Lattice Polynomials, 12312-Avoiding Partial Matchings and Even Trees

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Abstract. The lattice polynomials $L_{i,j}(x)$ are introduced by Hough and Shapiro as a weighted count of certain lattice paths from the origin to the point (i, j). In particular, $L_{2n,n}(x)$ reduces to the generating function of the numbers $T_{n,k} = \frac{1}{n}\binom{n-1+k}{n-1}\binom{2n-k}{n+1}$, which can be viewed as a refinement of the 3-Catalan numbers $T_n = \frac{1}{2n+1}\binom{3n}{n}$. In this paper, we establish a correspondence between 12312-avoiding partial matchings and lattice paths, and we show that the weighted count of such partial matchings with respect to the number of crossings in a more general sense coincides with the lattice polynomials $L_{i,j}(x)$. We also introduce a statistic on even trees, called the r-index, and show that the number of even trees with $2n$ edges and with r-index k equal to $T_{n,k}$.

Keywords: lattice polynomial, 12312-avoiding partial matching, even tree

AMS Classification: 05A15

1 Introduction

The lattice polynomials $L_{i,j}(x)$ are introduced by Hough and Shapiro [4] as a weighted count of lattice paths from the origin $(0, 0)$ to (i, j) consisting of unit east steps $(1, 0)$ and north steps $(0, 1)$ such that no step goes above the line $x = 2y$. To be more specific, a north step from (k, ℓ) to $(k, \ell + 1)$ is given a weight x if k is odd, and the other steps are assumed to have weight 1. The weight of a path is the product of the weights of all the steps. In particular, $L_{2n,n}(x)$ reduces to the generating function of the numbers

$$T_{n,k} = \frac{1}{n}\binom{n-1+k}{n-1}\binom{2n-k}{n+1},$$

which can be viewed as a refinement of the 3-Catalan numbers

$$T_n = \frac{1}{2n+1}\binom{3n}{n}.$$

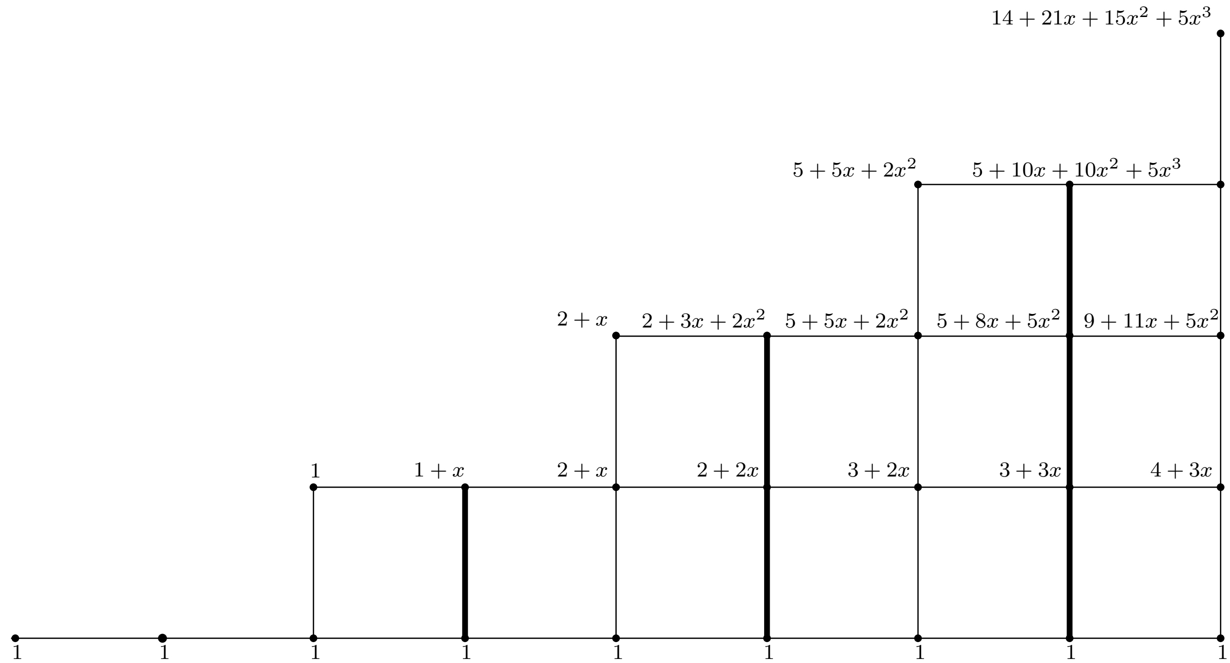


Figure 1.1: Lattice polynomials $L_{i,j}(x)$.

Figure 1.1 gives the first values of $L_{i,j}(x)$. Note that $L_{i,0}(x) = 1$ for any $i \geq 0$.

The lattice polynomials $L_{i,j}(x)$ are related to the descent polynomials on noncrossing trees. Let $d_n(k, j)$ be the number of noncrossing trees on $\{1, 2, \ldots, n\}$ with root degree j and descent number k. Then the descent polynomial is defined as

$$D_n(x, y) = \sum_{k,j} d_n(k, j) x^k y^j. \tag{1}$$

Hough and Shapiro [4] derived the following relation

$$L_{2n,n}(x) = D_{n+1}(x, 1), \tag{2}$$

$$L_{2n-1,j}(x) = [y^{n-j}]D_{n+1}(x, y), \quad \text{for } 0 \leq j \leq n-1, \tag{3}$$

where $[y^{n-j}]D_{n+1}(x, y)$ stands for the coefficient of y^{n-j} in $D_{n+1}(x, y)$. Hough [3] showed that

$$D_n(x, 1) = \sum_{k=0}^{n-2} \frac{1}{n-1} \binom{n-2+k}{n-2} \binom{2n-2-k}{n} x^k. \tag{4}$$

In view of (2), the above expression for $D_{n+1}(x, 1)$ can be considered as a formula for $L_{2n,n}(x)$.

This paper is motivated by the observation that the polynomials $D_n(x, 1)$ also arise in the context of pattern avoiding matchings. Let $T_{n,k}$ be the number of 12312-avoiding matchings on $\{1, 2, \ldots, 2n\}$ with k crossings. Chen, Mansour and Yan [1] have shown that

$$T_{n,k} = \frac{1}{n} \binom{n-1+k}{n-1} \binom{2n-k}{n+1}. \tag{5}$$

Denote by $T_n(x)$ the generating function of $T_{n,k}$, namely,

$$T_n(x) = \sum_{k=0}^{n-1} T_{n,k} x^k.$$

It is easy to check that

$$D_{n+1}(x, 1) = T_n(x),$$

which can be rewritten as

$$L_{2n,n}(x) = T_n(x). \tag{6}$$

We introduce a class of 12312-avoiding partial matchings with i points, denoted by $\mathcal{Q}_i(12312)$, and we establish a correspondence between lattice paths counted by the lattice polynomial $L_{i,j}(x)$ and the partial matchings in $\mathcal{Q}_i(12312)$ counted by a polynomial $Q_{i,j}(x)$. Let $q_{i,j,k}$ be the number of 12312-avoiding partial matchings in $\mathcal{Q}_i(12312)$ with j edges and k crossings, where the number of crossings of a partial matching is defined in a more general sense that an isolated point covered by an edge is also considered as a crossing. Then $Q_{i,j}(x)$ is the generating function of the numbers $q_{i,j,k}$ summed over k. In particular, we see that $Q_{2n,n}(x)$ equals $T_n(x)$, thus our correspondence gives a combinatorial interpretation of (6).

The second result of this paper is concerned with a refined enumeration of even trees. The number of even trees with $2n$ edges is known to be the 3-Catalan number T_n. We introduce the r-index of an even tree. We construct a correspondence between lattice paths from $(0, 0)$ to $(2n, n)$ with k north steps at odd positions and even trees with $2n$ edges and r-index k. Thus the lattice polynomial $L_{2n,n}(x)$ serves as a formula for the weighted count of even trees with respect to the r-index.

2 12312-avoiding partial matchings

In this section, we introduce the polynomials $Q_{i,j}(x)$ in connection with the enumeration of partial matchings with i points, j edges and k crossings. By constructing a bijection, we show that the polynomials $Q_{i,j}(x)$ coincide with the lattice polynomials $L_{i,j}(x)$.

We first recall some definitions. A (complete) matching on a set $[2n] = \{1, 2, \ldots, 2n\}$ is a graph with vertex set $[2n]$ and with n disjoint edges. We assume that the vertices or the points of a matching is arranged on a horizontal line in increasing order. Denote an edge of a matching by $e = (i, j)$ with $i < j$. Two edges $e = (i, j)$ and $e' = (i', j')$ form a crossing if $i < i' < j < j'$. Denote by $cr(M)$ the number of crossings of M.

A matching can be expressed by its canonical sequential form [5], or the Davenport-Schinzel sequence [6]. For a matching P on $[2n]$, denote the edges of P by $e_1 = (i_1, j_1)$, $e_2 = (i_2, j_2)$, $\ldots$, $e_n = (i_n, j_n)$, where $i_1 < i_2 < \cdots < i_n$. We write $P = a_1 a_2 \cdots a_{2n}$, where $a_i = j$ if i is an endpoint of the edge e_j. For example, the matching in Figure 2.1 can be expressed by the sequence 12324413.

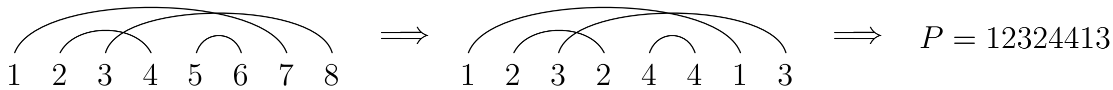


Figure 2.1: A matching and its canonical sequential form.

Let $\alpha = \alpha_1\alpha_2\cdots\alpha_k$ and $\pi = \pi_1\pi_2\cdots\pi_k$ be two sequences. We say α and π are order-isomorphic if for any $1 \le i, j \le k$, $\alpha_i < \alpha_j$ (resp. $\alpha_i = \alpha_j$, $\alpha_i > \alpha_j$) if and only if $\pi_i < \pi_j$ (resp. $\pi_i = \pi_j$, $\pi_i > \pi_j$). We say that a canonical sequential form P avoids a sequence π, or P is π-avoiding, if no subsequence of P is order-isomorphic to π. Such a sequence π is usually called a pattern. Denote by $\mathcal{M}_n(\pi)$ the set of matchings on $[2n]$ which avoid a pattern π. It has been shown by Chen, Mansour and Yan [1] that the number of 12312-avoiding matchings on $[2n]$ equals the 3-Catalan number, namely, $|\mathcal{M}_n(12312)| = T_n$.

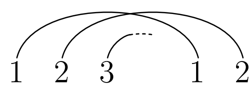


Figure 2.2: The pattern 12312.

A partial matching on $[m] = \{1, 2, \ldots, m\}$ can be viewed as a matching on a subset of $[m]$. For example, Figure 2.3 gives three partial matchings with 5 vertices.

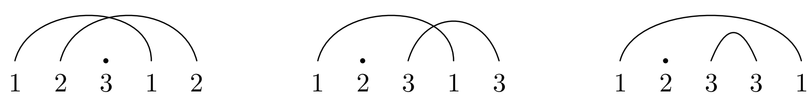


Figure 2.3: The partial matchings.

We shall consider a special class of partial matchings such that any edge covers at most one isolated point and the number of isolated points to the left of any edge cannot exceed the number of isolated points to the right of this edge. Denote this class of partial matchings on $[m]$ by $\mathcal{Q}_m$. For a partial matching $M \in \mathcal{Q}_m$, an edge $e = (i, j)$ and an isolated point k form a crossing if $i < k < j$. Denoted by $cr(M)$ the number of crossings of M, where we count the crossings formed by two edges, as well as one edge and an isolated point. For example, for the partial matchings in Figure 2.3, we have $cr(12312) = 3$, $cr(12313) = 2$, and $cr(12331) = 1$.

Let $\mathcal{Q}_m(\pi)$ denote the set of partial matchings in $\mathcal{Q}_m$ that avoid pattern π. We shall be concerned with a refined enumeration of the set $\mathcal{Q}_i(12312)$. Let $q_{i,j,k}$ be the number of partial matchings in the set $\mathcal{Q}_i(12312)$ with j edges and k crossings. Set

$$Q_{i,j}(x) = \sum_k q_{i,j,k}\, x^k, \tag{7}$$

where k ranges from 0 to $\lceil \frac{i-1}{2} \rceil$. In this notation, we have the following relation.

Theorem 2.1 *The polynomial $Q_{i,j}(x)$ equals the lattice polynomial $L_{i,j}(x)$.*

Proof. We proceed to give a procedure to generate all partial matchings in $\mathcal{Q}_m(12312)$. For this purpose, we introduce two operations, called the shifting operation and the lifting operation on partial matchings corresponding to the east and north steps in the lattice path.

The shifting operation is defined by adding an isolated point to the right of the last vertex of M. This operation corresponds to an east step $E = (1, 0)$ in the lattice path.

The lifting operation is defined by adding an edge in the middle of the partial matching M. More precisely, if there are $2k$ isolated points in M, then connect the k-th and $(k+1)$-st isolated points to form a new edge. If there are $2k+1$ isolated points in M, then connect the k-th and $(k+2)$-nd isolated points. This operation corresponds to a north step $N = (0, 1)$ in the lattice path.

We now describe the procedure to generate all partial matchings in $\mathcal{Q}_m(12312)$. Start with an empty set $\emptyset$ at the origin $(0, 0)$. Suppose that we have constructed a partial matching $M \in \mathcal{Q}_m(12312)$ at the position (i, j). Using the shifting operation or the lifting operation on M, we obtain a new partial matching M' at the position $(i+1, j)$ or $(i, j+1)$. Iterating this process, we generate all the partial matchings at a given position (i', j').

Figure 2.4 gives an illustration of the above procedure to generate partial matchings.

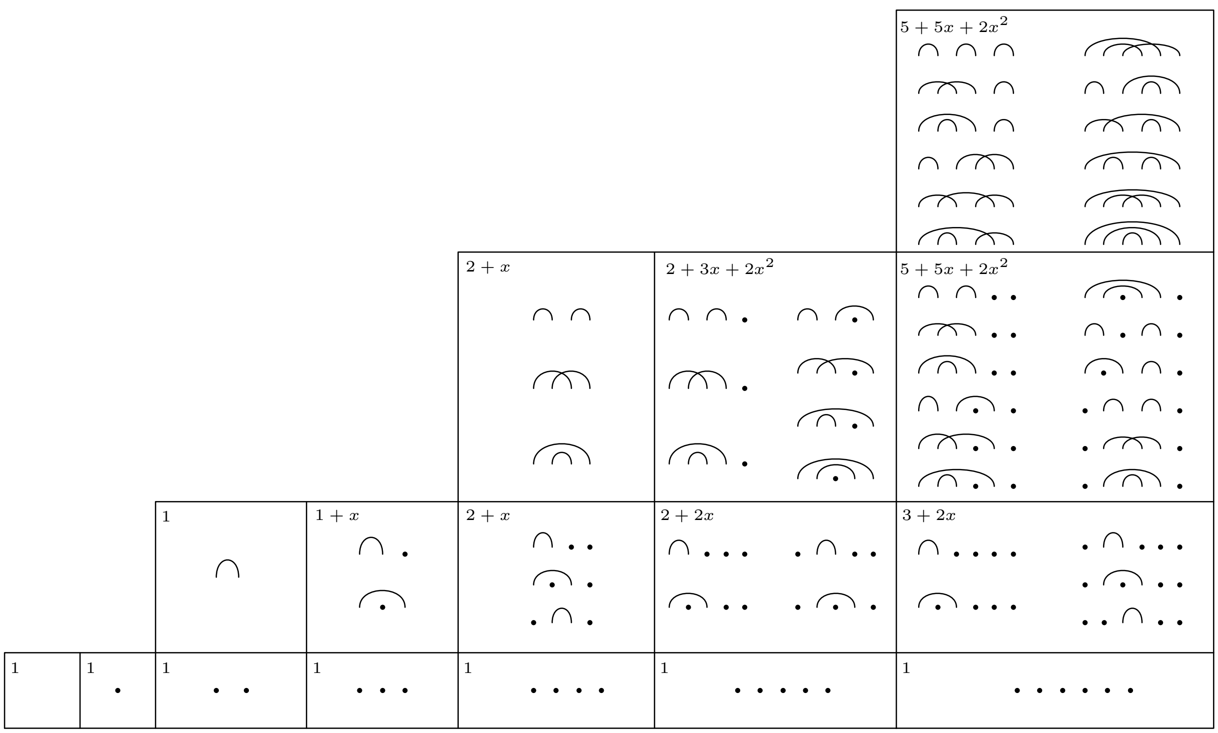


Figure 2.4: 12312-avoiding partial matchings and lattice polynomials.

Clearly, for a shifting operation, adding a new isolated point to the rightmost position does not change the number of crossings, i.e., $cr(M') = cr(M)$. For a lifting operation, if there are $2k$ isolated points in M, it is easily checked that connecting the k-th and $(k+1)$-st isolated points does not create any crossings, that is, $cr(M') = cr(M)$. If there are $2k+1$

isolated points in M, then connecting the k-th and $(k+2)$-nd isolated points creates a new crossing on M', that is, $cr(M') = cr(M) + 1$.

To show that $Q_{i,j}(x)$ equals the lattice polynomial $L_{i,j}(x)$, we need to verify the following facts.

Claim 1: After the shifting or lifting operation, the new partial matching M' still avoids the pattern 12312.

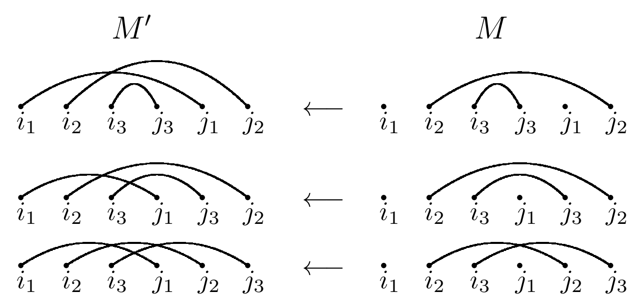
Claim 2: The two operations do not lead to the same partial matching.

Since the shifting operation is defined by just adding a new isolated point, it is obvious that M' still avoids the pattern 12312 after this operation.

Let us consider the lifting operation. Suppose to the contrary that M' is not 12312-avoiding, that is, there exists a subsequence of M' which is of pattern 12312. Assume that the new edge produced by lifting operation in M' is e. Since M is 12312-avoiding, the pattern 12312 in M' must involve the edge e. We have two cases.

Case 1. There are $2k$ isolated points in M. For any subsequence of M' which is of pattern 12312, the maximum point cannot be an isolated point; Otherwise there exits an edge in M that covers more than one point, contradicting the definition of $\mathcal{Q}_m$. Hence the subsequence of M' involves three edges (i_1, j_1), (i_2, j_2), (i_3, j_3), where $i_1 < i_2 < i_3 < j_1 < j_2$.

If $e = (i_1, j_1)$, according to the position of j_3, there are three possibilities as shown in the following figure. Obviously, the number of isolated points to the left of (i_2, j_2) is greater than the number of isolated points to the right of the edge (i_2, j_2), which contradicts the definition of $\mathcal{Q}_m$.



The same argument applies to the situations $e = (i_2, j_2)$ and $e = (i_3, j_3)$.

Case 2. There are $2k+1$ isolated points in M. It is easily seen that any subsequence of M' which is of pattern 12312 consists of two edges and one isolated point. If $e = (i_1, j_1)$ or $e = (i_2, j_2)$, there is always one edge of M covering more than one isolated point, which contradicts the definition of $\mathcal{Q}_m$. So Claim 1 is proved.

We turn to the proof of Claim 2. We use induction on i and j. It is easy to check the statement holds for small values of (i, j), see Figure 2.4. We assume that the claim holds for the positions $(i-1, j)$ and $(i, j-1)$.

Suppose that M_1' and M_2' are 12312-avoiding partial matchings in the position (i, j), and that M_1' is obtained from M_1 and M_2' is obtained from M_2. There are three cases for the positions of M_1 and M_2.

If both M_1 and M_2 are in the position $(i-1,j)$ or both M_1 and M_2 are in the position $(i,j-1)$, then by the induction hypothesis M_1 and M_2 are distinct. It follows that M_1' and M_2' are distinct.

We now consider the case that M_1 is in the position $(i-1,j)$ and M_2 is in the position $(i,j-1)$. The assumption implies that $i,j \geq 1$. Clearly, M_1' is obtained from M_1 via a shifting operation by adding an isolated point to the rightmost point of M_1. Evidently, the number of isolated points to the left of any edge in M_1' is less than the number of isolated points to the right of this edge. Since M_2' is obtained from M_2 via a lifting operation by adding an edge, we see that the number of isolated points to the left of this edge is equal to the number of isolated points to the right of this edge in M_2'. Therefore, in this case M_1' and M_2' are distinct. So we arrive at the conclusion that the partial matchings obtained at the position (i,j) are distinct. Thus the claim holds by induction.

It remains to show that the above procedure generates all partial matchings in $\mathcal{Q}_i$ with j edges. To this end, we give the reverse procedure to construct a lattice path from any partial matching in $\mathcal{Q}_i$ with j edges.

Let M be a 12312-avoiding partial matching in $\mathcal{Q}_i$ with j edges. The construction is recursive. More precisely, we shall construct a lattice path from (i,j) to the origin that will not go beyond the line $x = 2y$. We start at the position (i,j). If for any edge in M the number of isolated points to the left of this edge is less than the number of isolated points to the right of this edge, then there must be an isolated point in the end. We delete the last isolated point and continue to consider the position $(i-1,j)$. Note that in this case $i \geq 2j+1$ since there are isolated points in M, and thus $(i-1,j)$ does not go beyond the line $x = 2y$. Otherwise, among the edges in M for which the number of isolated points to the left equals the number of isolated points to the right, we can choose the unique edge e with the rightmost right endpoint. Then, we delete the edge e and continue to consider the position $(i,j-1)$. Obviously, $(i,j-1)$ does not exceed the line $x = 2y$. Iterating this procedure, we eventually get the required lattice path. ∎

The proof of the above theorem can be considered as a recursive construction of a correspondence between lattice paths counted by $L_{i,j}(x)$ and 12312-avoiding partial matchings counted by $Q_{i,j}(x)$.

3 The r-index of even trees

In this section, we define a statistic, which we call the r-index, on even trees. We shall show that the generating function for the number of even trees with $2n$ edges and with r-index equal to k coincides with the lattice polynomial $L_{2n,n}(x)$.

Recall that an even tree is a plane tree in which every vertex has an even number of children. The number of even trees with $2n$ edges equals the 3-Catalan number T_n. If a vertex has $2k$ children, we call the first k children left children and the last k children right children. So every vertex except for the root is either a left child or a right child. The r-index of an even tree T, denoted by $r(T)$, is defined as half of the sum of the degrees of right children, where the degree of a node is meant to be the number of its children. Define $R_{n,k}$ to be the number of even trees with $2n$ edges and with the r-index equal to k, and define the generating function of $R_{n,k}$ by

$$R_n(x) = \sum_{k=0}^{n-1} R_{n,k} x^k. \tag{8}$$

The following theorem gives a connection between lattice polynomials and even trees counted with respect to the number of edges and the r-index. The proof can be viewed as a recursive construction of a bijection.

Theorem 3.1 *For $n \geq 1$, $R_n(x)$ equals the lattice polynomial $L_{2n,n}(x)$.*

Proof. We proceed to present a procedure to generate even trees parallel to the construction of lattice paths counted by the lattice polynomials. To this end, we shall introduce two operations, called the shifting operation and the lifting operation, on even trees corresponding to the east and north steps of the lattice paths. These two operations can be viewed as the actions to shift the even trees in the lattice from left to right, and to lift the even trees.

Actually, we need an intermediate structure for the generation of even trees that corresponds to the intermediate points, namely, the points with even x-coordinates, for the generation of lattice polynomials. Such an intermediate structure will be represented by an even tree with a pair of dotted edges from the root to its first child and to its last child.

We first describe the shifting operation. We start with an empty set $\emptyset$ at origin $(0, 0)$. Let T be an even tree which may contain a pair of dotted edges as the outside edges of the root. The shifting operation is meant to transform T in a position (i, j) to an even tree T' in the position to the immediately right of T. There are two cases.

If T contains a pair of dotted edges, then T' is obtained from T by changing the dotted edges to regular edges. If T does not contain any dotted edges, then T' is obtained from T by adding a pair of dotted edges as the outside edges of the root.

Figure 3.1 gives an illustration of the two cases for the shifting operation, where A, B and C denote subtrees which may be empty, and S stands for the shifting operation.

The lifting operation is meant to transform an even tree T in position (i, j) to an even tree T' in the position immediately above T. Note that the position $(i, j + 1)$ cannot go beyond the line $x = 2y$.

There are also two cases for the lifting operation. If T contains a pair of dotted edges, namely, $i = 2m + 1$, then T' is obtained from T by moving the pair of edges of the root that are next to the dotted edges (along with the subtrees attached to these two edges) as outside edges to the last child of the root.

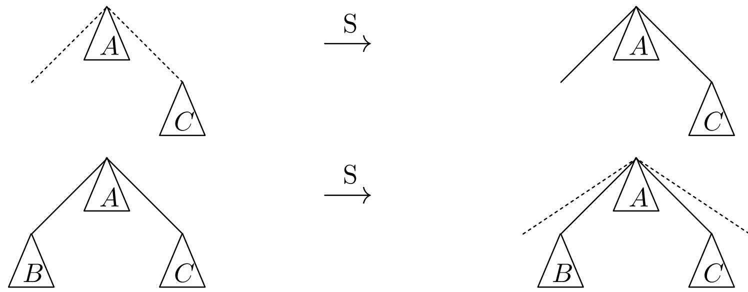


Figure 3.1: The shifting operation.

When T does not contain any dotted edges, namely, $i = 2m$, if $0 \leq j \leq m-2$, then T' is obtained from T by moving the pair of outside edges of the root (along with the subtrees attached to these two edges) as outside edges to the second child of the root of T, and if $j = m-1$, then let $T' = T$.

These two cases are illustrated in Figure 3.2, where A, B, C, D and E represent subtrees that may be empty and L stands for the lifting operation.

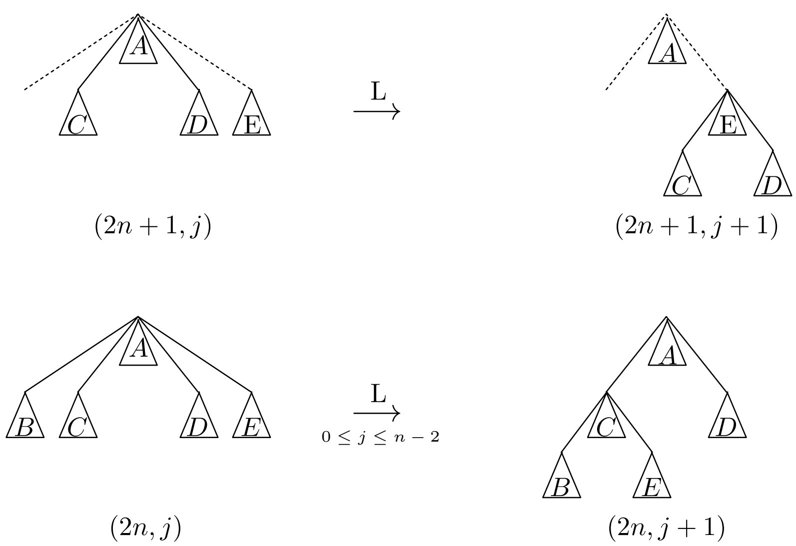


Figure 3.2: The lifting operation.

Note that in the above process, if T has a pair of dotted edges, then the degree of the first child of the root must be zero.

Obviously, in the case when T contains a pair of dotted edges, the lifting operation on T increases the r-index by one, that is, $r(T') = r(T) + 1$. It is also clear that the number of even trees generated at the position $(2n, n)$ equals the 3-Catalan numbers. However, it is still necessary to show that the above shifting and lifting operations do not generate any tree more than once at any point (i, j).

We use induction on i and j to complete the proof. It is easy to check the statement holds for small values of (i, j), see Figure 3.3. Assume that the claim holds for the positions $(i-1, j)$ and $(i, j-1)$.

Suppose that T_1' and T_2' are two even trees in the position (i, j) that are obtained from T_1 and T_2, respectively. There are three cases for the positions of T_1 and T_2.

Case 1. Both T_1 and T_2 are in the position $(i-1, j)$. In this case, we have $i \geq 1$. Clearly, T_1' and T_2' are obtained from T_1 and T_2 via the shifting operations. It is also easy to see that T_1' and T_2' are distinct.

Case 2. Both T_1 and T_2 are in the position $(i, j-1)$, where $j \geq 1$. In this case, T_1' and T_2' are obtained from T_1 and T_2 via the lifting operations. As can be seen from Figure 3.2, if T_1 and T_2 has dotted edges, then we consider the subtrees A_1, C_1, D_1, E_1 and A_2, C_2, D_2, E_2 of T_1 and T_2. Bear in mind that these subtrees may be empty. Since T_1 and T_2 are distinct, the corresponding subtrees cannot be all identical. After the lifting operation, the four subtrees are moved to different positions in T_1' and T_2'. It is easily seen that T_1' and T_2' are distinct.

If T_1 and T_2 do not have any dotted edges, we may consider the five subtrees A, B, C, D, E as shown in Figure 3.2. The same argument yields that T_1' and T_2' are distinct.

Case 3. T_1 is in the position $(i-1, j)$ and T_2 is in the position $(i, j-1)$. In this case, we have $i, j \geq 1$. Then T_1' is obtained from T_1 via a shifting operation and T_2' is obtained from T_2 via a lifting operation.

If T_1 has dotted edges, then the first child of the root is a leaf and the shifting operation transforms the dotted edges to regular edges. Therefore, the first child of root of T_1' is also a leaf. On the other hand, T_2 does not have any dotted edges. Applying the lifting operation to T_2, the first child of the root of T_2' becomes an internal vertex. Hence T_1' and T_2' are distinct.

If T_1 does not have any dotted edges, the shifting operation adds two dotted edges as outside edges to the root. Moreover, the last child of the root of T_1' is a leaf. Meanwhile, since T_2 has dotted edges, applying the lifting operation to T_2, the last child of the root of T_2' becomes an internal vertex. Thus, T_1' and T_2' are distinct.

Since the number of even trees with $2n$ edges equals the 3-Catalan number T_n, the above construction produces all even trees with $2n$ edges at the position $(2n, n)$. Hence we obtain a one-to-one correspondence. This completes the proof. ∎

For example, Figure 3.3 gives a sequence of shifting and lifting operations initially acting on the empty even tree in the origin $(0, 0)$. Figure 3.4 gives the first few steps to generate even trees.

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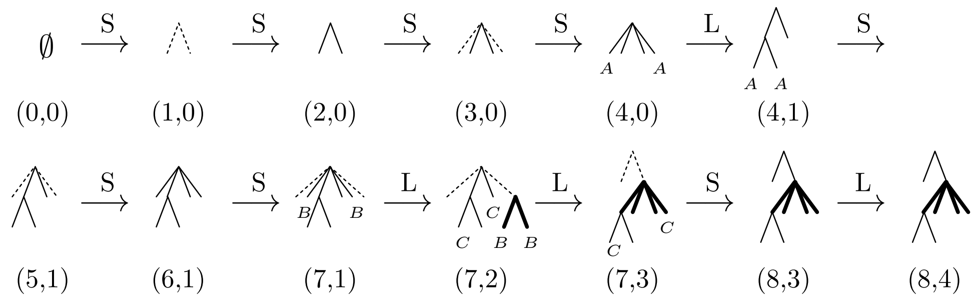

Figure 3.3: The actions of the shifting and lifting operations.

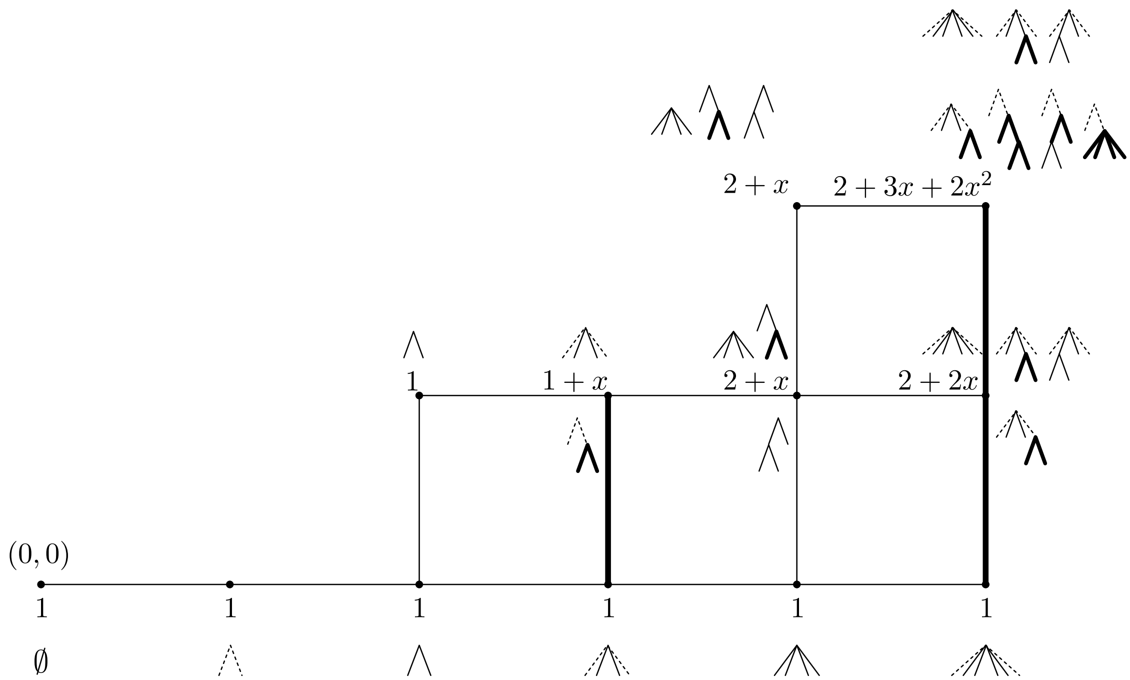

Figure 3.4: The generation of the polynomials $R_n(x)$.

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