REGULARITY RESULTS FOR FULLY NONLINEAR INTEGRO-DIFFERENTIAL OPERATORS WITH NONSYMMETRIC POSITIVE KERNELS

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Abstract. In this paper, we consider fully nonlinear integro-differential equations with possibly nonsymmetric kernels. We are able to find different versions of Alexandro-Backelman-Pucci estimate corresponding to three different cases: nonlinear equation with $0 < \sigma < 1$ (supercritical case) or $1 < \sigma < 2$ (subcritical case), and linear equation with $0 < \sigma < 2$ including $\sigma = 1$ (critical case). And we show a Harnack inequality, Hölder regularity, and $C^{1,\alpha}$-regularity of the solutions by obtaining decay estimates of their level sets in each cases.

1. Introduction

In this paper, we are going to consider the regularity of the visocosity solutions of *integro-differential operators* with possibly nonsymmetric kernel:

$$\mathcal{L}u(x) = \text{p.v.} \int_{\mathbb{R}^n} \mu(u, x, y) K(y)\, dy \tag{1.0.1}$$

where $\mu(u, x, y) = u(x + y) - u(x) - (\nabla u(x) \cdot y)\chi_{B_1}(y)$, which describes the infinitesimal generator of given purely jump processes, i.e. processes without diffusion or drift part [CS]. We refer the detailed definitions of notations to [KL]. Then we see that $\mathcal{L}u(x)$ is well-defined provided that $u \in C^{1,1}(x) \cap B(\mathbb{R}^n)$ where $B(\mathbb{R}^n)$ denotes *the family of all real-valued bounded functions defined on* $\mathbb{R}^n$. If K is symmetric (i.e. $K(-y) = K(y)$), then an odd function $\left[(\nabla u(x) \cdot y)\chi_{B_1}(y)\right]K(y)$ will be canceled in the integral, and so we have that

$$\mathcal{L}u(x) = \text{p.v.} \int_{\mathbb{R}^n} \left[u(x + y) + u(x - y) - 2u(x)\right] K(y)\, dy.$$

On the other hand, if K is not symmetric, the effect of $\left[(\nabla u(x) \cdot y)\chi_{B_1}(y)\right]K(y)$ persists and we can actually observe that the influence of this gradient term becomes stronger as we try to get an estimate in smaller regions.

Nonlinear integro-differential operators come from the stochastic control theory related with

$$\mathcal{I}u(x) = \sup_{\alpha} \mathcal{L}_\alpha u(x),$$

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or game theory associated with

$$\mathcal{I}u(x) = \inf_{\beta}\sup_{\alpha}\mathcal{L}_{\alpha\beta}u(x), \tag{1.0.2}$$

when the stochastic process is of Lèvy type allowing jumps; see [S, CS, KL]. Also an operator like $\mathcal{I}u(x) = \sup_{\alpha}\inf_{\beta}\mathcal{L}_{\alpha\beta}u(x)$ can be considered. Characteristic properties of these operators can easily be derived as follows;

$$\inf_{\alpha\beta}\mathcal{L}_{\alpha\beta}v(x) \le \mathcal{I}[u+v](x) - \mathcal{I}u(x) \le \sup_{\alpha\beta}\mathcal{L}_{\alpha\beta}v(x). \tag{1.0.3}$$

1.1. **operators.** In this section, we introduce a class of operators. All notations and the concepts of viscosity solution follows [KL] where a more general class of operators has been considered. Similar concepts can be found at [CS] for symmetric kernel.

For our purpose, we shall restrict our attention to the operators $\mathcal{L}$ where the measure $\mathfrak{m}$ is given by a positive kernel K which is not necessarily symmetric. That is to say, the operators $\mathcal{L}$ are given by

$$\mathcal{L}u(x) = \text{p.v.}\int_{\mathbb{R}^n}\mu(u,x,y)K(y)\,dy \tag{1.1.1}$$

where $\mu(u,x,y) = u(x+y) - u(x) - (\nabla u(x)\cdot y)\chi_{B_1}(y)$.

And we consider the class $\mathfrak{L}$ of the operators $\mathcal{L}$ associated with the measures $\mathfrak{m}$ given by positive kernels $K \in \mathcal{K}_0$ satisfying that

$$(2-\sigma)\frac{\lambda}{|y|^{n+\sigma}} \le K(y) \le (2-\sigma)\frac{\Lambda}{|y|^{n+\sigma}},\ 0<\sigma<2. \tag{1.1.2}$$

The maximal operator and the minimal operator with respect to $\mathfrak{L}$ are defined by

$$\mathcal{M}^+_{\mathfrak{L}}u(x) = \sup_{\mathcal{L}\in\mathfrak{L}}\mathcal{L}u(x) \ \text{ and } \ \mathcal{M}^-_{\mathfrak{L}}u(x) = \inf_{\mathcal{L}\in\mathfrak{L}}\mathcal{L}u(x). \tag{1.1.3}$$

For $x \in \Omega$ and a function $u : \mathbb{R}^n \to \mathbb{R}$ which is semicontinuous on $\overline{\Omega}$, we say that φ belongs to the function class $C^2_\Omega(u;x)^+$ (resp. $C^2_\Omega(u;x)^-$) and we write $\varphi \in C^2_\Omega(u;x)^+$ (resp. $\varphi \in C^2_\Omega(u;x)^-$) if there are an open neighborhood $U \subset \Omega$ of x and $\varphi \in C^2(U)$ such that $\varphi(x) = u(x)$ and $\varphi > u$ (resp. $\varphi < u$) on $U \setminus \{x\}$. We note that geometrically $u - \varphi$ having a local maximum at x in Ω is equivalent to $\varphi \in C^2_\Omega(u;x)^+$ and $u-\varphi$ having a local minimum at x in Ω is equivalent to $\varphi \in C^2_\Omega(u;x)^-$. For $x \in \Omega$ and $\varphi \in C^2_\Omega(u;x)^\pm$, we write

$$\mu(u,x,y;\nabla\varphi) = u(x+y) - u(x) - (\nabla\varphi(x)\cdot y)\chi_{B_1}(y),$$

and the expression for $\mathcal{L}_{\alpha\beta}\,u(x;\nabla\varphi)$ and $\mathcal{I}u(x;\nabla\varphi)$ may be written as

$$\mathcal{L}_{\alpha\beta}\,u(x;\nabla\varphi) = \int_{\mathbb{R}^n}\mu(u,x,y;\nabla\varphi)K_{\alpha\beta}(y)\,dy,$$
$$\mathcal{I}u(x;\nabla\varphi) = \inf_{\beta}\sup_{\alpha}\mathcal{L}_{\alpha\beta}\,u(x;\nabla\varphi),$$

where $K_{\alpha\beta} \in \mathcal{K}_0$. Then we see that $\mathcal{M}^-_{\mathfrak{L}}u(x;\nabla\varphi) \le \mathcal{I}u(x;\nabla\varphi) \le \mathcal{M}^+_{\mathfrak{L}}u(x;\nabla\varphi)$, and $\mathcal{M}^+_{\mathfrak{L}}u(x;\nabla\varphi)$ and $\mathcal{M}^-_{\mathfrak{L}}u(x;\nabla\varphi)$ have the following simple forms;

$$\begin{aligned}\mathcal{M}^+_{\mathfrak{L}}u(x;\nabla\varphi) &= (2-\sigma)\int_{\mathbb{R}^n} \frac{\Lambda\mu^+(u,x,y;\nabla\varphi) - \lambda\mu^-(u,x,y;\nabla\varphi)}{|y|^{n+\sigma}}\,dy,\\ \mathcal{M}^-_{\mathfrak{L}}u(x;\nabla\varphi) &= (2-\sigma)\int_{\mathbb{R}^n} \frac{\lambda\mu^+(u,x,y;\nabla\varphi) - \Lambda\mu^-(u,x,y;\nabla\varphi)}{|y|^{n+\sigma}}\,dy,\end{aligned} \tag{1.1.4}$$

where μ^+ and μ^- are given by

$$\mu^\pm(u,x,y;\nabla\varphi) = \max\{\pm\mu(u,x,y;\nabla\varphi), 0\}.$$

We note if $u \in C^{1,1}(x)$, then $\mathcal{I}u(x;\nabla\varphi) = \mathcal{I}u(x)$ and $\mathcal{M}^\pm_{\mathfrak{L}}u(x;\nabla\varphi) = \mathcal{M}^\pm_{\mathfrak{L}}u(x)$. We shall use these maximal and minimal operators to obtain regularity estimates.

Let $K(x) = \sup_\alpha K_\alpha(x)$ where K_α's are all the kernels of all operators in a class $\mathfrak{L}$. For any class $\mathfrak{L}$, we shall assume that

$$\int_{\mathbb{R}^n} (|y|^2 \wedge 1)\,K(y)\,dy < \infty. \tag{1.1.5}$$

Using the extremal operators, we provide a general definition of ellipticity for nonlocal equations. The following is a kind of operators of which the regularity result shall be obtained in this paper.

Definition 1.1.1. *Let $\mathfrak{L}$ be a class of linear integro-differential operators. Assume that* (1.1.5) *holds for $\mathfrak{L}$. Then we say that an operator $\mathcal{J}$ is* elliptic with respect to $\mathfrak{L}$, *if it satisfies the following properties:*

(*a*) *$\mathcal{J}u(x)$ is well-defined for any $u \in C^{1,1}[x] \cap B(\mathbb{R}^n)$.*
(*b*) *If $u \in C^{1,1}[\Omega] \cap B(\mathbb{R}^n)$ for an open $\Omega \subset \mathbb{R}^n$, then $\mathcal{J}u$ is continuous on Ω.*
(*c*) *If $u, v \in C^{1,1}[x] \cap B(\mathbb{R}^n)$, then we have that*

$$\mathcal{M}^-_{\mathfrak{L}}[u-v](x) \le \mathcal{J}u(x) - \mathcal{J}v(x) \le \mathcal{M}^+_{\mathfrak{L}}[u-v](x). \tag{1.1.6}$$

The concept of viscosity solutions and its comparison principle and stability properties can be found in [CS] for symmetric kernels and in [KL] for possibly nonsymmetric kernels. Kim and Lee [KL] considered much larger class of operators but prove the regularity of viscosity solutions only for $1 < \sigma < 2$.

1.2. **Main equation.** The natural Dirichlet problem for such a nonlocal operator $\mathcal{I}$. Let Ω be an open domain in $\mathbb{R}^n$. Given a function g defined on $\mathbb{R}^n \setminus \Omega$, we want to find a function u such that

$$\begin{cases}\mathcal{I}u(x) = 0 & \text{for any } x \in \Omega,\\ u(x) = g(x) & \text{for } x \in \mathbb{R}^n \setminus \Omega.\end{cases}$$

Note that the boundary condition is given not only on $\partial\Omega$ but also on the whole complement of Ω. This is because of the nonlocal character of the operator $\mathcal{I}$. From the stochastic point of view, it corresponds to the fact that a discontinuous Lèvy process can exit the domain Ω for the first time jumping to any point in $\mathbb{R}^n \setminus \Omega$.

In this paper, we shall concentrate mainly upon the regularity properties of viscosity solutions to an equation $\mathcal{I}u(x) = 0$. We shall briefly give a very general comparison principle from which existence of the solutions can be obtained in smooth domains. Since kernels of integro-differential operators are comparable to the kernel of the fractional Laplace operator $-(-\Delta)^{\sigma/2}$, the theory we want to develop can be understood as a theory of viscosity solutions for fully nonlinear operators of fractional order.

The differences between local and nonlocal operators have been discussed at [KL].

1.3. **Known results and Key Observations.** There are some known results about Harnack inequalities and Hölder estimates for integro-differential operators with positive symmetric kernels (see [J] for analytical proofs and [BBC], [BK1], [BK2],[BL], [KS], [SV] for probabilistic proofs). The estimates in all these previous results blow up as the index σ of the operator approaches 2. In this respect, they do not generalize to elliptic partial differential equations. However there is some known result on regularity results for fully nonlinear integro-differential equations associated with nonlinear integro-differential operators with positive symmetric kernels which remain uniform as the index σ of the operator approaches 2 (see [CS]). Therefore these results make the theory of integro-differential operators and elliptic differential operators become somewhat unified. For nonlinear integro-differential operators with possibly nonsymmetric kernels, the authors introduced larger classe of operators and proved Harnack inequalities and Hölder estimates when $1 < \sigma < 2$ (see [KL]).

In this paper, we are going to consider nonlinear integro-differential operators with possibly nonsymmetric kernels, when $0 < \sigma < 2$.

Throughout this paper we would like to briefly present the necessary definitions and then prove some regularity estimates. Our results in this paper are

• A nonlocal version of the Alexandroff-Backelman-Pucci estimate for fully nonlinear integro-differential equations.

• A Harnack inequality, Hölder regularity and an interior $C^{1,\alpha}$-regularity result for certain fully nonlinear integro-differential equations.

Key observations are the following:

• For the nonsymmetric case, $K(y)$ and $K(-y)$ can be chosen any of $\lambda/|y|^{n+\sigma}$ or $\Lambda/|y|^{n+\sigma}$. Therefore there could be an extra term $\displaystyle\int_{\mathbb{R}^n} \frac{\left|(\nabla u(x)\cdot y)\chi_{B_1}(y)\right|}{|y|^{n+\sigma}}dy$.

• The equation is not scaling invariant due to $|\chi_{B_1}(y)|$.

• Somehow the equation has a drift term, not only the diffusion term. The case $1 < \sigma < 2$ and the case $0 < \sigma \le 1$ require different technique due to the difference of the blow rate as $|y|$ approaches to zero and the decay rate as $|y|$ approaches to infinity. When $1 < \sigma < 2$, a controllable decay rate of kernel allows Hölder regularities in a larger class, which is invariant under an one-sided scaling i.e. if u is a solution of the homogeneous equation, then so is $u_\epsilon(x) = \epsilon^{-\sigma}u(\epsilon x)$ for $0 < \epsilon \leq 1$. Critical case ($\sigma = 1$) and supercritical case ($0 < \sigma < 1$) have been studied in [BBC] with different techniques due to the slow decay rate of the kernel as $|x| \to \infty$.

1.4. **Outline of Paper.** In Section 2, we show various nonlocal versions of the Alexandroff-Backelman-Pucci estimate to handle the difficulties caused by the gradient effect. It has different orders at subcritical, cirtical and supercritical cases. In Section 3, we construct a special function and apply A-B-P estimates to obtain the decay estimates of upper level sets which is essential in proving Hölder estimates in Section 4.2.

In Section 4, we prove a Harnack inequality which plays an important role in analysis. And then the Hölder estimates and an interior $C^{1,\alpha}$-estimates come from the arguments at [CS, KL].

2. A nonlocal Alexandroff-Bakelman-Pucci estimate

The Alexandroff-Bakelman-Pucci (A-B-P) estimate plays an important role in Krylov and Sofonov theory [KS] on Harnack inequality for linear uniformly elliptic equations with measurable coefficients. The concept of viscosity solution is given pointwise through touching test function; see [KL]. A-B-P estimate tells us that the maximum value is controlled by an integral quantity of the source term on the contact set, which will give us key lemma (Lemma 2.1.1) saying that the pointwise value of nonnegative function gives the lower bound of the measure of lower level set. We employ measure theoretical version of A-B-P estimate introduced at [CS] and extended to nonsymmetric case at [KL].

New A-B-P estimates below are two main differences from the arguments at [CS, KL].

- The operators considered at [CS, KL] are scaling invariant, but (1.1.1) doesn't have such property due to $\chi_{B_1}(y)$ in the gradient term. So we keep the size of the domain B_R at the following estimates.
- The control of bad set, Lemma 2.1.1, deteriorates as $R \to 0$ since $R^{\sigma-2}J_\sigma(R)$ goes to ∞ as $R \to 0$. A-B-P estimate will be used to prove key Lemma 3.2.1 where we have an extra term R to subdue the blow-up rate. But we have still $R \times R^{\sigma-2}J_\sigma(R) \approx R^{\sigma-1}$ (for $0 < \sigma < 1$) and $-log(R)$ (for $\sigma = 1$) which blows up when $0 < \sigma \leq 1$. So we introduced a different version of A-B-P estimate (Lemma 2.2.1) for $0 < \sigma < 1$ where we have better control of gradient term due to the intergrability of the kernel near the origin. For the critical case (i.e. $\sigma = 1$), we consider the linear equation, where the coefficent of the gradient effect, b_R, has a fixed direction on each small while we still doesn't know how to control the direction of b_R for the fully nonlinear case. Such consideration work for linear equation even $0 < \sigma < 2$. The corresponding A-B-P estimate is at Lemma 2.3.1.

Let $R \in (0, R_0]$ for some $R_0 \in (0,1)$ (in fact, the existence of R_0 was mentioned in [KL]) and let $u : \mathbb{R}^n \to \mathbb{R}$ be a function which is not positive outside the ball $B_{R/2}$ and is upper semicontinuous on $\overline{B}_R$. We consider its *concave envelope* Γ in B_{2R} defined as

$$\Gamma(x) = \begin{cases} \inf\{p(x) : p \in \Pi,\, p > u^+ \text{ in } B_{2R}\} & \text{in } B_{2R}, \\ 0 & \text{in } \mathbb{R}^n \setminus B_{2R}, \end{cases}$$

where Π is the family of all the hyperplanes in $\mathbb{R}^n$. Also we denote the *contact set* of u and Γ in B_R by $\mathcal{C}(u, \Gamma, B_R) = \{y \in B_R : u(y) = \Gamma(y)\}$ and set $\mathcal{C}^+(u, \Gamma, B_R; b) = \mathcal{C}(u, \Gamma, B_R) \cap \{y \in B_R : b \cdot \nabla\Gamma(y) \geq 0\}$ and $\mathcal{C}^-(u, \Gamma, B_R; b) = \mathcal{C}(u, \Gamma, B_R) \cap \{y \in B_R : -b \cdot \nabla\Gamma(y) \geq 0\}$ for $b \in \mathbb{R}^n$.

2.1. **A-B-P estimate with blow-up rate.**

Lemma 2.1.1. *Let $0 < \sigma < 2$ and $0 < R \leq R_0$. Let $u \leq 0$ in $\mathbb{R}^n \setminus B_R$ and let Γ be its concave envelope in B_{2R}. If $u \in \mathrm{B}(\mathbb{R}^n)$ is a viscosity subsolution to $\mathcal{M}^+_{\mathfrak{L}} u = -f$ on B_R where $f : \mathbb{R}^n \to \mathbb{R}$ is a function with $f > 0$ on $\mathcal{C}(u, \Gamma, B_R)$, then there exists some constant $C > 0$ depending only on n, λ and Λ (but not on σ) such that for any $x \in \mathcal{C}(u, \Gamma, B_R)$ and any $M > 0$ there is some $k \in \mathbb{N} \cup \{0\}$ such that*

$$\left|\overline{R}_k(x)\right| \leq C \frac{R^{\sigma-2}(f(x) + J_\sigma(R)|\nabla\Gamma(x)|)}{M} |R_k(x)| \tag{2.1.1}$$

where $\overline{R}_k(x) = \{y \in R_k(x) : \mu^-(u, x, y; \nabla\Gamma) \geq M_0 r_k^2\}$ and $R_k(x) = B_{r_k}(x) \setminus B_{r_{k+1}}(x)$ for $r_k = \varrho_0 2^{-\frac{1}{2-\sigma} - k} R$, $\varrho_0 = 1/(16\sqrt{n})$ and $J_\sigma(R)$ is $\frac{1}{1-\sigma}(1 - R^{1-\sigma})$ for $\sigma \in (0,1) \cup (1,2)$ and $-\log(R)$ for $\sigma = 1$. Here, $\nabla\Gamma(x)$ denotes any element of the superdifferential $\partial\Gamma(x)$ of Γ at x.

Remark. We note that $\nabla\Gamma(x) = \nabla u(x)$ for $x \in B_R$ if Γ and u are differentiable at $x \in B_R$. In this case, $\partial\Gamma(x)$ is a singleton set with element $\nabla u(x)$.

[*Proof of Lemma 2.1.1*] Let $0 < \sigma < 2$ and $0 < R \leq R_0$. Take any $x \in \mathcal{C}(u, \Gamma, B_R)$. Since u can be touched by a hyperplane from above at x, we see that $\nabla\varphi(x) = \nabla\Gamma(x)$ for some $\varphi \in \mathrm{C}^2_{B_R}(u; x)^+$. Thus $\mathcal{M}^+_{\mathfrak{L}} u(x; \nabla\Gamma)$ is well-defined and we have that

$$\mathcal{M}^+_{\mathfrak{L}} u(x; \nabla\Gamma) = (2 - \sigma) \int_{\mathbb{R}^n} \frac{\Lambda\mu^+(u, x, y; \nabla\Gamma) - \lambda\mu^-(u, x, y; \nabla\Gamma)}{|y|^{n+\sigma}}\, dy.$$

We note that $\mu(u, x, y; \nabla\Gamma) = u(x+y) - u(x) - (\nabla\Gamma(x) \cdot y)\chi_{B_1}(y) \leq 0$ for any $y \in B_R$ by the definition of concave envelope of u in B_{2R}. Since $\mu^+(u, x, y; \nabla\Gamma) \leq |\nabla\Gamma(x)||y|\chi_{B_1}(y)$ for any $y \in \mathbb{R}^n \setminus B_R$, we have that

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{\Lambda\mu^+(u, x, y; \nabla\Gamma)}{|y|^{n+\sigma}}\, dy &\leq \int_{B_1 \setminus B_R} \frac{\Lambda|\nabla\Gamma(x)||y|}{|y|^{n+\sigma}}\, dy \\ &= \omega_n \Lambda\, J_\sigma(R)|\nabla\Gamma(x)| \end{aligned} \tag{2.1.2}$$

where ω_n denotes the surface area of S^{n-1} and

$$J_\sigma(R) = \begin{cases} \frac{1}{1-\sigma}\left(1 - R^{1-\sigma}\right) & \text{for } \sigma \in (0,1)\cup(1,2), \\ -\log(R) & \text{for } \sigma = 1. \end{cases} \tag{2.1.3}$$

Here we see that $|J_\sigma(R)|$ is finite for $0 < \sigma < 2$. Thus it follows from simple calculation that

$$\begin{aligned} -f(x) &\le \mathcal{M}^+_{\mathfrak{L}} u(x;\nabla\Gamma) \\ &= (2-\sigma)\Big(\int_{\mathbb{R}^n} \frac{-\lambda\mu^-(u,x,y;\nabla\Gamma)}{|y|^{n+\sigma}}\,dy + \int_{\mathbb{R}^n} \frac{\Lambda\mu^+(u,x,y;\nabla\Gamma)}{|y|^{n+\sigma}}\,dy\Big) \\ &\le (2-\sigma)\int_{B_{r_0}(x)} \frac{-\lambda\mu^-(u,x,y;\nabla\Gamma)}{|y|^{n+\sigma}}\,dy + (2-\sigma)\omega_n\Lambda\, J_\sigma(R)\,|\nabla\Gamma(x)| \end{aligned}$$

for any $x \in \mathcal{C}(u,\Gamma,B_R)$, where $r_0 = \varrho_0 2^{-\frac{1}{2-\sigma}}R$. Splitting the above integral in the rings $R_k(x)$, we have that

$$f(x) \ge (2-\sigma)\lambda \sum_{k=0}^{\infty} \int_{R_k(x)} \frac{\mu^-(u,x,y;\nabla\Gamma)}{|y|^{n+\sigma}}\,dy - (2-\sigma)\omega_n\Lambda J_\sigma(R)|\nabla\Gamma(x)|. \tag{2.1.4}$$

Assume that the conclusion (2.1.1) does not hold, i.e. for any $C > 0$ there are some $x_0 \in \mathcal{C}(u,\Gamma,B_R)$ and $M_0 > 0$ such that

$$\left|\overline{R}_k(x_0)\right| > C\frac{R^{\sigma-2}\Big(f(x_0) + J_\sigma(R)|\nabla\Gamma(x_0)|\Big)}{M_0}|R_k(x_0)|$$

for all $k \in \mathbb{N}\cup\{0\}$. Since $-\mu \le \mu^-$ and $(2-\sigma)\frac{1}{1-2^{-(2-\sigma)}}$ remains bounded below for $\sigma \in (0,1]$, it thus follows from (2.1.4) that

$$\begin{aligned} \frac{f(x_0)}{2-\sigma} &\ge \lambda \sum_{k=0}^{\infty} \int_{R_k(x_0)} \frac{-\mu(u,x_0,y;\nabla\Gamma(x))}{|y|^{n+\sigma}}\,dy - \omega_n\Lambda J_\sigma(R)|\nabla\Gamma(x_0)| \\ &\ge c\sum_{k=0}^{\infty} M_0\,\frac{r_k^2}{r_k^\sigma}\,CR^{\sigma-2}\,\frac{f(x_0)+J_\sigma(R)|\nabla\Gamma(x_0)|}{M_0} - \omega_n\Lambda J_\sigma(R)|\nabla\Gamma(x_0)|. \end{aligned} \tag{2.1.5}$$

Thus this implies that

$$\begin{aligned} f(x_0) + (2-\sigma)\omega_n\Lambda J_\sigma(R)|\nabla\Gamma(x_0)| &\ge \frac{c\rho_0^2}{1-2^{-(2-\sigma)}}C(f(x_0) + J_\sigma(R)|\nabla\Gamma(x_0)|) \\ &\ge C(f(x_0) + (2-\sigma)J_\sigma(R)|\nabla\Gamma(x_0)|) \end{aligned}$$

for any $C > 0$. Taking C large enough, we obtain a contradiction. Hence we are done. □

Remark. Lemma 2.1.1 would hold for any particular choice of ϱ_0 by modifying C accordingly. The particular choice $\varrho_0 = 1/(16\sqrt{n})$ is convenient for the proofs in Section 3.

Lemma 2.1.2. [CS] *Let Γ be a concave function on $B_r(x)$ where $x \in \mathbb{R}^n$ and let $h > 0$. If $|\{y \in S_r(x) : \Gamma(y) < \Gamma(x) + (y-x)\cdot\nabla\Gamma(x) - h\}| \le \epsilon\,|S_r(x)|$ for any small $\epsilon > 0$ where $S_r(x) = B_r(x)\setminus B_{r/2}(x)$, then we have $\Gamma(y) \ge \Gamma(x) + (y-x)\cdot\nabla\Gamma(x) - h$ for any $y \in B_{r/2}(x)$.*

Corollary 2.1.3. *For any $\epsilon > 0$, there is a constant $C > 0$ such that for any function u with the same hypothesis as Lemma 2.1.1, there is some $r \in (0, \varrho_0 2^{-\frac{1}{2-\sigma}}R)$ such that*

$$\frac{|\{y \in S_r(x) : u(y) < u(x) + (y-x)\cdot\nabla\Gamma(x) - C\,R^{\sigma-2}(f(x) + J_\sigma(R)|\nabla\Gamma(x)|)r^2\}|}{|S_r(x)|} \le \epsilon,$$

$$\int_{\overline{Q}} g_\eta(\nabla\Gamma(y))\det[D^2\Gamma(y)]^-\,dy \le CR^{n(\sigma-2)}\sup_{y\in\overline{Q}}\Big(J_\sigma(R)^n + \eta^{-n}|f(y)|^n\Big)\,|Q|$$

for any $\eta > 0$ and any cube $Q \subset B_{r/4}(x)$ with diameter d such that $x \in \overline{Q}$ and $r/4 < d < r/2$, where $\varrho_0 = 1/(16\sqrt{n})$ and $g_\eta(z) = (|z|^{n/(n-1)} + \eta^{n/(n-1)})^{1-n}$.

Proof. The first part can be obtained by taking $M = CR^{\sigma-2}(f(x)+J_\sigma(R)|\nabla\Gamma(x)|)/\epsilon$ in Lemma 2.1.1. Also the second part follows as a consequence of Lemma 2.1.2 and concavity;

$$\begin{aligned}\det[D^2\Gamma(x)]^- &\le C(R^{\sigma-2}f(x) + R^{\sigma-2}J_\sigma(R)|\nabla\Gamma(x)|)^n\\ &\le 4^n C\frac{R^{n(\sigma-2)}J_\sigma(R)^n + \eta^{-n}R^{n(\sigma-2)}|f(x)|^n}{g_\eta(\nabla\Gamma(x))}.\end{aligned}$$

Thus we have $g_\eta(\nabla\Gamma(x))\det[D^2\Gamma(x)]^- \le 4^n C(R^{n(\sigma-2)}J_\sigma(R)^n + \eta^{-n}R^{n(\sigma-2)}|f(x)|^n)$.

Take any $y \in C(u,\Gamma,B_R)\cap Q$ where $Q \subset B_{r/4}(x)$ is a cube with diameter d such that $x \in \overline{Q}$ and $r/4 < d < r/2$. Similarly to the above, we can obtain that $g_\eta(\nabla\Gamma(\cdot))\det[D^2\Gamma(\cdot)]^- \le 4^n C(R^{n(\sigma-2)}J_\sigma(R)^n + \eta^{-n}R^{n(\sigma-2)}|f(\cdot)|^n)$ a.e. on Q because $\det[D^2\Gamma(\cdot)]^- = 0$ a.e. on $Q\setminus C(u,\Gamma,B_R)$ as in [CC]. Hence this implies the second part. □

We obtain a nonlocal version of Alexandroff-Bakelman-Pucci estimate in the following theorem.

Theorem 2.1.4. *Let u and Γ be functions as in Lemma 2.1.1. Then there exist a finite family $\{Q_j\}_{j=1}^m$ of open cubes Q_j with diameters d_j such that*

(*a*) *Any two cubes Q_i and Q_j do not intersect,* (*b*) $C(u,\Gamma,B_R) \subset \bigcup_{j=1}^m \overline{Q}_j$,

(*c*) $C(u,\Gamma,B_R)\cap\overline{Q}_j \ne \phi$ *for any Q_j,* (*d*) $d_j \le \varrho_0 2^{-\frac{1}{2-\sigma}}R$ *where* $\varrho_0 = 1/(16\sqrt{n})$,

(*e*) $\int_{\overline{Q}_j} g_\eta(\nabla\Gamma(y))\det(D^2\Gamma(y))^-\,dy \le CR^{n(\sigma-2)}(\sup_{\overline{Q}_j}(J_\sigma(R)^n + \eta^{-n}|f|^n)|Q_j|$,

(*f*) $|\{y \in 4\sqrt{n}\,Q_j : u(y) \ge \Gamma(y) - CR^{(\sigma-2)}(\sup_{\overline{Q}_j}(f + J_\sigma(R)|\nabla\Gamma|)d_j^2\}| \ge \eta_0|Q_j|$,

where the constants $C > 0$ and $\eta_0 > 0$ depend on n, Λ and λ (but not on σ).

Proof. In order to obtain such a family, we start by covering B_R with a tiling of cubes of diameter $\varrho_0 2^{-\frac{1}{2-\sigma}}R$. Then discard all those that do not intersect $C(u,\Gamma,B_R)$. Whenever a cube does not satisfy (e) and (f), we split it into 2^n

cubes of half diameter and discard those whose closure does not intersect $C(u,\Gamma,B_R)$. Now our goal is to prove that eventually all cubes satisfy (e) and (f) and this process ends after a finite number of steps.

Assume that the process does not finish in a finite number of steps. Then we can have an infinite nested sequence of cubes. The intersection of their closures will be a point $\hat{x}$. So we may choose a sequence $\{x_k\} \subset C(u,\Gamma,B_R)$ with $\lim_{k\to\infty} x_k = \hat{x}$. Since $u(x_k) = \Gamma(x_k)$ for all $k \in \mathbb{N}$, by the upper semicontinuity of u on $\overline{B}_R$ we have that $\Gamma(\hat{x}) = \limsup_{k\to\infty} u(x_k) \le u(\hat{x})$. Also we have that $u(\hat{x}) \le \Gamma(\hat{x})$ because $u \le \Gamma$ on B_{2R} by the definition of the concave envelope Γ in B_{2R}. Thus we obtain that $u(\hat{x}) = \Gamma(\hat{x})$. We will now get a contradiction by showing that eventually one of these cubes containing $\hat{x}$ will not split.

Take any $\epsilon > 0$. Then by Corollary 2.1.3 there is a radius $r \in (0, \varrho_0 2^{-\frac{1}{2-\sigma}} R)$ such that

$$\frac{|\{y \in S_r(\hat{x}) : u(y) < u(\hat{x}) + (y-\hat{x})\cdot\nabla\Gamma(\hat{x}) - CR^{\sigma-2}(f(\hat{x}) + J_\sigma(R)|\nabla\Gamma(\hat{x})|)r^2\}|}{|S_r(\hat{x})|} \le \epsilon,$$

$$\int_{\overline{Q}_j} g_\eta(\nabla\Gamma(y))\det[D^2\Gamma(y)]^- \, dy \le CR^{n(\sigma-2)} \sup_{y\in\overline{Q}_j}(J_\sigma(R)^n + \eta^{-n}|f(y)|^n)\,|Q_j|$$

for any $\eta > 0$ and a cube $Q_j \subset B_{r/4}(x)$ with diameter d_j such that $x \in \overline{Q}_j$ and $r/4 < d_j < r/2$. So we easily see that $\overline{Q}_j \subset B_{r/2}(\hat{x})$ and $B_r(\hat{x}) \subset 4\sqrt{n}\,Q_j$. We recall that $\Gamma(y) \le u(\hat{x}) + (y-\hat{x})\cdot\nabla\Gamma(\hat{x})$ for any $y \in B_{2R}$ because Γ is concave on B_{2R} and $\Gamma(\hat{x}) = u(\hat{x})$. Since d_j is comparable to r, it thus follows that

$$\begin{aligned}
&\Big|\{y \in 4\sqrt{n}\,Q_j : u(y) \ge \Gamma(y) - CR^{\sigma-2}\sup_{\overline{Q}_j}(f + J_\sigma(R)|\nabla\Gamma|)d_j^2\}\Big| \\
&\ge \Big|\{y \in 4\sqrt{n}\,Q_j : u(y) \ge u(\hat{x}) + (y-\hat{x})\cdot\nabla\Gamma(\hat{x}) - CR^{\sigma-2}(f(\hat{x}) + J_\sigma(R)|\nabla\Gamma(\hat{x})|)r^2\}\Big| \\
&\ge (1-\epsilon)\big|S_r(\hat{x})\big| \ge \eta_0|Q_j|.
\end{aligned}$$

Thus we proved (f). Moreover, (e) holds for Q_j because $\overline{Q}_j \subset B_{r/2}(\hat{x})$ and $B_r(\hat{x}) \subset 4\sqrt{n}\,Q_j$. Hence the cube Q_j would not split and the process must stop there. □

2.2. A-B-P estimate for $0 < \sigma < 1$.

Lemma 2.2.1. *Let $0 < \sigma < 1$ and $0 < R \le R_0$. Let $u \le 0$ in $\mathbb{R}^n \setminus B_R$ and let Γ be its concave envelope in B_{2R}. If $u \in \mathrm{B}(\mathbb{R}^n)$ is a viscosity subsolution to $\mathcal{M}^+_{\mathfrak{L}} u = -f$ on B_R where $f : \mathbb{R}^n \to \mathbb{R}$ is a function with $f > 0$ on $C(u,\Gamma,B_R)$, then there exist constants $C > 0$ depending only on n, λ and Λ (but not on σ) , and a vector b depending on n, λ, Λ and σ such that for any $x \in C^+(u,\Gamma,B_R;b)$ and any $M > 0$ there is some $k \in \mathbb{N}\cup\{0\}$ such that*

$$\big|\widetilde{R}_k(x)\big| \le C\frac{(R^{\sigma-2}f(x) + R^{-1}|\nabla\Gamma(x)|)}{M}|R_k(x)| \tag{2.2.1}$$

where $\widetilde{R}_k(x) = \{y \in R_k(x) : \mu^-(u, x, y; \nabla\Gamma) \geq M_0 r_k^2\}$ *and* $R_k(x) = B_{r_k}(x) \setminus B_{r_{k+1}}(x)$ *for* $r_k = \varrho_0 2^{-\frac{1}{2-\sigma}-k} R$, $\varrho_0 = 1/(16\sqrt{n})$. *Here,* $\nabla\Gamma(x)$ *denotes any element of the superdifferential* $\partial\Gamma(x)$ *of* Γ *at* x.

Proof. Let $0 < \sigma < 1$ and $0 < R \leq R_0$. We define the vector-valued function $b : B_R \to \mathbb{R}^n$ by

$$b(x) = -(2-\sigma) \int_{B_1} \frac{y(\Lambda\chi_{\mu>0} + \lambda\chi_{\mu\leq 0})}{|y|^{n+\sigma}}\, dy.$$

Then this function b is well defined for $0 < \sigma < 1$ and there exists some $x_1 \in B_R$ so that $|C^+(u, \Gamma, B_R; b)| \geq |C(u, \Gamma, B_R)|/2$ where $b = b(x_1)$.

Take any $x \in C^+(u, \Gamma, B_R; b)$. We now have that

$$\mathcal{M}^+_{\mathfrak{L}} u(x; \nabla\Gamma) = (2-\sigma) \int_{\mathbb{R}^n} \frac{\Lambda\mu^+(u, x, y; \nabla\Gamma) - \lambda\mu^-(u, x, y; \nabla\Gamma)}{|y|^{n+\sigma}}\, dy.$$

Set $\mu_R(u, x, y; \nabla\Gamma) = u(x+y) - u(x) - (\nabla\Gamma(x) \cdot y)\, \chi_{B_R}(y)$ and then define $\mu_R^\pm$ and $\mathcal{M}^\pm_{\mathfrak{L},R} u(x; \nabla\Gamma)$ by replacing μ by μ_R in the definition $\mathcal{M}^\pm_{\mathfrak{L}} u(x; \nabla\Gamma)$. We set

$$b_R(x) = (2-\sigma) \int_{B_1 \setminus B_R} \frac{y(\Lambda\chi_{\mu>0} + \lambda\chi_{\mu\leq 0})}{|y|^{n+\sigma}}\, dy.$$

Then we easily obtain that

$$|b_R(x) - b| \leq |b_R(x) - b(x)| + |b(x) - b| \leq (2-\sigma) C R^{1-\sigma}.$$

Then we have that

$$\begin{aligned}
\mathcal{M}^+_{\mathfrak{L}} u(x; \nabla\Gamma) &= \mathcal{M}^+_{\mathfrak{L},R} u(x; \nabla\Gamma) + (2-\sigma) \int_{B_1 \setminus B_R} \frac{\Lambda(\mu^+ - \mu_R^+) - \lambda(\mu^- - \mu_R^-)}{|y|^{n+\sigma}}\, dy \\
&\leq \mathcal{M}^+_{\mathfrak{L},R} u(x; \nabla\Gamma) - b_R(x) \cdot \nabla\Gamma(x) \\
&\leq \mathcal{M}^+_{\mathfrak{L},R} u(x; \nabla\Gamma) - b \cdot \nabla\Gamma(x) + (2-\sigma) C R^{1-\sigma} |\nabla\Gamma(x)| \\
&\leq \mathcal{M}^+_{\mathfrak{L},R} u(x; \nabla\Gamma) + (2-\sigma) C R^{1-\sigma} |\nabla\Gamma(x)| \\
&= (2-\sigma) \int_{\mathbb{R}^n} \frac{-\lambda\mu_R^-(u, x, y; \nabla\Gamma)}{|y|^{n+\sigma}}\, dy + (2-\sigma) C R^{1-\sigma} |\nabla\Gamma(x)|
\end{aligned}$$

because $b \cdot \nabla\Gamma(x) \geq 0$ from the assumption and $\mu_R(u, x, \cdot\,; \Gamma) \leq 0$ on B_R. The conclusion comes from similar arguments as Lemma 2.1.1. □

Now we have the following Corollary as Section (2.1).

Corollary 2.2.2. *Let* u *and* Γ *be functions as in Lemma 2.2.1 and* $0 < \sigma < 1$. *Then there exist a finite family* $\{Q_j\}_{j=1}^m$ *of open cubes* Q_j *with diameters* d_j *such that*

(*a*) *Any two cubes* Q_i *and* Q_j *do not intersect,* (*b*) $C^+(u, \Gamma, B_R; b) \subset \bigcup_{j=1}^m \overline{Q}_j$,

(*c*) $C^+(u, \Gamma, B_R; b) \cap \overline{Q}_j \neq \phi$ *for any* Q_j, (*d*) $d_j \leq \varrho_0 2^{-\frac{1}{2-\sigma}} R$ *for* $\varrho_0 = 1/(16\sqrt{n})$,

(*e*) $\int_{C^+(u,\Gamma,B_R;b)\cap Q_j} g_\eta(\nabla\Gamma(y)) \det(D^2\Gamma(y))^-\, dy \leq C(\sup_{\overline{Q}_j}(R^{-n} + \eta^{-n} R^{n(\sigma-2)} |f|^n)|Q_j|$, (*f*) $|\{y \in C^+(u,\Gamma,B_R;b) \cap 4\sqrt{n}\,Q_j : u(y) \ge \Gamma(y) - C(\sup_{\overline{Q}_j}(R^{(\sigma-2)}f + R^{-1}|\nabla\Gamma|)d_j^2\}|$

$\ge \eta_0|C^+(u,\Gamma,B_R;b) \cap Q_j|$,

where the constants $C > 0$ and $\eta_0 > 0$ depend on n, Λ and λ (but not on σ) and and b is some vector depending on n, λ, Λ and σ.

2.3. **A-B-P estimate for Linear Equation:** $0 < \sigma < 2$.

Lemma 2.3.1. *Let $0 < \sigma < 1$ and $0 < R \le R_0$. Let $u \le 0$ in $\mathbb{R}^n \setminus B_R$ and let Γ be its concave envelope in B_{2R}. If $u \in \mathrm{B}(\mathbb{R}^n)$ is a viscosity subsolution to $\mathcal{L}u = -f$ on B_R where $f : \mathbb{R}^n \to \mathbb{R}$ is a function with $f > 0$ on $C(u,\Gamma,B_R)$, then there exist constants $C > 0$ depending only on n, λ and Λ (but not on σ) , and a vector b depending on n, λ, Λ and σ such that for any $x \in C^+(u,\Gamma,B_R;b_R)$ and $M > 0$ there is some $k \in \mathbb{N} \cup \{0\}$ such that*

$$\left|\overline{R}_k(x)\right| \le C\frac{R^{\sigma-2}f(x)}{M}\,|R_k(x)| \tag{2.3.1}$$

where $b_R = (2-\sigma)\int_{B_1\setminus B_R} y\,K(y)\,dy$, $\overline{R}_k(x) = \{y \in R_k(x) : \mu^-(u,x,y;\nabla\Gamma) \ge M_0 r_k^2\}$ and $R_k(x) = B_{r_k}(x) \setminus B_{r_{k+1}}(x)$ for $r_k = \varrho_0 2^{-\frac{1}{2-\sigma}-k}R$ and $\varrho_0 = 1/(16\sqrt{n})$. Here, $\nabla\Gamma(x)$ denotes any element of the superdifferential $\partial\Gamma(x)$ of Γ at x.

Proof. Take any $x \in C^+(u,\Gamma,B_R;b_R)$. By Lemma 2.1.1 and Lemma 2.2.1 we have that

$$\mathcal{L}u(x;\nabla\Gamma) = \mathcal{L}_R u(x;\nabla\Gamma) - b_R \cdot \nabla\Gamma(x)$$

where $\mathcal{L}_R$ is the operator replaced μ by μ_R in the definition of $\mathcal{L}$ and

$$b_R = (2-\sigma)\int_{B_1\setminus B_R} y\,K(y)\,dy.$$

If $b_R = 0$, then we doesn't have the error term $R^{1-\sigma}|\nabla\Gamma(x)|$. If $b_R \ne 0$, then $\mathcal{L}u(x;\nabla\Gamma) \le \mathcal{L}_R u(x;\nabla\Gamma)$. Hence we conclude that

$$\mathcal{L}u(x;\nabla\Gamma) \le \mathcal{L}_R u(x;\nabla\Gamma)$$

for any $x \in C^+(u,\Gamma,B_R;b_R)$. The same argument as Lemma 2.2.1 gives us the conclusion. □

Now we have the following Corollary as Section 2.1, 2.2.

Corollary 2.3.2. *Let u and Γ be functions as in Lemma 2.3.1. Then there exist a finite family $\{Q_j\}_{j=1}^m$ of open cubes Q_j with diameters d_j such that*

(*a*) *Any two cubes Q_i and Q_j do not intersect*, (*b*) $C^+(u,\Gamma,B_R;b_R) \subset \bigcup_{j=1}^m \overline{Q}_j$,

(*c*) $C^+(u,\Gamma,B_R;b_R) \cap \overline{Q}_j \ne \phi$ *for any Q_j*, (*d*) $d_j \le \varrho_0 2^{-\frac{1}{2-\sigma}}R$ *for* $\varrho_0 = 1/(16\sqrt{n})$,

(*e*) $\int_{\overline{Q}_j} g_\eta(\nabla\Gamma(y))\det[D^2\Gamma(y)]^-\,dy \le CR^{n(\sigma-2)}\sup_{\overline{Q}_j}|f|^n|Q_j|$,

(*f*) $|\{y \in 4\sqrt{n}\,Q_j : u(y) \ge \Gamma(y) - CR^{(\sigma-2)}(\sup_{\overline{Q}_j} f)d_j^2\}| \ge \eta_0|Q_j|$,

where the constants $C > 0$ and $\eta_0 > 0$ depend on n, Λ and λ (but not on σ).

2.4. **Discussion of A-B-P estimates.** At this subsection, we are going to discuss motivations and differences of the A-B-P estimates at previous subsections.

Remark 2.4.1.

(1) *Key setp in the A-B-P estimate is the control of the volume of the gradient image, $|\nabla\Gamma(B_R)|$, in terms of $\frac{M_0}{R}$ for $M_0 = \sup_{B_R} u$. From the concavity $\Gamma(x)$, we have $B_{\frac{M_0}{R}} \subset \nabla\Gamma(B_R) = \nabla\Gamma(C(u,\Gamma,B_R))$ and then $\omega_n\left(\frac{M_0}{R}\right)^n \leq |\nabla\Gamma(C(u,\Gamma,B_R;b_R))|$ (see Lemma 9.2,* [GT]*). Similarly for a fixed vector $b_R \neq 0$, we have $\{z \in B_{\frac{M_0}{R}} : z\cdot b_R \geq 0\} \subset \nabla\Gamma(C^+(u,\Gamma,B_R;b_R))$ and then $\frac{1}{2}\omega_n\left(\frac{M_0}{R}\right)^n \leq |\nabla\Gamma(C^+(u,\Gamma,B_R;b_R))|$ This estimate depends only on the existence of a fixed nonzero vector q for a given ball B_R and it is independent of the size or direction of b_R. If $b_R = 0$, it recover the result at* [GT].

(2) *The different A-B-P estimates have been considered to control the effect of the gradient term $\nabla\Gamma(x)$ caused by the fact that the Kernel is not symmetric. And they will be used at Lemma 3.2.1 to prove the decay estimate of the upper level set of super-solutions. When we apply A-B-P estimate with blow-up rate, we have an extra term $(R^{\sigma-2}J_\sigma)^n|B_R| \approx 1$ (for $1<\sigma<2$), $-\log(R)$ (for $\sigma = 1$), and $R^{\sigma-1}$ (for $0<\sigma<1$) caused by $\nabla\Gamma(x)$. It is bounded only at $1<\sigma<2$. This is the main reason that we consider the other two A-B-P estimates.*

(3) *For $0<\sigma<1$, the extra term $R^{-1}|\nabla\Gamma(x)|$ looks optimal since the extra term at Lemma 3.2.1 at this case will become $(R^{-1}J_\sigma)^n|B_R| < C < \infty$ even for $0<\sigma<1$.*

(4) *For $\sigma = 1$, we can not find a suitable A-B-P estimate for the fully nonlinear equation since we doesn't know how to control $-\log(R)|\nabla\Gamma(x)|$, which shows up even after a direction b is chosen in B_R.*

(5) *For the linear case, the linearity of the equation creates only $b_R = (2-\sigma)\int_{B_1\setminus B_R} y\,K(y)\,dy$ which depends only on R, not the position x. Therefore $b_R\cdot\nabla\Gamma(x)$ is the only error caused by the fact that the Kernel is not symmetric. But if $b_R\cdot\nabla\Gamma(x)\geq 0$ (or, $x \in C^+(u,\Gamma,B_R;b_R)$), then $b_R\cdot\nabla\Gamma(x)$ goes away during the computation. And if the kernel is homogeneous of degree $-(n+\sigma)$, $b_R = (2-\sigma)J_\sigma(R)\int_{\partial B_1}\theta\,K(\theta)\,d\sigma_\theta$ for $\theta = \frac{x}{|x|}$ which has a fixed direction for $0<R\leq 1$. As we observed at (1), the lower bound of the volume of the gradient image $|\nabla\Gamma(C^+(u,\Gamma,B_R;b_R))|$ is independent of the choice of b_R. The main question is the control of the error term, $R^{\sigma-2}|\nabla\Gamma(x)|$ or $R^{-1}|\nabla\Gamma(x)|$ after the choice of b_R, which will be possible for $1<\sigma<2$ or $0<\sigma<1$ respectively, Lemma 3.2.1.*

3. Decay Estimate of Upper Level Sets

In this section, we are going to show the geometric decay rate of the upper level set of nonnegative solution u. The key Lemma 3.2.1 says that if a nonnegative function u has a value smaller than one in Q_R then the lower level set $\{x : u \le M\}$ has uniformly positive amount of measure $\nu|Q_R|$ which will be proven through ABP estimate. But the assumption of ABP estimate on a subsolution requires its special shape: it should be negative out side of Q_R and positive at some interior point. So we are going to construct a special function Ψ so that $\Psi - u$ meets the requirement of ABP estimate.

3.1. **Special functions.** The construction of the special function is based on the idea in [CS, KL]. Nontrivial finer computation has been done to take care of the influence of the gradient term and the lack of scaling.

Lemma 3.1.1. *There exist some $\sigma^* \in (0, 2)$ and $p > 0$ such that the function*

$$f(x) = \min\{2^p R^{-p}, |x|^{-p}\}$$

is a subsolution to $\mathcal{M}^-_{\mathfrak{L}_0} f(x) \ge 0$ for any $\sigma \in (\sigma^*, 2)$ and $x \in B^c_R$.

Proof. It is enough to show that there is some $\sigma^* \in (1, 2)$ so that

$$\mathcal{M}^-_{\mathfrak{L}_0} f(x) \ge 0 \tag{3.1.1}$$

for $x = R_1 e_n = (0, 0, \cdots, 0, R_1) \in \mathbb{R}^n$; for every other x with $|x| = R_1 \ge R$, the above inequality follows by rotation. In order to prove (3.1.1), we use the following elementary inequality that holds for any $a > b > 0$ and $p > 0$;

$$(a + b)^{-p} \ge a^{-p}\Big(1 - p\,\frac{b}{a} + \frac{p(p + 1)}{2!}\Big(\frac{b}{a}\Big)^2 - \frac{p(p + 1)(p + 2)}{3!}\Big(\frac{b}{a}\Big)^3\Big).$$

Using this inequality and $\mu(f, R_1 e_n, y) = R_1^{-p}\mu\,(f, e_n, \overline{y})$ for $\overline{y} = y/R_1$, we have that

$$\begin{aligned}
\mu(f, e_n, \overline{y}) &= |e_n + \overline{y}|^{-p} - 1 + p\,\overline{y}_n = (1 + |\overline{y}|^2 + 2\overline{y}_n)^{-p/2} - 1 + p\,\overline{y}_n \\
&\ge -\Big(\frac{p}{2} + 1\Big)|\overline{y}|^2 + \frac{p(p + 2)}{2}\,\overline{y}_n|\overline{y}|^2 + \frac{|\overline{y}|^2}{(1 + |\overline{y}|^2)^{p/2+1}} \\
&\quad + \frac{p(p + 2)}{2}\frac{\overline{y}_n^2}{(1 + |\overline{y}|^2)^{p/2+2}} - \frac{p(p + 2)(p + 4)}{6}\frac{\overline{y}_n^3}{(1 + |\overline{y}|^2)^{p/2+3}}
\end{aligned} \tag{3.1.2}$$

for any $y \in B_{\frac{1}{2}R}$. We choose some sufficiently large $p > 0$ so that

$$\frac{p(p + 2)}{2(1 + r^2)^{p/2+2}} \int_{S^{n-1}} \theta_n^2\, d\sigma(\theta) + \frac{\omega_n}{(1 + r^2)^{p/2+1}} - \Big(\frac{p}{2} + 1\Big)\omega_n = \delta_0(r) > 0 \tag{3.1.3}$$

for any sufficiently small $r > 0$. Since $\int_{S^{n-1}} \theta_n \, d\sigma(\theta) = \int_{S^{n-1}} \theta_n^3 \, d\sigma(\theta) = 0$, it follows from (1.1.3), (1.1.4), (3.1.2) and (3.1.3) that

$$\begin{aligned}
&\mathcal{M}^-_{\mathfrak{L}_0} f(e_n) \\
&\geq (2-\sigma)R_1^{-p}\Big(\int_{\mathbb{R}^n} \frac{\lambda\mu^+(f,e_n,y/R_1)}{|y|^{n+\sigma}}\,dy - \int_{\mathbb{R}^n} \frac{\Lambda\mu^-(f,e_n,y/R_1)}{|y|^{n+\sigma}}\,dy\Big) \\
&\geq (2-\sigma)R_1^{-p-\sigma}\Big(\lambda \int_{B_r} \frac{\mu(f,e_n,y)}{|y|^{n+\sigma}}\,dy - \Lambda \int_{\mathbb{R}^n\setminus B_r} \frac{\mu^-(f,e_n,y)}{|y|^{n+\sigma}}\,dy\Big) \\
&\geq (2-\sigma)R_1^{-p-\sigma}\Big(\frac{\lambda\delta_0(r)}{2-\sigma} - (2^p+1+p)\Lambda \int_{\mathbb{R}^n\setminus B_r} \frac{1}{|y|^{n+\sigma}}\,dy\Big) \\
&= R_1^{-p-\sigma}\Big(\lambda\delta_0(r) - (2^p+1+p)\Lambda\omega_n \frac{2-\sigma}{\sigma}\, r^{-\sigma}\Big)
\end{aligned}$$

for $r \in (0, \frac{1}{2}R)$, where ω_n denotes the surface measure of S^{n-1}. Thus we may take some sufficiently small $r \in (0, 1/2)$ and take some $\sigma^* \in (1,2)$ close enough to 2 in the above so that

$$\mathcal{M}^-_{\mathfrak{L}_0} f(e_n) \geq 0$$

for any $\sigma \in (\sigma^*, 2)$. Hence we complete the proof. □

Now we have the following Corollary as Corollary 4.1.2, [KL]. The only difference is that the influence of non-symmetry of the kernel is $p|y|\chi_{B_1}(y)$, not $p|y|$ in the proof since authors considered a larger class of operators for $1 < \sigma < 2$ at [KL].

Corollary 3.1.2. *Given any $\sigma_0 \in (0,2)$, there exist some $\delta > 0$ and $p > 0$ such that the function*

$$f(x) = \min\{\delta^{-p}R^{-p}, |x|^{-p}\}$$

is a subsolution to $\mathcal{M}^-_{\mathfrak{L}_0} f(x) \geq 0$ for any $\sigma \in (\sigma_0, 2)$ and $x \in B_R^c$.

Lemma 3.1.3. *Given any $\sigma_0 \in (0,2)$, there exists a function $\Psi \in \mathrm{B}(\mathbb{R}^n)$ such that*
(a) Ψ is continuous on $\mathbb{R}^n$, (b) $\Psi = 0$ on $B^c_{\sqrt{n}R}$,
(c) $\Psi > 2$ on Q_R, (d) $\mathcal{M}^-_{\mathfrak{L}_0}\Psi$ is continuous on $B_{\sqrt{n}R}$,
(e) $\mathcal{M}^-_{\mathfrak{L}_0}\Psi > -\psi/R^\sigma$ on $\mathbb{R}^n$ where ψ is a positive bounded function on $\mathbb{R}^n$ which is supported in $\overline{B}_{R/4}$, for any $\sigma \in (\sigma_0, 2)$.

Proof. We consider the function Ψ given by

$$\Psi = c \begin{cases} 0 & \text{in } \mathbb{R}^n \setminus B_{\sqrt{n}R}, \\ |x|^{-p} - (\sqrt{n})^{-p} & \text{in } B_{\sqrt{n}R} \setminus B_{R/2}, \\ P & \text{in } B_{R/2}, \end{cases}$$

where P is a quadratic paraboloid chosen so that Ψ is $C^{1,1}$ across $\partial B_{R/2}$. We now choose the constant c so that $\Psi(x) > 2$ for $x \in Q_R$ (recall that $Q_R \subset Q_{2R} \subset B_{\sqrt{n}R} \subset B_{2\sqrt{n}R}$). Since $\Psi \in C^{1,1}(B_{\sqrt{n}R})$, $\mathcal{M}^-_{\mathfrak{L}_0}\Psi$ is continuous on $B_{\sqrt{n}R}$. Also by Lemma 3.1.1 we see that $\mathcal{M}^-_{\mathfrak{L}_0}\Psi \geq 0$ on $B^c_{R/4}$. Hence this completes the proof. □

3.2. **Estimates in measure.** The main tool that shall be useful in proving Hölder estimates is a lemma that connects a pointwise estimate with an estimate in measure. The corresponding lemma in our context is the following.

Lemma 3.2.1. *Let $\sigma_0 \in (0,2)$. If $\sigma \in (\sigma_0, 2)$ and $R \in (0, R_0]$, then there exist some constants $\varepsilon_0 > 0$, $\nu \in (0,1)$ and $M > 1$ (depending only on $\sigma_0, \lambda, \Lambda$ and the dimension n) for which if $u \in \mathrm{B}(\mathbb{R}^n)$ is a viscosity supersolution to $\mathcal{M}^-_{\mathfrak{L}_0} u \leq \varepsilon_0/R^\sigma$ with $\sigma \neq 1$ or $\mathcal{L}u \leq \varepsilon_0/R^\sigma$ with $\sigma \in (\sigma_0, 2)$ on $B_{2\sqrt{n}R}$ such that $u \geq 0$ on $\mathbb{R}^n$ and $\inf_{Q_R} u \leq 1$, then $|\{u \leq M\} \cap Q_R| \geq \nu |Q_R|$*

Remark. We denote by $Q_r(x)$ an open cube $\{y \in \mathbb{R}^n : |y - x|_\infty \leq r/2\}$ and $Q_r = Q_r(0)$. If we set $Q = Q_r(x)$, then we denote by $sQ = Q_{sr}(x)$ for $s > 0$.

[*Proof of Lemma 3.2.1*]

(Case 1: $1 < \sigma < 2$). We consider the function $v := \Psi - u$ where Ψ is the special function constructed in Lemma 3.1.3. Then we easily see that v is upper semicontinuous on $\overline{B}_{2\sqrt{n}R}$ and v is not positive on $\mathbb{R}^n \setminus B_{\sqrt{n}R}$. Moreover, v is a viscosity subsolution to $\mathcal{M}^+_{\mathfrak{L}_0} v \geq \mathcal{M}^-_{\mathfrak{L}_0}\Psi - \mathcal{M}^-_{\mathfrak{L}_0} u \geq -(\psi + \varepsilon_0)/R^\sigma$ on $B_{2\sqrt{n}R}$. So we want to apply Theorem 2.1.4 (rescaled) to v. Let Γ be the concave envelope of v in $B_{4\sqrt{n}R}$. Since $\inf_{Q_R} u \leq 1$, $\inf_{Q_R}\Psi > 2$ and $Q_R \subset B_{2\sqrt{n}R}$, we easily see that $M_0 := \sup_{B_{2\sqrt{n}R}} v = v(x_0) > 1$ for some $x_0 \in B_{2\sqrt{n}R}$. We consider the function g whose graph is the cone in $\mathbb{R}^n \times \mathbb{R}$ with vertex (x_0, M_0) and base $\partial B_{6\sqrt{n}R}(x_0) \times \{0\}$. For any $\xi \in \mathbb{R}^n$ with $|\xi| < M_0/6\sqrt{n}R$, the hyperplane

$$H = \{(x, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R} : x_{n+1} = L(x) := M_0 + \xi \cdot (x - x_0)\}$$

is a supporting hyperplane for g at x_0 in $B_{6\sqrt{n}R}(x_0)$. Then H has a parallel hyperplane H' which is a supporting hyperplane for v in $B_{4\sqrt{n}R}$ at some point $x_1 \in B_{2\sqrt{n}R}$. By the definition of concave envelope, we see that H' is also the hyperplane tangent to the graph of Γ at x_1, so that $\xi = \nabla\Gamma(x_1)$. This implies that $B_{M_0/6\sqrt{n}R}(0) \subset \nabla\Gamma(B_{2\sqrt{n}R})$. Thus we have that

$$C(n)\log\left(\frac{(M_0/R)^n}{\eta^n}\right) \leq \int_{C(u,\Gamma,B_R)} g_\eta(\nabla\Gamma(y)) \det[D^2\Gamma(y)]^- \, dy, \tag{3.2.1}$$

where g_η is the function given in Corollary 2.1.3. We also observe as shown in [CC] that

$$\left|\nabla\Gamma\Big(B_{2\sqrt{n}R} \setminus C(v, \Gamma, B_{2\sqrt{n}R})\Big)\right| = 0. \tag{3.2.2}$$

Let $\{Q_j\}$ be the finite family of cubes given by Theorem 2.1.4 (rescaled on $B_{2\sqrt{n}R}$). Then it follows from (3.2.1), (3.2.2) and Theorem 2.1.4 that

$$
\begin{aligned}
\ln\Big(\frac{[(\sup_{B_{2\sqrt{n}R}} v)/R]^n}{\eta^n}+1\Big) &\leq C\int_{C(u,\Gamma,B_R)} g_\eta(\nabla\Gamma(y))\det[D^2\Gamma(y)]^-\,dy\\
&\leq C\Big(\sum_j \sup_{\overline{Q}_j}\Big((R^{\sigma-2}J_\sigma(R))^n+\eta^{-n}\big(R^{(\sigma-2)}(\psi+\varepsilon_0)/R^\sigma\big)^n|Q_j|\Big)\\
&\leq C\Big((R^{\sigma-2}J_\sigma(R))^n\sum_j|Q_j|+\eta^{-n}\sum_j\sup_{\overline{Q}_j}((\psi+\varepsilon_0)/R^2)^n|Q_j|\Big)\\
&\leq C\Big((R^{\sigma-1}J_\sigma(R))^n+\eta^{-n}\sum_j\sup_{\overline{Q}_j}((\psi+\varepsilon_0)/R^2)^n|Q_j|\Big).
\end{aligned}
\tag{3.2.3}
$$

Here we note that $K_R := Exp((R^{\sigma-1}J_\sigma(R))^n) \leq C < \infty$ for $1<\sigma<2$ and $R<1$. If we set $\eta=\Big(\sum_j \sup_{\overline{Q}_j}((\psi+\varepsilon_0)/R^2)^n|Q_j|\Big)^{1/n}$ in (3.2.3), then we have that

$$
\begin{aligned}
\sup_{B_{2\sqrt{n}R}} v &\leq CR\Big(\sum_j\sup_{\overline{Q}_j}((\psi+\varepsilon_0)/R^2)^n|Q_j|\Big)^{1/n}\\
&\leq C\varepsilon_0 + CR\Big(\sum_j\Big((\sup_{\overline{Q}_j}\psi)/R^2\Big)^n|Q_j|\Big)^{1/n}.
\end{aligned}
\tag{3.2.4}
$$

Since $\inf_{Q_R} u \leq 1$ and $\inf_{Q_R}\Psi > 2$, we see that $\sup_{B_{2\sqrt{n}R}} v > 1$. If we choose ε_0 small enough, the above inequality (3.2.4) implies that

$$
\frac{1}{2^{1/n}}R \leq C\Big(\sum_j\Big(\sup_{Q_j}\psi\Big)^n|Q_j|\Big)^{1/n}.
$$

We recall from the proof of Lemma 3.1.3 that ψ is supported on $\overline{B}_{R/4}$ and bounded on $\mathbb{R}^n$. Thus the above inequality becomes

$$
\frac{1}{2}|Q_R| \leq C\Big(\sum_{Q_j\cap B_{R/4}\neq\phi}|Q_j|\Big),
$$

which provides a lower bound for the sum of the volumes of the cubes Q_j intersecting $B_{R/4}$ as follows;

$$
\sum_{Q_j\cap B_{R/4}\neq\phi}|Q_j| \geq c|Q_R|.
\tag{3.2.5}
$$

Since $\mathrm{diam}(Q_j) \leq \rho_0 2^{-\frac{1}{2-\sigma}}R \leq \rho_0 R$ for any $\sigma\in(\sigma_0,2)$, the cube $4\sqrt{n}Q_j$ is contained in $B_{R/2}$ for any Q_j with $Q_j\cap B_{R/4}\neq\phi$. Set $M_1=\sup_{B_{R/2}}(\Psi-\Gamma)$.

Then by Theorem 2.1.4 we have that
(3.2.6)
$$\left|\{y \in 4\sqrt{n}Q_j : v(x) \geq \Gamma(y) - CR^{\sigma-2}\sup_{\overline{Q}_j}((\psi+\varepsilon_0)/R^\sigma + J_\sigma(R)|\nabla\Gamma|)d_j^2\}\right| \geq \eta_0|Q_j|$$

and $d_j^2 \leq \rho_0^2 R^2$. Then the family $\mathfrak{F} = \{4\sqrt{n}Q_j : Q_j \cap B_{R/4} \neq \phi\}$ is an open covering of the union

$$D := \bigcup_{Q_j \cap B_{R/4} \neq \phi} \overline{Q}_j.$$

Now we may take a subcovering of $\mathfrak{F}$ with finite overlapping number (depending only on the dimension n) which covers the set D. Thus it follows from (3.2.5) and (3.2.6) that

$$\left|\{x \in B_{R/2} : v(x) \geq \Gamma(x) - C\rho_0^2\}\right| \geq c\eta_0|Q_R|,$$

because $R^{\sigma-2}\sup_{\overline{Q}_j}((\psi+\varepsilon_0)/R^\sigma + J_\sigma(R)|\nabla\Gamma|)d_j^2 \leq \rho_0^2 \sup_{\overline{Q}_j}((\psi+\varepsilon_0)+|\nabla\Gamma|) \leq C\rho_0^2$ for any $R \in (0, R_0]$. So we have that

$$\left|\{x \in B_{R/2} : u(x) \leq M_1 + C\rho_0^2\}\right| \geq c\eta_0|Q_R|.$$

Taking $M = M_1 + C\rho_0^2 > 1$, we conclude that $|\{u \leq M\} \cap Q_R| \geq \nu|Q_R|$ where $\nu = c\eta_0$, because $B_{R/2} \subset Q_R$.

(Case 2: $0 < \sigma < 1$) Now we are going to apply A-B-P estimate (Corollary 2.2.2). We can observe

$$\begin{aligned}\int_{C^+(u,\Gamma,B_R;b)} g_\eta(\nabla\Gamma(y))\det[D^2\Gamma(y)]^-\,dy &\geq \int_{B_{\tilde{M}}\cap\{b\cdot x\geq 0\}} g\\ &\geq \frac{1}{2}\int_{B_{\tilde{M}}} g \geq \ln c_0\Big(\frac{[(\sup_{B_{2\sqrt{n}R}} v)/R]^n}{\eta^n} + 1\Big)\end{aligned}$$

for $\tilde{M} = \sup_{B_{2\sqrt{n}R}} v)/R$. Then we have

$$\begin{aligned}\Big(\frac{[(\sup_{B_{2\sqrt{n}R}} v)/R]^n}{\eta^n} + 1\Big) &\leq C\Big((R^{-1})^n\sum_j |Q_j| + \eta^{-n}\sum_j \sup_{\overline{Q}_j}((\psi+\varepsilon_0)/R^2)^n|Q_j|\Big)\\ &\leq C\Big(1 + \eta^{-n}\sum_j \sup_{\overline{Q}_j}((\psi+\varepsilon_0)/R^2)^n|Q_j|\Big)\end{aligned}$$

If we follow the same argument as in (Case1), we have the conclusion.

(Case 3: Linear equations with $0 < \sigma < 2$) If apply A-B-P estimate (Corollary 2.3.2) and follow the exactly same argument as (Case 2) with b_R instead b, we have the conclusion. □

We split Q_R into 2^n cubes of half side. We do the same splitting step with each one of these 2^n cubes and we continue this process. The cubes obtained in this way are called *dyadic cubes*. If Q is a dyadic cube different from Q_R, then we say that $\widetilde{Q}$ is the *predecessor* of Q if Q is one of 2^n cubes obtained from splitting $\widetilde{Q}$.

Lemma 3.2.2. [CC] *Let A, B be measurable sets with $A \subset B \subset Q_R$. If $\delta \in (0,1)$ is some number such that (a) $|A| \leq \delta$ and (b) $\widetilde{Q} \subset B$ for any dyadic cube Q with $|A \cap Q| > \delta |Q|$, then $|A| \leq \delta |B|$.*

The following lemma is a consequence of Lemma 3.2.1 and Lemma 3.2.2.

Lemma 3.2.3. *Given $\sigma_0 \in (0,2)$, let $\sigma \in (\sigma_0, 2)$ and $R \in (0, R_0]$. Let $\varepsilon_0 > 0$ be the constant in Lemma 3.2.1. If $u \in \mathrm{B}(\mathbb{R}^n)$ is a viscosity supersolution to $\mathcal{M}^-_{\mathfrak{L}_0} u \leq \varepsilon_0 / R^\sigma$ with $\sigma \neq 1$ or $\mathcal{L} u \leq \varepsilon_0 / R^\sigma$ with $\sigma \in (\sigma_0, 2)$ on $B_{2\sqrt{n}R}$ such that $u \geq 0$ on $\mathbb{R}^n$ and $\inf_{Q_R} u \leq 1$, then there are universal constants $C > 0$ and $\varepsilon_* > 0$ such that*

$$\Big|\{u > t\} \cap Q_R\Big| \leq C\, t^{-\varepsilon_*} |Q_R|, \ \forall\, t > 0.$$

Remark 3.2.4. *We note that $B_{R/2} \subset Q_R \subset Q_{3R} \subset B_{3\sqrt{n}R/2} \subset B_{2\sqrt{n}R}$.*

Proof. First, we shall prove that

$$\Big|\{u > M^k\} \cap Q_R\Big| \leq (1-\nu)^k |Q_R|, \ \forall\, k \in \mathbb{N}, \tag{3.2.7}$$

where $\nu > 0$ is the constant as in Lemma 3.2.1 and $M > 1$ is the constant chosen in Lemma 3.2.1

If $k = 1$, then it has been done in Lemma 3.2.1. Assume that the result (3.2.7) holds for $k-1$ $(k \geq 2)$ and let

$$A = \{u > M^k\} \cap Q_R \ \text{ and } \ B = \{u > M^{k-1}\} \cap Q_R.$$

If we can show that $|A| \leq (1-\nu)|B|$, then (3.2.7) can be obtained for k. To show this, we apply Lemma 3.2.3. By Lemma 3.2.1, it is clear that $A \subset B \subset Q_R$ and $|A| \leq |\{u > M\} \cap Q_R| \leq (1-\nu)|Q_R|$. So it remain only to prove (b) of Lemma 3.2.2; that is, we need to show that if $Q = Q_{2^{-i}R}(x_0)$ is a dyadic cube satisfying

$$|A \cap Q| > (1-\nu)|Q| \tag{3.2.8}$$

then $\widetilde{Q} \subset B$. Indeed, we suppose that $\widetilde{Q} \not\subset B$ and take $x_* \in \widetilde{Q}$ such that

$$u(x_*) \leq M^{k-1}. \tag{3.2.9}$$

We now consider the transformation $x = x_0 + y, \ y \in Q_{2^{-i}R}, \ x \in Q = Q_{2^{-i}R}(x_0)$ and the function $v(y) = u(x)/M^{k-1}$. If we can show that v satisfies the hypothesis of Lemma 3.2.1, then we have that $\nu|Q| < |\{u(x) \leq M^k\} \cap Q|$, and thus $|Q \setminus A| > \nu|Q|$ which contradicts (3.2.8).

To complete the proof, we consider once again the transformation

$$x = x_0 + z, \ z \in B_{\frac{\sqrt{n}}{2^i}R}, \ x \in B_{\frac{\sqrt{n}}{2^{i-1}}R}(x_0) \subset B_{2\sqrt{n}R}$$

and the function $v(z) = u(x)/M^{k-1}$. It now remains to show that v satisfies the hypothesis of Lemma 3.2.1. We now take any $\varphi \in \mathrm{C}^2_{2\sqrt{n}2^{-i}R}(v; z)^-$. If we set $\psi = M^{k-1}\varphi(\cdot - x_0)$, then we observe that

$$\varphi \in C^2_{B_{2\sqrt{n}2^{-i}R}}(v;z)^- \quad\Leftrightarrow\quad \psi \in C^2_{B_{2\sqrt{n}2^{-i}R}(x_0)}(u;x_0+z)^-.$$

Since $B_{2\sqrt{n}2^{-i}R}(x_0) \subset B_{2\sqrt{n}R}$, we have that

$$\begin{aligned}\mathcal{M}^-_{\mathfrak{L}_0} v(z;\nabla\varphi) &\le \mathcal{L}v(z;\nabla\varphi)\\ &= \frac{1}{M^{k-1}}\int_{\mathbb{R}^n} \mu(u, x_0+z, y; \nabla\psi)K(y)\,dy\\ &:= \frac{1}{M^{k-1}}\mathcal{L}u(x_0+z;\nabla\psi)\end{aligned}$$

for any $\mathcal{L} \in \mathfrak{L}$. Taking the infimum of the right-hand side in the above inequality, we obtain that

$$\mathcal{M}^-_{\mathfrak{L}_0} v(z;\nabla\varphi) \le \frac{1}{M^{k-1}}\mathcal{M}^-_{\mathfrak{L}_0} u(x_0+z;\nabla\psi).$$

Thus we have that

$$\mathcal{M}^-_{\mathfrak{L}_0} v(z;\nabla\varphi) \le \frac{\varepsilon_0}{(2^{-i}R)^\sigma},$$

because $\mathcal{M}^-_{\mathfrak{L}_0} u \le \frac{\varepsilon_0}{R^\sigma}$ on $B_{2\sqrt{n}R}$. Also it is obvious that $v \ge 0$ on $\mathbb{R}^n$ and we see from (3.2.9) that $\inf_Q v \le 1$. Finally the result follows immediately from (3.2.7) by taking $C = (1-\nu)^{-1}$ and $\varepsilon_* > 0$ so that $1-\nu = M^{-\varepsilon_*}$. Hence we complete the proof. □

By a standard covering argument we obtain the following theorem.

Theorem 3.2.5. *For any $\sigma_0 \in (0,2)$, let $\sigma \in (\sigma_0, 2)$ be given. If $u \in \mathrm{B}(\mathbb{R}^n)$ is a viscosity supersolution to $\mathcal{M}^-_{\mathfrak{L}_0} u \le \frac{\varepsilon_0}{R^\sigma}$ with $\sigma \neq 1$ or $\mathcal{L}u \le \varepsilon_0/R^\sigma$ with $\sigma \in (\sigma_0,2)$ on B_{2R} such that $u \ge 0$ on $\mathbb{R}^n$ and $u(0) \le 1$ where ε_0 is the constant given in Lemma 3.2.1, then there are universal constants $C > 0$ and $\varepsilon_* > 0$ such that*

$$\left|\{u > t\} \cap B_R\right| \le C\,t^{-\varepsilon_*}|B_R|,\ \forall\, t > 0.$$

In contrast to symmetric cases, we note that we can not obtain the following theorem by rescaling the above theorem because our cases are not scaling invariant. We note that Theorem 3.2.6 on $r \in (0,1)$ shall be applied to obtain a Harnack inequality, and also Theorem 3.2.6 on $r \in [R, 2R]$ will be used to prove Hölder estimates and an interior $C^{1,\alpha}$-regularity.

Theorem 3.2.6. *For any $\sigma_0 \in (0,2)$, let $\sigma \in (\sigma_0,2)$ be given, and let $x \in \mathbb{R}^n$ and $r \in (0,2]$. If $u \in \mathrm{B}(\mathbb{R}^n)$ is a viscosity supersolution to $\mathcal{M}^-_{\mathfrak{L}_0} u \le c_0$ with $\sigma \neq 1$ or $\mathcal{L}u \le \varepsilon_0/R^\sigma$ with $\sigma \in (\sigma_0, 2)$ on $B_{2r}(x)$ such that $u \ge 0$ on $\mathbb{R}^n$, then there are universal constants $\varepsilon_* > 0$ and $C > 0$ such that*

$$\left|\{u > t\} \cap B_r(x)\right| \le C\,r^n\Big(u(x) + c_0 R^\sigma\, r^\sigma\Big)^{\varepsilon_*} t^{-\varepsilon_*},\ \forall\, t > 0.$$

Proof. Let $x \in \mathbb{R}^n$ and set $v(z) = u(z+x)/q$ for $z \in B_{2r}$ where $q = u(x) + c_0 r^\sigma/\varepsilon_0$. Take any $\varphi \in C^2_{B_{2r}}(v;z)^-$. If we set $\psi = q\,\varphi(\cdot - x)$, then we see that $\psi \in C^2_{B_{2r}(x)}(u; z+x)$. Thus by the change of variables we have that

$$\begin{aligned}\mathcal{M}^-_{\mathfrak{L}_0} v(z;\nabla\varphi) &\leq \mathcal{L}v(z;\nabla\varphi)\\ &= \frac{1}{q}\int_{\mathbb{R}^n} \mu(z+x,y;\nabla\psi)\,K(y)\,dy\\ &:= \frac{1}{q}\mathcal{L}u(z+x;\nabla\psi)\end{aligned}$$

for any $\mathcal{L} \in \mathfrak{L}_0$. Taking the infimum of the right-hand side in the above inequality, we get that

$$\mathcal{M}^-_{\mathfrak{L}_0} v(z;\nabla\varphi) \leq \frac{1}{q}\mathcal{M}^-_{\mathfrak{L}_0} u(rz+x;\nabla\psi) \leq \frac{\varepsilon_0}{r^2}.$$

Thus we have that $\mathcal{M}^-_{\mathfrak{L}_0} v \leq \frac{\varepsilon_0}{r^2}$ on B_{2r}. Applying Theorem 3.2.5 to the function v, we complete the proof. □

4. Regularity Theory

4.1. **Harnack inequality.**

Theorem 4.1.1. *For a given $\sigma_0 \in (0,2)$, let $\sigma_0 < \sigma < 2$. If $u \in \mathrm{B}(\mathbb{R}^n)$ is a positive function such that*

$$\mathcal{M}^-_{\mathfrak{L}_0} u \leq \frac{C_0}{R^\sigma} \quad \textit{and} \quad \mathcal{M}^+_{\mathfrak{L}_0} u \geq -\frac{C_0}{R^\sigma} \quad \textit{with } \sigma \neq 1 \textit{ on } B_{2R}$$

or

$$\mathcal{L}u \leq \frac{C_0}{R^\sigma} \quad \textit{and} \quad \mathcal{L}u \geq -\frac{C_0}{R^\sigma} \quad \textit{with } \sigma_0 < \sigma < 2 \textit{ on } B_{2R}$$

in the viscosity sense, then there is some constant $C > 0$ depending only on λ, Λ, n and σ_0 such that

$$\sup_{B_{R/2}} u \leq C\Big(\inf_{B_{R/2}} u + C_0\Big).$$

Proof. Let $\hat{x} \in B_{R/2}$ be a point so that $\inf_{B_{R/2}} u = u(\hat{x})$. Then it is enough to show that

$$\sup_{B_{R/2}} u \leq C\Big(u(\hat{x}) + C_0\Big).$$

Without loss of generality, we may assume that $u(\hat{x}) \leq 1$ and $C_0 = 1$ by dividing u by $u(\hat{x}) + C_0$. Let $\varepsilon_* > 0$ be the number given in Theorem 3.2.6 and let $\beta = n/\varepsilon_*$. We now set $s_0 = \inf\{s > 0 : u(x) \leq s(1 - |x|/R)^{-\beta},\ \forall\, x \in B_R\}$. Then we see that $s_0 > 0$ because u is positive on $\mathbb{R}^n$. Also there is some $x_0 \in B_R$ such that $u(x_0) = s_0(1 - |x_0|/R)^{-\beta} = s_0\left(\frac{\mathrm{d}_0}{R}\right)^{-\beta}$ where $\mathrm{d}_0 = \mathrm{d}(x_0, \partial B_R) \leq R$.

To finish the proof, we have only to show that s_0 can not be too large because $u(x) \leq C_1(1 - |x|/R)^{-\beta} \leq C$ for any $x \in B_{R/2}$ if $C_1 > 0$ is some constant with $s_0 \le C_1$. Assume that s_0 is very large. Then by Theorem 3.2.5 we have that

$$\left|\{u \ge u(x_0)/2\} \cap B_R\}\right| \le \left|\frac{2}{u(x_0)}\right|^{\varepsilon_*} |B_R| \le C s_0^{-\varepsilon_*} \mathrm{d}_0^n.$$

Since $|B_r| = C\mathrm{d}_0^n$ for $r = \mathrm{d}_0/2 < R$, we easily obtain that

$$\left|\{u \ge u(x_0)/2\} \cap B_r(x_0)\}\right| \le \left|\frac{2}{u(x_0)}\right|^{\varepsilon_*} \le C s_0^{-\varepsilon_*} |B_r|. \tag{4.1.1}$$

In order to get a contradiction, we estimate $|\{u \le u(x_0)/2\} \cap B_{\delta r}(x_0)|$ for some very small $\delta > 0$ (to be determined later). For any $x \in B_{2\delta r}(x_0)$, we have that $u(x) \le s_0(\mathrm{d}_0 - \delta \mathrm{d}_0/R)^{-\beta} \le u(x_0)(1-\delta)^{-\beta}$ for $\delta > 0$ so that $(1-\delta)^{-\beta}$ is close to 1. We consider the function

$$v(x) = (1-\delta)^{-\beta} u(x_0) - u(x).$$

Then we see that $v \ge 0$ on $B_{2\delta r}(x_0)$, and also $\mathcal{M}^-_{\mathfrak{L}_0} v \le \frac{1}{R^\sigma}$ on $B_{\delta r}(x_0)$ because $\mathcal{M}^+_{\mathfrak{L}_0} u \ge -\frac{1}{R^\sigma}$ on $B_{\delta r}(x_0)$. We now want to apply Theorem 3.2.6 to v. However v is not positive on $\mathbb{R}^n$ but only on $B_{\delta r}(x_0)$. To apply Theorem 3.2.6, we consider $w = v^+$ instead of v. Since $w = v + v^-$, we have that $\mathcal{M}^-_{\mathfrak{L}_0} w \le \mathcal{M}^-_{\mathfrak{L}_0} v + \mathcal{M}^+_{\mathfrak{L}_0} v^- \le \frac{1}{R^\sigma} + \mathcal{M}^+_{\mathfrak{L}_0} v^-$ on $B_{\delta r}(x_0)$. Since $v^- \equiv 0$ on $B_{2\delta r}(x_0)$, if $x \in B_{\delta r}(x_0)$ then we have that $\mu(v^-, x, y; \nabla\varphi) = v^-(x+y)$, $y \in B_{\delta r}(x_0)$ and $\varphi \in \mathrm{C}^2_{B_{\delta r}(x_0)}(v^-; x)^+$. Take any $\varphi \in \mathrm{C}^2_{B_{\delta r}(x_0)}(v^-; x)^+$ and any $x \in B_{\delta r}(x_0)$. Since $x + B_{\delta r} \subset B_{2\delta r}(x_0)$, we thus have that

$$\begin{aligned}
&\mathcal{M}^-_{\mathfrak{L}_0} w(x; \nabla\varphi) \\
&\quad \le \frac{1}{R^2} + (2-\sigma) \int_{\mathbb{R}^n} \frac{\Lambda \mu^+(v^-, x, y; \nabla\varphi) - \lambda \mu^-(v^-, x, y; \nabla\varphi)}{|y|^{n+\sigma}}\, dy \\
&\quad \le \frac{1}{R^\sigma} + (2-\sigma) \int_{\{y \in \mathbb{R}^n : v(x+y) < 0\}} \frac{-\Lambda\, v(x+y)}{|y|^{n+\sigma}}\, dy \\
&\quad \le \frac{1}{R^\sigma} + (2-\sigma)\Lambda \int_{\mathbb{R}^n \setminus B_{\delta r}} \frac{\left(u(x+y) - (1-\delta)^{-\beta} u(x_0)\right)_+}{|y|^{n+\sigma}}\, dy.
\end{aligned} \tag{4.1.2}$$

We consider the function $h_c(x) = c(1 - |x|^2/R^2)_+$ for $c > 0$ and we set

$$c_1 = \sup\{c > 0 : u(x) \ge h_c(x),\ \forall\, x \in \mathbb{R}^n\}.$$

Then there is some $x_1 \in B_R$ such that $u(x_1) = c_1(1 - |x_1|^2/R^2)$ and we see that $c_1 \leq 4/3$ because $u(\hat{x}) \leq 1$. Since $\nabla h_{c_1}(x) = -\frac{2c_1 x}{R^2}$, we have that

$$
\begin{aligned}
&(2-\sigma)\int_{\mathbb{R}^n} \frac{\mu^-(u,x_1,y;\nabla h_{c_1})}{|y|^{n+\sigma}}\,dy \\
&\leq (2-\sigma)\int_{\mathbb{R}^n} \frac{(h_{c_1}(x_1+y) - h_{c_1}(x_1) - y\cdot\nabla h_{c_1}(x_1)\chi_{B_1}(y))_-}{|y|^{n+\sigma}}\,dy \\
&\leq C(2-\sigma)\int_{B_R} \frac{|y|^2/R^2}{|y|^{n+\sigma}}\,dy + C(2-\sigma)\int_{B_1\setminus B_R} \frac{|y|/R}{|y|^{n+\sigma}}\,dy \\
&\leq \frac{C(2-\sigma_0)}{R^\sigma}
\end{aligned}
\tag{4.1.3}
$$

for some constant $C > 0$ which is independent of σ, and so we have that

$$\Lambda(2-\sigma)\int_{\mathbb{R}^n} \frac{\mu^-(u,x_1,y;\nabla h_{c_1})}{|y|^{n+\sigma}}\,dy \leq \frac{C}{R^\sigma}.$$

Since $\mathcal{M}^-_{\mathfrak{L}_0} u(x_1) \leq \frac{1}{R^\sigma}$ on B_{2R}, by (4.1.3) we have that

$$
\begin{aligned}
\frac{1}{R^\sigma} &\geq \mathcal{M}^-_{\mathfrak{L}_0} u(x_1;\nabla h_{c_1}) \\
&\geq \lambda(2-\sigma)\int_{\mathbb{R}^n} \frac{\mu^+(u,x_1,y;\nabla h_{c_1})}{|y|^{n+\sigma}}\,dy \\
&\quad - \Lambda(2-\sigma)\int_{\mathbb{R}^n} \frac{\mu^-(u,x_1,y;\nabla h_{c_1})}{|y|^{n+\sigma}}\,dy.
\end{aligned}
$$

Thus we obtain that $(2-\sigma)\displaystyle\int_{\mathbb{R}^n} \frac{\mu^+(u,x_1,y;\nabla h_{c_1})}{|y|^{n+\sigma}}\,dy \leq \frac{C}{R^\sigma}$ for a constant $C > 0$ which is independent of σ. We may assume that $(1-\delta)^{-\beta}u(x_0) = (1-\delta)^{-\beta}s_0(1-|x_0|/R)^{-\beta} \geq 4$ because s_0 was very large and $(1-\delta)^{-\beta}$ was close to $1/R$. Since $\delta r < R$, by the change of variables we have that

$$
\begin{aligned}
&(2-\sigma)\Lambda\int_{B^c_{\delta r}} \frac{\big(u(x+y) - (1-\delta)^{-\beta}u(x_0)\big)_+}{|y|^{n+\sigma}}\,dy \\
&\leq C(2-\sigma)\Lambda\int_{\mathbb{R}^n} \frac{\big(u(x_1+y) - 4/R\big)_+}{|y|^{n+\sigma}}\,dx \\
&\leq (2-\sigma)\int_{\mathbb{R}^n} \frac{\mu^+(u,x_1,y;\nabla h_{c_1})}{|y|^{n+\sigma}}\,dy \leq \frac{C}{R^\sigma}
\end{aligned}
$$

for any $x \in B_{\delta r}(x_0)$. Thus by (4.1.2) we obtain that

$$\mathcal{M}^-_{\mathfrak{L}_0} w(x) \leq \frac{C}{R^\sigma} \leq \frac{C}{(\delta r)^\sigma} \quad \text{on } B_{\delta r}(x_0)\,.$$

Since $u(x_0) = s_0(\mathrm{d}_0/R)^{-\beta} = 2^{-\beta}s_0(r/R)^{-\beta}$ and $\beta\varepsilon_* = n$, applying Theorem 3.2.6 we have that

$$\begin{aligned}\left|\{u \le u(x_0)/2\} \cap B_{\delta r/2}(x_0)\right| &= \left|\{w \ge u(x_0)((1-\delta)^{-\beta} - 1/2)\} \cap B_{\delta r/2}(x_0)\right|\\ &\le C(\delta r)^n\Big[((1-\delta)^{-\beta}-1)u(x_0) + C(\delta r)^{-n-\sigma}(\delta r)^{\sigma}\Big]^{\varepsilon_*}\Big[u(x_0)((1-\delta)^{-\beta}-1/2)\Big]^{-\varepsilon_*}\\ &\le C(\delta r)^n\Big[((1-\delta)^{-\beta}-1)^{\varepsilon_*} + \delta^{-n\varepsilon_*}s_0^{-\varepsilon_*}\Big].\end{aligned}$$

We now choose $\delta > 0$ so small enough that $C(\delta r)^n((1-\delta)^{-\beta}-1)^{\varepsilon_*} \le |B_{\delta r/2}(x_0)|/4$. Since δ was chosen independently of s_0, if s_0 is large enough for such fixed δ then we get that $C(\delta r)^n\delta^{-n\varepsilon_*}s_0^{-\varepsilon_*} \le |B_{\delta r/2}(x_0)|/4$. Therefore we obtain that $\left|\{u \le u(x_0)/2\} \cap B_{\delta r/2}(x_0)\right| \le |B_{\delta r/2}(x_0)|/2$. Thus we conclude that

$$\begin{aligned}\left|\{u \ge u(x_0)/2\} \cap B_r(x_0)\right| &\ge \left|\{u \ge u(x_0)/2\} \cap B_{\delta r/2}(x_0)\right|\\ &\ge \left|\{u > u(x_0)/2\} \cap B_{\delta r/2}(x_0)\right|\\ &\ge \left|B_{\delta r/2}(x_0)\right| - \left|B_{\delta r/2}(x_0)\right|/2\\ &= \left|B_{\delta r/2}(x_0)\right|/2 = C|B_r|,\end{aligned}$$

which contradicts (4.1.1) if s_0 is large enough. Thus we complete the proof. □

4.2. **Hölder estimates.** In this subsection, we obtain Hölder regularity result. The following technical lemma is very useful in proving it. As in [CS, KL], its proof can be derived from Theorem 3.2.6.

Lemma 4.2.1. *For $\sigma_0 \in (1,2)$, let $\sigma \in (\sigma_0, 2)$ be given. If u is a bounded function with $|u| \le 1/2$ on $\mathbb{R}^n$ such that*

$$\mathcal{M}^-_{\mathfrak{L}_0} u \le \frac{C_0}{R^\sigma} \quad \text{and} \quad \mathcal{M}^+_{\mathfrak{L}_0} u \ge -\frac{C_0}{R^\sigma} \quad \text{with } \sigma \ne 1 \text{ on } B_{2R}$$

or

$$\mathcal{L}u \le \frac{C_0}{R^\sigma} \quad \text{and} \quad \mathcal{L}u \ge -\frac{C_0}{R^\sigma} \quad \text{with } \sigma_0 < \sigma < 2 \text{ on } B_{2R}$$

in the viscosity sense where $\varepsilon_0 > 0$ is some sufficiently small constant, then there is some universal constant $\alpha > 0$ (depending only on λ, Λ, n and σ_0) such that $u \in C^\alpha$ at the origin. More precisely,

$$|u(x) - u(0)| \le C\,\frac{|x|^\alpha}{R^\alpha}$$

for some universal constant $C > 0$ depending only on α.

Lemma 4.2.1 and a simple rescaling argument give the following theorem as in [CS, KL].

Theorem 4.2.2. *For any $\sigma_0 \in (0,2)$, let $\sigma \in (\sigma_0, 2)$ be given. If u is a bounded function on $\mathbb{R}^n$ such that*

$$\mathcal{M}^-_{\mathfrak{L}_0} u \le \frac{C_0}{R^\sigma} \quad \text{and} \quad \mathcal{M}^+_{\mathfrak{L}_0} u \ge -\frac{C_0}{R^\sigma} \quad \text{with } \sigma \ne 1 \text{ on } B_{2R}$$

or

$$\mathcal{L}u \leq \frac{C_0}{R^\sigma} \quad \textit{and} \quad \mathcal{L}u \geq -\frac{C_0}{R^\sigma} \quad \textit{with } \sigma_0 < \sigma < 2 \textit{ on } B_{2R}$$

in the viscosity sense, then there is some constant $\alpha > 0$ (*depending only on* λ, Λ, n *and* σ_0) *such that*

$$\|u\|_{C^\alpha(B_{R/2})} \leq \frac{C}{R^\alpha}\Big(\|u\|_{L^\infty(\mathbb{R}^n)} + C_0\Big)$$

where $C > 0$ *is some universal constant depending only on* α*.*

4.3. $C^{1,\alpha}$**-estimates.** If we apply Theorem 4.2.2 on the Hölder difference quotients which satifies the same class of operators as the solution, we will have the following interior $C^{1,\alpha}$-estimate as in [CS, CC]. For $R \in (0, R_0]$, we set $\|u\|^*_{C^{1,\alpha}(B_R)} = \|u\|_{L^\infty(B_R)} + R\|Du\|_{L^\infty(B_R)} + R^{1+\alpha}\|Du\|_{C^\alpha(B_R)}$.

Theorem 4.3.1. *For* $\sigma_0 \in (0, 2)$*, let* $\sigma \in (\sigma_0, 2)$ *be given. Then there is some* $\varrho_1 > 0$ (*depending on* $\lambda, \Lambda, \sigma_0$ *and the dimension* n) *so that if* $\mathcal{I}$ *is a nonlocal elliptic operator with respect to* $\mathfrak{L}_0^1$ *in the sense of Definition 3.1 and* $u \in \mathrm{B}(\mathbb{R}^n)$ *is a viscosity solution to* $\mathcal{I}u = 0$ *with* $\sigma \neq 1$ *or* $\mathcal{L}u = 0$ *with* $0 < \sigma < 2$ *on* B_1*, then there is a universal constant* $\alpha > 0$ (*depending only on* $\lambda, \Lambda, \sigma_0$ *and the dimension* n) *such that*

$$\|u\|^*_{C^{1,\alpha}(B_{R/2})} \leq C\Big(\|u\|_{L^\infty(\mathbb{R}^n)} + R^\sigma|\mathcal{J}0|\Big)$$

for some constant $C > 0$ *depending on* $\lambda, \Lambda, \sigma_0, n$ *and the constant given in* (10.1) (*where we denote by* $\mathcal{J}0$ *the value we obtain when we apply* $\mathcal{I}$ *with* $\sigma \neq 1$ *or* $\mathcal{L}$ *with* $0 < \sigma < 2$ *to the constant function that is equal to zero*).

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