

ON UPPER LIMITS FOR GRAVITATIONAL RADIATION

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Abstract

A procedure with a Bayesian approach for calculating upper limits to gravitational wave bursts from coincidence experiments with multiple detectors is described.

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1 Introduction

After the initial experiments with room temperature resonant detectors, the new generation of cryogenic gravitational wave (GW) antennas entered long term data taking operation in 1990 (EXPLORER [1]), in 1991 (ALLEGRO [2]), in 1993 (NIOBE [3]), in 1994 (NAUTILUS [4]) and in 1997 (AURIGA [5]).

Searches for coincident events between detectors have been performed. Between EXPLORER and NAUTILUS and between EXPLORER and NIOBE in the years 1995 and 1996 [6]. Between ALLEGRO and EXPLORER with data recorded in 1991 [7]. In both cases no significative coincidence excesses were found and an upper limit to GW bursts was calculated [7].

However, the upper limit determination has been done under the hidden hypothesis that the signal-to-noise ratio (SNR) is very large. According to theoretical estimations the signals expected from cosmic GW sources are extremely feeble, so small that extremely sensitive detectors are needed. In fact, according to present knowledge, the detectors available today have not yet reached the sensitivity to detect even a few events per year.

Thus it is important to study the problem of the upper limit determination in the cases the SNRs of the observed *events* are not large. In order to do this we have to discuss our definition of *event*.

The raw data from a resonant GW detector are filtered with a filter matched to short bursts [8]. We describe now in more detail the procedure used for the GW detectors of the Rome group, EXPLORER and NAUTILUS.

After the filtering of the raw-data, *events* are extracted as follows. Be $x(t)$ the filtered output of the electromechanical transducer which converts the mechanical vibrations of the bar in electrical signals. This quantity is normalized, using the detector calibration, such that its square gives the energy innovation E_f of the oscillation for each sample, expressed in kelvin units. In absence of signals, for well behaved noise due only to the thermal motion of the bar and to the electronic noise of the amplifier, the distribution of $x(t)$ is normal with zero mean. The variance (average value of the square of $x(t)$) is called *effective temperature* and is indicated with T_{eff} . The distribution of $x(t)$ is

$$f(x) = \frac{1}{\sqrt{2\pi T_{eff}}} e^{-\frac{x^2}{2T_{eff}}} \quad (1)$$

For extracting *events* (in absence of signals the events are just due to noise) we set a threshold in terms of a critical ratio defined by

$$CR = \frac{|x| - \langle |x| \rangle}{\sigma(|x|)} = \frac{\sqrt{SNR_f} - \sqrt{\frac{2}{\pi}}}{\sqrt{1 - \frac{2}{\pi}}} \quad (2)$$

where $\sigma(|x|)$ is the standard deviation of $|x|$ and we put

$$SNR_f = \frac{E_f}{T_{eff}} \quad (3)$$

The threshold is set at a value CR such to obtain, in presence of thermal and electronic noise alone, a number of events which can be easily exchanged among the other groups who participate to the data exchange. For about one hundred *events* per day the threshold corresponds to an energy $E_t = 19.5 T_{eff}$.

We calculate now the theoretical probability to detect a signal with a given SNR, in presence of a well behaved Gaussian noise. We put $y = (s + x)^2$ where $s \equiv \sqrt{SNR}$ is the signal we look for and x is the gaussian noise. We obtain easily [9]

$$probability(SNR) = \int_{SNR_t}^{\infty} \frac{1}{\sqrt{2\pi y}} e^{-\frac{(SNR+y)}{2}} \cosh(\sqrt{y \cdot SNR}) dy \quad (4)$$

We put $SNR_t = 19.5$ for the present EXPLORER and NAUTILUS detectors.

2 Upper limit determination

We consider M detectors and search for M-fold coincidences over a total period of time t_m during which all detectors are in operation. Be \bar{n} the average number of accidental coincidences (due to chance) and n_c the number of coincidences which are found within a given time window.

For events which have a Poissonian distribution in time the expected average number of M-fold accidental coincidences is given [14] by

$$\bar{n} = Mw^{M-1} \prod_k^{1,M} n_k \quad (5)$$

where n_k is the event density of the k^{th} detector.

The accidental coincidence distribution can be estimated experimentally by proper shifting [10] the event occurrence times of each detector. In the case of Poissonian distribution the average number of the M-fold accidental coincidences coincides with that given by eq. 5. The comparison between n_c and \bar{n} allows to reach some conclusion about the detection of GW or to establish an upper limit to their existence.

In paper [7] and in the previous paper [11] the upper limit has been estimated as follows. It has been found that, for various energy levels of the observed events, the number n_c was smaller than or did not exceeded significantly \bar{n} . Such numbers n_c , one for each energy level, were used for calculating the upper limit. A Poissonian distribution of the number of the observed events was considered together with the hypothesis of an isotropic distribution in the sky of the GW sources. The value of h (adimensional perturbation of the metric tensor) was then derived *from* the energy levels, using the detector cross-section for gravitational waves.

This procedure can be objected on two points:

a)The most important point is that, as shown in [12], for SNR small and up to values of a few dozens, the energy of an event is *not* the energy of the GW absorbed by the detector. This means that we cannot deduce the value of h directly from the energy levels of the observed events;

b) In addition, the efficiency of detection, again for SNR values up to one or two dozens, is rather smaller than unity, and this changes the upper limit, particularly at small SNR.

We introduce a new procedure for estimating the upper limit, which circumvents the difficulties indicated in the above two points.

The problem to determine the upper limit has been discussed in several papers. In particular in paper [15], as indicated by the PDG, [16] and, more recently, in paper [17]. According to [17] the upper limit can be calculated using the relative belief updating ratio

$$R(n_{GW}, n_c, \bar{n}) = e^{-n_{GW}} \left(1 + \frac{n_{GW}}{\bar{n}}\right)^{n_c} \quad (6)$$

referring to a given period t_m of data taking. This function is proportional to the likelihood and it allows to infer the probability to have n_{GW} signals for given priors (using the Bayes's theorem). It has already been used in High Energy Physics [18, 19].

We calculate the upper limit by solving the equation

$$R(n_{GW}, n_c, \bar{n}) = 0.05 \quad (7)$$

We remark that 5% does not represent a probability but it is an useful way to put a limit independently on the priors¹⁾.

Eq. 7 has a very interesting solution. Putting $n_c = 0$ we find $n_{GW} = 2.99$, independent on the value of the background \bar{n} . If we use the calculations of ref. [15] we find that, for $n_c = 0$ and $\bar{n} = 0$, the upper limit is 3.09 (almost identical to the previous one) *but* it decreases for increasing \bar{n} . The reason for this different behavior is due to the non-Bayesian character of the calculations made in [15], as we discuss in the following.

Suppose we have $n_c = 0$ and $\bar{n} \neq 0$. This certainly means that the number of accidentals, whose average value can be determined with any desired accuracy, has undergone a fluctuation. For larger \bar{n} values, smaller is the (*a priori*) probability that such fluctuation occur. Thus one could reason that it is less likely that a number n_{GW} be associated to a large value of \bar{n} , since the observation gave $n_c = 0$.

According to the Bayesian approach instead, as discussed in [17], one cannot ignore the fact that the observation $n_c = 0$ had already being made at the time the estimation of the upper limit is considered. The Bayesian approach requires that, given $n_c = 0$ and $\bar{n} \neq 0$, one evaluate the *chance* that a number n_{GW} of signals exist. This *chance* of a possible signal is referred to the observation already made and, rather obviously, it cannot depend on the previous fluctuation of the background, since the presence of a signal cannot be related to the background due to the detector. Mathematically, it is easy to demonstrate, using the results obtained in [17], that due to the Poissonian character of the number of accidentals this *relative chance* (for $n_c = 0$) is indeed independent on \bar{n} .

It can be seen, comparing the results of [15] with those of [17], that the Bayesian upper limits are for all values of n_c and \bar{n} (except $n_c = \bar{n} = 0$), greater than those obtained

¹⁾ To avoid confusion we shall continue to use the words *upper limit*, although it would be more appropriate to call it *standard sensitivity bound* [20].

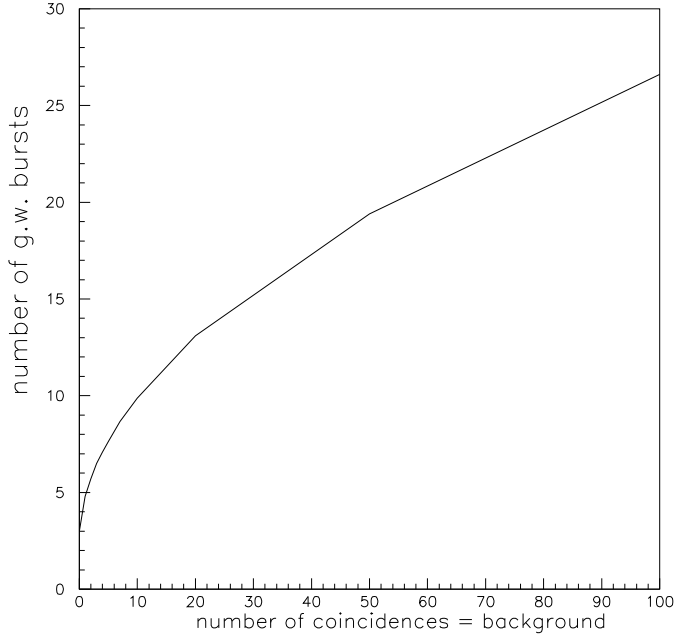


Figure 1: Number of GW signals expected for the sensitivity limit of 5% versus the number of coincidences equal to the average number of accidentals.

with the non-Bayesian procedure. In our opinion the Bayesian approach has to be preferred, and so we do in this paper.

If we have $n_c \neq 0$ then we apply eq. 6. It is interesting to show the result for the case $n_c = \bar{n} \neq 0$ for the standard sensitivity bound of 5%. The result is given in fig.1 We note that for $n_c = \bar{n}$ and $n_{GW} \ll \bar{n}$ eq. 6 can be approximated with

$$n_{GW} \approx \sqrt{6 \bar{n}} \quad (8)$$

From the result shown in fig.1 it appears evident that the lowest upper limit is obtained for $n_c \sim \bar{n} \sim 0$. In order to obtain $\bar{n} \sim 0$ one can raise the threshold used for determining the events. However in doing this one diminish the efficiency of detection, as shown in eq.4. Whether the procedure to raise the threshold is convenient or not, it depends on the numerical effects of the two competing operations. Certainly for large GW signals, when the detection efficiency is always unity, it is much better to have a threshold that gives $\bar{n} = 0$. For smaller signals one has to consider specific cases. However it can be seen that in the most interesting cases it is better to raise the threshold until we get $\bar{n} \sim 0$. This will be shown in the section where we reconsider the upper limit obtained with ALLEGRO and EXPLORER in 1991 [7].

In the estimation of the upper limit we consider the efficiency of detection, which we indicate with $\epsilon_k(SNR)$ where k refers to the k^{th} detector. For EXPLORER and NAUTILUS the theoretical efficiency is obtained from eq. 4.

We must relate the h values of the GW to the energy E absorbed by the detectors.

We have to consider that the absorbed energy depends on the direction of the impinging GW and on its polarization. For taking care of the various polarization we use the average value dividing the cross section by a factor of two. We then have [13]

$$h = 1.13 \cdot 10^{-17} \sqrt{E} \quad (9)$$

with the energy E expressed in kelvin unit. This formula is valid only if the GW arrives perpendicularly to the detector axis ($\theta = 90^\circ$). For a given direction we calculate the absorbed energy using the $\sin(\theta)^4$ dependency. We also consider that for an isotropic distribution of sources the number of possible GW impinging directions is proportional to $\sin(\theta)^2$.

The procedure for calculating the upper limit is accomplished thru the following points:

- a) consider various values of h ;
- b) assume an isotropic distribution of the GW sources;
- c) for each direction θ and for each h calculate the absorbed energy $E(\theta)$ by means of eq. 9 and the $\sin^4(\theta)$ dependency;
- d) for each detector calculate the SNR for the adsorbed energy by taking into consideration the noise $T_{eff,k}$:

$$SNR_k(\theta) = \frac{E(\theta)}{T_{eff,k}}, \quad k = 1, \dots, M \quad (10)$$

- e) using the individual efficiencies $\epsilon_k(SNR_k(\theta))$ consider the total efficiency $\epsilon_t(\theta) = \prod_k^{1,M} \epsilon_k(SNR_k(\theta))$;
- f) integrate $\epsilon_t(\theta)$ over θ with the weight $\sin^2(\theta)$, because of the assumed isotropic distribution of the sources;
- g) from eq.6, given n_c and \bar{n} , we obtain n_{GW} . We then divide n_{GW} by the result of point f) and obtain for each value of h the upper limit during the measuring time t_m .

We remark that in this case we have not used the energy of the observed events, as done instead previously [11, 7].

The total efficiency is calculated with the following eq. 11.

$$\epsilon_{tot}(h) = \frac{\int_0^{\frac{\pi}{2}} \prod_k^{1,M} \epsilon_k(SNR_k(\theta)) \sin^2(\theta) d\theta}{\frac{\pi}{4}} \quad (11)$$

For more clarity we show in table 1 some of the steps needed for our calculation, using two parallel detectors and $n_c = 0$. We use the efficiency given by eq. 4, valid for a well behaved noise²⁾.

3 Ricalculation of the upper limit with the data of ALLEGRO and EXPLORER in 1991

In a previous paper [7] the upper limit for GW bursts was calculated, using the data recorded by ALLEGRO and EXPLORER in 1991. We wish now to recalculate the upper limit according the considerations discussed in this paper.

Table 1: Procedure for calculating the upper limit with two detectors. We assume that one detector has noise $T_{eff} = 1 \text{ mK}$, the other one has noise $T_{eff} = 2 \text{ mK}$. For each value of h we give: maximum energy adsorbed by the detector (for $\sin^4(\theta) = 1$), SNR and efficiency of detection for each detector, total weighted efficiency (having considered an isotropic distribution of the GW sources. Due to the angular weighting $\epsilon_{total} < \epsilon_A \epsilon_B$). The upper limit is given by $\frac{2.99}{\epsilon_{total}}$.

h 10^{18}	E_{abs} [mK]	Detector A		Detector B		ϵ_{tot} %	upper limit
		SNR_A	ϵ_A %	SNR_B	ϵ_B %		
2	31	31	0.88	15.5	0.32	0.12	16
3	70	70	1	35	0.93	0.56	4.3
4	126	126	1	63	1	0.76	3.6
10	783	783	1	392	1	0.95	3.1

In 1991 the EXPLORER data filtering was done differently from that described in this Introduction. For both ALLEGRO and EXPLORER the output of the electromechanical transducer was sent to lock-ins referred to the frequencies of the resonant modes. Then the outputs of the lock-ins (in phase and in quadrature) were filtered searching for delta-like signals and combined for obtaining the energy innovation, which we still indicate with E_f . In this case the probability to have an event (above threshold SNR_t) due to a signal with given SNR is obtained (see ref. [2, 12]) with the following equation:

$$probability(SNR) = \int_{SNR_t}^{\infty} e^{-(SNR+y)} I_0(2\sqrt{y \cdot SNR}) dy \quad (12)$$

Here $y = \frac{E_f}{T_{eff}}$, I_0 is the modified Bessel function of order zero, and the noise temperature T_{eff} is the average value of the energy innovation E_f .

We recall that in a time period of 123 days 70 coincidences were found with a background of 59.3. For extracting the events the ALLEGRO threshold was $SNR_t = 11.5$ with a noise temperature $T_{eff} \sim 8 \text{ mK}$. For EXPLORER the threshold was $SNR_t = 10$ also with $T_{eff} \sim 8 \text{ mK}$. Applying eq.6 we find an upper limit of $n_{GW} = 37$ over the 123 days.

According to the previous considerations we can raise the event threshold, say for EXPLORER, in order to reduce the number of accidentals. For instance, for a threshold $SNR_t = 24$ we get $n_c = 1$ and $\bar{n} = 0.74$, obtaining, from eq. 6, the value $n_{GW} = 4.8$.

Thus the procedure for calculating the upper limit with the Bayesian approach when we have data at various thresholds, including cases with n_c and \bar{n} different from zero, is the following.

Start with n_c and \bar{n} for various thresholds and use eq.6 for obtaining n_{GW} at each threshold. Calculate the upper limit for various values of h as shown in the previous section. For each h take as upper limit the smallest value among those obtained by varying

²⁾ The real data often show a non gaussian behaviour. In this case the efficiency differs from the theoretical one given by eq.4, but one can easily make use of the efficiency experimentally measured.

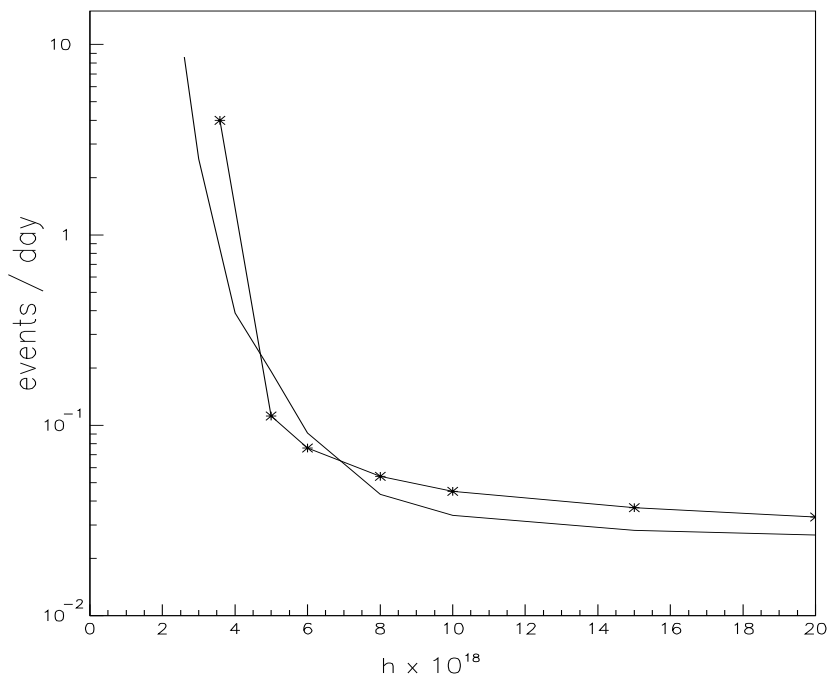


Figure 2: The asterisks indicate the upper limit calculated in [7]. The other line indicates the upper limit evaluated with the Bayesian approach.

the threshold. Clearly at large h values, when we get $n_c = 0$, the upper limit is, for the entire period of time, $n_{GW} = 2.99$.

The result is shown in fig.2 together with that obtained previously in [7]. It turns out that the two upper limits are similar.

The reason for this is due to the fact that in applying the previous algorithm [7] we started from an energy level higher than the largest energy of the detected (accidental) coincidences, thus obtaining, at this level ($n_c = \bar{n} = 0$) an upper limit of 3.09 very close to the value 2.99 obtained with the Bayesian approach. The similarity of the results at lower h values is accidental. In the previous algorithm the increase at lower h is due only to the increase of the number \bar{n} of accidentals. In the present algorithm the increase is due to the smaller efficiency of detection and to the increase in n_{GW} which roughly goes with $\sqrt{\bar{n}}$ (eq.8).

In spite of the similar numerical results, we believe that the procedure proposed here which does not extract the value of h from the energy levels of the accidental coincidences and it uses the Bayesian approach is methodologically more correct.

4 Discussion

The best upper limit which can be obtained with an array of M identical parallel detectors in M^{pl} coincidence cannot go below the value 2.99, because this is the upper limit [17] when one finds *zero* coincidences independently on the background.

The basic advantage in using many detectors comes from the fact that with many detectors it is easier to obtain $\bar{n} \sim 0$, and thus (in absence of GW) $n_c = 0$. Because of

the Poisson distributions, the average number of accidental coincidences for M detectors in a time window $\pm w$ is given by eq.5. On the time scale of 1 second ($w=1$ s) it turns out that $n_k \ll 1$. By increasing the number of detectors one obtains smaller values of \bar{n} , thus approaching the requirement to have $n_c = 0$ and then the lowest possible upper limit.

This is certainly true at large h values, where the detection efficiency for all detectors is unity. The result, as shown in fig.2, is a plateau. Instead it might be convenient at low h values to use the two most sensitive detectors, in order to have the largest possible efficiency of detection. The overall upper limit is then obtained by taking the smallest ones among the values of the various upper limit determinations.

The above procedure can be easily adjusted to the more general case of any distribution of the GW sources, and of non-parallel detectors.

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