$ is just the crossed product of $C_0(\Sigma) \rtimes \Gamma$ by the modular automorphism group σ of the associated von Neumann algebra

$$C_0(P) \rtimes \Gamma = (C_0(\Sigma) \rtimes \Gamma) \rtimes_\sigma \mathbb{R} \ . \tag{93}$$

By homotopy we can deform σ continuously into the trivial action. For $\lambda \in [0, 1]$, let $\sigma_t^\lambda = \sigma_{\lambda t}$, $\forall t \in \mathbb{R}$. Then $\sigma^1 = \sigma$, $\sigma^0 = \mathrm{Id}$ and

$$(C_0(\Sigma) \rtimes \Gamma) \rtimes_{\mathrm{Id}} \mathbb{R} = C_0(\Sigma) \rtimes \Gamma \otimes C_0(\mathbb{R}) \ . \tag{94}$$

Next, the coordinate system $(z, \overline{z})$ of Σ gives a smooth volume form $\frac{dz \wedge d\overline{z}}{2i}$ together with a representative of σ, whose action on the subalgebra $C_c^\infty(\Sigma) \rtimes \Gamma$ is

$$\sigma_t(f U_\psi^*) = f |\psi'|^{2it} U_\psi^* \ , \qquad f \in C_c^\infty(\Sigma), \psi \in \Gamma \ , \tag{95}$$

and accordingly

$$\sigma_t^\lambda(fU_\psi^*) = f|\psi'|^{2i\lambda t} U_\psi^* \ . \tag{96}$$

Remark that the algebra $(C_0(\Sigma) \rtimes \Gamma) \rtimes_{\sigma^\lambda} \mathbb{R}$ is equal to the crossed product $C_0(P) \rtimes_\lambda \Gamma$ obtained from the following deformed action of Γ on P:

$$\begin{aligned} z &\to \psi(z) \qquad \overline{z} \to \overline{\psi(z)} \\ r &\to r - \frac{1}{2}\lambda \ln |\psi'(z)|^2 \qquad \psi \in \Gamma \ . \end{aligned} \tag{97}$$

Hence for any $\lambda \in [0,1]$, one has a Thom isomorphism

$$\Phi^\lambda : K_0(C_0(\Sigma) \rtimes \Gamma) \to K_1(C_0(P) \rtimes_\lambda \Gamma) \ , \tag{98}$$

and Φ^0 is just the connecting map $K_0(C_0(\Sigma) \rtimes \Gamma) \to K_1(S(C_0(\Sigma) \rtimes \Gamma))$. We introduce also the family $\{[P]^\lambda\}_{\lambda \in [0,1]}$ of cyclic cocycles

$$[P]^\lambda(a_0^\lambda, ..., a_3^\lambda) = \int_P a_0^\lambda da_1^\lambda ... da_3^\lambda \ , \qquad \forall a_i^\lambda \in C_c^\infty(P) \rtimes_\lambda \Gamma \ . \tag{99}$$

One has $[P]^1 = [P]$ and $[P]^0 = [\Sigma]\#[\mathbb{R}] \in (C_c^\infty(\Sigma) \rtimes \Gamma) \otimes C_c^\infty(\mathbb{R})$, where

$$[\Sigma](a_0, a_1, a_2) = \int_\Sigma a_0 da_1 da_2 \qquad \forall a_i \in C_c^\infty(\Sigma) \rtimes \Gamma \ . \tag{100}$$

Moreover for any element $[e] \in K_0(C_0(\Sigma) \rtimes \Gamma)$ such that $\Phi^\lambda([e])$ is in the domain of definition of $[P]^\lambda$, the pairing

$$\langle \Phi^\lambda([e]), [P]^\lambda \rangle \tag{101}$$

depends continuously upon λ. Next for any $\lambda \in (0,1]$, consider the vertical diffeomorphism of P whose action on the coordinates $(z, \overline{z}, r)$ reads

$$\tilde{\lambda}(z) = z \qquad \tilde{\lambda}(\overline{z}) = \overline{z} \qquad \tilde{\lambda}(r) = \lambda r \ . \tag{102}$$

Thus for $\lambda \neq 0$ one has an algebra isomorphism

$$\chi_\lambda : C_c^\infty(P) \rtimes_\lambda \Gamma \to C_c^\infty(P) \rtimes \Gamma \tag{103}$$

by setting

$$\chi_\lambda(fU_\psi^*) = f \circ \tilde{\lambda} \ U_\psi^* \qquad \forall f \in C_c^\infty(P), \psi \in \Gamma \ . \tag{104}$$

For any $\lambda \neq 0$,

$$(\chi_\lambda)_* \circ \Phi^\lambda = \Phi^1 \ , \tag{105}$$

$$(\chi_\lambda)^*[P]^1 = [P]^\lambda \ . \tag{106}$$

Eq.(105) comes from the unicity of the Thom map (cf. [1]), and (106) is obvious. Thus $\langle \Phi^\lambda([e]), [P]^\lambda \rangle$ is constant for $\lambda \neq 0$, and by continuity at 0

$$\langle \Phi^1([e]), [P] \rangle = \langle [e], [\Sigma] \rangle \ . \tag{107}$$

This shows that the image of $[P]$ by Thom isomorphism is the cyclic 2-cocycle $[\Sigma]$ corresponding to the fundamental class of Σ. In exactly the same way we show that the image of $[c_1]$ is the cyclic 2-cocycle τ defined, for $a_i = f_i U^*_{\psi_i} \in C^\infty_c(\Sigma) \rtimes \Gamma$, by

$$\tau(a_0, a_1, a_2) = \int_\Sigma a_0 (da_1 \partial \ln \psi'_2\, a_2 + \partial \ln \psi'_1\, a_1 da_2) \ , \tag{108}$$

with $\partial = dz\partial_z$. Note that in the decomposition of the differential on Σ, $d = \partial + \overline{\partial}$, both ∂ and $\overline{\partial}$ commute with the pullbacks by the conformal transformations $\psi \in \Gamma$.

So far we have considered a *flat* Riemann surface and the constructions we made were relative to a coordinate system $(z, \overline{z})$. We shall now remove this unpleasant feature by using the Morita equivalence [5]. In order to understand the general situation, let us first treat the particular case of the Riemann sphere $S^2 = \mathbb{C} \cup \{\infty\}$. We consider an open covering of the sphere by two planes: $S^2 = U_1 \cup U_2$, $U_1 = \mathbb{C}$, $U_2 = \mathbb{C}$, together with the glueing function g:

$$\begin{aligned} g : U_1 \backslash \{0\} \quad &\rightarrow \quad U_2 \backslash \{0\} \\ z \quad &\mapsto \quad \frac{1}{z} \ . \end{aligned} \tag{109}$$

The pseudogroup of conformal transformations Γ_0 generated by $\{U^*_g, U_g\}$ acts on the disjoint union $\Sigma = U_1 \amalg U_2$, which is flat. Then S^2 is described by the groupoid $\Sigma \rtimes \Gamma_0$. If Γ is a pseudogroup of local transformations of S^2, there exists a pseudogroup Γ' containing Γ_0, acting on Σ and such that the crossed product $C^\infty(S^2) \rtimes \Gamma$ is Morita equivalent to $C^\infty_c(\Sigma) \rtimes \Gamma'$. The latter splits into four parts: it is the direct sum, for $i, j = 1, 2$, of elements of the form $f_{ij} U^*_{\psi_{ij}}$ with

$$\psi_{ij} : U_i \rightarrow U_j \quad \text{and} \quad \mathrm{supp} f_{ij} \subset \mathrm{Dom} \psi_{ij} \ . \tag{110}$$

For convenience, we adopt a matricial notation for any generic element $b \in C^\infty_c(\Sigma) \rtimes \Gamma'$:

$$b = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \ , \qquad b_{ij} = f_{ij} U^*_{\psi_{ij}} \ . \tag{111}$$

Now the Morita equivalence is explicitly realized through the following idempotent $e \in C^\infty_c(\Sigma) \rtimes \Gamma'$:

$$e = \begin{pmatrix} {\rho_1}^2 & \rho_1 \rho_2 U^*_g \\ U_g \rho_2 \rho_1 & U_g {\rho_2}^2 U^*_g \end{pmatrix} \ , \qquad e^2 = e \ , \tag{112}$$

where $\{\rho_i\}_{i=1,2}$ is a partition of unity relative to the covering $\{U_i\}$:

$$\rho_1 \in C^\infty_c(U_1) \ , \quad {\rho_1}^2 + {\rho_2}^2 = 1 \text{ on } S^2 = U_1 \cup \{\infty\} \ . \tag{113}$$

The reduction of $C^\infty_c(\Sigma) \rtimes \Gamma'$ by e is the subalgebra

$$(C^\infty_c(\Sigma) \rtimes \Gamma')_e = \{ b \in C^\infty_c(\Sigma) \rtimes \Gamma' / b = be = eb \} \ . \tag{114}$$

Its elements are of the form

$$ebe = \begin{pmatrix} \rho_1 c \rho_1 & \rho_1 c \rho_2 U_g^* \\ U_g \rho_2 c \rho_1 & U_g \rho_2 c \rho_2 U_g^* \end{pmatrix} \tag{115}$$

with $c = \rho_1 b_{11} \rho_1 + \rho_2 U_g^* b_{21} \rho_1 + \rho_1 b_{12} U_g \rho_2 + \rho_2 U_g^* b_{22} U_g \rho_2$. Then c can be considered as an element of $C^\infty(S^2) \rtimes \Gamma$ under the identification $S^2 = U_1 \cup \{\infty\}$. $(C_c^\infty(\Sigma) \rtimes \Gamma')_e$ and $C^\infty(S^2) \rtimes \Gamma$ are isomorphic through the map

$$\begin{aligned} \theta : C^\infty(S^2) \rtimes \Gamma &\longrightarrow (C_c^\infty(\Sigma) \rtimes \Gamma')_e \\ a &\longmapsto \begin{pmatrix} \rho_1 a \rho_1 & \rho_1 a \rho_2 U_g^* \\ U_g \rho_2 a \rho_1 & U_g \rho_2 a \rho_2 U_g^* \end{pmatrix} . \end{aligned} \tag{116}$$

We are ready to compute the pullbacks of $[\Sigma]$ and $\tau \in HC^2(C_c^\infty(\Sigma) \rtimes \Gamma')$ by θ. This yields the following cyclic 2-cocycles on $C^\infty(S^2) \rtimes \Gamma$:

$$\begin{aligned} \theta^*[\Sigma] &= [S^2] , \\ (\theta^*\tau)(a_0, a_1, a_2) &= \int_{S^2} a_0 \Big(da_1 (\partial \ln \psi_2' \, a_2 + [a_2, {\rho_2}^2 \partial \ln g']) \\ &\quad + (\partial \ln \psi_1' a_1 + [a_1, {\rho_2}^2 \partial \ln g']) \Big) \\ &\quad - \int_{S^2} a_2 a_0 a_1 d({\rho_2}^2) \partial \ln g' , \end{aligned} \tag{117}$$

with $a_i = f_i U_{\psi_i}^* \in C^\infty(S^2) \rtimes \Gamma$. In formula (117), $S^2 = U_1 \cup \{\infty\}$ is gifted with the coordinate shart $(z, \overline{z})$ of U_1, which makes sense to $\psi_i'(z) = \partial_z \psi_i(z)$ and $g'(z) = \partial_z g(z) = -1/z^2$, but gives singular expressions at 0 and ∞. We can overcome this difficulty by introducing a smooth volume form $\nu = \rho(z, \overline{z}) \frac{dz \wedge d\overline{z}}{2i}$ on S^2. The associated modular automorphism group σ^ν leaves $C^\infty(S^2) \rtimes \Gamma$ globally invariant and is expressed in the coordinates $(z, \overline{z})$ by

$$\sigma_t^\nu(f U_\psi^*) = \left(\frac{\nu \circ \psi}{\nu} \right)^{it} f U_\psi^* = \left(\frac{\rho \circ \psi}{\rho} |\partial_z \psi|^2 \right)^{it} f U_\psi^* , \qquad \forall t \in \mathbb{R} . \tag{118}$$

Define the derivation δ^ν on $C^\infty(S^2) \rtimes \Gamma$

$$\begin{aligned} \delta^\nu(f U_\psi^*) &\equiv -i[\partial, \frac{d}{dt} \sigma_t^\nu](f U_\psi^*)|_{t=0} \\ &= [\partial, \ln (\frac{\rho \circ \psi}{\rho} |\partial_z \psi|^2)](f U_\psi^*) \qquad (119) \\ &= \partial \ln \psi' \, f U_\psi^* - [\partial \ln \rho, f U_\psi^*] . \qquad (120) \end{aligned}$$

One has

$$\partial \ln \psi' \, f U_\psi^* + [f U_\psi^*, {\rho_2}^2 \partial \ln g'] = \delta^\nu(f U_\psi^*) + [\partial \ln \rho - {\rho_2}^2 \partial \ln g', f U_\psi^*] , \tag{121}$$

where the 1-form $\omega = \partial \ln \rho - {\rho_2}^2 \partial \ln g'$ is globally defined, nowhere singular on S^2. Let $R^\nu = \partial \overline{\partial} \ln \rho$ be the curvature 2-form associated to the Kähler metric $\rho dz \otimes d\overline{z}$. One has the commutation rule

$$(\overline{\partial} \delta^\nu + \delta^\nu \overline{\partial}) a = [R^\nu, a] \qquad \forall a \in C^\infty(S^2) \rtimes \Gamma . \tag{122}$$

Simple algebraic manipulations show that the following 2-cochain

$$\tau^\nu(a_0,a_1,a_2) = \int_{S^2} a_0(da_1\delta^\nu a_2 + \delta^\nu a_1 da_2) + \int_{S^2} a_2 a_0 a_1 R^\nu \qquad (123)$$

is a cyclic cocycle. Moreover, τ^ν is cohomologous to $\theta^*\tau$. To see this, let φ be the cyclic 1-cochain

$$\varphi(a_0,a_1) = \int_{S^2} (a_0 da_1 - a_1 da_0)\omega \ . \qquad (124)$$

Then for all $a_i \in C^\infty(S^2) \rtimes \Gamma$,

$$\begin{aligned}(\tau^\nu - \theta^*\tau)(a_0,a_1,a_2) &= -\int_{S^2} (a_0 da_1 a_2 + a_2 da_0 a_1 + a_1 da_2 a_0)\omega \\ &= b\varphi(a_0,a_1,a_2) \ . \end{aligned} \qquad (125)$$

It is clear now that the construction of characteristic classes for an arbitrary (non flat) Riemann surface Σ follows exactly the same steps as in the above example. Using an open cover with partition of unity, one gets the desired cyclic cocycles by pullback. Choose a smooth measure ν on Σ, then the associated modular group is

$$\sigma_t^\nu(fU_\psi^*) = \left(\frac{\nu\circ\psi}{\nu}\right)^{it} fU_\psi^* \qquad fU_\psi^* \in C_c^\infty(\Sigma) \rtimes \Gamma \ . \qquad (126)$$

The corresponding derivation

$$D^\nu(fU_\psi^*) = \ln\left(\frac{\nu\circ\psi}{\nu}\right) fU_\psi^* \qquad (127)$$

allows to define the noncommutative differential

$$\delta^\nu = [\partial, D^\nu] \ . \qquad (128)$$

Then the characteristic classes of the groupoid $\Sigma \rtimes \Gamma$ are given by $[\Sigma]$ and $[\tau^\nu] \in HC^2(C_c^\infty(\Sigma) \rtimes \Gamma)$, where τ^ν is given by eq.(123) with S^2 replaced by Σ.

In the case $\Gamma = \mathrm{Id}$, the crossed product reduces to the commutative algebra $C_c^\infty(\Sigma)$ for which ($\delta^\nu = 0$)

$$\tau^\nu(a_0,a_1,a_2) = \int_\Sigma a_0 a_1 a_2 R^\nu \qquad (129)$$

is just the image of the cyclic 0-cocycle

$$\tau_0^\nu(a) = \int_\Sigma a R^\nu \qquad (130)$$

by the suspension map in cyclic cohomology $S: HC^*(C_c^\infty(\Sigma)) \to HC^{*+2}(C_c^\infty(\Sigma))$. Thus the periodic cyclic cohomology class of τ^ν corresponds in de Rham homology to the cap product

$$\frac{1}{2\pi i}[\tau^\nu] = c_1(\kappa) \cap [\Sigma] \quad \in H_0(\Sigma) \qquad (131)$$

of the first Chern class of the holomorphic tangent bundle κ by the fundamental class. This motivates the following definition:

Definition 2 *Let Σ be a Riemann surface without boundary and Γ a discrete pseudogroup acting on Σ by local conformal transformations. Let ν be a smooth volume form on Σ, and σ^ν the associated modular automorphism group leaving $C_c^\infty(\Sigma) \rtimes \Gamma$ globally invariant. Then the Euler class $e(\Sigma \rtimes \Gamma)$ is the class of the following cyclic 2-cocycle on $C_c^\infty(\Sigma) \rtimes \Gamma$*

$$\frac{1}{2\pi i}\tau^\nu(a_0, a_1, a_2) = \frac{1}{2\pi i}\int_\Sigma (a_2 a_0 a_1 R^\nu + a_0(da_1 \delta^\nu a_2 + \delta^\nu a_1 da_2)) \ , \qquad (132)$$

where δ^ν is the derivation $-i[\partial, \frac{d}{dt}\sigma_t^\nu|_{t=0}]$, and R^ν is the curvature of the Kähler metric determined by ν and the complex structure of Σ. Moreover, this cohomology class is independent of ν.

Now if $\Gamma = \mathrm{Id}$, the operator Q of section II defines an element of the K-homology of $\Sigma \times \mathbb{R}^2$. It corresponds to the tensor product of the classical Dolbeault complex $[\overline{\partial}]$ of Σ by the signature complex $[\sigma]$ of the fiber $\mathbb{R}^2$, so that its Chern character in de Rham homology is the cup product

$$\begin{aligned} \mathrm{ch}_*(Q) &= \mathrm{ch}_*([\overline{\partial}])\#\mathrm{ch}_*([\sigma]) \\ &= ([\Sigma] + \frac{1}{2}c_1(\kappa) \cap [\Sigma])\#2[\mathbb{R}^2] \quad \in H_*(\Sigma \times \mathbb{R}^2) \end{aligned} \qquad (133)$$

which yields, by Thom isomorphism, the homology class on Σ

$$2[\Sigma] + c_1(\kappa) \cap [\Sigma] \qquad \in H_*(\Sigma) \ . \qquad (134)$$

Next for any Γ, we know from the last section that the Chern character of the Dolbeault K-cycle, expressed in the periodic cyclic cohomology of $C_c^\infty(\Sigma) \rtimes \Gamma$, is a linear combination of $[\Sigma]$ and $e(\Sigma \rtimes \Gamma)$. Thus we deduce immediately the following generalisation of the Riemann-Roch theorem:

Theorem 3 *Let Σ be a Riemann surface without boundary and Γ a discrete pseudogroup acting on Σ by local conformal mappings without fixed point. The Chern character of the Dolbeault K-cycle is represented by the following cyclic 2-cocycle on $C_c^\infty(\Sigma) \rtimes \Gamma$*

$$\mathrm{ch}_*(Q) = 2[\Sigma] + e(\Sigma \rtimes \Gamma) \ . \qquad (135)$$

□

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