Gravitational Solitons and Monodromy Transform Approach to Solution of Integrable Reductions of Einstein Equations

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November 15, 1999

Abstract

In this paper the well known Belinskii and Zakharov soliton generating transformations of the solution space of vacuum Einstein equations with two-dimensional Abelian groups of isometries are considered in the context of the so called "monodromy transform approach", which provides some general base for the study of various integrable space - time symmetry reductions of Einstein equations. Similarly to the scattering data used in the known spectral transform, in this approach the monodromy data for solution of associated linear system characterize completely any solution of the reduced Einstein equations, and many physical and geometrical properties of the solutions can be expressed directly in terms of the analytical structure on the spectral plane of the corresponding monodromy data functions. The Belinskii and Zakharov vacuum soliton generating transformations can be expressed in explicit form (without specification of the background solution) as simple (linear-fractional) transformations of the corresponding monodromy data functions with coefficients, polynomial in spectral parameter. This allows to determine many physical parameters of the generating soliton solutions without (or before) calculation of all components of the solutions. The similar characterization for electrovacuum soliton generating transformations is also presented.

Keywords: Solitons, Einstein equations, monodromy transform
PACS: 02.30.Jr, 04.20.Jb, 04.40.Nr

1 Introduction

The existence of very rich, integrable structure of Einstein equations, at least for space-times with two-dimensional Abelian isometry groups, have been conjectured by different authors many years ago, but the real discoveries of beautiful integrability properties of these equations and effective procedures for construction of their solutions have been started more than twenty years ago in the papers of Belinskii and Zakharov [1, 2]. In these papers the inverse scattering methods have been developed for Einstein equations for vacuum gravitational fields. In particular, the solution of the entire problem had been reduced to some matrix Riemann - Hilbert problem and soliton generating (dressing) technique had been suggested for calculation of vacuum gravitational solitons on an arbitrary chosen (vacuum) background. Later numerous investigations of integrable reductions of Einstein equations (for vacuum or in the presence of electromagnetic and some other matter fields) have been made by many authors, using different powerful ideas of the modern theory of completely integrable systems. A number of different more or less general approaches were developed and many other interesting results were obtained.[1]

Here we consider Belinskii and Zakharov vacuum soliton generating transformations in the context of so called "monodromy transform" approach. This approach, developed by the author in [4] - [7], provides some general base for the analysis of all known integrable reductions of Einstein equations. In this approach any local solution of reduced Einstein equations is characterized by a set of monodromy data of the fundamental solution of some associated spectral problem.[2] The direct and inverse problems of such monodromy transform possess unambiguous solutions.[3] It is remarkable, that many physical and geometrical properties of the solutions can be expressed directly in terms of the analytical structure of these monodromy data on the spectral plane. A lot of the known physically interesting solutions possess very simple, rational structures of these monodromy data functions. As it will be shown below, it is also convenient to characterize the Belinskii and Zakharov vacuum soliton generating procedure in terms of the corresponding transformation of the monodromy data of the background solution into the monodromy data of the (multi-) soliton solution. These transformations possess an explicit and remarkably simple, linear-fractional form with coefficients, polynomial in the spectral parameter of the orders less or equal to a number of solitons. This allows to calculate various physical parameters of the generating soliton solutions without (or before) their complete calculation. A generalization of these transformations for the case of electrovacuum solitons, found in [9], is also presented.

[1]Avoiding a detail citation, we refer the readers to the references in a few papers cited here, but mainly – to a large and very useful F.J.Ernst's collection of related references and abstracts, accessible throw http://pages.slic.com/gravity.

[2]Though the similar constructions surely can be realized for any of the gauge equivalent null curvature representations of the field equations, there were used in this approach a spectral plane w, which is covered twice by the Belinskii and Zakharov spectral plane λ, and different spectral problem (first constructed by Kinnersley and Chitre [8] in the context of their group - theoretic approach), which is gauge equivalent to the Belinskii and Zakharov one. The reason for the first difference is the absence of differentiation in the transformed linear system with respect to a spectral parameter w, and for the second one - the simplest monodromy properties of solutions on this spectral plane.

[3]The linear singular integral equation with a scalar kernel, which solves the inverse problem of this monodromy transform, as well as its equivalent regularization – a (quasi-) Fredholm equation of the second kind have been presented in [4, 5] and [7] respectively.

2 Belinskii and Zakharov gravitational solitons

The dynamical part of vacuum Einstein equations for the space-time metrics

$$ds^2 = f(x^1, x^2)\eta_{\mu\nu}dx^\mu dx^\nu + g_{ab}(x^1, x^2)dx^a dx^b \tag{1}$$

where $\mu, \nu = 1, 2$; $a, b = 3, 4$; $f > 0$ and $\eta_{\mu\nu} = \mathrm{diag}\{\epsilon_1, \epsilon_2\}$ with $\epsilon_1 = \pm 1$, $\epsilon_2 = \pm 1$, can be reduced to the form (used in [1, 2] in a bit different notation):

$$\left\{\begin{array}{l} \eta^{\mu\nu}\partial_\mu(\alpha\,\partial_\nu \mathbf{g}\cdot\mathbf{g}^{-1}) = 0 \\ \mathbf{g}^T = \mathbf{g}, \quad \det\mathbf{g} = \epsilon\alpha^2 \end{array}\right. \left\| \begin{array}{c} \epsilon \equiv -\epsilon_1\epsilon_2 \\ \partial_\mu\partial_\nu\alpha = 0 \end{array} \right\| \begin{array}{ll} \beta: & \partial_1\beta = \epsilon_1\partial_2\alpha, \\ & \partial_2\beta = -\epsilon_2\partial_1\alpha \end{array} \tag{2}$$

where $\mathbf{g} = \|g_{ab}\|$ is a real symmetric 2×2 - matrix, and the given above definition of ϵ is implied by the Lorentz signature of the metric (1). Therefore, $\epsilon = 1$ corresponds to a hyperbolic case and $\epsilon = -1$ is the elliptic case. Note, that the linear "harmonic" equation for α, which follows immediately from the trace of the equation for $\mathbf{g}$ in (2), provides the existence of the defined above function β, "harmonically" conjugated to α. It is convenient to use farther these geometrically defined functions as new coordinates in the linear combinations $\xi = \beta + j\alpha$, $\eta = \beta - j\alpha$, where $j = 1$ for $\epsilon = 1$ and $j = i$ for $\epsilon = -1$.

In these notation the Belinskii and Zakharov spectral problem reads as

$$\left\{\begin{array}{l} D_\xi\mathbf{\Psi}_{BZ} = \dfrac{\mathbb{V}_\xi}{\lambda - j\alpha}\mathbf{\Psi}_{BZ} \\ D_\eta\mathbf{\Psi}_{BZ} = \dfrac{\mathbb{V}_\eta}{\lambda + j\alpha}\mathbf{\Psi}_{BZ} \end{array}\right. \left| \begin{array}{l} D_\xi = \partial_\xi - \dfrac{\lambda}{\lambda - j\alpha}\dfrac{\partial}{\partial\lambda} \\ D_\eta = \partial_\eta - \dfrac{\lambda}{\lambda + j\alpha}\dfrac{\partial}{\partial\lambda} \end{array} \right| \begin{array}{l} \mathbb{V}_\xi = -j\alpha\,\partial_\xi\mathbf{g}\cdot\mathbf{g}^{-1} \\ \mathbb{V}_\eta = j\alpha\,\partial_\eta\mathbf{g}\cdot\mathbf{g}^{-1} \end{array} \tag{3}$$

with additional "reduction" conditions imposed on the solutions of (3):

$$\mathbf{\Psi}^T_{BZ}(\lambda \to \frac{\epsilon\alpha^2}{\lambda})\cdot\mathbf{g}^{-1}\cdot\mathbf{\Psi}_{BZ} = \mathbf{K}_{BZ}(w), \quad \mathbf{K}^T_{BZ} = \mathbf{K}_{BZ}, \quad D_\xi w = D_\eta w = 0 \tag{4}$$

For the construction of vacuum solitons, based on the form (3), (4) of the spectral problem, the dressing transformation $\mathbf{\Psi}_{BZ}(\xi,\eta,\lambda) = \chi_{BZ}(\xi,\eta,\lambda)\cdot\overset{o}{\mathbf{\Psi}}_{BZ}(\xi,\eta,\lambda)$ was used in [1, 2] with the soliton ansatz[4]

$$\chi_{BZ} = \mathbb{I} + \sum_{k=1}^{2N}\frac{\mathbb{R}_k(\xi,\eta)}{\lambda - \mu_k(\xi,\eta)}, \qquad \chi^{-1}_{BZ} = \mathbb{I} + \sum_{k=1}^{2N}\frac{\mathbb{S}_k(\xi,\eta)}{\lambda - \nu_k(\xi,\eta)} \tag{5}$$

where $\mathbb{I}$ is the identity matrix, the 2×2-matrix residues at the poles as well as the pole trajectories μ_k and ν_k are real or constitute complex conjugated pairs. Then in [1, 2] all constraints on these functions, which follow from (3), had been successfully solved and all metric components of the generating soliton solutions have been explicitly expressed in terms of a number of integration constants and the matrix $\overset{o}{\mathbf{\Psi}}_{BZ}(\xi,\eta,\lambda)$, characterizing arbitrarily chosen "background" solution. (See also [3] for more compact, determinant form of these soliton solutions).

[4]In the case $\epsilon = 1$ the number of solitons (poles) can be odd as well, but we shall not consider this case here.

3 Vacuum w - solitons.

The function w, which was used as the new spectral parameter in the mentioned above monodromy transform approach, have been introduced also in [1, 2], as a solution of the equations $D_\xi w = D_\eta w = 0$, such that $2w = \lambda + 2\beta + \epsilon\alpha^2/\lambda$. Thus, the spectral plane λ covers twice the spectral plane w. The Kinnersley - Chitre-like linear system [8] which substitute in our approach the linear system (3), and the gauge transformation between these systems in our notation are

$$\left\{\begin{array}{l} \partial_\xi \mathbf{\Psi} = \dfrac{\mathbf{U}(\xi,\eta)}{2i(w-\xi)}\mathbf{\Psi} \\ \partial_\eta \mathbf{\Psi} = \dfrac{\mathbf{V}(\xi,\eta)}{2i(w-\eta)}\mathbf{\Psi} \end{array}\right. \quad \Bigg\| \quad \begin{array}{l} \mathbf{\Psi}_{BZ}(\xi,\eta,\lambda) = \mathbf{A}_\otimes \cdot \mathbf{\Psi}(\xi,\eta,w) \\ \mathbf{A}_\otimes = \mathbf{g}\cdot\varepsilon + i\lambda\mathbb{I}, \quad \varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{array} \tag{6}$$

The following two groups of conditions constitute a spectral problem, based on the linear system (6) and equivalent to the reduced vacuum equations (2) [9, 5]:

$$\left\{\begin{array}{l} 2i(w-\xi)\partial_\xi \mathbf{\Psi} = \mathbf{U}(\xi,\eta)\cdot\mathbf{\Psi} \\ 2i(w-\eta)\partial_\eta \mathbf{\Psi} = \mathbf{V}(\xi,\eta)\cdot\mathbf{\Psi} \end{array}\right. \quad \Bigg\| \quad \begin{array}{ll} \operatorname{rank}\mathbf{U} = 1, & \operatorname{tr}\mathbf{U} = i, \\ \operatorname{rank}\mathbf{V} = 1, & \operatorname{tr}\mathbf{V} = i, \end{array} \tag{7}$$

$$\left\{\begin{array}{l} \mathbf{\Psi}^\dagger \cdot \mathbf{W} \cdot \mathbf{\Psi} = \mathbf{K}(w) \\ \mathbf{K}^\dagger(w) = \mathbf{K}(w) \end{array}\right. \quad \Bigg\| \quad \frac{\partial \mathbf{W}}{\partial w} = 4i\varepsilon, \quad \varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{8}$$

For the construction of the Belinskii and Zakharov vacuum solitons, based on the spectral problem (7), (8), we use similar dressing transformation $\mathbf{\Psi}(\xi,\eta,w) = \chi(\xi,\eta,w)\cdot \overset{o}{\mathbf{\Psi}}(\xi,\eta,w)$ with a bit different, than in (5), soliton ansatz [5]

$$\chi = X(w)\left(\mathbf{I} + \sum_{k=1}^N \frac{\mathbf{R}_k(\xi,\eta)}{w - w_k}\right), \qquad \chi^{-1} = \frac{1}{X(w)}\left(\mathbf{I} + \sum_{k=1}^N \frac{\mathbf{S}_k(\xi,\eta)}{w - \widetilde{w}_k}\right), \tag{9}$$

where $X(w) = \prod_{k=1}^N \left(\frac{w-w_k}{w-\widetilde{w}_k}\right)^{\frac{1}{2}}$, the gauge condition $\mathbf{K}(w) = \overset{o}{\mathbf{K}}(w)$ is used, the constant pole locations w_k and $\widetilde{w}_k$ for each $k = 1, 2, \ldots, N$ are the pairs of different real or complex conjugated constants, [6] and the values, having "o" overhead, correspond to the background solution. Again, similarly to the original Belinskii and Zakharov construction of λ-solitons, all constraints, which follow from (7), (8) for the matrix residues in (9) can be solved [9, 5], and for any choice of the background solution the generating N-soliton solution can be expressed in terms of the background $\overset{o}{\mathbf{\Psi}}(\xi,\eta,w)$ and a set of integration constants (four real constants per each of the poles w_k).

[5]For the case of real poles an alternative construction of the Belinskii and Zakharov solitons in the context of the spectral problem (7), (8), which corresponds to the choice $X(w) = 1$ in (9) and $\mathbf{K}(w) = \prod_{k=1}^N \left(\frac{w-\widetilde{w}_k}{w-w_k}\right) \overset{o}{\mathbf{K}}(w)$, had been presented in [10].

[6]It is useful to note, that the number N of w-poles (solitons) in (9) corresponds to $2N$ solitons (λ-poles) in (5); each real pole with $w_k \neq \widetilde{w}_k$ is equivalent to a pair of real λ - poles in (5), while each of the complex w-poles with $\widetilde{w}_k = \overline{w_k}$ correspond to a pair of complex conjugated to each other λ-poles in (5).

4 Electrovacuum w - solitons

One of the important features of the spectral problem (7), (8) is that very small changes of its structure lead to a matrix problem, equivalent to the space-time symmetry reduced electrovacuum Einstein - Maxwell equations [9, 5, 7]:

$$\left\{\begin{array}{l} 2i(w-\xi)\partial_\xi \mathbf{\Psi} = \mathbf{U}(\xi,\eta)\cdot\mathbf{\Psi} \\ 2i(w-\eta)\partial_\eta \mathbf{\Psi} = \mathbf{V}(\xi,\eta)\cdot\mathbf{\Psi} \end{array}\right. \quad \Bigg\| \quad \begin{array}{ll} \operatorname{rank}\mathbf{U}=1, & \operatorname{tr}\mathbf{U}=i, \\ \operatorname{rank}\mathbf{V}=1, & \operatorname{tr}\mathbf{V}=i, \end{array} \tag{10}$$

$$\left\{\begin{array}{l} \mathbf{\Psi}^\dagger\cdot\mathbf{W}\cdot\mathbf{\Psi} = \mathbf{K}(w) \\ \mathbf{K}^\dagger(w) = \mathbf{K}(w) \end{array}\right. \quad \Bigg\| \quad \frac{\partial \mathbf{W}}{\partial w} = 4i\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{W}^{55}=1 \tag{11}$$

where the unknown matrix variables $\mathbf{\Psi}(\xi,\eta,w)$, $\mathbf{U}(\xi,\eta)$, $\mathbf{V}(\xi,\eta)$ and $\mathbf{W}(\xi,\eta,w)$ are now 3×3 - matrices and $\mathbf{W}^{55}$ denotes the lower right component of $3\times$ 3-matrix $\mathbf{W}$ and $\mathbf{\Psi}^\dagger(\xi,\eta,w) \equiv \overline{\mathbf{\Psi}^T(\xi,\eta,\overline{w})}$. With the use of (10), (11) the electrovacuum soliton solutions have been constructed using the same dressing transformation $\mathbf{\Psi} = \chi\cdot\overset{o}{\mathbf{\Psi}}$ with a more particular, than (9), soliton ansatz [9]:

$$\chi = \mathbf{I} + \sum_{k=1}^{N}\frac{\mathbf{R}_k(\xi,\eta)}{w-w_k}, \qquad \chi^{-1} = \mathbf{I} + \sum_{k=1}^{N}\frac{\mathbf{S}_k(\xi,\eta)}{w-\widetilde{w}_k} \tag{12}$$

where the poles of χ are complex conjugated to the poles of χ^{-1}, i.e. for each $k=1,2,\ldots,N$ we have $\widetilde{w}_k = \overline{w_k}$. This leads to electrovacuum generalization (which includes six real parameters per each w-pole of χ) of the Belinskii and Zakharov vacuum solitons with complex conjugated poles, while a generalization of vacuum solitons with real poles does not arise in this way. [7]

5 Monodromy data of the solutions

The analysis, suggested below, is based on the monodromy transform approach, developed in [4] - [7]. In this approach any local solutions of reduced vacuum or electrovacuum Einstein equations is characterized unambiguously by a finite set of functional parameters, which are functions of the spectral parameter w only and which admit a simple interpretation as the monodromy data on the spectral plane for the fundamental solution $\mathbf{\Psi}(\xi,\eta,w)$ of the spectral problem

[7]Many electrovacuum solutions which generalize vacuum solitons with real poles and on some specially chosen backgrounds (e.g., on the Minkowski background) can be constructed as the analytical continuations of soliton solutions with complex poles in the space of their constant parameters. This complex analytical continuation is quite similar to the known one, which relates the "underextreme" and "overextreme" parts of the Kerr - Newman family of solutions. Another way for construction of such solutions is a direct integration of the integral equation, which the spectral problem (10), (11) can be reduced to. This integral equation, derived in [4, 5], can be solved and the solutions with arbitrarily large number of free constant parameters can be constructed explicitly for rational values of some special functional parameters in the kernel of this equation, called as "monodromy data", and expressed in terms of a number of arbitrary rational functions of spectral parameter w [5, 11].

(10), (11), corresponding to a given solution of the field equations under consideration. In order to define the monodromy data, we need, first of all, to fix some gauge freedom, existing in (10), (11). For this we impose at some chosen "initial" space-time point (ξ_0, η_0)[8] the universal "normalization" conditions for the metric components (say, $\mathbf{g}(\xi_0, \eta_0) = \epsilon_0 \,\mathrm{diag}\,\{1, \epsilon\alpha_0^2\}$, where $\epsilon_0 = \pm 1$), which determine the value of $\mathbf{W}_o(w) \equiv \mathbf{W}(\xi_0, \eta_0, w)$ as

$$\mathbf{W}_o(w) = 4i(w - \beta_0) \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} -4\epsilon_0\epsilon\alpha_0^2 & 0 & 0 \\ 0 & -4\epsilon_0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{13}$$

where $\alpha_0 = (\xi_0 - \eta_0)/2j$, $\beta_0 = (\xi_0 + \eta_0)/2$. The normalization condition, used for the value of $\mathbf{\Psi}(\xi_0, \eta_0, w)$, which determines then the "normalized" value of $\mathbf{K}(w)$, and the corresponding gauge transformations can be chosen in the form

$$\begin{array}{l} \mathbf{\Psi}(\xi_0, \eta_0, w) = \mathbf{I} \\[2ex] \mathbf{K}(w) = \mathbf{W}_o(w) \end{array} \quad \left\| \quad \begin{array}{l} \mathbf{\Psi} \to \mathbf{\Psi} \cdot \mathbf{C}(w), \qquad \mathbf{C}(w) = \mathbf{\Psi}^{-1}(\xi_0, \eta_0, w) \\[1ex] \begin{array}{l} \mathbf{\Psi} \to \mathbf{A}\,\mathbf{\Psi}\,\mathbf{A}^{-1}, \\ \mathbf{W} \to (\mathbf{A}^\dagger)^{-1}\,\mathbf{W}\,\mathbf{A}^{-1}, \end{array} \quad \mathbf{A}(w) = \begin{pmatrix} SL(2,R) & \begin{matrix} 0 \\ 0 \end{matrix} \\ a^3 \quad a^4 & 1 \end{pmatrix} \end{array} \right. \tag{14}$$

where the 3×3 - matrix $\mathbf{A}$ is constant. The complex constants a^3 and a^4 change the additive constants in the definitions of the components of a complex electromagnetic potentials and $SL(2, R)$ part of $\mathbf{A}$ corresponds to a linear transformation of the coordinates x^3, x^4 in (1).

A detail analysis [4, 5] of the analytical structure on the spectral plane of the solutions of (10), (11) shows the existence of some universal properties of $\mathbf{\Psi}(\xi, \eta, w)$.[9] In particular, it is holomorphic function of w everywhere outside four algebraic branchpoints and the cut $L = L_+ + L_-$ joining these points, as it is shown on Fig. 1. It turns out, that the behaviour of $\mathbf{\Psi}$ near the branchpoints can be described by the monodromy matrices $\mathbf{T}_\pm(w)$, which characterize the linear transformations of $\mathbf{\Psi}$, continued analytically along the paths $T_\pm$, rounding one of the branchpoints and joining different edges of L_+ or L_- respectively:

$$\mathbf{\Psi} \quad \xrightarrow{T_\pm} \quad \widetilde{\mathbf{\Psi}} = \mathbf{\Psi} \cdot \mathbf{T}_\pm(w), \qquad \mathbf{T}_\pm(w) = \mathbf{I} - 2\frac{\mathbf{l}_\pm(w) \otimes \mathbf{k}_\pm(w)}{(\mathbf{l}_\pm(w) \cdot \mathbf{k}_\pm(w))}. \tag{15}$$

It is remarkable, that these matrices, satisfying the identities $\mathbf{T}_\pm^2(w) \equiv \mathbf{I}$, are independent of the space-time coordinates ξ, η. The structure (15) allows to express $\mathbf{T}_\pm$ in terms of the four complex projective vectors $\mathbf{k}_\pm(w)$ and $\mathbf{l}_\pm(w)$, but it was found in [4], that (11) relate unambiguously $\mathbf{l}_\pm(w)$ and $\mathbf{k}_\pm^\dagger(w)$ with the same suffices. Therefore, both $\mathbf{T}_\pm$ are determined completely by four scalar functions, which parametrize the components of two projective vectors $\mathbf{k}_\pm(w)$:

$$\mathbf{k}_\pm(w) = \{1\,, \mathbf{u}_\pm(w)\,,\, \mathbf{v}_\pm(w)\,\} \tag{16}$$

[8] Actually, this is a point in the orbit space of the space-time isometry group.

[9] From now, the functions $\mathbf{\Psi}$, $\mathbf{W}$, $\mathbf{K}$ are considered in the gauges, fixed as in (13), (14).

The functions $\mathbf{u}_\pm(w)$, $\mathbf{v}_\pm(w)$, which domains of holomorphicity are shown on Fig. 1, are called as monodromy data. These data characterize unambiguously any local solution of the field equations near some chosen "initial" or "reference" point (ξ_0, η_0), however, it is useful to keep in mind, that for a given solution these data are dependent upon the choice of this point. Note, that for pure vacuum case $\mathbf{v}(w) \equiv 0$.

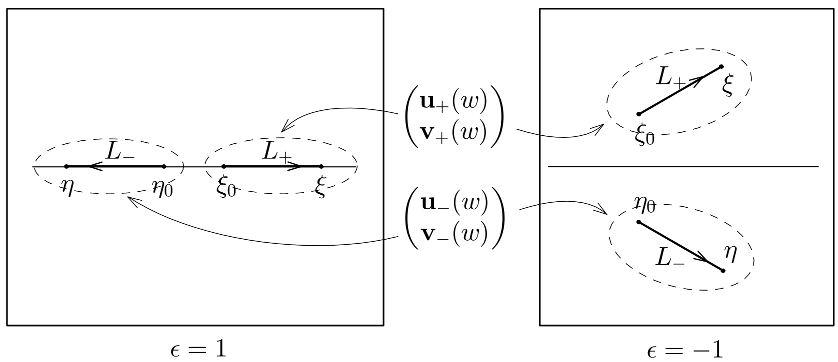

Figure 1: The singular points of $\mathbf{\Psi}$ and the structure of the cut $L = L_+ + L_-$ on the spectral plane as well as the domains, where the monodromy data functions $\mathbf{u}_\pm(w)$ and $\mathbf{v}_\pm(w)$ are defined and holomorphic, are shown here for the hyperbolic ($\epsilon = 1$) and elliptic ($\epsilon = -1$) cases separately.

6 Soliton generating transformations in terms of the monodromy data

Besides the definition of the monodromy data for any local vacuum or electrovacuum solution, described in the previous section, we recall here that these data enter explicitly into the local structure of $\mathbf{\Psi}$ near the cuts $L_\pm$. This structure can be described by the expressions [5]:

$$\mathbf{\Psi} = \lambda_\pm^{-1}\,\psi_\pm(\xi,\eta,w) \otimes \mathbf{k}_\pm(w) + \mathbf{M}_\pm(\xi,\eta,w) \tag{17}$$

where $\lambda_+ = \sqrt{(w-\xi)/(w-\xi_0)}$, $\lambda_- = \sqrt{(w-\eta)/(w-\eta_0)}$ and the column-vectors $\psi_\pm$, the row-vectors $\mathbf{k}_\pm$ and the matrices $\mathbf{M}_\pm$ are holomorphic at the points of the cut L_+ or L_-, respectively to their suffices. In accordance with these expressions, the monodromy data can be calculated from the branching parts of the components of $\mathbf{\Psi}$ on the cuts $L_\pm$. We use now the expressions (17) for calculation of the transformations of the vectors $\mathbf{k}_\pm$ and therefore, of the monodromy data, induced by the soliton generating transformations.

It is easy to see, that any dressing transformation $\mathbf{\Psi} = \chi \cdot \overset{o}{\mathbf{\Psi}}(\xi,\eta,w)$ for the normalized $\mathbf{\Psi}$ - functions reads

$$\mathbf{\Psi} = \mathbf{A} \cdot \chi \cdot \overset{o}{\mathbf{\Psi}} \cdot \chi_0^{-1} \cdot \mathbf{A}^{-1} \tag{18}$$

where $\overset{o}{\mathbf{\Psi}}$ is the normalized background solution, $\chi_0(w) = \chi(\xi_0, \eta_0, w)$ and the constant matrix $\mathbf{A}$ should be chosen afterwards for normalization of the metric components and the matrix $\mathbf{W}$. Let us consider now the expression (18) near the cuts $L_\pm$, using there the local representations (17) for $\mathbf{\Psi}$ and $\overset{o}{\mathbf{\Psi}}$. These expressions imply the following transformations of the projective vectors $\mathbf{k}(w)$:

$$\mathbf{k}_\pm(w) = \overset{o}{\mathbf{k}}_\pm(w) \cdot \chi_0^{-1}(w) \cdot \mathbf{A}^{-1}. \tag{19}$$

Now it is easy to calculate the monodromy data $\mathbf{u}_\pm(w)$, $\mathbf{v}_\pm(w)$ for solitons, as the ratios of the components of the projective vectors (19), using one of the soliton ansatz (9) or (12). The result can be presented in the linear-fractional form (for vacuum background $\overset{o}{\mathbf{v}}_\pm(w) \equiv 0$ and for vacuum solitons $\mathbf{v}_\pm(w) \equiv 0$):

$$\mathbf{u}_\pm(w) = \frac{\mathcal{U}_0 + \mathcal{U}_1\,\overset{o}{\mathbf{u}}_\pm(w) + \mathcal{U}_2\,\overset{o}{\mathbf{v}}_\pm(w)}{\mathcal{Q}_0 + \mathcal{Q}_1\,\overset{o}{\mathbf{u}}_\pm(w) + \mathcal{Q}_2\,\overset{o}{\mathbf{v}}_\pm(w)}, \;\; \mathbf{v}_\pm(w) = \frac{\mathcal{V}_0 + \mathcal{V}_1\,\overset{o}{\mathbf{u}}_\pm(w) + \mathcal{V}_2\,\overset{o}{\mathbf{v}}_\pm(w)}{\mathcal{Q}_0 + \mathcal{Q}_1\,\overset{o}{\mathbf{u}}_\pm(w) + \mathcal{Q}_2\,\overset{o}{\mathbf{v}}_\pm(w)},$$

where all coefficients are polynomials in w, which orders do not exceed the number of w-solitons N (or $2N$ in the Belinskii and Zakharov formalism).

Acknowledgments

This work was partly supported by the British Engineering and Physical Sciences Research Council and the Russian Foundation for Basic Research Grants 99-01-01150, 99-02-18415.

References

[1] V.A.Belinskii and V.E.Zakharov, Sov. Phys. JETP **48** (1978) 985

[2] V.A.Belinskii and V.E.Zakharov, Sov. Phys. JETP **50** (1979) 1

[3] G.A. Alekseev, Sov. Phys. Dokl. (USA) **26** (1981) 158

[4] G.A. Alekseev, Sov. Phys. Dokl. (USA) **30** (1985) 565

[5] G.A. Alekseev, Proceedings of the Steklov Inst. of Math. **3** (1988) 215

[6] G.A. Aleksejev, Proceedings of 8th Int. Workshop NEEDS'92, World Scientific (1992) 5

[7] G.A. Alekseev, gr-qc/9911045, submitted to World Scientific.

[8] W. Kinnersley and D.M. Chitre, J.Math.Phys. **19** (1978) 1927

[9] G.A. Alekseev, Pis'ma JETP, **32** (1980) 301

[10] S. Miccichè and J.B. Griffiths, Class. Quantum Grav., **17**, 1, (2000); gr-qc/990974

[11] G.A. Alekseev, in Abstracts of contributed papers, 13th Intern. Conf. on General relativity and gravitation (Cordoba, Argentina, 1992), (1993) 3.