CORE COMPACTNESS AND DIAGONALITY IN SPACES OF OPEN SETS

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1. Introduction

Definitions and notations concerning convergence structures follow [1] and are gathered as an appendix at the end of these notes. If X is a topological space, we denote by $\mathcal{O}_X$ the set of its open subsets. Ordered by inclusion, it is a complete lattice in which the *Scott convergence* (in the sense of, for instance, [9]) is given by

$$U \in \lim \mathcal{F} \iff U \subseteq \bigcup_{F \in \mathcal{F}} \operatorname{int}\left(\bigcap_{O \in F} O\right), \tag{1.1}$$

where $\mathcal{F}$ is a filter on $\mathcal{O}_X$; and its topological modification, the *Scott topology*, has open sets composed of compact families ([1]). $\mathcal{O}_X$ can be identified with the set $C(X,\$)$ of continuous functions from X to the Sierpiński space $\$$ because the indicator function of $A \subseteq X$ is continuous if and only if A is open. Via this identification, the convergence (1.1) coincides with the continuous convergence $[X,\$]$ on $C(X,\$)$, and its topological modification $T[X,\$]$ coincides with the Scott topology.

On the other hand, for a general convergence space X, the underlying set of $[X,\$]$ can still be identified with the collection $\mathcal{O}_X$ of open subsets of X (or TX), but the characterization (1.1) of convergence in $[X,\$]$ (when interpreted as a convergence on $\mathcal{O}_X$) needs to be modified. Recall that a family $\mathcal{S}$ of subsets of a convergence space X is a *cover* if every convergent filter on X contains an element of the family $\mathcal{S}$. In general, $U \in \lim_{[X,\$]} \mathcal{F}$ if the family $\{\bigcap_{O\in F} O : F \in \mathcal{F}\}$ is a cover of U (for the induced convergence).

Given a convergence space X, it is known (e.g., [19], [6]) that the following are equivalent:

$$\forall Y,\ T(X \times Y) \leq X \times TY; \tag{1.2}$$

$$T(X \times [X,\$]) \leq X \times T[X,\$]; \tag{1.3}$$

$$[X,\$] = T[X,\$]. \tag{1.4}$$

[1]See the Appendix for definitions

Let us call a convergence space X satisfying this condition T-*dual*. In the case where X is topological, the latter is well-known to be equivalent to core compactness of X (e.g., [11], [18]). Recall that a topological space X is *core compact* if for every x and $O \in \mathcal{O}(x)$, there is $U \in \mathcal{O}(x)$ such that every open cover of O has a finite subfamily that covers U. In [6], a *convergence space* is called *core compact* if whenever $x \in \lim \mathcal{F}$, there is $\mathcal{G} \leq \mathcal{F}$ with $x \in \lim \mathcal{G}$ and for every $G \in \mathcal{G}$ there is $G' \in \mathcal{G}$ such that G' is compact at G. A convergence space is called T-*core compact* if whenever $x \in \lim \mathcal{F}$ and $U \in \mathcal{O}_{TX}(x)$, there is $F \in \mathcal{F}$ that is compact at U. It is shown in [6] that

(1.5)
$$X \text{ is core compact } \Longrightarrow X \text{ is } T\text{-dual } \Longrightarrow X \text{ is } T\text{-core compact.}$$

The three notions clearly coincide if X is topological. However, so far, it was not known whether they do in general. At the end of the paper, we provide an example (Example 22) of a convergence space that is T-dual but not core compact.

It was observed in [10] that if X is topological, then so is $[[X,\$],\$]$. Therefore $[X,\$]$ is then T-core compact, which makes $[X,\$]$ for X topological but not core compact a natural candidate to distinguish core-compactness from T-core compactness. This however fails, in view of Proposition 1 below. In the next section, we also investigate under what condition $T[X,\$]$ (that is $\mathcal{O}_X$ with the Scott topology) is core compact. This question, while natural in itself, is motivated by its connection with the (now recently solved) problem [7, Problem 1.2] of finding a completely regular infraconsonant topological space that is not consonant (see section 3 for definitions). We observe in Section 3 that X is infraconsonant whenever $T[X,\$]$ is core compact and we prove more generally that X is infraconsonant if and only if the Scott topology on $\mathcal{O}_X \times \mathcal{O}_X$ for the product order coincides with the product of the Scott topologies at the point (X,X) (Theorem 13). Infraconsonance was introduced while studying the Isbell topology on the set of real-valued continuous functions over a topological space. In fact a completely regular space X is infraconsonant if and only if the Isbell topology on the set of real-valued continuous functions on X is a group topology [5, Corollary 4.6]. On the other hand, the fact that the Scott topology on the product does not coincide in general with the product of the Scott topologies has been at the origin of a number of problems and errors (e.g., [9, p.197]). Therefore, Theorem 13 provides new motivations to investigate infraconsonance.

In section 4, we show that, for a topological space X, despite the fact that $[X,\$]$ is topological whenever it is pretopological, $[X,\$]$ does not need to be diagonal in general. Diagonality of $[X,\$]$ is characterized in terms of a variant of core-compactness that do not need to coincide with core-compactness.

2. CORE-COMPACTNESS OF $\mathcal{O}_X$

As $[X,\$]$ can be identified with $\mathcal{O}_X$ for any convergence space X, the space $[[X,\$],\$]$ has as underlying set the set of Scott-open subsets of $\mathcal{O}_X$, that is, if X is topological, the set $\kappa(X)$ of openly isotone compact families on X. Note that the family

$$\{U^+ := \{\mathcal{A} \in \kappa(X) : U \in \mathcal{A}\} : U \in \mathcal{O}_X\}$$

forms a subbase for a topology on $\kappa(X)$, called *Stone topology*.

As observed in [8, Proposition 5.2], when X is topological, the convergence $[X,\$]$ is based in filters of the form

$$\mathcal{O}^\natural(\mathcal{P}) := \{\mathcal{O}(P) : P \in \mathcal{P}\}, \tag{2.1}$$

where $\mathcal{P}$ is an *ideal subbase* of open subsets of X, that is, such that there is $P \in \mathcal{P}$ with $\bigcup_{Q\in\mathcal{P}_0} Q \subseteq P$ whenever $\mathcal{P}_0$ is a finite subfamily of $\mathcal{P}$. More precisely, for every filter $\mathcal{F}$ on $[X,\$]$ with $U \in \lim_{[X,\$]} \mathcal{F}$ there is an open cover $\mathcal{P}$ of U that forms an ideal subbase, such that $U \in \lim_{[X,\$]} \mathcal{O}^\natural(\mathcal{P})$ and $\mathcal{O}^\natural(\mathcal{P}) \leq \mathcal{F}$.

Note also that

$$\mathcal{A} \subseteq \mathcal{B} \Longrightarrow \mathcal{A} \in \lim_{[[X,\$],\$]} \{\mathcal{B}\}^\uparrow, \tag{2.2}$$

for every $\mathcal{A}$ and $\mathcal{B}$ in $\kappa(X)$. In particular if O is $[[X,\$],\$]$-open, $\mathcal{A} \in O$ and $\mathcal{A} \subseteq \mathcal{B} \in \kappa(X)$ then $\mathcal{B} \in O$.

Proposition 1. *If X is topological, then $[X,\$]$ is core compact, so that $[[X,\$],\$]$ is topological. More specifically, $[[X,\$],\$]$ can be identified with the space $\kappa(X)$ with the Stone topology.*

Proof. Let $U \in \lim_{[X,\$]} \mathcal{O}^\natural(\mathcal{P})$ for an ideal subbase $\mathcal{P}$ of open subsets of X. Then for each $P \in \mathcal{P}$, the set $\mathcal{O}(P)$ is a compact subset of $[X,\$]$ because $P \in \lim_{[X,\$]} \mathcal{O}(P)$. Indeed, $P = \operatorname{int}\left(\bigcap_{O\in\mathcal{O}(P)} O\right)$.

U^+ is $[[X,\$],\$]$-open for each $U \in \mathcal{O}_X$. Indeed, if $\mathcal{A} \in U^+ \cap \lim_{[[X,\$],\$]} \mathcal{F}$ then $\left\{\bigcap_{\mathcal{B}\in F} \mathcal{B} : F \in \mathcal{F}\right\}$ is a cover of $\mathcal{A}$ (in the sense of convergence) so that there is $F \in \mathcal{F}$ with $\bigcap_{\mathcal{B}\in F} \mathcal{B} \in \{U\}^\uparrow$ because $U \in \lim_{[X,\$]} \{U\}^\uparrow \cap \mathcal{A}$. In other words, $F \subseteq U^+$, so that $U^+ \in \mathcal{F}$.

Conversely, if O is $[[X,\$],\$]$-open and $\mathcal{A} \in O$, there is $U \in \mathcal{A}$ such that $U^+ \subseteq O$. Otherwise, for each $U \in \mathcal{A}$, there is $\mathcal{B} \in \kappa(X)$ with $U \in \mathcal{B}$ and $\mathcal{B} \notin O$. In that case, $\widehat{U} := \{\mathcal{B} \in \kappa(X) : U \in \mathcal{B}, \mathcal{B} \notin O\} \neq \emptyset$

for all $U \in \mathcal{A}$. Note also that in view of 2.2, $\mathcal{B}_U \cap \mathcal{B}_V \in \widehat{U} \cap \widehat{V}$ whenever $\mathcal{B}_U \in \widehat{U}$ and $\mathcal{B}_V \in \widehat{V}$. Therefore $\left\{\bigcap_{i\in I} \widehat{U}_i : U_i \in \mathcal{A} : \operatorname{card} I < \infty\right\}$ is a filter-base generating a filter $\mathcal{F}$. This filter converges to $\mathcal{A}$ in $[[X,\$],\$]$. To show that, we need to see that $\left\{\bigcap_{\mathcal{B}\in\widehat{U}} \mathcal{B} : U \in \mathcal{A}\right\}$ is a cover of $\mathcal{A}$ for $[X,\$]$. In view of the form 2.1 of a base for $[X,\$]$, it is enough to show that if $U_0 \in \mathcal{A}$ and $\mathcal{P}$ is an ideal subbase of open subsets of X covering U_0, then there is $A \in \mathcal{A}$ with $\bigcap_{\mathcal{B}\in\widehat{A}} \mathcal{B} \in \mathcal{O}^{\natural}(\mathcal{P})$. Because $U_0 \subseteq \bigcup_{P\in\mathcal{P}} P$ and $\mathcal{A}$ is a compact family, there is a finite subfamily $\mathcal{P}_0$ of $\mathcal{P}$ such that $\bigcup_{P\in\mathcal{P}_0} P \in \mathcal{A}$. Since $\mathcal{P}$ is an ideal subbase, there is $P \in \mathcal{P} \cap \mathcal{A}$. Then $\mathcal{O}(P) \subseteq \bigcap_{\mathcal{B}\in\widehat{P}} \mathcal{B}$, which concludes the proof that $\mathcal{A} \in \lim_{[[X,\$],\$]} \mathcal{F}$. On the other hand, $O \notin \mathcal{F}$, which contradicts the fact that O is open for $[[X,\$],\$]$. □

In order to investigate when $T[X,\$]$, that is, $\mathcal{O}_X$ with the Scott topology, is core-compact, we will need notions and results from [6]. The concrete endofunctor Epi_T of the category of convergence spaces (and continuous maps) is defined (on objects) by

$$\operatorname{Epi}_T X = i^{-}[T[X,\$],\$]$$

where $i : X \to [[X,\$],\$]$ is defined by $i(x)(f) = f(x)$. In view of [6, Theorem 3.1]

$$W \geq \operatorname{Epi}_T X \iff T[X,\$] \geq [W,\$], \tag{2.3}$$

where $X \geq W$ have the same underlying set. In particular, X is T-dual if and only if $X \geq \operatorname{Epi}_T X$. A convergence space X is *epitopological* if $i : X \to [[X,\$],\$]$ is initial (in the category **Conv** of convergence spaces and continuous maps). Epitopologies form a reflective subcategory **Epi** of **Conv** and the (concrete) reflector is given (on objects) by $\operatorname{Epi} X = i^{-}[[X,\$],\$]$. Because $[\operatorname{Epi} X,\$] = [X,\$]$, we assume from now on that every space is epitopological. Observe that a topological space is epitopological. Note that if $[X,\$]$ is T-dual, then $\operatorname{Epi} X = X$ is topological. Therefore, in contrast to Proposition 1, $[X,\$]$ is not T-dual if X is not topological.

Proposition 2. *Let X be an epitopological space. Then X is topological if and only if $[X,\$]$ is T-dual.*

Note also that $\operatorname{Epi} X \leq \operatorname{Epi}_T X$ and that $\operatorname{Epi}_T \circ \operatorname{Epi} = \operatorname{Epi}_T$, so that Epi_T restricts to an expansive endofunctor of **Epi**. By iterating this functor, we obtain the coreflector on T-dual epitopologies. More precisely, if F is an expansive concrete endofunctor of **C**, we define the transfinite sequence of functors F^{α} by $F^1 = F$ and $F^{\alpha}X =$

$F\left(\bigvee_{\beta<\alpha} F^{\beta}X\right)$. For each epitopological space X, there is $\alpha(X)$ such that

$$\operatorname{Epi}_T^{\alpha(X)} X = \operatorname{Epi}_T^{\alpha(X)+1} X := d_T X.$$

Proposition 3. *The class of T-dual epitopologies is concretely coreflective in* ***Epi*** *and the coreflector is d_T.*

Proof. The class of T-dual convergences is closed under infima because

$$\left[\bigwedge_{i\in I} X_i, Z\right] = \bigvee_{i\in I}[X_i, Z].$$

Indeed, if each X_i is T-dual, then

$$\left[\bigwedge_{i\in I} X_i, \$\right] = \bigvee_{i\in I}[X_i, \$] = \bigvee_{i\in I} T[X_i, \$] \leq T\left(\bigvee_{i\in I}[X_i, \$]\right) = T\left(\left[\bigwedge_{i\in I} X_i, \$\right]\right),$$

and $\bigwedge_{i\in I} X_i$ is T-dual. The functor Epi_T is expansive on **Epi** and therefore, so is d_T. Moreover, $d_T X$ is T-dual for each epitopological space X because

$$[d_T X, \$] = [\operatorname{Epi}_T^{\alpha(X)+1} X, \$] \leq T[\operatorname{Epi}_T^{\alpha(X)} X, \$] = T[d_T X, \$].$$

Therefore, for each epitopological space X, there exists the coarsest T-dual convergence $\overline{X}$ finer than X. By definition $X \leq \overline{X} \leq d_T X$. Then $[\overline{X}, \$] \leq [X, \$]$ and $[\overline{X}, \$]$ is topological, so that $[\overline{X}, \$] \leq T[X, \$]$. But $\operatorname{Epi}_T X$ is the coarsest convergence with this property. Therefore $\operatorname{Epi}_T X \leq \overline{X} = \operatorname{Epi}_T \overline{X}$ and $d_T X \leq \overline{X}$. □

Proposition 4. *If X is a core compact topological space, so is $T[X, \$]$.*

Proof. Under these assumptions, $[X, \$] = T[X, \$]$ and $[X, \$]$ is T-dual by Proposition 1. Therefore $T[X, \$]$ is a core compact topology. □

However, if X is a non-topological T-dual convergence space ([2]), then $[X, \$] = T[X, \$]$ is *not* core compact, by Proposition 2. In other words, we have:

Proposition 5. *If X is T-dual then X is topological if and only if $[X, \$] = T[X, \$]$ is core compact.*

In particular, $d_T X$ is topological if and only if $[d_T X, \$]$ is core compact.

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[2]Such convergences exist: take for a instance a non-locally compact Hausdorff regular topological k-space. Then $X = TK_hX$ but $X < K_hX$ so that K_hX is non-topological.

Theorem 6. *If $X \geq Td_T X$ then $T[X,\$]$ is core compact if and only if X is a core compact topological space.*

Proof. We already know that if X is a core compact topological space then $[X,\$] = T[X,\$]$ and that $[X,\$]$ is core compact by Proposition 1. Conversely, if $T[X,\$]$ is core compact then $[T[X,\$],\$]$ is topological, so that $\operatorname{Epi}_T X$ is topological. Under our assumptions, we have

$$X \geq Td_T X \geq T \operatorname{Epi}_T X = \operatorname{Epi}_T X,$$

so that X is T-dual. Therefore $[X,\$] = T[X,\$]$ is core compact and, in view of Proposition 2, X is topological, and T-dual, hence a core compact topological space. □

Note that, at least among Hausdorff topological spaces, Theorem 6 generalizes [16, Corollary 3.6] that states that if X is first countable, then X is core compact if and only if $T[X,\$]$ is core compact. Indeed, the locally compact coreflection KX of a Hausdorff topological space is T-dual so that $d_T X \leq KX$. Hence if X is a Hausdorff topological k-space, that is $X = TKX$, (in particular a first-countable space) then $X \geq Td_T X$. We will see in the next section that similarly, if X is a consonant topological space, then $T[X,\$]$ is core compact if and only if X is locally compact.

Problem 7. Are there completely regular non locally compact topological spaces X such that $T[X,\$]$ is core compact?

Of course, in view of the observations above, such a space cannot be a k-space or consonant.

3. Core compact dual, Consonance, and infraconsonance

A topological space is *consonant* [4] if every Scott open subset $\mathcal{A}$ of $\mathcal{O}_X$ is compactly generated, that is, there are compact subsets $(K_i)_{i\in I}$ of X such that $\mathcal{A} = \bigcup_{i\in I} \mathcal{O}(K_i)$. A space is *infraconsonant* [7] if for every Scott open subset $\mathcal{A}$ of $\mathcal{O}_X$ there is a Scott open set $\mathcal{C}$ such that $\mathcal{C} \vee \mathcal{C} \subseteq \mathcal{A}$, where $\mathcal{C} \vee \mathcal{C} := \{C \cap D : C, D \in \mathcal{C}\}$. The notion's importance stems from Theorem 8 below. If the set $C(X,Y)$ of continuous functions from X to Y is equipped with the Isbell topology ([3]), we denote it $C_\kappa(X,Y)$, while $C_k(X,Y)$ denotes $C(X,Y)$ endowed with the compact-open topology. Note that $C_\kappa(X,\$) = T[X,\$]$.

[3]whose sub-basic open sets are given by

$$[\mathcal{A}, U] := \{f \in C(X,Y) : \exists A \in \mathcal{A},\ f(A) \subseteq U\},$$

where $\mathcal{A}$ ranges over openly isotone compact families on X and U ranges over open subsets of Y.

Theorem 8. [5] *Let X be a completely regular topological space. The following are equivalent:*

1. *X is infraconsonant;*
2. *addition is jointly continuous at the zero function in $C_\kappa(X,\mathbb{R})$;*
3. *$C_\kappa(X,\mathbb{R})$ is a topological vector space;*
4. *$\cap : T[X,\$] \times T[X,\$] \to T[X,\$]$ is jointly continuous.*

On the other hand, if X is consonant then $C_\kappa(X,\mathbb{R}) = C_k(X,\mathbb{R})$ so that consonance provides an obvious sufficient condition for $C_\kappa(X,\mathbb{R})$ to be a topological vector space. Hence Theorem 8 becomes truly interesting if completely regular examples of infraconsonant non consonant spaces can be provided [7, Problem 1.2]. The first author recently obtained the first example of this kind. The following results show that a space answering positively Problem 7 would necessarily be infraconsonant and non-consonant and might provide an avenue to construct new examples.

Theorem 9. *If X is topological and $T[X,\$]$ is core compact then X is infraconsonant.*

Proof. [7, Lemma 3.3] shows the equivalence between (1) and (4) in Theorem 8, and that the implication (4)$\Longrightarrow$(1) does not require any separation. Therefore, it is enough to show that $\cap : T[X,\$]\times T[X,\$] \to T[X,\$]$ is continuous. Since X is topological, $[X,\$]$ is T-dual by Proposition 1. In view of (1.2)

$$T([X,\$]\times[X,\$]) \leq [X,\$]\times T[X,\$]$$

so that $T([X,\$]\times[X,\$]) \leq T([X,\$]\times T[X,\$])$. If $T[X,\$]$ is core compact, hence T-dual then $T([X,\$]\times T[X,\$]) \leq T[X,\$]\times T[X,\$]$ so that

$$T([X,\$]\times[X,\$]) \leq T[X,\$]\times T[X,\$]. \tag{3.1}$$

Therefore the continuity of $\cap : [X,\$]\times[X,\$] \to [X,\$]$ implies that of $\cap : T([X,\$]\times[X,\$]) \to T[X,\$]$ because T is a functor, and in view of (3.1), that of $\cap : T[X,\$]\times T[X,\$] \to T[X,\$]$. □

Theorem 10. *Let X be a topological space. If $C_k(X,\$)$ is core compact then X is locally compact.*

Proof. If X is not locally compact, then $C_k(X,\$) \ngeq [X,\$]$ (e.g., [18, 2.19]) so that there is $U_0 \in \mathcal{O}_X$ with $U_0 \notin \lim_{[X,\$]} \mathcal{N}_k(U_0)$. Therefore, there is $x_0 \in U_0$ such that $x_0 \notin \operatorname{int}\left(\bigcap_{V\in\mathcal{O}(K)} V\right)$ whenever K is a compact subset of X with $K \subseteq U_0$. In other words, for each such K and for each $U \in \mathcal{O}(x_0)$ there is $V_U \in \mathcal{O}(K)$ and $x_U \in U \setminus V_U$.

Then $C_k(X,\$)$ is not core compact at U_0. Indeed, there is $U_0 \in \mathcal{O}(x_0)$ such that for every compact set K with $K \subseteq U_0$, the k-open set $\mathcal{O}(K)$ is not relatively compact in $\mathcal{O}(x_0)$. To see that, consider the cover $\mathcal{S} := \{\mathcal{O}(x_U) : U \in \mathcal{O}(x_0)\}$ of $\mathcal{O}(x_0)$. No finite subfamily of $\mathcal{S}$ covers $\mathcal{O}(K)$ because for any finite choice of $U_1, \ldots, U_n$ in $\mathcal{O}(x_0)$, we have $W := \cap_{i=1}^{i=n} V_{U_i} \in \mathcal{O}(K)$ but $W \notin \cup_{i=1}^{i=n} \mathcal{O}(x_{U_i})$. □

Note that a Hausdorff topological space X is locally compact if and only if it is core compact, and that the Scott open filter topology on $\mathcal{O}(X)$ then coincides with $C_k(X,\$)$ (e.g., [9, Lemma II.1.19]). Hence Theorem 10 could also be deduced (for the Hausdorff case) from [16, Corollary 3.6].

Corollary 11. *If X is a consonant topological space such that $T[X,\$]$ is core compact, then X is locally compact.*

4. Scott topology of the product versus product of Scott topologies

We now turn to a new characterization of infraconsonance, which motivates further the systematic investigation of the notion.

Proposition 12. *$T([X,\$]^2)$ is the Scott topology on $\mathcal{O}_X \times \mathcal{O}_X$.*

Theorem 13. *A space X is infraconsonant if and only if the product $(T[X,\$])^2$ of the Scott topologies on $\mathcal{O}_X$ and the Scott topology $T([X,\$]^2)$ on the product $\mathcal{O}_X \times \mathcal{O}_X$ coincide at (X,X).*

Lemma 14. *A subset $\mathcal{S}$ of $\mathcal{O}_X \times \mathcal{O}_X$ is $[X,\$]^2$-open if and only if*
(1) *$\mathcal{S} = \mathcal{S}^{\uparrow}$, that is, if $(U,V) \in \mathcal{S}$ and $U \subseteq U'$, $V \subseteq V'$ then $(U',V') \in \mathcal{S}$;*
(2) *$\mathcal{S}$ is coordinatewise compact, that is,*

$$\left(\bigcup_{i\in I} O_i, \bigcup_{j\in J} V_j\right) \in \mathcal{S} \Longrightarrow \exists I_0 \in [I]^{<\omega}, J_0 \in [J]^{<\omega} : \left(\bigcup_{i\in I_0} O_i, \bigcup_{j\in J_0} V_j\right) \in \mathcal{S}$$

Proof. Assume $\mathcal{S}$ is $[X,\$]^2$-open and let $(U,V) \in \mathcal{S}$ and $U \subseteq U'$, $V \subseteq V'$. Then $(U,V) \in \lim_{[X,\$]^2}\{(U',V')\}^{\uparrow}$ so that $(U',V') \in \mathcal{S}$. Assume now that $\left(\bigcup_{i\in I} O_i, \bigcup_{j\in J} V_j\right) \in \mathcal{S}$. Then $\{\mathcal{O}\left(\bigcup_{i\in F} O_i\right) : F \in [I]^{<\infty}\}$ is a filter-base for a filter γ on $\mathcal{O}_X$ such that $\bigcup_{i\in I} O_i \in \lim_{[X,\$]} \gamma$ and $\{\mathcal{O}\left(\bigcup_{j\in D} V_j\right) : D \in [J]^{<\infty}\}$ is a filter-base for a filter η on $\mathcal{O}_X$ such that $\bigcup_{j\in J} V_j \in \lim_{[X,\$]} \eta$. Hence $\mathcal{S} \in \gamma \times \eta$ because $\mathcal{S}$ is $[X,\$]^2$-open. Therefore, there are finite subsets I_0 of I and J_0 of J such that $\mathcal{O}\left(\bigcup_{i\in I_0} O_i\right) \times \mathcal{O}\left(\bigcup_{j\in J_0} V_j\right) \subseteq \mathcal{S}$, so that $\left(\bigcup_{i\in I_0} O_i, \bigcup_{j\in J_0} V_j\right) \in \mathcal{S}$.

Conversely, assume that $\mathcal{S}$ satisfies the two conditions of the Lemma and $(U,V) \in \mathcal{S} \cap \lim_{[X,\$]^2}(\gamma \times \eta)$. Since $U \subseteq \bigcup_{\mathcal{G}\in\gamma} \operatorname{int}\left(\bigcap_{G\in\mathcal{G}} G\right)$ and $V \subseteq \bigcup_{\mathcal{H}\in\eta} \operatorname{int}\left(\bigcap_{H\in\mathcal{H}} H\right)$, we have, by the first condition, that $\left(\bigcup_{\mathcal{G}\in\gamma} \operatorname{int}\left(\bigcap_{G\in\mathcal{G}} G\right), \bigcup_{\mathcal{H}\in\eta} \operatorname{int}\left(\bigcap_{H\in\mathcal{H}} H\right)\right) \in \mathcal{S}$. By the second condition, there are $\mathcal{G}_1, \ldots, \mathcal{G}_k \in \gamma$ and $\mathcal{H}_1, \ldots, \mathcal{H}_n \in \eta$ such that

$$\left(\bigcup_{i=1}^{k} \operatorname{int}\left(\bigcap_{G\in\mathcal{G}_i} G\right), \bigcup_{j=1}^{n} \operatorname{int}\left(\bigcap_{H\in\mathcal{H}_j} H\right)\right) \in \mathcal{S}.$$

Therefore $\left(\operatorname{int}\left(\bigcap_{G\in\bigcap_{i=1}^{k}\mathcal{G}_i} G\right), \operatorname{int}\left(\bigcap_{H\in\bigcap_{j=1}^{n}\mathcal{H}_j} H\right)\right) \in \mathcal{S}$ so that

$$\left(\bigcap_{i=1}^{k} \mathcal{G}_i, \bigcap_{j=1}^{n} \mathcal{H}_j\right) \subseteq \mathcal{S},$$

and $\mathcal{S} \in \gamma \times \eta$. □

Proof of Proposition 12. In view of Lemma 14, every $[X,\$]^2$-open subset of $\mathcal{O}_X \times \mathcal{O}_X$ is Scott open. Conversely, consider a Scott open subset $\mathcal{S}$ of $\mathcal{O}_X \times \mathcal{O}_X$. We only have to check that $\mathcal{S}$ statisfies the second condition in Lemma 14. Let $(\bigcup_{i\in I} O_i, \bigcup_{j\in J} V_j) \in \mathcal{S}$. The set $D := \{\left(\bigcup_{i\in I_0} O_i, \bigcup_{j\in J_0} V_j\right) : I_0 \in [I]^{<\omega}, J_0 \in [J]^{<\omega}\}$ is a directed subset of $\mathcal{O}_X \times \mathcal{O}_X$ (for the coordinatewise inclusion order) whose supremum is $(\bigcup_{i\in I} O_i, \bigcup_{j\in J} V_j)$. As $\mathcal{S}$ is Scott-open, there are finite subsets I_0 of I and J_0 of J such that $\left(\bigcup_{i\in I_0} O_i, \bigcup_{j\in J_0} V_j\right) \in \mathcal{S}$. □

Lemma 15. *If* $\mathcal{A} \in \kappa(X)$ *then* $\mathcal{S}_\mathcal{A} := \{(U,V) \in \mathcal{O}_X \times \mathcal{O}_X : U \cap V \in \mathcal{A}\}^{\uparrow}$ *is* $[X,\$]^2$*-open.*

Proof. Let $(\bigcup_{i\in I} O_i, \bigcup_{j\in J} V_j) \in \mathcal{S}_\mathcal{A}$. Then

$$(\bigcup_{i\in I} O_i) \cap (\bigcup_{j\in J} V_j) = \bigcup_{(i,j)\in I\times J} O_i \cap V_j \in \mathcal{A}.$$

By compactness of $\mathcal{A}$, there is a finite subset I_0 of I and a finite subset J_0 of J such that $\bigcup_{(i,j)\in I_0\times J_0} O_i \cap V_j \in \mathcal{A}$, so that $(\bigcup_{i\in I_0} O_i, \bigcup_{j\in J_0} V_j) \in \mathcal{S}_\mathcal{A}$. In view of Lemma 14, $\mathcal{S}_\mathcal{A}$ is $[X,\$]^2$-open. □

Lemma 16. *If* $\mathcal{S}$ *is* $[X,\$]^2$*-open, then*

$$\downarrow \mathcal{S} := \mathcal{O}_X(\{U \cup V : (U,V) \in \mathcal{S}\})$$

is a compact family on X*.*

Proof. If $U\cup V\subseteq\bigcup_{i\in I}O_i$ for some $(U,V)\in\mathcal{S}$ then $\left(\bigcup_{i\in I}O_i,\bigcup_{i\in I}O_i\right)\in\mathcal{S}$ so that, in view of Lemma 14, there is a finite subset I_0 of I such that $\left(\bigcup_{i\in I_0}O_i,\bigcup_{i\in I_0}O_i\right)\in\mathcal{S}$. Hence $\bigcup_{i\in I_0}O_i\in\downarrow\mathcal{S}$. □

Proof of Theorem 13. Suppose that X is infraconsonant. Note that $(T[X,\$])^2\leq T([X,\$]^2)$ is always true, so that we only have to prove the reverse inequality at (X,X). Consider an $[X,\$]^2$-open neighborhood $\mathcal{S}$ of (X,X). By Lemma 16, the family $\downarrow\mathcal{S}$ is compact. By infraconsonance, there is $\mathcal{C}\in\kappa(X)$ with $\mathcal{C}\vee\mathcal{C}\subseteq\downarrow\mathcal{S}$. Note that

$$\mathcal{C}\times\mathcal{C}\subseteq\mathcal{S},$$

because if $(C_1,C_2)\in\mathcal{C}\times\mathcal{C}$ then $C_1\cap C_2\in\downarrow\mathcal{S}$ so that $C_1\cap C_2\supseteq U\cup V$ for some $(U,V)\in\mathcal{S}$, and therefore $(C_1,C_2)\in\mathcal{S}$.

Conversely, assume that $\mathcal{N}_{[X,\$]^2}(X,X)=\mathcal{N}_{T[X,\$]^2}(X,X)$ and let $\mathcal{A}\in\kappa(X)$. By Lemma 15, $\mathcal{S}_\mathcal{A}\in\mathcal{N}_{[X,\$]^2}(X,X)$ so that $\mathcal{S}_\mathcal{A}\in\mathcal{N}_{T[X,\$]^2}(X,X)$. In other words, there are families $\mathcal{B}$ and $\mathcal{C}$ in $\kappa(X)$ such that $\mathcal{B}\times\mathcal{C}\subseteq\mathcal{S}_\mathcal{A}$. In particular $\mathcal{D}:=\mathcal{B}\cap\mathcal{C}$ belongs to $\kappa(X)$ and satisfies $\mathcal{D}\times\mathcal{D}\subseteq\mathcal{S}_\mathcal{A}$. By definition of $\mathcal{S}_\mathcal{A}$, we have that $\mathcal{D}\vee\mathcal{D}\subseteq\mathcal{A}$ and X is infraconsonant. □

5. Topologicity, pretopologicity and diagonality of $[X,\$]$

A convergence space X is *diagonal* if for every selection $\mathcal{S}[\cdot]:X\to\mathbb{F}X$ with $x\in\lim_X\mathcal{S}[x]$ for all $x\in X$ and every filter $\mathcal{F}$ with $x_0\in\lim_X\mathcal{F}$ the filter

$$\mathcal{S}[\mathcal{F}]:=\bigcup_{F\in\mathcal{F}}\bigcap_{x\in F}\mathcal{S}[x] \tag{5.1}$$

converges to x_0. If this property only holds when $\mathcal{F}$ is additionally principal, we say that X is $\mathbb{F}_1$-*diagonal.* Of course, every topology is diagonal. In fact a convergence is topological if and only if it is both pretopological and diagonal (e.g., [3]).

In order to compare our condition for diagonality of $[X,\$]$ with core-compactness, we first rephrase the condition of core-compactness.

Lemma 17. *A topological space is core compact if and only if for every $x\in X$, every $U\in\mathcal{O}(x)$ and every family $\mathbb{H}$ of filters on X, we have*

$$\forall\mathcal{H}\in\mathbb{H}:\operatorname{adh}\mathcal{H}\cap U=\emptyset\Longrightarrow x\notin\operatorname{adh}\bigwedge_{\mathcal{H}\in\mathbb{H}}\mathcal{H}. \tag{5.2}$$

Proof. If X is core compact, then there is $V\in\mathcal{O}(x)$ which is relatively compact in U. If $\operatorname{adh}\mathcal{H}\cap U=\emptyset$ for every $\mathcal{H}\in\mathbb{H}$, then $U\subseteq\bigcup_{H\in\mathcal{H}}(\operatorname{cl}H)^c$ so that, by relative compactness of V in U there is, for each $\mathcal{H}\in\mathbb{H}$, a set $H_\mathcal{H}\in\mathcal{H}$ with $V\cap\operatorname{cl}H_\mathcal{H}=\emptyset$. Then $\bigcup_{\mathcal{H}\in\mathbb{H}}H_\mathcal{H}\in\bigwedge_{\mathcal{H}\in\mathbb{H}}\mathcal{H}$ but $\bigcup_{\mathcal{H}\in\mathbb{H}}H_\mathcal{H}\cap V=\emptyset$ so that $x\notin\operatorname{adh}\bigwedge_{\mathcal{H}\in\mathbb{H}}\mathcal{H}$.

Conversely, if (5.2) is true, consider the family $\mathbb{H} := \{\mathcal{H} \in \mathbb{F}X : \operatorname{adh}\mathcal{H} \cap U = \emptyset\}$. In view of (5.2), $x \notin \operatorname{adh} \bigwedge_{\mathcal{H}\in\mathbb{H}} \mathcal{H}$ so that there is $V \in \mathcal{O}(x)$ such that $V \notin \left(\bigwedge_{\mathcal{H}\in\mathbb{H}} \mathcal{H}\right)^{\#}$. Now V is relatively compact in U because any filter than meshes with V cannot be in $\mathbb{H}$ and has therefore adherence point in U. □

Recall that $[X,\$] = P[X,\$]$ if and only if X is T-core compact, and that, if X is topological, $[X,\$]$ is topological whenever it is pretopological. While the latest follows for instance from the results of [6], it seems difficult to find an elementary argument in the literature, which is why we include the following proposition, which also illustrates the usefulness of Lemma (17).

Proposition 18. *If X is topological and $[X,\$]$ is pretopological, then $[X,\$]$ is topological.*

Proof. We will show that under these assumptions, X satisfies (5.2). Let $x \in X$ and $U \in \mathcal{O}(x)$. Let $\mathbb{H}$ be a family of filters satisfying the hypothesis of (5.2). Let $\mathcal{H} \in \mathbb{H}$. Consider the filter base $\mathcal{H}^* := \{\mathcal{O}(X \setminus \operatorname{cl}(H)) \colon H \in \mathcal{H}\}$ on $[X,\$]$. Since $\operatorname{adh}(\mathcal{H}) \cap U = \emptyset$, it follows that $U \in \lim \mathcal{H}^*$. Since $[X,\$]$ is pretopological, $U \in \lim \bigwedge_{\mathcal{H}\in\mathbb{H}} \mathcal{H}^*$. In particular, there exist, for each $\mathcal{H} \in \mathbb{H}$, a $H_{\mathcal{H}} \in \mathcal{H}$ such that

$$\begin{aligned} x \in \operatorname{int}\left(\bigcap \bigcup_{\mathcal{H}\in\mathbb{H}} \mathcal{O}(X \setminus \operatorname{cl}(H_{\mathcal{H}}))\right) &= \operatorname{int}\left(\bigcap_{\mathcal{H}\in\mathbb{H}} (X \setminus \operatorname{cl}(H_{\mathcal{H}}))\right) \\ &= \operatorname{int}\left(X \setminus (\bigcup_{\mathcal{H}\in\mathbb{H}} \operatorname{cl}(H_{\mathcal{H}}))\right) \\ &\subseteq X \setminus \operatorname{cl}(\bigcup_{\mathcal{H}\in\mathbb{H}} H_{\mathcal{H}}). \end{aligned}$$

Thus, $x \notin \operatorname{adh}(\bigwedge_{\mathcal{H}\in\mathbb{H}})$. □

In other words, if $[X,\$]$ is pretopological it is also diagonal, provided that X is topological. We will see that even if X is topological, $[X,\$]$ is not always diagonal. Moreover it can be diagonal without being pretopological. On the other hand, we do not know if $[X,\$]$ can be pretopological but not diagonal (equivalently, if X can be T-core compact but not T-dual).

We call a topological space *injectively core compact* if for every $x \in X$ and $U \in \mathcal{O}(x)$ the conclusion of (5.2) holds for every family $\mathbb{H}$ of filters such that there is an injection $\theta : \mathbb{H} \to \mathcal{O}(U)$ satisfying $\operatorname{adh}\mathcal{H} \cap \theta(\mathcal{H}) = \emptyset$ for each $\mathcal{H} \in \mathbb{H}$. As such a family $\mathbb{H}$ clearly satisfies the premise of (5.2), every core compact space is in particular injectively core compact.

Theorem 19. *Let X be a topological space. The following are equivalent:*

1. *X is injectively core compact;*
2. *$[X,\$]$ is diagonal;*
3. *$[X,\$]$ is $\mathbb{F}_1$-diagonal.*

Proof. (1)$\Longrightarrow$(2): Let $\mathcal{S}[\centerdot]:\mathcal{O}_X\to\mathbb{F}\mathcal{O}_X$ be a selection for $[X,\$]$ and let $U\in\lim_{[X,\$]}\mathcal{F}$. If $x\in U$, there is $F\in\mathcal{F}$ such that $x\in\operatorname{int}\left(\bigcap_{O\in F}O\right):=V$. Note that $F\subseteq\mathcal{O}(V)$. For each $O\in F$, consider the filter $\mathcal{H}_O$ on X generated by $\{\operatorname{cl}_X\left(\bigcup_{W\in S}W^c\right):S\in\mathcal{S}[O]\}$. Because $O\in\lim_{[X,\$]}\mathcal{S}[O]$, we have that $\operatorname{adh}_X\mathcal{H}_O\cap O=\emptyset$. Because X is injectively core compact and $\mathbb{H}:=\{\mathcal{H}_O:O\in F\}$ satisfies the required condition (with $\theta(\mathcal{H}_O)=O$), we conclude that $x\notin\operatorname{adh}_X\bigwedge_{O\in F}\mathcal{H}_O$. In other words, there is $H\in\bigwedge_{O\in F}\mathcal{H}_O$ such that $x\notin\operatorname{cl}_X H$, that is, $x\in\operatorname{int}_X H^c$. Therefore, for each $O\in F$ there is $S_O\in\mathcal{S}[O]$ such that

$$x\in\operatorname{int}\left(\bigcap_{O\in F}\operatorname{int}\left(\bigcap_{W\in S_O}W\right)\right)\subseteq\operatorname{int}\left(\bigcap_{W\in\bigcup_{O\in F}S_O}W\right).$$

In other words, there is $F\in\mathcal{F}$ and $M\in\bigwedge_{O\in F}\mathcal{S}[O]$ such that $x\in\operatorname{int}_X\left(\bigcap_{W\in M}W\right)$, that is, $U\in\lim_{[X,\$]}\mathcal{S}[\mathcal{F}]$.

(2)$\Longrightarrow$(3) is clear. (3)$\Longrightarrow$(1): Suppose X is not injectively core compact. Then there is $x\in X$, $U\in\mathcal{O}(x)$ and a family $\mathbb{H}$ of filters on X with an injective map $\theta\colon\mathbb{H}\to\mathcal{O}(U)$ such that $\theta(\mathcal{H})\cap\operatorname{adh}_X\mathcal{H}=\emptyset$ for each $\mathcal{H}\in\mathbb{H}$ but $x\in\operatorname{adh}_X\bigwedge_{\mathcal{H}\in\mathbb{H}}\mathcal{H}$. Define a relation $\sim$ on $\mathbb{H}$ by $H_1\sim H_2$ provided that the collections $\{\operatorname{cl}(H)\colon H\in\mathcal{H}_1\}$ and $\{\operatorname{cl}(H)\colon H\in\mathcal{H}_2\}$ both generate the same filter. Clearly, $\sim$ is an equivalance relation. Let $\mathbb{H}^*\subseteq\mathbb{H}$ be such that $\mathbb{H}^*$ contains exactly one element of each equivalance class of $\sim$. For each $\mathcal{H}\in\mathbb{H}^*$ let $\mathcal{H}^*$ be the filter with base $\{\operatorname{cl}(H)\colon H\in\mathcal{H}\}$. Let $\mathbb{J}=\{\mathcal{H}^*\colon\mathcal{H}\in\mathbb{H}^*\}$.

Define $\theta^*\colon\mathbb{J}\to\mathcal{O}(U)$ so that $\theta^*(\mathcal{J})=\theta(\mathcal{H})$, where $\mathcal{H}\in\mathbb{H}^*$ is such that $\mathcal{J}=\mathcal{H}^*$. It is easily checked that θ^* is injective. Since $\operatorname{adh}(\mathcal{H}^*)=\operatorname{adh}(\mathcal{H})$ for every $\mathcal{H}\in\mathbb{H}^*$, we have $\theta^*(\mathcal{J})\cap\operatorname{adh}(\mathcal{J})=\emptyset$. It is also easy to check that $x\in\operatorname{adh}\left(\bigwedge_{\mathcal{J}\in\mathbb{J}}\mathcal{J}\right)$.

For each $\mathcal{J}\in\mathbb{J}$, the filter $\widetilde{\mathcal{J}}$ generated on $\mathcal{O}_X$ by the filter-base $\{O_X(X\setminus J)\colon J\in\mathcal{J}\}$ converges to $\theta^*(\mathcal{J})$. Consider now the subset $\theta^*(\mathbb{J})$ of $\mathcal{O}(U)\subseteq\mathcal{O}_X$ and the selection $\mathcal{S}[\centerdot]:\mathcal{O}_X\to\mathbb{F}\mathcal{O}_X$ defined by $\mathcal{S}[\theta(\mathcal{J})]=\widetilde{\mathcal{J}}$ for each $\mathcal{J}\in\mathbb{J}$ and $\mathcal{S}[O]=\{O\}^{\uparrow}$ for $O\notin\theta^*(\mathbb{J})$. This is indeed a well-defined selection because θ^* is injective.

Notice that $U\in\lim_{[X,\$]}\theta^*(\mathbb{J})$ because $\theta^*(\mathbb{J})\subseteq\mathcal{O}(U)$. Let $L\in\mathcal{S}[\theta^*(\mathbb{J})]$. We may pick from each $\mathcal{J}\in\mathbb{J}$ a closed set $J_{\mathcal{J}}\in\mathcal{J}$ such that $\bigcup_{\mathcal{J}\in\mathbb{J}}\mathcal{O}_x(X\setminus J_{\mathcal{J}})\subseteq L$. Let V be an open neighborhood of x. Since

$x \in \mathrm{adh}_X \bigwedge_{\mathcal{J}\in\mathbb{J}} \mathcal{J}$ and $\bigcup_{\mathcal{J}\in\mathbb{J}} J_{\mathcal{J}} \in \bigwedge_{\mathcal{J}\in\mathbb{J}} \mathcal{J}$, there is an $\mathcal{J}_0 \in \mathbb{J}$ such that $V \cap J_{\mathcal{J}_0} \neq \emptyset$. Since $V \nsubseteq X \setminus J_{\mathcal{J}_0}$ and $X \setminus J_{\mathcal{J}_0} \in \mathcal{O}_X(X \setminus J_{J_0}) \subseteq L$, $V \nsubseteq \bigcap \mathcal{O}_X(X \setminus J_{\mathcal{J}_0})$. Since $\mathcal{O}_X(X \setminus J_{\mathcal{J}_0}) \subseteq L$, $V \nsubseteq \bigcap L$. Since V was an arbitrary neighborhood of x, $x \notin \mathrm{int}(\bigcap L)$. Thus, $U \notin \mathcal{S}[\theta^*(\mathbb{J})]$. Therefore, $[X, \$]$ is not $\mathbb{F}_1$-diagonal. □

A cardinal number κ is *regular* if a union of less than κ-many sets of cardinality less than κ has cardinality less than κ. A strong limit cardinal κ is a cardinal for which $\mathrm{card}(2^A) < \kappa$ whenever $\mathrm{card}(A) < \kappa$. A *strongly inaccessible cardinal* is a regular strong limit cardinal. Uncountable strongly inaccessible cardinals cannot be proved to exist within ZFC, though their existence is not known to be inconsistent with ZFC. Let us denote by (*) the assumption that such a cardinal exist.

Example 20 (A Hausdorff space X such that $[X, \$]$ is diagonal but not pretopological under (*))**.** Assume that κ is a (uncountable) strong limit cardinal. Let X be the subspace of $\kappa \cup \{\kappa\}$ endowed with the order topology, obtained by removing all the limit ordinals but κ. Since X is a non locally compact Hausdorff topological space, $[X, \$]$ is not pretopological. To show that X is injectively core compact, we only need to consider $x = \kappa$ and $U \in \mathcal{O}(\kappa)$ in the definition, because κ is the only non-isolated point of X. Let $\mathbb{H}$ be a family of filters on X admitting an injective map $\theta : \mathbb{H} \to \mathcal{O}(U)$ such that $\mathrm{adh}\,\mathcal{H} \cap \theta(\mathcal{H}) = \emptyset$ for each $\mathcal{H} \in \mathbb{H}$. Let β be the least element of U. For each $\mathcal{H} \in \mathbb{H}$ there is $H_{\mathcal{H}} \in \mathcal{H}$ such that $\beta \notin H_{\mathcal{H}}$ so that $\mathrm{card}(H) < \beta$. Moreover, $\mathrm{card}\,\mathbb{H} \leq \mathrm{card}\,\mathcal{O}(U) = 2^\beta$. Since κ is a strong limit cardinal, $\mathrm{card}\,\mathbb{H} < \kappa$. Since κ is regular, $\bigcup_{\mathcal{H}\in\mathbb{H}} H_{\mathcal{H}} < \kappa$ so that $\kappa \notin \mathrm{adh} \bigwedge_{\mathcal{H}\in\mathbb{H}} \mathcal{H}$.

We do not know if the existence of large cardinals is necessary for the construction of a Hausdorff space X such that $[X, \$]$ is diagonal and not pretopological. However, we can construct in ZFC a T_0 space X such that $[X, \$]$ is diagonal and not pretopological.

Example 21 (A T_0 space X such that $[X, \$]$ is diagonal but not pretopological.)**.** Let $\mathbb{Z}$ stand for integers, $\mathfrak{c}$ be the cardinality of the continuum, and $\mathfrak{c}^+$ be the cardinal successor of $\mathfrak{c}$. Let ∞ be a point that is not in $\mathfrak{c}^+ \times \mathbb{Z}$ and $X = \{\infty\} \cup (\mathfrak{c}^+ \times \mathbb{Z})$. For each $(\alpha, n) \in \mathfrak{c}^+ \times \mathbb{Z}$ define $S_{\alpha,n} = \{(\beta, k) \colon \alpha \leq \beta \text{ and } n \leq k\}$. For each $\alpha \in \mathfrak{c}^+$, let $T_\alpha = \{(\beta, k) \colon \alpha \leq \beta \text{ and } k \in \mathbb{Z}\} \cup \{\infty\}$. Topologize X by declaring all sets of the form T_α and $S_{\alpha,n}$ to be sub-basic open sets.

We show that X is not core compact at ∞. Let U be a neighborhood of ∞. There is an α such that $T_\alpha \subseteq U$. Notice that $T_{\alpha+1} \cup \{S_{0,n} \colon n \in \mathbb{Z}\}$

is a cover of X but no finite subcollection covers T_α. Thus, X is not core compact at ∞. In particular, $[X, \$]$ is not pretopological.

Let $(\alpha, n) \in X \setminus \{\infty\}$. Let U be an open neighborhood of (α, n). Since $(\alpha, n) \in U$ it follows from the way we chose our sub-base that $T_{\alpha,n} \subseteq U$. Since (α, n) has a minimal open neighborhood, X is core compact at (α, n).

Let V be an open neighborhood of ∞. There is an α such that $T_\alpha \subseteq V$. Let $U \subseteq X$ be an open superset of V. For every $n \in \mathbb{Z}$ $U \cap (\mathfrak{c}^+ \times \{n\}) \neq \emptyset$. For each $n \in \mathbb{Z}$ define $\alpha_n = \min\{\beta\colon (\beta, n) \in U$. Notice that $\{\beta\colon \alpha_n \leq \beta\} \times \{n\} = U \cap (\mathfrak{c}^+ \times \{n\})$ and $\alpha_n \leq \alpha$. Since each open superset of V will determine a unique sequence $(\alpha_n)_{n\in z}$, it follows that the open supersets of V can injectively be mapped into the countable sequences on $\{\beta\colon \beta \leq \alpha\} \times \mathbb{Z}$. Since $\{\beta\colon \beta \leq \alpha\} \times \mathbb{Z}$ has cardinality at most $\mathfrak{c}$, $\{\beta\colon \beta \leq \alpha\} \times \mathbb{Z}$ has at most $\mathfrak{c}$-many countable sequences. Thus, V has most $\mathfrak{c}$-many supersets.

Let V be an open neighborhood of ∞, $\mathbb{H}$ be a collection of filters, and $\theta\colon \mathbb{H} \to \mathcal{O}_X(V)$ be an injection such that $\operatorname{adh}(\mathcal{H}) \cap \theta(\mathcal{H}) = \emptyset$ for every $\mathcal{H} \in \mathbb{H}$. Since V has at most $\mathfrak{c}$-many open supersets, $|\mathbb{H}| \leq \mathfrak{c}$. Let $\mathcal{H} \in \mathbb{H}$. Since $\infty \notin \operatorname{adh} \mathcal{H}$, there is an $\alpha_{\mathcal{H}} \in \mathfrak{c}^+$ such that $\operatorname{adh}(\mathcal{H}) \cap T_{\alpha_{\mathcal{H}}} = \emptyset$. Let $\alpha = (\sup_{\mathcal{H}\in\mathbb{H}} \alpha_{\mathcal{H}}) + 1 < \mathfrak{c}^+$. It is easy to check that, $\operatorname{adh}\left(\bigwedge_{\mathcal{H}\in\mathbb{H}} \mathcal{H}\right) \cap T_\alpha = \emptyset$. Thus, X is injectively core compact at ∞.

Since X is injectively core compact at each point, $[X, \$]$ is diagonal, by Theorem 19.

Example 22 (A T-dual convergence space that is not core compact)**.** Consider a partition $\{\mathbb{A}_n : n \in \omega\}$ of the set ω^* of free ultrafilters on ω satisfying the condition that for every infinite subset S of ω and every $n \in \omega$, there is $\mathcal{U} \in \mathbb{A}_n$ with $S \in \mathcal{U}$. Let $M := \{m_n : n \in \omega\}$ be disjoint from ω and let $X := \omega \cup M$. Define on X the finest convergence in which $\lim\{m_n\}^\uparrow = M$ for all $n \in \omega$, and each free ultrafilter $\mathcal{U}$ on ω converges to m_n (and m_n only), where n is defined by $\mathcal{U} \in \mathbb{A}_n$.

Claim. X is not core compact.

Proof. Let $m_n \in M$ and $\mathcal{U} \in \mathbb{A}_n$. Pick $S \subseteq \omega$, $S \in \mathcal{U}$, and $k \neq n$. For every $U \in \mathcal{U}$ there is $\mathcal{W} \in \mathbb{A}_k$ such that $U \in \mathcal{W}$. But $\lim \mathcal{W} = \{m_k\}$ is disjoint from S. □

Claim. X is T-core compact, and therefore $[X, \$]$ is pretopological.

Proof. For each $m_n \in M$, the set M is included in every open set containing m_n because $m_n \in \bigcap_{k\in\omega} \lim\{m_k\}^\uparrow$. If $\mathcal{U}$ is a non-trivial convergent ultrafilter in X then $\lim \mathcal{U} = \{m_n\}$ for some $n \in \omega$. For any

$S \in \mathcal{U}$, $S \cap \omega$ is infinite and any free ultrafilter $\mathcal{W}$ on $S \cap \omega$ belongs to one of the element $\mathbb{A}_k$ of the partition, so that $\lim \mathcal{W} = \{m_k\}$ intersects M, and therefore any open set containing m_n. □

Claim. $[X, \$]$ is diagonal.

Proof. Let $\mathcal{S}[\centerdot] : \mathcal{O}_X \to \mathbb{F}\mathcal{O}_X$ be a selection for $[X, \$]$ and let $U \in \lim_{[X,\$]} \mathcal{F}$. Now, $\{\bigcap F \colon F \in \mathcal{F}\}$ is a (convergence) cover of U.

Let $x \in U$ and $\mathcal{D}$ be a filter on X such that $x \in \lim \mathcal{D}$. There is an $F \in \mathcal{F}$ and a $D \in \mathcal{D}$ such that $D \subseteq \bigcap F := V$.

Assume $x \in \omega$, in which case $\mathcal{D} = \{x\}^{\uparrow}$. In particular, $x \in O$ for every $O \in F$. For every $O \in F$ there is a $T_O \in S[O]$ such that $x \in \bigcap T_O$. Now, $x \in \bigcap \bigcap_{O \in F} T_O \in \mathcal{S}[F]$. So, $\bigcap \bigcap_{O \in F} T_O \in \{x\}^{\uparrow} = \mathcal{D}$.

Assume $x \in M$. In this case, $M \cap O \neq \emptyset$ for all $O \in F$ and, by definition of the convergence on X, $M \subseteq O$ for all $O \in F$. Since $O \in \lim_{[X,\$]} \mathcal{S}[O]$ and $M \subseteq O$, there is $S \in \mathcal{S}[O]$ such that $x \in \bigcap S$, and, since each element of S is open, $M \subseteq \bigcap S$. If there is no $S \in \mathcal{S}[O]$ such that $O \subseteq \bigcap S$ then the filter $\mathcal{H}$ generated by $\{(O \cap \omega) \setminus \bigcap S : S \in \mathcal{S}[O], S \subseteq S_0\}$ is non degenerate. Notice that it is not free, for otherwise there would be an $n \in \omega$ and $\mathcal{U} \in \mathbb{A}_n$ with $\mathcal{U} \geq \mathcal{H}$. But $m_n \in \lim \mathcal{U} \cap O$, and there would be $S \in \mathcal{S}[O]$ such that $\bigcap S \in \mathcal{U}$, which is not possible. Therefore there is $y \in \bigcap_{S \in \setminus S[O]} (O \setminus \bigcap S)$ which contradicts $O \in \lim_{[X,\$]} \mathcal{S}[O]$. Hence, there is $S_0 \in \mathcal{S}[O]$ such that $O \subseteq \bigcap S_0$. Now, $D \subseteq \bigcap F \subseteq \bigcap_{O \in F} \bigcap S_O$. In particular, $\bigcap_{O \in F} \bigcap S_O \in \mathcal{D}$.

Thus, $\{\bigcap J \colon J \in \mathcal{S}[\mathcal{F}]\}$ is a cover of U, and $[X, \$]$ is diagonal. □

Therefore $[X, \$]$ is pretopological and diagonal, hence topological, and X is T-dual.

6. Appendix: convergence spaces

A family $\mathcal{A}$ of subsets of a set X is called *isotone* if $B \in \mathcal{A}$ whenever $A \in \mathcal{A}$ and $A \subseteq B$. We denote by $\mathcal{A}^{\uparrow}$ the smallest isotone family containing $\mathcal{A}$, that is, the collection of subsets of X that contain an element of $\mathcal{A}$. If $\mathcal{A}$ and $\mathcal{B}$ are two families of subsets of X we say that $\mathcal{B}$ is *finer than* $\mathcal{A}$, in symbols $\mathcal{A} \leq \mathcal{B}$, if for every $A \in \mathcal{A}$ there is $B \in \mathcal{B}$ such that $B \subseteq A$. Of course, if $\mathcal{A}$ and $\mathcal{B}$ are isotone, then $\mathcal{A} \leq \mathcal{B} \iff \mathcal{A} \subseteq \mathcal{B}$. This defines a partial order on isotone families, in particular on the set $\mathbb{F}X$ of filters on X. Every family $(\mathcal{F}_\alpha)_{\alpha \in I}$ of filters on X admits an infimum

$$\bigwedge_{\alpha \in I} \mathcal{F}_\alpha := \bigcap_{\alpha \in I} \mathcal{F}_\alpha = \left\{ \bigcup_{\alpha \in I} F_\alpha : F_\alpha \in \mathcal{F}_\alpha \right\}^{\uparrow}.$$

On the other hand the supremum even of a pair of filters may fail to exist. We call *grill of* $\mathcal{A}$ the collection $\mathcal{A}^{\#} := \{H \subseteq X : \forall A \in \mathcal{A},\ H \cap A \neq \emptyset\}$. It is easy to see that $\mathcal{A} = \mathcal{A}^{\#\#}$ if and only if $\mathcal{A}$ is isotone. In particular $\mathcal{F} = \mathcal{F}^{\#\#} \subseteq \mathcal{F}^{\#}$ if $\mathcal{F}$ is a filter. We say that two families $\mathcal{A}$ and $\mathcal{B}$ of subsets of X *mesh,* in symbols $\mathcal{A}\#\mathcal{B}$, if $\mathcal{A} \subseteq \mathcal{B}^{\#}$, equivalently if $\mathcal{B}^{\#} \subseteq \mathcal{A}$. The supremum of two filters $\mathcal{F}$ and $\mathcal{G}$ exists if and only if they mesh, in which case $\mathcal{F} \vee \mathcal{G} = \{F \cap G : F \in \mathcal{F}, G \in \mathcal{G}\}^{\uparrow}$. An infinite family $(\mathcal{F}_\alpha)_{\alpha \in I}$ of filters has a supremum $\bigvee_{\alpha \in I} \mathcal{F}_\alpha$ if pairwise suprema exist and for every $\alpha, \beta \in I$ there is $\gamma \in I$ with $\mathcal{F}_\gamma \geq \mathcal{F}_\alpha \vee \mathcal{F}_\beta$.

A *convergence* ξ on a set X is a relation between X and the set $\mathbb{F}X$ of filters on X, denoted $x \in \lim_\xi \mathcal{F}$ whenever x and $\mathcal{F}$ are in relation, satisfying that $x \in \lim_\xi \{x\}^{\uparrow}$ for every $x \in X$, and $\lim_\xi \mathcal{F} \subseteq \lim_\xi \mathcal{G}$ whenever $\mathcal{F} \leq \mathcal{G}$. The pair (X, ξ) is called a *convergence space.* A function $f : (X, \xi) \to (Y, \sigma)$ between two convergence space is *continuous* if

$$x \in \lim_\xi \mathcal{F} \Longrightarrow f(x) \in \lim_\sigma f(\mathcal{F}),$$

where $f(\mathcal{F})$ is the filter $\{f(F) : F \in \mathcal{F}\}^{\uparrow}$ on Y. If ξ and τ are two convergences on the same set X, we say that ξ is *finer than* τ, in symbols $\xi \geq \tau$, if $\lim_\xi \mathcal{F} \subseteq \lim_\tau \mathcal{F}$ for every $\mathcal{F} \in \mathbb{F}X$. This defines a partial order on the set of convergence structures on X, which defines a complete lattice for which supremum $\vee_{i \in I} \xi_i$ and infimum $\wedge_{i \in I} \xi_i$ of a family $\{\xi_i : i \in I\}$ of convergences are defined by

$$\lim_{\vee_{i \in I} \xi_i} \mathcal{F} = \bigcap_{i \in I} \lim_{\xi_i} \mathcal{F},$$

$$\lim_{\wedge_{i \in I} \xi_i} \mathcal{F} = \bigcup_{i \in I} \lim_{\xi_i} \mathcal{F}.$$

Every topology can be identified with a convergence, in which $x \in \lim \mathcal{F}$ if $\mathcal{F} \geq \mathcal{N}(x)$, where $\mathcal{N}(x)$ is the neighborhood filter of x for this topology. A convergence obtained this way is called *topological.* Moreover, a function $f : X \to Y$ between two topological spaces is continuous in the usual topological sense if and only if it is continuous in the sense of convergence. On the other hand, every convergence determines a topology in the following way: A subset C of a convergence space (X, ξ) is *closed* if $\lim_\xi \mathcal{F} \subseteq C$ for every filter $\mathcal{F}$ on X with $C \in \mathcal{F}$. A subset O is *open* if its complement is closed, that is, if $O \in \mathcal{F}$ whenever $\lim_\xi \mathcal{F} \cap O \neq \emptyset$. The collection of open subsets for a convergence ξ is a topology $T\xi$ on X, called *topological modification of* ξ. The topology $T\xi$ is the finest topological convergence coarser than ξ. If $f : (X, \xi) \to (Y, \tau)$ is continuous, so is $f : (X, T\xi) \to (Y, T\tau)$.

In other words, T is a concrete endofunctor of the category **Conv** of convergence spaces and continuous maps.

Continuity induces canonical notions of subspace convergence, product convergence, and quotient convergence. Namely, if $f : X \to Y$ and Y carries a convergence τ, there is the coarsest convergence on X making f continuous (to (Y, τ)). It is denoted $f^{-}\tau$ and called *initial convergence for f and τ*. For instance if $S \subseteq X$ and (X, ξ) is a convergence space, the *induced convergence by ξ on S* is by definition $i^{-}\xi$ where i is the inclusion map of S into X. Similarly, if $\{(X_i, \xi_i) : i \in I\}$ is a family of convergence space, then the product convergence $\Pi_{i\in I}\xi_i$ on the cartesian product $\Pi_{i\in I}X_i$ is the coarsest convergence making each projection $p_j : \Pi_{i\in I}X_i \to X_j$ continuous. In other words, $\Pi_{i\in I}\xi_i = \vee_{i\in I}p_i^{-}\xi_i$. In the case of a product of two factors (X, ξ) and (Y, τ), we write $\xi \times \tau$ for the product convergence on $X \times Y$.

Dually, if $f : X \to Y$ and (X, ξ) is a convergence space, there is the finest convergence on Y making f continuous (from (X, ξ)). It is denoted $f\xi$ and called *final convergence for f and ξ*. If $f : (X, \xi) \to Y$ is a surjection, the associated *quotient convergence on Y* is $f\xi$. Note that if ξ is a topology, the quotient topology is not $f\xi$ but $Tf\xi$.

The functor T is a reflector. In other words, the subcategory **Top** of **Conv** formed by topological spaces and continuous maps is closed under initial constructions. Note however that the functor T does not commute with initial constructions. In particular $T\xi \times T\tau \le T(\xi \times \tau)$ but the reverse inequality is genrally not true. Similarly, if $i : S \to (X, \xi)$ is an inclusion map, $i^{-}(T\xi) \le T(i^{-}\xi)$ but the reverse inequality may not hold. A convergence ξ is *pretopological* or a *pretopology* if $\lim_\xi \bigwedge_{\alpha\in I} \mathcal{F}_\alpha = \bigcap_{\alpha\in I} \lim_\xi \mathcal{F}_\alpha$. Of course, every topology is a pretopology, but not conversely. For any convergence ξ there is the finest pretopology $P\xi$ coarser than ξ. Moreover, $x \in \lim_{P\xi} \mathcal{F}$ if and only if $\mathcal{F} \ge \mathcal{V}_\xi(x)$ where $\mathcal{V}_\xi(x) := \bigwedge_{x\in\lim_\xi \mathcal{F}} \mathcal{F}$ is called *vicinity filter of x*. The subcategory **PrTop** of **Conv** formed by pretopological spaces and continuous maps is reflective (closed under initial constructions). Moreover, in contrast with topologies, the reflector P commutes with subspaces. However, like T, it does not commute with products.

The *adherence* $\operatorname{adh}_\xi \mathcal{F}$ of a filter $\mathcal{F}$ on a convergence space (X, ξ) is by definition

$$\operatorname{adh}_\xi \mathcal{F} := \bigcup_{\mathcal{H}\#\mathcal{F}} \lim_\xi \mathcal{H} = \bigcup_{\mathcal{U}\in\mathbb{U}(\mathcal{F})} \lim_\xi \mathcal{U},$$

where $\mathbb{U}X$ denotes the set of ultrafilters on X and $\mathbb{U}(\mathcal{F})$ denotes the set of ultrafilters on X finer than the filter $\mathcal{F}$. We write $\operatorname{adh}_\xi A$ for

$\mathrm{adh}_\xi\{A\}^\uparrow$. Note that in a convergence space adh_ξ may not be idempotent on subsets of A. In fact a pretopology is a topology if and only if adh is idempotent on subsets. We reserve the notations cl and int to topological closure and interior operators.

A family $\mathcal{A}$ of subsets of X is *compact at a family $\mathcal{B}$ for ξ if*

$$\mathcal{F}\#\mathcal{A} \Longrightarrow \mathrm{adh}_\xi \mathcal{F}\#\mathcal{B}.$$

We call a family *compact* if it is compact at itself. In particular, a subset A of X is *compact* if $\{A\}$ is compact, and *compact at* $B \subseteq X$ if $\{A\}$ is compact at $\{B\}$.

Given a class $\mathbb{D}$ of filters, a convergence is called *based in* $\mathbb{D}$ or $\mathbb{D}$-*based* if for every convergent filter $\mathcal{F}$, say $x \in \lim \mathcal{F}$, there is a filter $\mathcal{D} \in \mathbb{D}$ with $\mathcal{D} \leq \mathcal{F}$ and $x \in \lim \mathcal{D}$. A convergence is called *locally compact* if every convergent filter contains a compact set, and *hereditarily locally compact* if it is based in filters with a filter-base composed of compact sets. For every convergence, there is the coarsest locally compact convergence $K\xi$ that is finer than ξ and the coarsest hereditarily locally compact convergence $K_h\xi$ that is finer than ξ. Both K and K_h are concrete endofunctors of **Conv** that are also coreflectors.

If $A \subseteq X$ and (X,ξ) is a convergence space, then $\mathcal{O}(A)$ denotes the collection of open subsets of X that contain A and if $\mathcal{A}$ is a family of subsets of X then $\mathcal{O}(\mathcal{A}) := \bigcup_{A\in\mathcal{A}} \mathcal{O}(A)$. A family is called *openly isotone* if $\mathcal{A} = \mathcal{O}(\mathcal{A})$. Note that in a topological space X, an openly isotone family $\mathcal{A}$ of open subsets of X is compact if and only if, whenever $\bigcup_{i\in I} O_i \in \mathcal{A}$ and each O_i is open, there is a finite subset J of I such that $\bigcup_{i\in J} O_i \in \mathcal{A}$.

If (X,ξ) and (Y,σ) are two convergence spaces, $C(X,Y)$ or $C(\xi,\sigma)$ deonte the set of continuous maps from X to Y. The coarsest convergence on $C(X,Y)$ making the evaluation map $e : X \times C(X,Y) \to Y$, $e(x,f) = f(x)$, jointly continuous is called *continuous convergence* and denoted $[X,Y]$ or $[\xi,\sigma]$. Explicitely,

$$f \in \lim_{[X,Y]} \mathcal{F} \iff \forall_{x\in X} \forall_{\mathcal{G}\in\mathbb{F}X : x\in\lim_\xi \mathcal{G}}\ f(x) \in \lim_\sigma e\left(\mathcal{G}\times\mathcal{F}\right).$$

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