MULTI-POINT GREEN'S FUNCTIONS FOR SLE AND AN ESTIMATE OF BEFFARA

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ABSTRACT. In this paper we define and prove of the existence of the multi-point Green's function for SLE–a normalized limit of the probability that an SLE_κ curve passes near to a pair of marked points in the interior of a domain. When $\kappa < 8$ this probability is non-trivial, and an expression can be written in terms two-sided radial SLE. One of the main components to our proof is a refinement of a bound first provided by Beffara in [3]. This work contains a proof of this bound independent from the original.

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1. INTRODUCTION

The Schramm-Loewner evolution (SLE) is a random process first introduced by Oded Schramm in [12] as a candidate for scaling limits of models from statistical physics which are believed to be conformally invariant. Since its introduction, SLE has been rigorously established as the scaling limit for a number of these processes, including the loop-erased random walk [9], the

Research of the first author supported by NSF grant DMS-0907143.

percolation exploration process [14], and the interface of the Gaussian free field [13]. For a general introduction to SLE see, for example, [6, 7, 15].

Chordal SLE_κ for $\kappa > 0$ in the upper half-plane ($\mathbb{H}$) is a one-parameter family of non-crossing random curves $\gamma : [0,\infty) \to \overline{\mathbb{H}}$ with $\gamma(0) = 0$ and $\gamma(\infty^-) = \infty$. Depending on κ, the geometry of the curve has several different phases. When $0 < \kappa \leq 4$, the curves are simple (no self intersections). When $\kappa > 4$, the curves are no longer simple, but they remain non-crossing. When $\kappa \geq 8$, the curve is spacefilling, passing through every point in $\overline{\mathbb{H}}$.

Due to the strong dependence on the history of the curve forced by the curves being non-crossing and the measure being conformally invariant, the process $\gamma(t)$ is highly non-Markovian. This makes estimating the probability that the process passes close to a series of marked points in $\mathbb{H}$ difficult when $\kappa < 8$, as is required, for example, in the proof of the almost sure Hausdorff dimension of SLE_κ given by Beffara in [3].

When trying to understand the probability that SLE_κ gets near to some point $z \in \mathbb{H}$ it is convenient to consider the conformal radius of z in $H_t := \mathbb{H} \setminus \gamma[0,t]$, which we denote by $\Upsilon_t(z)$, instead of the Euclidean distance (see Section 2.1 for the definition). This change does little to the geometry of the problem being considered since the conformal radius differs from the Euclidean distance by at most a universal multiplicative constant.

The Green's function for SLE_κ from 0 to ∞ in $\mathbb{H}$ for $\kappa < 8$ is a form of normalized probability of passing near to a point in $\mathbb{H}$. It is defined by

$$\lim_{\varepsilon\to 0} \varepsilon^{d-2}\mathbb{P}\{\Upsilon_\infty(z) < \varepsilon\} = c_* G_{\mathbb{H}}(z;0,\infty)$$

where $d := 1 + \kappa/8$ is the Hausdorff dimension of the SLE_κ and c_* is some known constant depending on κ. The Green's function was first computed in [11] (although they neither used this name nor definition), and the exact formula found there is given in Section 2.1. The existence of the limit requires some argument, and a form of it is proven in Lemma 2.10.

We wish to show that analogously that

$$\lim_{\varepsilon,\delta\to 0} \varepsilon^{d-2}\delta^{d-2}\mathbb{P}\{\Upsilon_\infty(z) < \varepsilon;\ \Upsilon_\infty(w) < \delta\}$$

exists and can be written as

$$c_*^2 G_{\mathbb{H}}(z;0,\infty)\mathbb{E}_z^*[G_{H_{T_z}}(w;z,\infty)] + c_*^2 G_{\mathbb{H}}(w;0,\infty)\mathbb{E}_w^*[G_{H_{T_w}}(z;w,\infty)]$$

where $\mathbb{E}_z^*$ is the expectation of a particular form of SLE called *two-sided radial SLE*, which can be understood as chordal SLE conditioned to pass though the point z, and $G_{H_{T_z}}$ is the Green's function for SLE in the domain remaining at the time it does so. The form of the limit as the sum of two similar terms comes from the two possible orders that the curve can pass through z and w, and each term individually can be thought of as an ordered Green's function.

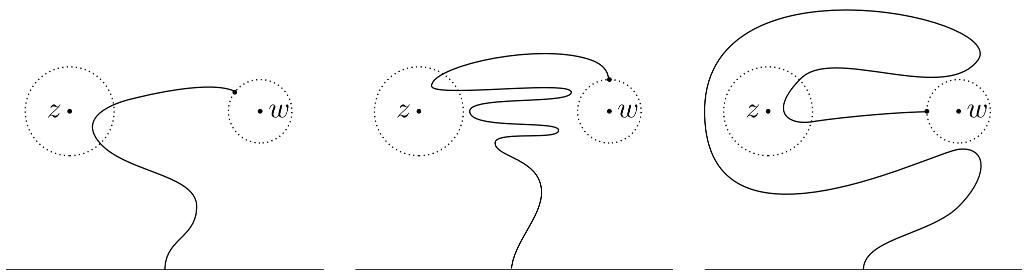

FIGURE 1. We wish to show that curves that get near z then near w concentrates on curves like those in the left image. Estimating the probability of such curves is easy by repeated application of the Green's function. However, such simple estimation gives the same order of magnitude to curves like those in the center image. This issue can be overcome as long as getting near to w before z decreases the probability that the SLE gets even closer to w later on. This is often the case, however the right image shows an example where it is not. In this case, once the curve gets near to z, it is essentially guaranteed to pass near w. Controlling for these issues forms the bulk of this work.

To prove this result, we will use techniques similar to those used in [3], where Beffara (in slightly different notation) established the estimate that there exists some $c > 0$ such that for any two points $z, w \in \mathbb{H}$ with $|z|, |w| \geq 1$

$$\mathbb{P}\{\Upsilon_\infty(z) < \varepsilon;\ \Upsilon_\infty(w) < \varepsilon\} < c\,\varepsilon^{2(d-2)}|z-w|^{2-d}.$$

The reason that similar techniques are relevant to this problem comes from the fact that the difficulty in both of these propositions lies in proving a rigorous version of the following heuristic statement: an SLE curve that first passes through z and then through w will do so directly, which is to say by getting near to z without becoming very near w and then heading to w. Figure 1 demonstrates some of the issues which can occur which make this a tricky statement to make rigorous.

In the process of proving the existence of the multi-point Green's function for SLE, we also obtain an independent proof of a mild generalization of Beffara's estimate—that there exists a $c > 0$ such that for any $z, w \in \mathbb{H}$ with $|z|, |w| \geq 1$

$$\mathbb{P}\{\Upsilon_\infty(z) < \varepsilon;\ \Upsilon_\infty(w) < \delta\} < c\,\varepsilon^{d-2}\delta^{d-2}|z-w|^{2-d}.$$

While it may be possible to derive some of the lemmas we require directly from the proof in [3], we include a complete proof of them, along with Beffara's original estimate, so that the proof of our main result is completely self contained.

It is worth noting that Beffara's estimate itself immediately yields an upper bound on the multi-point Green's function. For a lower bound, and an application of the multi-point Green's function to the proof of the existence of the "natural parametrization" of SLE, a parametrization of SLE by what can be thought of as a form d-dimensional arc length, see [10].

The paper is structured as follows. Section 2.1 begins by establishing the notation used throughout the paper, and to provide a few simple deterministic and random bounds required in the proofs that follow. Section 2.4 then gives a brief introduction to two-sided radial SLE and collects the facts about this process that we require to show the existence of the multi-point Green's function. Section 3 provides a proof of the existence of the multi-point Green's function assuming an estimate derived from our proof of Beffara's estimate. The rest of the paper is then dedicated to our independent proof of Beffara's estimate where the various bounds required are split into topological lemmas, probabilistic lemmas, and the combinatorial lemma required to assemble the complete result. The proof of one of the topological lemmas is left to the appendix as the result is intuitive and the formal proof of it does little to aid the understanding of our main results.

Throughout this paper we fix $\kappa < 8$ and constants implicitly depend on κ.

2. Preliminaries

2.1. **Notation.** We set

$$a = \frac{2}{\kappa}, \quad d = 1 + \frac{\kappa}{8} = 1 + \frac{1}{4a},$$

$$\beta = \frac{\kappa}{8} + \frac{8}{\kappa} - 2 = 4a + \frac{1}{4a} - 2 > 0,$$

$$\lambda = \frac{8}{\kappa} - 1 = 4a - 1 > 0.$$

The Green's function for chordal SLE_κ (from 0 to ∞ in $\mathbb{H}$) is

$$G(x + iy) = G(re^{i\theta}) = r^{d-2} \sin^{\lambda} \theta = y^{d-2} \sin^{\beta} \theta.$$

The Green's function can be defined for other simply connected domains as we now demonstrate. If D is a simply connected domain, z_1, z_2 are distinct boundary points, let $\Phi_D : D \to \mathbb{H}$ be a conformal transformation with $\Phi_D(z_1) = 0, \Phi_D(z_2) = \infty$. This is unique up to a final dilation. If $w \in D$, we define

$$S_D(w; z_1, z_2) = \sin \arg \Phi_D(w),$$

which is independent of the choice of Φ_D and gives a conformal invariant. We let $\Upsilon_D(w)$ be (twice the) conformal radius of w in D, that is, if $f : \mathbb{D} \to D$ is a conformal transformation with $f(0) = w$, then $\Upsilon_D(w) = 2\,|f'(0)|$. This satisfies the scaling rule

$$\Upsilon_{f(D)}(f(w)) = |f'(w)|\,\Upsilon_D(w).$$

It is easy to check that $\Upsilon_{\mathbb{H}}(x+iy) = y$, and, more generally,

$$\Upsilon_D(w) = \frac{\operatorname{Im}(\Phi_D(w))}{|\Phi_D'(w)|}.$$

The Green's function for SLE_κ from z_1 to z_2 in D is defined by

$$G_D(w; z_1, z_2) = \Upsilon(w)^{d-2}\, S(w; z_1, z_2)^\beta.$$

It satisfies the scaling rule

$$G_D(w; z_1, z_2) = |f'(w)|^{2-d}\, G_{f(D)}(f(w); f(z_1), f(z_2)).$$

For a sketch of a proof that the Green's function so defined satisfies the limit claimed in the introduction, see Lemma 2.10.

Let $\operatorname{inrad}_D(w) = \operatorname{dist}(w, \partial D)$ denote the inradius. Using the Koebe 1/4-theorem, we know that

$$\frac{1}{2}\operatorname{inrad}_D(w) \le \Upsilon_D(w) \le 2\operatorname{inrad}_D(w). \tag{1}$$

Therefore,

$$G_D(w; z_1, z_2) \asymp \operatorname{inrad}_D(w)^{d-2}\, S_D(w; z_1, z_2)^\beta,$$

where we write $f_1 \asymp f_2$ if there exists some constant c such that $f_1 \le c\, f_2$ and $f_2 \le c\, f_1$. We write

$$\partial D = \partial_1 D \cup \partial_2 D \cup \{z_1, z_2\}$$

where $\partial_1 D, \partial_2 D$ denote the two open arcs of ∂D with endpoints z_1, z_2. Let $\hat{S}_D(w; z_1, z_2)$ be the minimum of the harmonic measures of $\partial_1 D, \partial_2 D$ from w. This is a conformal invariant, and a simple computation in $\mathbb{H}$ shows that

$$\hat{S}_D(w; z_1, z_2) = \frac{1}{\pi}\,\min\left\{\arg \Phi_D(w), \pi - \arg \Phi_D(w)\right\},$$

and hence

$$\hat{S}_D(w; z_1, z_2) \asymp S_D(w; z_1, z_2),$$

and

$$G_D(w; z_1, z_2) \asymp \operatorname{inrad}_D(w)^{d-2}\, \hat{S}_D(w; z_1, z_2)^\beta.$$

To bound the harmonic measure, it is often useful to use the Beurling estimate. We recall it here; for a proof see, for example, [2, Chapter V]. Let B_t be a standard Brownian motion and τ_D denote the first exit time of some domain D for this Brownian motion.

Proposition 2.1 (Beurling Estimate)**.** *There is a constant $c > 0$ such that if $z \in \mathbb{D}$ and K is a connected compact subset of $\overline{\mathbb{D}}$ with $0 \in K$ and $K \cap \partial\mathbb{D} \neq \emptyset$, then*

$$\mathbb{P}^z\{B[0,\tau_{\mathbb{D}}] \cap K = \emptyset\} \leq c\,|z|^{1/2}.$$

We may derive from this the following consequence.

Proposition 2.2. *There is a constant $c > 0$ such that if K is a connected compact subset of $\overline{\mathbb{H}}$ with $K \cap \mathbb{R} \neq \emptyset$ and $z_0 \in \mathbb{H}$, $\varepsilon > 0$ are such that $B_\varepsilon(z_0) \cap K \neq \emptyset$ then for $w \in \mathbb{H}$,*

$$\mathbb{P}^w\{B[0,\tau_{\mathbb{H}\setminus K}] \cap B_\varepsilon(z_0) \neq \emptyset\} \leq c\,\left[\frac{\varepsilon}{|z_0 - w|}\right]^{1/2}.$$

Proof. Consider the map

$$g(z) := \frac{\varepsilon}{z - z_0}, \qquad g : \mathbb{C} \setminus B_\varepsilon(z_0) \to \mathbb{D}.$$

Let $K' = g([\mathbb{C} \setminus \mathbb{H}] \cup [K \setminus B_\varepsilon(z_0)])$, and note that K' is a connected compact subset of $\mathbb{D}$ with $0 \in K'$ and $K' \cap \partial\mathbb{D} \neq \emptyset$. Thus by Proposition 2.1 we know

$$\mathbb{P}^{g(w)}\{B[0,\tau_{\mathbb{D}}] \cap K' = \emptyset\} \leq c\,|g(w)|^{1/2}.$$

which by the conformal invariance of Brownian motion, and the definition of g is the desired statement. ■

If $j = 1, 2$, let $\Delta_{D,j}(w; z_1, z_2)$ be the infimum of all s such that there exists a curve $\eta : [0,1) \to D$ contained in the disk of radius s about w with $\eta(0) = w, \eta(1^-) \in \partial_j D$. Note that

$$\mathrm{inrad}_D(w) = \min\left\{\Delta_{D,1}(w; z_1, z_2), \Delta_{D,2}(w; z_1, z_2)\right\}.$$

We let

$$\Delta^*_D(w; z_1, z_2) = \max\left\{\Delta_{D,1}(w; z_1, z_2), \Delta_{D,2}(w; z_1, z_2)\right\}.$$

The Beurling estimate implies that there is a $c < \infty$ such that the probability a Brownian motion starting at w reaches distance $\Delta^*_D(w; z_1, z_2)$ before leaving D is bounded above by

$$c\left[\frac{\mathrm{inrad}_D(w)}{\Delta^*_D(w; z_1, z_2)}\right]^{1/2}.$$

Therefore,

$$S_D(w; z_1, z_2) \asymp \hat{S}_D(w; z_1, z_2) \leq c\left[\frac{\mathrm{inrad}_D(w)}{\Delta^*_D(w; z_1, z_2)}\right]^{1/2}, \tag{2}$$

which gives us the upper bound

$$G_D(w; z_1, z_2) \leq c\,\mathrm{inrad}_D(w)^{d-2+\frac{\beta}{2}}\,\Delta^*_D(w; z_1, z_2)^{-\frac{\beta}{2}}.$$

We will also need a fact which is a form of continuity of the Green's function under a small perturbation of the domain. First consider the following two lemmas on the conformal radius.

Lemma 2.3. *Let $\mathcal{B}_r$ denote the closed disk of radius e^{-r} about the origin. Suppose D is a simply connected subdomain of $\mathbb{D}$ containing the origin and $e^{-r} < \mathrm{inrad}_D(0)$. Suppose B_t is a Brownian motion starting at the origin and let*

$$\tau_D = \inf\{t : B_t \notin D\}, \quad \tau_{\mathbb{D}} = \inf\{t : B_t \notin \mathbb{D}\}, \quad \sigma_{r,D} = \inf\{t \geq \tau_D : B_t \in \mathcal{B}_r\}.$$

Then,

$$\mathbb{P}\{\tau_D < \sigma_{r,D} < \tau_{\mathbb{D}}\} = -r \, \log \Upsilon_D(0).$$

Proof. Let $f : D \to \mathbb{D}$ be the conformal transformation with $f'(0) > 0$ fixing 0; then $-\log \Upsilon_D(0) = \log f'(0)$. Let $g(z) = \log[|f(z)|/|z|]$ which is a bounded harmonic function on D, and hence

$$\log f'(0) = g(0) = \mathbb{E}[g(B_{\tau_D})] = -\mathbb{E}[\log |B_{\tau_D}|].$$

For $e^{-r} \leq |w| < 1$, $-\frac{1}{r}\log|w|$ is the probability that a Brownian motion starting at w hits $\mathcal{B}_r$ before leaving the $\mathbb{D}$. Thus,

$$\mathbb{P}\{\tau_D < \sigma_{r,D} < \tau_{\mathbb{D}}\} = r\,\mathbb{E}\Big[-\frac{1}{r}\log|B_{\tau_D}|\Big] = r\,\log f'(0) = -r\,\log \Upsilon_D(0).$$

■

Lemma 2.4. *There exists a $c > 0$ such that for any two simply connected domains $D_1 \subseteq D_2$ and a point $w \in D_1 \cap D_2$ then*

$$0 \leq \Upsilon_{D_2}(w) - \Upsilon_{D_1}(w) \leq c\,\mathrm{diam}(D_2 \setminus D_1)$$

Proof. Without loss of generality, we assume $\mathrm{inrad}(D_2) = 1$. If $\mathrm{inrad}(D_1) \leq 7/8$, then $\mathrm{diam}(D_2 \setminus D_1) \geq 1/8$, and we can use the estimate $\mathrm{inrad}(D) \asymp \Upsilon(D)$. if $\mathrm{inrad}(D_1) \geq 7/8$, then we can use the previous lemma, conformal invariance, and the Koebe-(1/4) theorem to see $\Upsilon_{D_2}(w) - \Upsilon_{D_1}(w)$ is comparable to the probability that a Brownian motion starting at w hits $D_2 \setminus D_1$ and returns to $\mathcal{B} = B_{1/16}(w)$, the disk of radius $1/16$ about w without leaving D_2. Using the Beurling estimate, we see the probability of hitting $D_2 \setminus D_1$ is bounded above by $c\,\mathrm{diam}(D_2 \setminus D_1)^{1/2}$ and using it again the probability of getting back to $\mathcal{B}$ before leaving D is bounded by $c\,\mathrm{diam}(D_2 \setminus D_1)^{1/2}$. ■

We will need some notion of closeness of two nested domains before we can state our lemma.

Definition. Given two simply connected domains $D_1 \subseteq D_2 \subseteq \mathbb{H}$ with marked boundary points $z_1 \in \partial D_1$ and $z_2 \in \partial D_2$, we say (D_1, z_1) and (D_2, z_2) are *R-close near z* if the following holds. Let $B_R^{(i)}(z)$ denote the connected component of $B_R(z) \cap D_i$ which contains z. Then,

- $z_1 \in \partial B_R^{(1)}(z)$,
- $z_2 \in \partial B_R^{(2)}(z)$, and
- $D_2 \setminus D_1 \subseteq B_R(z)$.

Lemma 2.5. *There exists $c > 0$ such that the following holds. Suppose $z, w \in \mathbb{H}$, $D_1 \subseteq D_2 \subseteq \mathbb{H}$ are simply connected domains, and $z_1 \in \partial D_1$, $z_2 \in \partial D_2$. If*

- $z, w \in D_1 \cap D_2$,
- (D_1, z_1) *and* (D_2, z_2) *are R-close near z for* $R \leq \operatorname{inrad}_{D_1}(w) \wedge \frac{1}{2}|z - w|$,
- $\infty \in \partial D_1 \cap \partial D_2$,

then

$$|G_{D_1}(w; z_1, \infty) - G_{D_2}(w; z_2, \infty)| \leq c \operatorname{inrad}_{D_1}(w)^{d-2-\frac{\beta\wedge 1}{2}} R^{\frac{\beta\wedge 1}{2}}.$$

Proof. Recall that

$$G_D(w; z_1, z_2) = \Upsilon_D(w)^{d-2} S_D(w; z_1, z_2)^{\beta}$$

where $S(w; z_1, z_2)$ is the sine of the argument of w after applying the unique (up to scaling) conformal map, Φ_D, that sends D to $\mathbb{H}$ while sending z_1 to 0 and z_2 to ∞. Writing, as before,

$$\partial D = \partial_1 D \cup \{z_1\} \cup \partial_2 D \cup \{z_2\}$$

where the union is written in counter-clockwise order, this argument is a conformally invariant and can be computed by

$$\arg \Phi_D(w) = \pi \cdot \mathbb{P}^w\{B_\tau \in \partial_2 D\} \quad \text{where} \quad \tau = \inf\{t \ : \ B_t \in \partial D\}$$

where $\mathbb{P}^w$ is the probability for a standard Brownian motion started at w.

Consider our case. Write

$$\partial D_1 = \partial_1 D_1 \cup \{z_1\} \cup \partial_2 D_1 \cup \{\infty\} \ \text{ and } \ \partial D_2 = \partial_1 D_2 \cup \{z_2\} \cup \partial_2 D_2 \cup \{\infty\}$$

again with the union written in counter-clockwise order. Note that the condition that (D_1, z_1) and (D_2, z_2) are R-close near z implies that

$$\partial_1 D_1 \setminus B_R(z) = \partial_1 D_2 \setminus B_R(z) \ \text{ and } \ \partial_2 D_1 \setminus B_R(z) = \partial_2 D_2 \setminus B_R(z). \tag{3}$$

Define

$$\tau_1 = \inf\{t \ : \ B_t \in \partial D_1\} \ \text{ and } \ \tau_2 = \inf\{t \ : \ B_t \in \partial D_2\}$$

and note that $\tau_1 \leq \tau_2$.

We may write that

$$\begin{aligned} |\arg \Phi_{D_1}(w) - \arg \Phi_{D_2}(w)| &= |\pi \cdot \mathbb{P}^w\{B_{\tau_1} \in \partial_2 D_1\} - \pi \cdot \mathbb{P}^w\{B_{\tau_2} \in \partial_2 D_2\}| \\ &\leq 2\pi \cdot \mathbb{P}^w\{B_t \in B_R(z) \text{ for some } t \leq \tau_2\}. \end{aligned}$$

where the last line follows since, if considered path-wise, the Brownian motion must enter $B_R(z)$ if it is to hit a different side of the boundary in D_1 versus D_2 by equation (3). By the Beurling estimate (Proposition 2.2),

$$|\arg \Phi_{D_1}(w) - \arg \Phi_{D_2}(w)| \leq c \left(\frac{R}{|z-w|} \right)^{1/2}.$$

By noting that $\mathrm{inrad}_{D_1}(w) \leq c\,|z-w|$ by the choice of R and the definition of R-close, and splitting into the cases when $\beta \geq 1$ versus $\beta < 1$ we see

$$|S_{D_1}(w; z_1, \infty)^\beta - S_{D_2}(w; z_2, \infty)^\beta| \leq c \left(\frac{R}{\mathrm{inrad}_{D_1}(w)} \right)^{(\beta \wedge 1)/2}.$$

Consider the term involving the conformal radius. By using Lemma 2.4 and recalling that $d-2<0$ and $\Upsilon_{D_1}(w) \leq \Upsilon_{D_2}(w)$, we see

$$\begin{aligned}|\Upsilon_{D_2}(w)^{d-2} - \Upsilon_{D_1}(w)^{d-2}| &\leq (d-2)\Upsilon_{D_1}(w)^{d-3}|\Upsilon_{D_2}(w) - \Upsilon_{D_1}(w)| \\ &\leq c\Upsilon_{D_1}(w)^{d-2} \left(\frac{R}{\mathrm{inrad}_{D_1}(w)} \right).\end{aligned}$$

Combining these, noting that $R < \mathrm{inrad}_{D_1}(w)$, gives

$$\begin{aligned}&|G_{D_1}(w; z_1, \infty) - G_{D_2}(w; z_2, \infty)| \\ &\quad \leq |\Upsilon_{D_1}(w)^{d-2} S_{D_1}(w; z_1, \infty)^\beta - \Upsilon_{D_1}(w)^{d-2} S_{D_2}(w; z_2, \infty)^\beta| \\ &\qquad + |\Upsilon_{D_1}(w)^{d-2} S_{D_2}(w; z_2, \infty)^\beta - \Upsilon_{D_2}(w)^{d-2} S_{D_2}(w; z_2, \infty)^\beta| \\ &\quad \leq c\Upsilon_{D_1}(w)^{d-2} \left(\frac{R}{\mathrm{inrad}_{D_1}(w)} \right)^{(\beta \wedge 1)/2} + c\Upsilon_{D_1}(w)^{d-2} \left(\frac{R}{\mathrm{inrad}_{D_1}(w)} \right) \\ &\quad \leq c\, \mathrm{inrad}_{D_1}(w)^{d-2-\frac{\beta \wedge 1}{2}} R^{\frac{\beta \wedge 1}{2}}.\end{aligned}$$

as desired. ■

2.2. **Schramm-Loewner evolution.** The chordal Schramm-Loewner evolution with parameter κ (from 0 to ∞ in $\mathbb{H}$ parametrized so that the half-plane capacity grows at rate $a = 2/\kappa$) is the random curve $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ with $\gamma(0) = 0$ satisfying the following. Let H_t denote the unbounded component of $\mathbb{H} \setminus \gamma(0,t]$, and let g_t be the unique conformal transformation of H_t onto $\mathbb{H}$ with $g_t(z) - z \to 0$ as $z \to \infty$. Then g_t satisfies the Loewner differential equation

$$\partial_t g_t(z) = \frac{a}{g_t(z) - U_t}, \quad g_0(z) = z, \tag{4}$$

where $U_t = -B_t$ is a standard Brownian motion. For $z \in \overline{\mathbb{H}} \setminus \{0\}$, the solution of this initial value problem exists up to time $T_z \in (0, \infty]$.

Suppose $z \in \mathbb{H}$ and let

$$Z_t = Z_t(z) = X_t + iY_t = g_t(z) - U_t.$$

Then the Loewner differential equation becomes the SDE

$$dZ_t = \frac{a}{Z_t}\,dt + dB_t.$$

Let

$$S_t = S_t(z) = S_{H_t}(z;\gamma(t),\infty) = \sin\arg Z_t,$$
$$\Upsilon_t = \Upsilon_t(z) = \Upsilon_{H_t}(z;\gamma(t),\infty) = \frac{Y_t}{|g_t'(z)|},$$
$$M_t = M_t(z) = G_{H_t}(z;\gamma(t),\infty) = \Upsilon_t^{d-2}\,S_t^{\beta}.$$

Either by direct computation or by using the Schwarz lemma, we can see that Υ_t decreases in t and hence we can define $\Upsilon = \Upsilon_{T_z-}$. If $0<\kappa\leq 4$, the SLE paths are simple and with probability one $T_z=\infty$. If $4<\kappa<8$, $T_z<\infty$ and by (1) we know

$$\Upsilon \asymp \operatorname{dist}\left[z,\gamma(0,T_z]\cup\mathbb{R}\right] = \operatorname{dist}\left[z,\gamma(0,\infty)\cup\mathbb{R}\right]. \tag{5}$$

Using Itô's formula, we can see that M_t is a local martingale satisfying

$$dM_t = \frac{a\,X_t}{X_t^2+Y_t^2}\,M_t\,dB_t.$$

We will need the following estimate for SLE; see [1] for a proof. By a *crosscut* in D we will mean a simple curve $\eta:(0,1)\to D$ with $\eta(0^+),\eta(1^-)\in\partial D$. We call $\eta(0^+),\eta(1^-)$ the endpoints of the crosscut.

Proposition 2.6. *There exists $c<\infty$ such that if η is a crosscut in $\mathbb{H}$ with $-\infty<\eta(1^-)\leq\eta(0^+)=-1$, then the probability that an SLE_κ curve from 0 to ∞ intersects η is bounded above by $c\,\operatorname{diam}(\eta)^{\lambda}$ where λ is as defined in Section 2.1.*

2.3. **Radial parametrization.** In order to prove the existence of multi-point Green's functions, we will need to study the behavior of the SLE curve from the perspective of $z\in\mathbb{H}$. To do so, it is useful to parametrize the curve so that the conformal radius seen from z decays deterministically. We fix $z\in\mathbb{H}$ and let

$$\sigma(t) = \inf\left\{s : \Upsilon_s = e^{-2at}\right\}.$$

Under this parametrization, the total lifetime of the curve is $\log(\Upsilon_0/\Upsilon)/2a$. Let $\Theta_t = \arg Z_{\sigma(t)}(z), \hat S_t = S_{\sigma(t)}(z) = \sin\Theta_t$. Using Itô's formula one can see that Θ_t satisfies

$$d\Theta_t = (1-2a)\,\cot\Theta_t\,dt + d\hat W_t,$$

where $\hat W_t$ is a standard Brownian motion. Since $a>1/4$, comparison to a Bessel process shows that solutions to this process leave $(0,\pi)$ in finite time. This reflects that fact that chordal SLE_κ does not reach z for $\kappa<8$ and hence $\Upsilon>0$. Let

$$\hat M_t = M_{\sigma(t)}(z) = e^{-2as(d-2)}\,\hat S_t^{\beta} = e^{-(2a-\frac{1}{2})t}\,\hat S_t.$$

This is a time change of a local martingale and hence is a local martingale; indeed, Itô's formula gives

$$d\hat{M}_t = (4a-1)\,\cot\Theta_t\,d\hat{W}_t.$$

Using Girsanov's theorem, see for example [4], we can define a new probability measure $\mathbb{P}^*$ which corresponds to paths "weighted locally by the local martingale $\hat{M}_t$". For the time being, we treat this as an arbitrary change of measure, however in Section 2.4 we will see that is precisely the change of measure which gives two-sided radial SLE. Intuitively, $\hat{M}_t$ weights more heavily those paths whose continuations are likely to get much closer to z. For more examples of the application of Girsanov's theorem to the study of SLE, and a general outline of the way Girsanov's theorem is used below, see [8].

In this weighting,

$$d\hat{W}_t = (4a-1)\,\cot\Theta_t\,dt + dW_t,$$

where W_t is a standard Brownian motion with respect to $\mathbb{P}^*$. In particular,

$$d\Theta_t = 2a\,\cot\Theta_t\,dt + dW_t. \tag{6}$$

Since $2a > 1/2$, we can see by comparison with a Bessel process that with respect to $\mathbb{P}^*$, the process stays in $(0,\pi)$ for all times. Using this we can show that $\hat{M}_t$ is actually a martingale, and the measure $\mathbb{P}^*$ can be defined by

$$\mathbb{P}^*[V] = \hat{M}_0^{-1}\,\mathbb{E}[\hat{M}_t\,1_V] \quad\text{for}\quad V\in\mathcal{F}_t,$$

where $\mathcal{F}_t$ denotes the σ-algebra generated by $\{\hat{W}_s : 0\le s\le t\}$. Much of the analysis of SLE_κ as it gets close to z uses properties of the simple SDE (6). Recall that we assume that $a > 1/4$ and all constants can depend on a.

Lemma 2.7. *There exists $c<\infty$ such that if Θ_t satisfies* (6) *with $\Theta_0 = x \in (0,\pi/2)$, then if $0<y<1$ and*

$$\tau = \inf\left\{t : \Theta_t \in \{y,\pi/2\}\right\},$$

then

$$\mathbb{P}^*\{\Theta_\tau = y\} \le c\,(y/x)^{1-4a}.$$

Proof. Itô's formula shows that $F(\Theta_{t\wedge\tau})$ is a $\mathbb{P}^*$-martingale where

$$F(s) = \int_s^{\pi/2} (\sin u)^{-4a}\,du, \qquad \frac{F''(s)}{F'(s)} = -4a\,\cot s.$$

Note that $F(\pi/2)=0$ and

$$F(s) \sim \frac{s^{1-4a}}{1-4a}, \qquad s\to 0^+.$$

The optional sampling theorem implies that

$$F(x) = \mathbb{P}^*\{\Theta_\tau = y\}\,F(y).$$

■

Lemma 2.8. *The invariant density for the SDE* (6) *is*

$$f(x) = C_{4a} \sin^{4a} x, \quad 0 < x < \pi, \quad C_{4a} := \left[\int_0^{\pi} \sin^{4a} x \right]^{-1}. \tag{7}$$

Proof. This can be quickly verified and is left to the reader. ■

One can use standard techniques for one-dimensional diffusions to discuss the rate of convergence to the equilibrium distribution. We will state the one result that we need; see [10] for more details. If F is a nonnegative function on $(0, \pi)$. let

$$I_F := C_{4a} \int_0^{\pi} F(x) \, \sin^{4a} x \, dx.$$

An important fact is that the implicit constant in $O(e^{-ut})$ is independent of the starting point x.

Lemma 2.9. *There exists $u < \infty$ such that for every $t_0 > 0$ there exists $c < \infty$ such that if F is a nonnegative function with $I_F < \infty$ and $t \geq t_0$.*

$$|\mathbb{E}[F(\Theta_t)] - I_F| \leq c \, e^{-ut} \, I_F.$$

An important case for us is $F(x) = [\sin x]^{-\beta} = \sin^{2-4a-\frac{1}{4a}} x$. Let

$$c_* = I_F = \frac{C_{4a}}{C_{2-\frac{1}{4a}}} = \frac{\int_0^{\pi} \sin^{1-\frac{1}{4a}} x \, dx}{\int_0^{\pi} \sin^{4a} x \, dx}. \tag{8}$$

We will take advantage of this uniform bound to give a concrete estimate on how well the Green's function approximates the probability of getting near a point.

Lemma 2.10. *There exists $u > 0$ such that if D is a simply connected domain and z_1, z_2 are points in its boundary, $r \leq 3/4$, γ is an SLE_κ curve from z_1 to z_2, $w \in D$, and D_∞ denotes the connected component of $D \setminus \gamma(0, \infty)$ containing w, then*

$$\mathbb{P}\left\{ \Upsilon_{D_\infty}(w) \leq r \cdot \Upsilon_D(w) \right\} = c_* \, r^{d-2} \, S_D(w; z_1, z_2)^{\beta} \, \left[1 + O(r^u)\right],$$

where c_* is as defined in (8). *In particular, there exists $c < \infty$ such that for all $r \leq 3/4$,*

$$\mathbb{P}\left\{ \Upsilon_{D_\infty}(w) \leq r \cdot \Upsilon_D(w) \right\} \leq c \, r^{d-2} \, S_D(w; z_1, z_2)^{\beta}.$$

Proof. By conformal invariance we may assume $\Upsilon_D(w) = 1$ and define t by $r = e^{-2at}$. Let $\sigma = \inf\{s : \Upsilon_s = r\}$. Then,

$$\begin{aligned}
\mathbb{P}\{\sigma < \infty\} &= \mathbb{E}\big[1\{\sigma < \infty\}\big] \\
&= r^{2-d}\,\mathbb{E}\big[\hat{M}_t\,\hat{S}_t^{-\beta}\big] \\
&= r^{2-d}\,S_D(w; z_1, z_2)^{\beta}\,\mathbb{E}^*\big[\hat{S}_t^{-\beta}\big] \\
&= c_*\,r^{2-d}\,S_D(w; z_1, z_2)^{\beta}\,[1 + O(e^{-ur})].
\end{aligned}$$

■

Using (1) and (2), we immediately get the following lemma which is in the form that we will use.

Lemma 2.11. *There exists $C < \infty$, such that if D is a simply connected domain and z_1, z_2 are points in its boundary, $r \leq 3/4$, and γ is an SLE_κ curve from z_1 to z_2, then*

$$\mathbb{P}\left\{\mathrm{dist}[w, \gamma[0,\infty)] \leq r \cdot \mathrm{inrad}_D(w)\right\} \leq C r^{d-2} \left[\frac{\mathrm{inrad}_D(w)}{\Delta_D^*(w; z_1, z_2)}\right]^{\beta/2}.$$

2.4. **Two-sided radial** SLE**.** We call SLE_κ under the measure $\mathbb{P}^*$ in the previous subsection *two-sided radial SLE_κ* (from 0 to ∞ through z in $\mathbb{H}$ stopped when it reaches z). Roughly speaking it is chordal SLE_κ conditioned to go through z (stopped when it reaches z). Of course this is an event of probability zero, so we cannot define the process exactly this way. We may provide a direct definition by driving the Loewner equation by the process defined in equation (6) rather than a standard Brownian motion. This definition uses the radial parametrization. We could also just as well use the capacity parametrization, in which case with probability one $T_z < \infty$.

One may justify the definition above examining its relationship to SLE_κ conditioned to get close to z. This next proposition is just a restatement of the definition of the measure $\mathbb{P}^*$ when restricted to curves stopped at a particular stopping time.

Proposition 2.12. *Suppose γ is a chordal SLE_κ path from 0 to ∞ and $z \in \mathbb{H}$. For $\varepsilon \leq \mathrm{Im}(z)$, let $\rho_\varepsilon = \inf\{t : \Upsilon_t(z) = \varepsilon\}$. Let μ, μ_1 be the two measures on $\{\gamma(t) : 0 \leq t \leq \rho_\varepsilon\}$ corresponding to chordal SLE_κ restricted to the event $\{\rho_\varepsilon < \infty\}$ and two-sided radial SLE_κ through z. Then μ, μ_1 are mutually absolutely continuous with respect to each other with Radon-Nikodym derivative*

$$\frac{d\mu_1}{d\mu} = \frac{G_{H_{\rho_\varepsilon}}(z; \gamma(\rho_\varepsilon), \infty)}{G_{\mathbb{H}}(z; 0, \infty)} = \frac{\varepsilon^{d-2}\,S_{\rho_\varepsilon}(z)^{\beta}}{G_{\mathbb{H}}(z; 0, \infty)}.$$

This proposition seems to indicate that there is a still a significant difference between two-sided radial SLE_κ going though z and SLE_κ conditioned to get within a specific distance. However, by using the methods of Lemma 2.9 we get the following improvement.

Proposition 2.13. *There exists $u > 0, c < \infty$ such that the following is true. Suppose γ is a chordal SLE_κ path from 0 to ∞ and $z \in \mathbb{H}$. For $\varepsilon \leq \mathrm{Im}(z)$, let $\rho_\varepsilon = \inf\{t : \Upsilon_t(z) = \varepsilon\}$. Suppose $\varepsilon' < 3\varepsilon/4$. Let μ_1, μ_2 be the two probability measures on $\{\gamma(t) : 0 \leq t \leq \rho_\varepsilon\}$ corresponding to chordal SLE_κ conditioned on the event $\{\rho_{\varepsilon_2} < \infty\}$ and two-sided radial SLE_κ through z. Then μ_1, μ_2 are mutually absolutely continuous with respect to each other and the Radon-Nikodym derivative satisfies*

$$\left|\frac{d\mu_2}{d\mu_1} - 1\right| \leq c\,(\varepsilon'/\varepsilon)^u.$$

From the definition, it is easy to show that there is a subsequence $t_n \uparrow T_z$ with $\gamma(t_n) \to z$. Stronger than this is true and in [5] it is proven that, for $0 < \kappa < 8$, with probability one, two-sided radial gives a curve, by which we mean that with probability one $\gamma(T_z-) = z$. The following lemma, used in this proof, can be found in [5].

Lemma 2.14. *Let $\rho_\varepsilon = \inf\{t \ : \ \Upsilon_t(z) = \varepsilon\}$. Fix $\eta < R$ and $z \in \mathbb{H}$, then there exists some $\alpha > 0$ and c depending only on z such that*

$$\mathbb{P}^*\{\gamma[\xi_{\rho_\varepsilon}, T_z] \not\subseteq B_R(z)\} \leq c\left(\frac{\eta}{R}\right)^\alpha.$$

We will also need this bound in a conditional form. In order to prove the conditional form, we need the following lemma.

Lemma 2.15. *Let $\rho_\varepsilon = \inf\{t \ : \ \Upsilon_t(z) = \varepsilon\}$. There exists $c < \infty$, such that if $z \in \mathbb{H}$ and $\varepsilon \leq \mathrm{Im}(z)/2$, $0 < \theta_0 \leq \pi/2$,*

$$\mathbb{P}\{S_{\rho_\varepsilon}(z) < \sin(\theta_0) \mid \rho_\varepsilon < \infty\} \leq c\,\theta_0^{1-\frac{\kappa}{8}}.$$

Proof. First note that by Lemma 2.12 and Lemma 2.10 we have that

$$\mathbb{P}\{S_{\rho_\varepsilon}(z) < \sin(\theta_0) \mid \rho_\varepsilon < \infty\} \leq c\,\mathbb{E}^*[S_{\rho_\varepsilon}^{-\beta}(z)\mathbb{1}\{S_{\rho_\varepsilon}(z) < \sin(\theta_0)\}].$$

By applying the techniques from Lemma 2.9 with the function

$$F(\theta) = \sin(\theta)^{-\beta}\mathbb{1}\{\sin(\theta) < \sin(\theta_0)\}$$

and noting that the integral is

$$\int_0^\pi \sin(\theta)^{-\beta}\mathbb{1}\{\sin(\theta) < \sin(\theta_0)\}\sin^{4a} d\theta = 2\int_0^{\theta_0} \sin(\theta)^{2-\frac{\kappa}{8}} = O\big(\theta_0^{1-\frac{\kappa}{8}}\big)$$

we get the result. ∎

Lemma 2.16. *Let $\rho_\varepsilon = \inf\{t \ : \ \Upsilon_t(z) = \varepsilon\}$. Fix $\varepsilon < \eta < R < 1$ and $z \in \mathbb{H}$, then there exists some c depending only on z and $\alpha > 0$ such that*

$$\mathbb{P}\{\gamma[\rho_\eta, \rho_\varepsilon] \not\subseteq B_R(z) \mid \rho_\varepsilon < \infty\} \leq c\left(\frac{\eta}{R}\right)^\alpha.$$

Proof. We apply Lemma 2.15 with the above to see that

$$\begin{aligned}
&\mathbb{P}\{\gamma[\rho_\eta, \rho_\varepsilon] \not\subseteq B_R(z) \mid \rho_\varepsilon < \infty\} \\
&\quad = \mathbb{P}\{\gamma[\rho_\eta, \rho_\varepsilon] \not\subseteq B_R(z); S_{\rho_\varepsilon}(z) \geq \sin(\theta) \mid \rho_\varepsilon < \infty\} \\
&\quad\quad + \mathbb{P}\{\gamma[\rho_\eta, \rho_\varepsilon] \not\subseteq B_R(z); S_{\rho_\varepsilon}(z) < \sin(\theta) \mid \rho_\varepsilon < \infty\} \\
&\quad \leq c\mathbb{E}^*[S_{\rho_\varepsilon}^{-\beta}(z)\mathbb{1}\{\gamma[\rho_\eta, \rho_\varepsilon] \not\subseteq B_R(z); S_{\rho_\varepsilon}(z) \geq \sin(\theta)\}] + c\theta^{1-\frac{\kappa}{8}} \\
&\quad \leq c\theta^{-\beta}\mathbb{P}^*\{\gamma[\rho_\eta, \rho_\varepsilon] \not\subseteq B_R(z)\} + c\theta^{1-\frac{\kappa}{8}} \\
&\quad \leq c\theta^{-\beta}\mathbb{P}^*\{\gamma[\rho_\eta, T_z] \not\subseteq B_R(z)\} + c\theta^{1-\frac{\kappa}{8}} \\
&\quad \leq c\theta^{-\beta}(\eta/R)^\alpha + c\theta^{1-\frac{\kappa}{8}}
\end{aligned}$$

where c is being used generically. Thus by an appropriate choice of θ, for example

$$\theta = (\eta/R)^{\frac{\alpha}{2\beta}},$$

we get the desired bound. ∎

3. Multi-point Green's function

In this subsection we consider two distinct points $z, w \in \mathbb{H}$. To simplify notation, we write

$$\xi = \xi_\varepsilon = \xi_{z,\varepsilon} = \inf\{t : \Upsilon_t(z) \leq \varepsilon\},$$

$$\chi = \chi_\delta = \xi_{w,\delta} = \inf\{t : \Upsilon_t(w) \leq \delta\}.$$

Although we will write ξ, χ, it is important to remember that these quantities depend on $z, \varepsilon, w, \delta$. We let $\mathbb{P}, \mathbb{E}$ denote probabilities and expectations for SLE_κ from 0 to ∞ in $\mathbb{H}$ and $\mathbb{P}^*, \mathbb{E}^*$ for the corresponding quantities for two-sided radial through z. The multi-point Green's function,

$$G(z, w) = G_{\mathbb{H}}(z, w; 0, \infty)$$

roughly corresponds to the probability that SLE in $\mathbb{H}$ from 0 to ∞ goes through z and then through w. This quantity is not symmetric. Although we do not have a closed from for this quantity, we can define it precisely.

Definition. The multi-point Green's function $G(z, w)$ is defined by

$$G(z, w) = G(z)\,\mathbb{E}^*\left[G_H(w; z, \infty)\right],$$

where H is the unbounded component of $\mathbb{H} \setminus \gamma(0, T_z]$.

In [11], the exact formula for $G_{\mathbb{H}}(z;0,\infty)$ was found by considering the martingale $G_{H_t}(z,\gamma(t),\infty)$ and then using Itô's formula and scaling to find the ODE that it satisfies, which could then be explicitly solved. When attempting the same technique here, a three variable PDE results, which does not immediately seem to admit a closed form solution.

The justification for this definition comes from the following theorem. Implicit in the statement, is that the limit can be taken along any sequence of ε, δ going to zero.

Theorem 1. *If $z, w \in \mathbb{H}$, then*

$$\lim_{\varepsilon,\delta \to 0+} \varepsilon^{d-2}\, \delta^{d-2}\, \mathbb{P}\{\xi < \chi < \infty\} = c_*^2\, G(z,w),$$

where c_* is as defined in (8).

When

$$d = \left(1 + \frac{\kappa}{8}\right) \wedge 2$$

is the dimension rather than simply $d = 1 + \kappa/8$, this theorem still defines an interesting quantity for $\kappa \geq 8$. Since the curve is space filling for $\kappa \geq 8$, the limit is trivial and

$$\lim_{\varepsilon,\delta \to 0+} \varepsilon^{d-2}\, \delta^{d-2}\, \mathbb{P}\{\xi < \chi < \infty\} = \mathbb{P}\{\xi_0 < \chi_0\} = c_* G(z,w).$$

This agrees with the above definition of $G(z,w)$ since we may take two-sided radial through z for $\kappa \geq 8$ to be the measure on γ stopped at the time the curve passes through z and

$$G_D(w; z_1, z_2) = \mathbb{1}\{w \in D\}.$$

Since this case requires no further work, we will continue to assume that $\kappa < 8$.

We will need one lemma that will follow from our work on Beffara's estimate, which we will prove in Section 4.

Lemma 3.1. *There exists $\alpha > 0$, such that if $z, w \in \mathbb{H}$, then there exists $c = c_{z,w} < \infty$, such that for all $\varepsilon, \delta, r > 0$,*

$$\mathbb{P}\{\xi < \chi < \infty; \mathrm{inrad}_\xi(w) \leq r\} \leq c\, \varepsilon^{2-d} \delta^{2-d} r^\alpha.$$

More precise results than this are obtained in this paper, but this is all that is required in this section. Before going through the details of the proof, we briefly sketch the argument. To estimate

$$\mathbb{P}\{\xi < \chi < \infty\},$$

we wish to show that this probability is carried mostly on curves which get within ε of z in conformal radius before decreasing the conformal radius of w much at all. To show that the curves which do not do this are negligible, we use Lemma 3.1, obtained from our proof of Beffara's estimate.

On the event that the curves stays bounded away from w, we know the Green's function for getting to w stays uniformly bounded, allowing us to use convergence of the conditioned measures $\mathbb{E}[\,\cdot \mid \xi < \infty]$ to $\mathbb{E}^*[\,\cdot\,]$, the two-sided radial measure, as measures on the SLE curve up until some fixed conformal radius $\eta \gg \varepsilon$.

This would be everything if it were not for the fact that the tip of the curves (the portion very near z) under the conditioned measure versus the two-sided radial measure have very different distribution. To handle this, we use Lemmas 2.14 and 2.16 to show that under both measures the tip stays close z most of the time in Euclidean distance, and then Lemma 2.5 tells us that the Green's function for getting to w is insensitive to these changes.

To aid in the understanding of the proof, Figure 2 shows diagrammatically the various distances considered and the approximate shape of a curve in the main term.

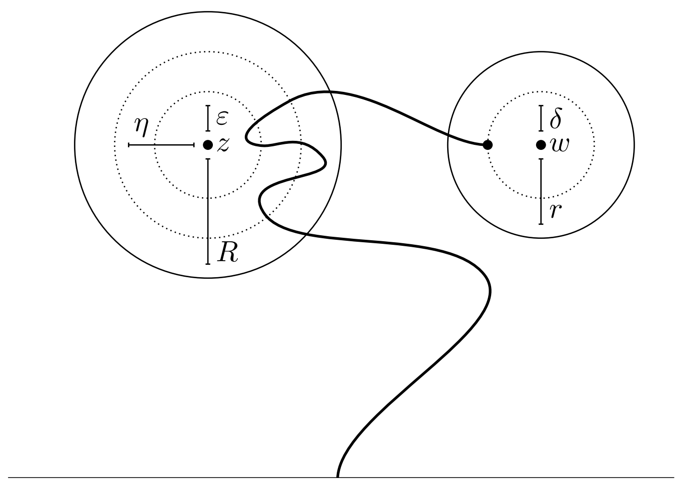

FIGURE 2. A diagram of the proof of Theorem 1. Dotted circles represent conformal radii and solid circles refer to geometric radii. The bold curve gives an example of the approximate shape of a curve contributing to the leading order event.

Proof of Theorem 1 given Lemma 3.1. We first split according to how close we get to w before getting close to z. Fixing some $r < |z - w|/2$, by Lemma 3.1 we see that for some $\alpha > 0$

$$\begin{aligned}\mathbb{P}\{\xi < \chi < \infty\} &= \mathbb{P}\{\xi < \chi < \infty; \mathrm{inrad}_\xi(w) > r\} \\ &\quad + \mathbb{P}\{\xi < \chi < \infty; \mathrm{inrad}_\xi(w) < r\} \\ &= \mathbb{P}\{\xi < \chi < \infty; \mathrm{inrad}_\xi(w) > r\} + O(\varepsilon^{2-d}\delta^{2-d}r^\alpha).\end{aligned}$$

Let $\mathcal{F}_\xi$ denote the σ-algebra generated by the stopping time ξ. By Lemma 2.10 we can see that if $\delta \le r/2$,

$$\begin{aligned}&\mathbb{P}\{\xi < \chi < \infty; \operatorname{inrad}_\xi(w) > r \mid \mathcal{F}_\xi\} \\ &\quad = \mathbb{1}\{\xi < \infty; \operatorname{inrad}_\xi(w) > r\}\, c_*\, \delta^{2-d} G_{H_\xi}(w; \gamma(\xi), \infty)[1 + O((\delta/r)^u)].\end{aligned}$$

Using Lemma 2.10 again, this implies

$$\begin{aligned}&c_*^{-2}\, \varepsilon^{d-2} \delta^{d-2}\, G_{\mathbb{H}}(z; 0, \infty)^{-1}\, \mathbb{P}\{\xi < \chi < \infty; \operatorname{inrad}_\xi(w) > r\} \\ &\quad = [1 + O(\varepsilon^u + (\delta/r)^u)]\, \mathbb{E}[G_{H_\xi}(w; \gamma(\xi), \infty) \mathbb{1}\{\operatorname{inrad}_\xi(w) > r\} \mid \xi < \infty].\end{aligned}$$

For simplicity of notation, for a stopping time τ, we let

$$\mathbb{E}_\tau[\,\cdot\,] = \mathbb{E}[\,\cdot \mid \tau < \infty] \text{ and } G_{\tau,r} = G_{H_\tau}(w; \gamma(\tau), \infty) \mathbb{1}\{\operatorname{inrad}_\tau(w) > r\}.$$

and hence we may rewrite this as

$$\begin{aligned}&\mathbb{P}\{\xi < \chi < \infty; \operatorname{inrad}_\xi(w) > r\} \\ &\quad = c_*^2 \varepsilon^{2-d} \delta^{2-d} G_{\mathbb{H}}(z; 0, \infty)[1 + O(\varepsilon^u + (\delta/r)^u)] \mathbb{E}_\xi[G_{\xi,r}].\end{aligned}$$

We wish to transform this expression from the conditioned measure to the two-sided radial measure, and from considering the situation at time ξ (the time it first gets within conformal radius ε) to T_z (the time under the two-sided radial measure that z is first contained in the boundary of H_{T_z}). To do so we will pass through a series of steps.

Fix some η, R so that $\varepsilon < \eta < R < |z - w|/2$. We wish to control the difference

$$\begin{aligned}|\mathbb{E}_\xi[G_{\xi,r}] - \mathbb{E}_\xi[G_{\xi_\eta,r}]| \le\ & \mathbb{E}_\xi[|G_{\xi,r} - G_{\xi_\eta,r}| \mathbb{1}\{\gamma[\xi_\eta, \xi] \subseteq B_R(z)\}] \\ &+ \mathbb{E}_\xi[|G_{\xi,r} - G_{\xi_\eta,r}| \mathbb{1}\{\gamma[\xi_\eta, \xi] \not\subseteq B_R(z)\}].\end{aligned}$$

By Lemma 2.5 and the fact that the inradius about w cannot decrease between ξ_η and ξ if $\gamma[\xi_\eta, \xi] \subseteq B_R(z)$ we see that

$$\mathbb{E}_\xi[|G_{\xi,r} - G_{\xi_\eta,r}| \mathbb{1}\{\gamma[\xi_\eta, \xi] \subseteq B_R(z)\}] = O(r^{d-2-(\beta\wedge 1)/2} R^{(\beta\wedge 1)/2}).$$

On the second term, the difference is no bigger than $O(r^{d-2})$ on an event, which by Lemma 2.16 is $O((\eta/R)^{\alpha'})$ for some $\alpha' > 0$. Putting it all together yields

$$|\mathbb{E}_\xi[G_{\xi,r}] - \mathbb{E}_\xi[G_{\xi_\eta,r}]| = O(r^{d-2-(\beta\wedge 1)/2} R^{(\beta\wedge 1)/2} + r^{d-2} (\eta/R)^{\alpha'}).$$

By Lemma 2.13, we know for events in $\mathcal{F}_{\xi_\eta}$ we have

$$\left|\frac{\mathrm{d}\mathbb{P}^*}{\mathrm{d}\mathbb{P}_\xi} - 1\right| = O((\varepsilon/\eta)^u)$$

and hence we have

$$|\mathbb{E}_\varepsilon[G_{\xi_\eta,r}] - \mathbb{E}^*[G_{\xi_\eta,r}]| = O(r^{d-2} (\varepsilon/\eta)^u).$$

Analogously to before, consider splitting the difference

$$|\mathbb{E}^*[G_{\xi_\eta,r}] - \mathbb{E}^*[G_{T_z,r}]| \leq \mathbb{E}^*[|G_{\xi_\eta,r} - G_{T_z,r}|\mathbb{1}\{\gamma[\xi_\eta, T_z] \subseteq B_R(z)\}] \\ + \mathbb{E}^*[|G_{\xi_\eta,r} - G_{T_z,r}|\mathbb{1}\{\gamma[\xi_\eta, T_z] \not\subseteq B_R(z)\}].$$

By Lemma 2.5 and the fact that the inradius about w cannot decrease between ξ_η and T_z if $\gamma[\xi_\eta, T_z] \subseteq B_R(z)$, we again see

$$\mathbb{E}^*[|G_{\xi_\eta,r} - G_{T_z,r}|\mathbb{1}\{\gamma[\xi_\eta, T_z] \subseteq B_R(z)\}] = O(r^{d-2-(\beta\wedge 1)/2} R^{(\beta\wedge 1)/2})$$

The second term is on an event which is by Lemma 2.14 is $O((\eta/R)^{\alpha'})$, and hence we have again that

$$|\mathbb{E}^*[G_{\xi_\eta,r}] - \mathbb{E}^*[G_{T_z,r}]| = O(r^{d-2-(\beta\wedge 1)/2} R^{(\beta\wedge 1)/2} + r^{d-2}(\eta/R)^{\alpha'}).$$

We may easily see that

$$\mathbb{P}^*\{\mathrm{inrad}_{T_z}(w) = 0\} \leq \sum_{k\geq 1} \mathbb{P}^*\{\mathrm{inrad}_{\xi_{1/k}}(w) = 0\} = 0$$

by the fact that $\mathbb{P}^*$ is absolutely continuous with respect to $\mathbb{P}$ until the stopping time $\xi_{1/k}$ combined with that fact that two-sided radial SLE generates a curve with probability one. Hence, since $G_{T_z}(w; z, \infty) \geq 0$, we have that

$$\mathbb{E}^*[G_{T_z}(w; z, \infty)\mathbb{1}\{\mathrm{inrad}_{T_z}(w) > r\}] \to \mathbb{E}^*[G_{T_z}(w; z, \infty)] \quad \text{as} \quad r \to 0.$$

Combining all these terms and by combining exponents, we see there exists some $b > 0$ such that

$$\varepsilon^{d-2}\delta^{d-2}\mathbb{P}\{\xi < \chi < \infty\} = c_*^2 G_{\mathbb{H}}(z; 0, \infty)[1 + O(\varepsilon^b + (\delta/r)^b)]\mathbb{E}^*[G_{T_z,r}] \\ + O(r^b + (R/r)^b + (R/r)^b(\eta/R)^b + (\varepsilon/r)^b(\varepsilon/\eta)^b).$$

Thus by choosing r, η, and R so that as $\varepsilon, \delta \to 0$ we also have

$$r \to 0, \quad \delta/r \to 0, \quad \varepsilon/r \to 0, \quad R/r \to 0, \quad \eta/R \to 0, \quad \varepsilon/\eta \to 0$$

we see that

$$\varepsilon^{d-2}\delta^{d-2}\mathbb{P}\{\xi < \chi < \infty\} \to c_*^2 G_{\mathbb{H}}(z; 0, \infty)\mathbb{E}^*[G_{T_z}(w; z, \infty)]$$

as desired. ■

This same argument generalizes to show that we can define higher-order Green's functions of SLE as well (those that give normalized probabilities for passing through k marked points in the interior) and that the resulting multi-point Green's functions can be written in terms of expectations under the two-sided radial measure of lower order Green's functions, for instance

$$\varepsilon_1^{d-2}\varepsilon_2^{d-2}\varepsilon_3^{d-2}\mathbb{P}\{\xi_{\varepsilon_1,z_1} < \xi_{\varepsilon_2,z_2} < \xi_{\varepsilon_3,z_3}\} \to c_*^3 G_{\mathbb{H}}(z_1; 0, \infty)\mathbb{E}^*[G_{H_{T_{z_1}}}(z_2, z_3; z, \infty)]$$

where $\mathbb{E}^*$ is the two sided radial measure passing through z_1.

Note that we may obtain the multi-point Green's function as defined in the introduction by summing this over the case where it gets near to z then w and the case where it gets near to w then z.

The remainder of this paper is dedicated to provides a proof of Lemma 3.1 and a sharpened version of Beffara's estimate.

4. Proof of Beffara's estimate and Lemma 3.1

To complete our proof of the existence of multi-point Green's functions we require a proof of Lemma 3.1. We also wish to prove Befarra's estimate which is the following theorem.

Theorem 2 (Beffara's Estimate)**.** *There exists a $c > 0$ such that for all $z, w \in \mathbb{H}$ with $|z|, |w| \geq 1$ we have that*

$$\mathbb{P}\{\Upsilon_\infty(z) < \varepsilon, \Upsilon_\infty(w) < \delta\} \leq c\varepsilon^{2-d}\delta^{2-d}|z-w|^{d-2} \tag{9}$$

The hard work will be establishing the result when z, w are far apart. We will prove the following. We use the notation introduced in Section 2.1. For later convenience, we write this proposition in terms of the usual radius rather than the conformal radius, but it is easy to convert to conformal radius using the Koebe-1/4 theorem.

Proposition 4.1. *For every $0 < \theta < \infty$, there exists $c < \infty$, such that if $z, w \in \mathbb{H}$ with*

$$|z|, |w| \geq \theta \quad and \quad |z - w| \geq \theta/9,$$

then

$$\mathbb{P}\{\Delta_\infty(z) \leq \varepsilon, \Delta_\infty(w) \leq \delta\} \leq c\,\varepsilon^{2-d}\,\delta^{2-d}.$$

Proof of Theorem 2 given Proposition 4.1. Note that if (9) holds for all ε, δ for (z, w) with constant c, then it also holds for all (ε, δ) for (rz, rw) with constant $r^{d-2}\,c$. Hence by scaling and Proposition 4.1, it suffices to prove the estimate when $|z - w| = 1/2$, $|z|, |w| \geq 9/2$ and $\varepsilon, \delta \leq 1/8$. We assume this. For a curve to get within ε of z or δ of w, it will first need to get within distance one of z. By Lemma 2.10, it does so with probability at most $c|z|^{d-2}$. Let $\tau = \inf\{t \ : \ |\gamma(t) - z| \leq 1\}$ be the first such time. Apply the map $f = b\,g_\tau$ to the entire picture where $g_\tau : H_\tau \to \mathbb{H}$ is the usual Loewner map and the constant $b > 0$ is chosen so that $|f'(z)| = 1$. Using the distortion theorem, Koebe-1/4 theorem, and the growth theorem, we can find universal constants $c_1 < c_2$ such that the following estimates holds:

$$\begin{gathered} c_1 \leq |f'(w)| \leq c_2, \\ \mathrm{Im} f(z), \mathrm{Im} f(w) \geq c_1, \\ |f(z) - f(w)| \geq c_1. \\ |f(z) - f(z')| \leq c_2\,\varepsilon \quad \text{if } |z - z'| \leq \varepsilon, \end{gathered}$$

$$|f(w) - f(w')| \leq c_2\, \delta \quad \text{if } |w - w'| \leq \delta$$

Hence, given $\tau < \infty$, the conditional probability that $\Delta_\infty(z) \leq \varepsilon, \Delta_\infty(w) \leq \delta$ is bounded above by the supremum of

$$\mathbb{P}\left\{\Delta_\infty(z') \leq c_2\, \varepsilon, \Delta_\infty(w') \leq c_2\, \delta\right\},$$

where the supremum is over all z', w' satisfying

$$\operatorname{Im}(z'), \operatorname{Im}(w') \geq c_1, \qquad |z - w| \geq c_1.$$

Proposition 4.1 implies that this is bounded above by a constant time $\varepsilon^{2-d}\, \delta^{2-d}$. ■

By an analogous argument to how we obtained Theorem 2 from Proposition 4.1, we may obtain Lemma 3.1 from Proposition 4.2.

Proposition 4.2. *For every $0 < \theta < \infty$, there exists $c < \infty$ and $\alpha > 0$ such that if $z, w \in \mathbb{H}$ with*

$$|z|, |w| \geq \theta \quad \textit{and} \quad |z - w| \geq \theta/9,$$

then for $\rho > \delta$

$$\mathbb{P}\{\Delta_\infty(z) \leq \varepsilon, \Delta_\infty(w) \leq \delta, \Delta_\sigma(w) \leq \rho\} \leq c\varepsilon^{2-d}\delta^{2-d}\rho^\alpha. \tag{10}$$

where $\sigma = \inf\{t \;:\; \Delta_t(z) \leq \varepsilon \;\; \text{or} \;\; \Delta_t(w) \leq \delta\}$.

This proposition will follow immediately from the work required to show Proposition 3.1.

To prove the proposition, we will show that there exists a $c < \infty$ such that (10) holds if $|z - w| \geq 2\sqrt{2}$ and $|z|, |w| \geq 1$. By scaling one can easily deduce the result for all $\theta > 0$ with a θ-dependent constant. We fix z, w with $|z - w| \geq 2\sqrt{2}$ and $|z|, |w| \geq 1$, and denote by $\mathcal{I}$ some fixed vertical or diagonal line such that

$$\operatorname{dist}(z, \mathcal{I}), \operatorname{dist}(w, \mathcal{I}) \geq 1$$

and z, w lie in different components of $\mathbb{H} \setminus \mathcal{I}$. We will further assume, without loss of generality, that z is in the component of $\mathbb{H} \setminus \mathcal{I}$ which contains arbitrarily large negative real numbers in it's boundary (more informally that z is in the left component).

4.1. **An excursion measure estimate.** Our main result will require an estimate of the "distance" between two boundary arcs in a simply connected domain. We will use excursion measure to gauge the distance; we could also use extremal distance, but we find excursion measure more convenient.

Suppose η is a crosscut in $\mathbb{H}$ with $-\infty < \eta(1^-) \leq \eta(0^+) \leq -1$. Let H_η denote the unbounded component of $\mathbb{H} \setminus \eta$. Let $\mathcal{E}(\eta) = \mathcal{E}_{\mathbb{H}_\eta}(\eta, [0, \infty))$ denote the excursion measure between η and $[0, \infty)$ in H_η, the definition of which we now recall (see [6, Section 5.2] for more details). If $z \in H_\eta$, let $h_\eta(z)$ be the probability that a Brownian motion starting at z exits H_η at η. For $x \geq 0$, let $\partial_y h_\eta(x)$ denote the partial derivative. Then

$$\mathcal{E}(\eta) = \int_0^\infty \partial_y h_\eta(x)\, dx.$$

The excursion measure $\mathcal{E}_D(V_1, V_2)$ is defined for any domain and boundary arcs V_1, V_2 in a similar way and is a conformal invariant. If V_2 is smooth, then we can compute $\mathcal{E}_D(V_1, V_2)$ by a similar integral

$$\mathcal{E}_D(V_1, V_2) = \int_{V_2} \partial_{\mathbf{n}} h_{V_1}(z)\, |dz|,$$

where $\mathbf{n}$ denotes the inward normal. We need the following easy estimate.

Lemma 4.3. *There exist c_1, c_2 such that if η is as in Proposition 2.6 with* $\mathrm{diam}(\eta) \leq 1/2$*, then*

$$c_1 \,\mathrm{diam}(\eta) \leq \mathcal{E}(\eta) \leq c_2 \,\mathrm{diam}(\eta).$$

Sketch of proof. In fact, we get an estimate

$$\partial_y h_\eta(x) \asymp \frac{\mathrm{diam}(\eta)}{(x+1)^2}.$$

The key estimate used here is the fact that that if $\mathrm{Re}(z) \geq 0$,

$$h_\eta(z) \asymp \frac{\mathrm{Im}(z)\,\mathrm{diam}(\eta)}{(|z|+1)^2}.$$

■

Lemma 4.4. *There exists a $C < \infty$ such that the following is true. Suppose $H \subset \mathbb{C}$ is a half plane bounded by the line $L = \partial H$, D is a simply connected subdomain of $\mathbb{H}$ and $z \in \partial D$ with $d(z, L) > \frac{1}{2}$. Suppose I is a subinterval of $L \cap \partial D$. Then for every $\varepsilon < \frac{1}{2}$, the excursion measure between I and $V := \partial D \cap \{w : |w - z| \leq \varepsilon\}$ is bounded above by $C\varepsilon^{1/2}$.*

Proof. Without loss of generality we assume that $H = \mathbb{H}, z = i/2$. Let $h(w)$ denote the probability that a Brownian motion starting at w exits D at V. Then the excursion measure is exactly

$$\int_I \partial_y h(x)\, dx.$$

Hence it suffices to give an estimate

$$\partial_y h(x) \leq c\,\varepsilon^{1/2}\,[1 \wedge x^{-2}]. \tag{11}$$

For $|x| \leq 4$, this follows from the Beurling estimate. For $|x| \geq 4$, we first consider the excursion "probability" to reach $\mathrm{Re}(w) = x/2$. By the gambler's ruin estimate, this is bounded by $O(|x|^{-1})$. Then we need to consider the probability that a Brownian motion starting at z' with $\mathrm{Re}(z') = x/2$ reaches the disk of radius 1 about z without leaving D. By comparison with the same probability in the domain $\mathbb{H}$, we see that this is bounded above by $O(|x|^{-1})$. Finally from there we need to hit V which contributes a factor of $O(\varepsilon^{1/2})$ by the Beurling estimate. Combining these estimates gives (11). ■

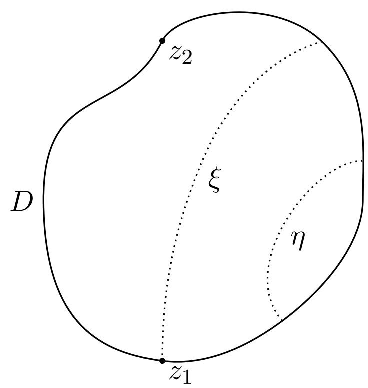

FIGURE 3. The setup for Lemma 4.5.

Lemma 4.5. *There exists $c > 0$ such that the following holds. Let D be a simply connected domain, and let γ be a chordal SLE_κ path from z_1 to z_2 in D. Let $\eta : (0,1) \to D$ be a crosscut in D. Let $\xi : (0,1) \to D$ be another crosscut with $\xi(0^+) = z_1$, and let D_1, D_2 denote the components of $D \setminus \xi$. Suppose $\eta \subset D_1$ and $z_2 \in \partial D_2$. Then,*

$$\mathbb{P}\{\gamma(0,\infty) \cap \eta(0,1) \neq \emptyset\} \leq c\, \mathcal{E}_D(\eta,\xi)^\lambda.$$

See Figure 3 for a diagram of the setup of this lemma.

Proof. By conformal invariance, we may assume that $D = \mathbb{H}$, $z_1 = 0, z_2 = \infty$, and it suffices to prove the result when $\mathcal{E}_D(\eta,\xi) \leq 1$ in which case the endpoints of η are nonzero. Without loss of generality we assume that they lie on the negative real axis with $\eta(1^-) \leq \eta(0^+) < 0$,. Then monotonicity of the excursion measure implies that

$$\mathcal{E}_D(\eta,\xi) \geq \mathcal{E}_D(\eta).$$

Lemma 4.3 implies that $\mathcal{E}_D(\eta) \asymp \mathrm{diam}(\eta)$ if $\mathcal{E}_D(\eta) \leq 1$, and the result then follows from Proposition 2.6. ■

4.2. **Topological lemmas.** The most challenging portion of this proof is gaining simultaneous control of the distances to the near and far edges of the curve. Luckily, we may eliminate a number of hard cases of the computations that follow by purely topological means. For clarity of presentation, we have isolated these topological lemmas here in a separate section. Let $z, w, \mathcal{I}$ be as in

Section 4. We call γ a non-crossing curve (from 0 to ∞ in $\mathbb{H}$) if is generated by the Loewner equation (4) with some driving function U_t, and, as before, we let H_t be the unbounded component of $\mathbb{H} \setminus \gamma(0,t]$ and $\partial_1 H_t, \partial_2 H_t$ be the preimages (considered as prime ends) under g_t of $(-\infty, U_t)$ and (U_t, ∞). We call a simple curve $\omega : (0,\infty) \to H_t$ with $\omega(0^+) = \gamma(t)$ and $\omega(\infty) = \infty$ an *infinite crosscut of* H_t. Such curves can be obtained as preimages under g_t of simple curves from U_t to ∞ in $\mathbb{H}$. An important observation is that infinite crosscuts of H_t separate $\partial_1 H_t$ from $\partial_2 H_t$.

We now define a particular crosscut of H_t contained in $\mathcal{I}$ that separates z from w.

Definition. Let γ be a non-crossing curve and let $\mathcal{I}_t = \mathcal{I} \setminus \gamma(0,t]$. We denote by $I_t = I_t(\mathcal{I}, z, w, \gamma)$ the unique open interval contained in $\mathcal{I}$ such that the following four properties hold. For any $t \leq t'$ we have:

- I_t is a connected component of $\mathcal{I}_t$,
- $I_{t'} \subseteq I_t$,
- $H_t \setminus I_t$ has exactly two connected components, one containing z and one containing w, and
- $I_t = I_{t'}$ whenever $\gamma(t,t'] \cap \mathcal{I} = \emptyset$.

We let H_t^z, H_t^w denote the components of $H_t \setminus I_t$ that contain z and w respectively.

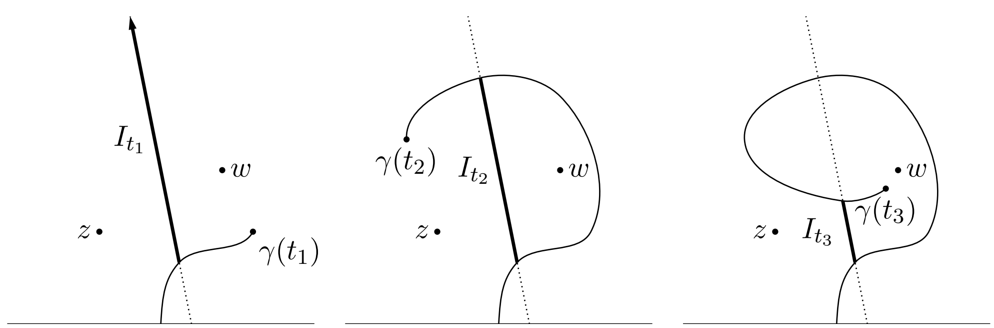

FIGURE 4. A few steps showing the behavior of I_t for some times $0 < t_1 < t_2 < t_3$.

Seeing that this notion is well defined is non-trivial, despite the intuitive nature what it should be (see Figure 4). To avoid breaking the flow of the document, the proof that it is well defined has been deferred to Appendix A.

Lemma 4.6. *Suppose γ is a non-crossing curve with $z, w \notin \gamma(0,\infty)$ and $I_t = I_t(\mathcal{I}, z, w, \gamma)$ as above. Suppose $\gamma(t) \in \overline{I_t}$. If I_t is not bounded, then*

$$\Delta^*_{H_t}(z, \gamma(t), \infty) \geq 1, \qquad \Delta^*_{H_t}(w, \gamma(t), \infty) \geq 1.$$

Proof. Suppose I_t is not bounded. Then I_t is an infinite crosscut of H_t. Suppose that $\Delta^*_{H_t}(z,\gamma(t),\infty) < 1$. Then there is a crosscut η contained in a disc of radius strictly less than one centered on z which has one end point in $\partial_1 H_t$ and one end point in $\partial_2 H_t$. Hence η must intersect I_t. However, $\mathrm{dist}(z, I_t) \geq \mathrm{dist}(z,\mathcal{I}) \geq 1$ which is a contradiction. Therefore, I_t is bounded. ∎

Lemma 4.7. *Suppose γ is a non-crossing curve with $z, w \notin \gamma(0,\infty)$ and $I_t = I_t(\mathcal{I}, z, w, \gamma)$ as above. Suppose $\gamma(t) \in \overline{I_t}$. If I_t is bounded and H^z_t is bounded then*

$$\Delta^*_{H_t}(z,\gamma(t),\infty) \geq 1.$$

Proof. Suppose I_t is bounded, H^z_t is bounded, and $\Delta^*_{H_t}(z,\gamma(t),\infty) < 1$. Then there is a crosscut η of H^z_t which has one end point in $\partial_1 H_t$ and one end point in $\partial_2 H_t$. Since H^z_t is bounded and $\gamma(t) \in \overline{I_t}$ we may find an infinite crosscut ω of H_t that never enters H^z_t (take a simple curve from ∞ in H_t until it first hits I_t and then continue the curve along I_t to reach $\gamma(t)$). Since η and ω do not intersect, we get a contradiction. ∎

Given these simple observations, we can restrict the manner in which the various distances to the curve can be decreased.

Lemma 4.8. *Suppose γ is a non-crossing curve with $z, w \notin \gamma(0,\infty)$ and $I_t = I_t(\mathcal{I}, z, w, \gamma)$ as above. Suppose t_0 is a time so that $\gamma(t_0) \in \overline{I_{t_0}}$. Let $\zeta = \inf\{t > t_0 \mid \gamma(t) \in I_{t^-}\}$. Then at most one of the following holds:*

- $\Delta_{H_\zeta,1}(z,\gamma(\zeta),\infty) < \Delta_{H_{t_0},1}(z,\gamma(t_0),\infty) \wedge 1$, *or*
- $\Delta_{H_\zeta,2}(z,\gamma(\zeta),\infty) < \Delta_{H_{t_0},2}(z,\gamma(t_0),\infty) \wedge 1$.

Proof. If $\zeta = t_0$, the above statement follows immediately, so we may assume $\zeta > t_0$. Consider the non-crossing loop $\ell = \gamma[t_0,\zeta] \cup L$ where L is the line connecting $\gamma(\zeta)$ and $\gamma(t_0)$. Partition $\mathbb{H}$ into two sets, the infinite component of $\mathbb{H} \setminus \ell$, which we will denote by A_∞, and the union of the finite components of $\mathbb{H} \setminus \ell$ which we will denote by A_0. The point z is either in A_∞ or A_0. As the cases are similar, assume $z \in A_\infty$. Since ℓ is a non-crossing loop, we either have a curve $\eta : [0,1) \to A_\infty$ with $\eta(0) = z$ and $\eta(1^-) \in \partial_1 H_\zeta$ or $\eta(1^-) \in \partial_2 H_\zeta$, but not both. This yields that only one of the $\Delta_{H_\zeta,j}(z,\gamma(\zeta),\infty)$ could have decreased past the minimum of 1 and its previous value.

∎

Lemma 4.9. *Suppose γ is a non-crossing curve with $z, w \notin \gamma(0,\infty)$ and $I_t = I_t(\mathcal{I}, z, w, \gamma)$ as above. Suppose t_0 is a time so that $\gamma(t_0) \in \overline{I_{t_0}}$, and let $\zeta = \inf\{t > t_0 \mid \gamma(t) \in I_{t^-}\}$. Suppose for some $s < 1$,*

$$\Delta^*_\zeta(z) \leq s < \Delta^*_{t_0}(z).$$

Then $\Delta_{t_0}(z) \leq s$, and $H^w_{t_0}$ and H^w_ζ are bounded.

Proof. By the previous lemma, we have that either $\Delta^1_\zeta(z) \geq \Delta^1_{t_0}(z) \wedge 1$ or $\Delta^2_\zeta(z) \geq \Delta^2_{t_0}(z) \wedge 1$. This implies that $\Delta^*_\zeta(z) \geq \Delta_{t_0}(z) \wedge 1$, and hence $\Delta_{t_0}(z) \wedge 1 \leq s$ which is the first assertion.

We now prove that H^w_ζ is bounded. Assume first that both H^w_ζ and H^z_ζ are unbounded. Then I_ζ is unbounded and by Lemma 4.6 we have that

$$\Delta^*_{H_\zeta}(z, \gamma(t), \infty) \geq 1$$

which is a contradiction. Thus one of H^w_ζ or H^z_ζ is bounded. If H^z_ζ is bounded, then by Lemma 4.7 we have

$$\Delta^*_{H_\zeta}(z, \gamma(t), \infty) \geq 1$$

which is again a contradiction. Thus H^w_ζ is bounded, as desired.

By the definition of ζ and I_t, we know $\gamma(t_0, \zeta)$ is contained in precisely one of $H^z_{t_0}$ or $H^w_{t_0}$. Since

$$\Delta^*_\zeta(z) < 1 \leq \Delta^*_{t_0}(z)$$

by assumption, we know $\gamma(t_0, \zeta) \subseteq H^z_{t_0}$. Assume that $H^w_{t_0}$ were unbounded. Then there is a curve η from w to ∞ contained in $H^w_{t_0}$. Since H^w_ζ is bounded $\eta \cap \partial H^w_\zeta$ is non-empty. By definition,

$$\partial H^w_\zeta \subseteq \gamma(0, t_0] \cup \gamma(t_0, \zeta] \cup I_\zeta.$$

We now show η cannot intersect any of the three sets on the right. Since η is in $H^w_{t_0}$, we know $\eta \cap (\gamma(0, t_0] \cup I_{t_0}) = \emptyset$ and moreover, since $I_\zeta \subseteq I_{t_0}$, that $\eta \cap I_\zeta = \emptyset$. Since $\gamma(t_0, \zeta) \subseteq H^z_{t_0}$, we know $\eta \cap \gamma(t_0, \zeta) = \emptyset$. Thus we have a contradiction, and $H^w_{t_0}$ must be bounded, as desired. ■

4.3. **Main *SLE* estimates.** We now use the above topological restrictions to help us establish the needed *SLE* estimates. Let T_z (resp. T_w) denote the first time that z (resp. w) is not in H_t and let $T = T_z \wedge T_w$ denote the first time that one of z, w is not in H_t. Note that if the curve to approach z and w to within ε and δ as desired, it must occur before $T_z \vee T_w$. We also define the following recursive set of stopping times. Let $\tau_0 = 0$. Given $\tau_j < T$, define $\hat{\tau}_j$ as the infimum over times $t > \tau_j$ such that

$$\Delta_t(z) \leq \frac{1}{2}\Delta_{\tau_j}(z) \text{ or } \Delta_t(w) \leq \frac{1}{2}\Delta_{\tau_j}(w).$$

Given this, let τ_{j+1} be the infimum over times $t > \hat{\tau}_j$ such that $\gamma(t) \in \overline{I_{\hat{\tau}_j}}$. These times are understood to be infinite when past T and hence at least one of the points can no longer be approached by the curve. The sequence of stopping times $\{\tau_k\}_{k \geq 0}$ are called renewal times. We let $R_{k+1} = 0$ if $\tau_{k+1} < \infty$ and $\Delta_{\tau_{k+1}}(z) \leq \frac{1}{2}\Delta_{\tau_k}(z)$; in this case, we can see that $\Delta_{\tau_{k+1}}(w) > \frac{1}{2}\Delta_{\tau_k}(w)$. If $\tau_{k+1} < \infty$ and $\Delta_{\tau_{k+1}}(w) \leq \frac{1}{2}\Delta_{\tau_k}(w)$, we set $R_{k+1} = 1$. We set $R_{k+1} = \infty$ if $\tau_{k+1} = \infty$. Let $\mathcal{F}_k = \mathcal{F}_{\tau_k}$.

Lemma 4.10. *There exist $c < \infty, \alpha > 0$ such that for all $k \geq 0$, $r \leq 1/2$,,*

$$\mathbb{P}\left\{R_{\tau_{k+1}} = 0; \Delta_{\tau_{k+1}}(z) \leq r\,\Delta_{\tau_k}(z) \;\mid\; \mathcal{F}_k\right\} \leq c\,1\{\tau_k < T\}\,\Delta_{\tau_k}(z)^{\alpha}\,r^{2-d}. \tag{12}$$

Proof. We assume $\tau_k < T$ and we write $\tau = \tau_k$, $\xi = \xi(z; r\Delta_\tau(z))$. First, consider the event that either I_τ is not bounded or both I_τ and H^z_τ are bounded. By Lemma 4.7 we have $\Delta^*(z) \geq 1$. Thus by Lemma 2.11, we get

$$\mathbb{P}\{\xi < \infty \mid \mathcal{F}_k\} \leq c\,r^{2-d}\,\Delta_\tau(z)^{\beta/2}.$$

Suppose that I_τ is bounded and H^w_τ is bounded. We split into two cases: $\Delta^*_\tau(z) \leq \sqrt{\Delta_\tau(z)}$ and $\Delta^*_\tau(z) > \sqrt{\Delta_\tau(z)}$. If $\Delta^*_\tau(z) > \sqrt{\Delta_\tau(z)}$, then Lemma 2.11 implies

$$\mathbb{P}\{\xi < \infty \mid \mathcal{F}_k\} \leq c\,r^{2-d}\,\Delta_\tau(z)^{\beta/4}.$$

Suppose $\Delta^*_\tau(z) \leq \sqrt{\Delta_\tau(z)}$. Then there exist simple curves $\eta_1, \eta_2 : [0,1) \to H^z_\tau$ contained in the disk of radius $2\Delta^*_\tau(z)$ about z with $\eta^j(0) = z$ and $\eta^j(1+) \in \partial_j H_\tau$. At the time ξ we can consider the line segment L from $\gamma(\xi)$ to z. There exists a crosscut of H_ξ, $\hat{\eta}$, contained in $L \cup \eta_1$ or in $L \cup \eta_2$, one of whose endpoints is $\gamma(\xi)$, that disconnects I_ξ from infinity. Using Lemma 4.4, we see that

$$\mathcal{E}_{H_\xi}(\hat{\eta}, I_\xi) \leq c\,\Delta^*_\tau(z)^{1/2} \leq c\,\Delta_\tau(z)^{1/4}.$$

Thus, using Lemma 4.5 we see that

$$\mathbb{P}\left\{\xi < \tau_{k+1} < \infty \mid \mathcal{F}\right\} \leq c\,\Delta_\tau(z)^{\lambda/4}\,\mathbb{P}\left\{\xi < \infty \mid \mathcal{F}\right\} \leq c\,r^{2-d}\,\Delta_\tau(z)^{\lambda/4}.$$

∎

Remark. The proof of the last lemma was not difficult given the estimates we have derived. However, it is useful to summarize the basic idea. If $\Delta^*_\tau(z)$ is not too small, then it suffices to estimate

$$\mathbb{P}\left\{R_{\tau_{k+1}} = 0; \Delta_{\tau_{k+1}}(z) \leq r\,\Delta_{\tau_k}(z) \;\mid\; \mathcal{F}_k\right\}$$

by

$$\mathbb{P}\left\{\xi < \infty \mid \mathcal{F}_k\right\}.$$

However, if $\Delta^*_{\tau_k}(z)$ is not much bigger than $\Delta_{\tau_k}(z)$ this estimate is not sufficient. In this case, we need to use

$$\mathbb{P}\left\{\xi < \infty \mid \mathcal{F}_k\right\}\,\mathbb{P}\left\{\tau_{k+1} < \infty \mid \mathcal{F}_k, \xi < \infty\right\}.$$

Now that we have a good bound on the probability that the near side gets closer, we must also provide a bound limiting the probability that the far side can get closer.

Lemma 4.11. *There exists $c < \infty$ such that for all $k \geq 0$, $s \leq 1/4$, if*

$$\xi^* = \inf\{t > \tau_k \mid \Delta^*_t(z) \leq s\} \quad \textit{and} \quad \eta^* = \inf\{t > \xi^* \mid \gamma(t) \in I_{t^-}\},$$

then

$$\mathbb{P}\{\eta^* < \infty, \Delta^*_{\eta^*}(z) \leq s \mid \Delta^*_{\tau_k}(z) > s, \mathcal{F}_{\tau_k}\} \leq cs^{\lambda/2}$$

Proof. Assume $\Delta^*_{\tau_k}(z) > s$. If $\eta^* < \infty$ we may define

$$\varpi = \sup\{t < \eta^* \mid \gamma(t) \in I_{t^-}\}$$

to be the previous time that γ crossed I_{t^-} before η^*. Note that $\tau_k \leq \varpi < \xi^* < \eta^*$ and $\Delta^*_{\varpi}(z) > s$. By considering the two times ϖ and η^* in Lemma 4.9 we see that H^w_ϖ is bounded.

Consider the situation at time ξ^*. By the definition of the stopping times, there must be a curve $\nu : (0,1) \to H_{\xi^*}$ which contains z, is never more than distance $2s$ from z, has $\nu(0^+) \in \partial_1 H_{\xi^*}$ and $\nu(1^-) \in \partial_2 H_{\xi^*}$ such that ν separates I_{ξ^*} and hence w from infinity. Since ν is at least distance $1/2$ from I_{ξ^*} we know from Lemma 4.4 that the excursion measure between ν and I_{ξ^*} in H_{ξ^*} is bounded above by $Cs^{1/2}$. Then an application of Lemma 4.5 tells us that the probability of γ returning to I_{ξ^*} is bounded above by $Cs^{\lambda/2}$ which gives the lemma. ■

The following two lemmas combine the methods of the above two bounds.

Lemma 4.12. *There exist $c < \infty, \alpha > 0$ such that for all $k \geq 0$, $r \leq 1/2$, $s \leq 1/4$,*

$$\begin{aligned}\mathbb{P}\big\{R_{\tau_{k+1}} = 0; \Delta_{\tau_{k+1}}(z) \leq r\,\Delta_{\tau_k}(z); \Delta^*_{\tau_{k+1}}(w) \leq s \;\mid\; \mathcal{F}_k\big\} \\ \leq c\,1\{\tau_k < T\}\,\Delta_{\tau_k}(z)^\alpha\,\left[s^\alpha + \mathbb{1}\{\Delta^*_{\tau_k}(w) \leq s\}\right] r^{2-d}.\end{aligned}$$

Proof. If $\Delta^*_{\tau_k}(w) \leq s$ then the desired statement reduces to Lemma 4.10. Thus, we may assume that $\Delta^*_{\tau_k}(w) > s$.

Let $\zeta^* = \zeta^*_k$ be the infimum over times $t > \tau_k$ so that $\Delta^*_t(w) \leq s$ and $\gamma(t) \in I_{t^-}$. Let $\sigma = \sigma_k = \inf\{t > \tau_k \mid \Delta_t(z) \leq r\Delta_{\tau_k}(z)\}$. If $\Delta^*_{\tau_k}(w) > s, \Delta^*_{\tau_{k+1}}(w) \leq s$, and $\sigma < \infty$, then $\zeta^* < \sigma$ since the curve γ would need to intersect I_σ before approaching w and hence would force the renewal time τ_{k+1} before ζ_k.

By the same argument as in Lemma 4.11 we know if $\Delta^*_{\tau_k}(w) > s$ and $\zeta^* < \infty$, there is a time ω, $\tau_k \leq \omega < \zeta^*$ for which there is a curve connecting $\partial_1 H_\omega$ to $\partial_2 H_\omega$ passing through $\gamma(\omega)$ contained in a disk of radius $2s$ about w separating I_κ from infinity. Then, by Lemma 4.5, we have that

$$\mathbb{P}\{\zeta^* < \infty \mid \Delta^*_{\tau_k}(z) > s, \mathcal{F}_{\tau_k}\} \leq cs^\alpha.$$

By Lemma 4.9 we know $H^z_{\zeta^*}$ is bounded. Lemma 4.7 implies that $\Delta^*_{\zeta^*}(z) = 1$, and hence by Lemma 4.4

$$\mathbb{P}\{R_{\tau_{k+1}} = 0; \Delta_{\tau_{k+1}} \leq r\Delta_{\tau_k}(z) \mid \mathcal{F}_{\zeta^*}, \zeta^* < \infty\} \leq c\mathbb{1}\{\tau_k < T\}\Delta_{\zeta^*}(z)^\alpha r^{2-d}.$$

Combining the above two bounds gives the desired result. ■

Lemma 4.13. *There exist $c < \infty, \alpha > 0$ such that for all $k \geq 0$, $r \leq 1/2$, $s > 0$,*

$$\mathbb{P}\big\{R_{\tau_{k+1}} = 0; \Delta_{\tau_{k+1}}(z) \leq r\, \Delta_{\tau_k}(z); \Delta^*_{\tau_{k+1}}(z) \leq s \ \mid\ \mathcal{F}_k\big\}$$
$$\leq c\, 1\{\tau_k < T\}\, \Delta_{\tau_k}(z)^\alpha\, \left[s^\alpha + 1\{\Delta^*_{\tau_k}(z) \leq s\}\right]\, r^{2-d}.$$

Proof. If $\Delta^*_{\tau_k}(z) \leq s$ or $s \geq 1/4$, the conclusion reduces to Lemma 4.10. Thus we may assume that $\Delta^*_{\tau_k}(z) > s, s \leq 1/4$. Let E denote the event

$$E = \left\{R_{\tau_{k+1}} = 0; \Delta_{\tau_{k+1}}(z) \leq r\, \Delta_{\tau_k}(z); \Delta^*_{\tau_{k+1}}(z) \leq s; \Delta^*_{\tau_k}(z) > s\right\}.$$

Let

$$\sigma = \inf\{t \mid \Delta_t(z) \leq r\Delta_{\tau_k}(z)\}.$$

and note that on the event E,

$$\tau_{k+1} = \inf\{t > \sigma \mid \gamma(t) \in I_{t^-}\}.$$

Define ξ to be the infimum over times $t \geq \sigma$ such that there is a curve $\eta : (0,1) \to H_t$ with $\eta(0^+) = \gamma(t)$ and $\eta(1^-) \in \partial H_t$ with η contained entirely in the ball of radius $2s$ about z, and η separating I_t from ∞.

We now claim that given $\mathcal{F}_\sigma$ either $\xi < \tau_{k+1}$ or $\Delta^*_{\tau_{k+1}}(z) > s$. To see this, suppose neither holds. Since $\Delta^*_{\tau_{k+1}}(z) \leq s$, for every $s < s' \leq 2s \leq 1/2$, there is a crosscut η of $H_{\tau_{k+1}}$ going through z whose endpoints are in $\partial_1 H_{\tau_{k+1}}, \partial_2 H_{\tau_{k+1}}$, respectively and which is contained in the disk of radius s' about z. By Lemma 4.9 we know η must disconnect $I_{\tau_{k+1}}$ from ∞ since $H^w_{\tau_{k+1}}$ must be bounded. We can choose such an η such that at least one end point of η is not in $\gamma[0, \tau_k]$, for otherwise all such η would be a crosscuts of H_{τ_k} separating w from infinity which would imply that $\Delta^*_{\tau_k}(z) \leq s$.

Let $\zeta = \sup\{t \leq \tau_{k+1} \mid \gamma(t) \in \overline{\eta}\} > \tau_k$ and note that $\tau_k < \zeta < \tau_{k+1}$. If $\zeta \geq \sigma$ we are done since this η demonstrates that $\xi < \tau_{k+1}$.

Thus assume $\zeta < \sigma$. In this case we will construct another curve which satisfies what we want at the time σ. Since $\zeta < \sigma$ we know the curve η defined above disconnects I_σ from infinity in H_σ. By the definition of σ as the first time that $\Delta_\sigma(z) \leq r\Delta_{\tau_k}(z)$, the straight open line segment, L, from $\gamma(\sigma)$ to z is contained in H_σ. Additionally, since $\Delta_\sigma(z) \leq \Delta^*_\sigma(z) \leq s$ we know $\eta(0,1) \cup L$ is contained entirely in the ball of radius $2s$ about z. Thus we may find a curve $\hat{\eta}$ contained in $\eta(0,1) \cup L$ which separates I_σ from infinity in H_σ with $\eta(0^+) = \gamma(t)$ and $\eta(1^-) \in \partial H_t$ and with η contained entirely in the ball of radius $2s$ about z, proving that $\xi = \sigma < \tau_{k+1}$. Thus we have reached a contradiction.

On the event E we know $\Delta^*_{\tau_{k+1}}(z) \leq s$ and thus the above argument tells us $\xi < \tau_{k+1}$. We have therefore shown that if $\Delta^*_{\tau_k}(z) > s, s \leq 1/4$, then

$$\mathbb{P}\left(E \mid \mathcal{F}_k\right) \leq \mathbb{P}\left\{\sigma \leq \xi < \tau_{k+1} < \infty \mid \mathcal{F}_k\right\}.$$

We now argue as in the second part of the proof of Lemma 4.10 that $\mathbb{P}\{\sigma < \infty \mid \mathcal{F}_k\} \leq c\,\Delta_{\tau_k}(z)^\alpha$ and $\mathbb{P}\{\tau_{k+1} < \infty \mid \mathcal{F}_\xi\} \leq c\, s^\alpha$.

∎

4.4. **Combinatorial estimates.** We have now completed the bulk of the probabilistic estimates. Most of what remains is a combinatorial argument to sum up the bounds proven above across all possible ways that the SLE curve may approach z and w in turn.

Without loss of generality, assume that $\delta = 2^{-m}$ and $\varepsilon = 2^{-n}$ and let

$$\xi_z = \xi_{z,\varepsilon} = \inf\{t : \Delta_t(z) \leq 2^{-n}\}, \quad \xi_w = \xi_{w,\delta} = \inf\{t : \Delta_t(w) \leq 2^{-m}\},$$

$$\xi = \xi_z \vee \xi_w = \inf\{t : \Delta_t(z) \leq 2^{-n}, \Delta_t(w) \leq 2^{-m}\}.$$

Let σ be the minimal τ_k such that $\Delta_{\tau_k}(z) < 2^{-n+1}$ or $\Delta_{\tau_k}(w) < 2^{-m+1}$. Let k_σ be the index so that $\sigma = \tau_{k_\sigma}$. If such a renewal time does not exist, let $k_\sigma = \infty$ and $\sigma = \infty$. Let $\xi = \inf\{t \mid \Delta_t(z) < \varepsilon, \Delta_t(w) < \delta\}$. Note that if ξ is finite, then so is σ.

Let $V_{z,k}, V_z$ denote the events (and their indicator functions)

$$V_{z,k} = \{k_\sigma = k, R_\sigma = 0\}\,,\ V_z = \bigcup_{k=1}^{\infty} V_{z,k}.$$

We define V_w analogously. By the definition of σ, on the event the event V_z,

$$\Delta_{\tau_{k_\sigma - 1}}(z) \geq 2^{-n+1}, \quad \Delta_{\tau_{k_\sigma - 1}}(w) \geq 2^{-n+1}, \quad \Delta_\sigma(z) < 2^{-n+1}.$$

Also,

$$\Delta_\sigma(w) > 2^{-n},$$

for if $\Delta_\sigma(w) \leq 2^{-n}$, there would have been a renewal time after $\tau_{k_\sigma - 1}$ but before $\tau_k = \sigma$. Note that

$$\{\xi < \infty\} \subset [V_z \cap \{\xi_w < \infty\}] \cup [V_w \cap \{\xi_z < \infty\}]\,.$$

We will concentrate on the event $V_z \cap \{\xi_w < \infty\}$; similar arguments handle the event $V_w \cap \{\xi_z < \infty\}$.

Define the integers (i_l, j_l) by stating that at the renewal time τ_l,

$$2^{-i_l} < \Delta_{\tau_l}(z) \leq 2^{-i_l+1},\ 2^{-j_l} < \Delta_{\tau_l}(w) \leq 2^{-j_l+1}.$$

If $\sigma < \infty$, we write $(i_\sigma, j_\sigma) = (i_{k_\sigma}, j_{k_\sigma})$. On the event $k_\sigma = k, R_\sigma = 0$, there is a finite sequence of ordered triples

$$\pi = [(i_0, j_0, 0), (i_1, j_1, R_1), \ldots, (i_{k-1}, j_{k-1}, R_{k-1}), (i_k, j_k, R_k) = (i_\sigma, j_\sigma, 0)]\,,$$

$$i_l, j_l \in \{1, 2, 3, \ldots\}, \quad R_l \in \{0, 1\}.$$

We have the following properties for $0 \leq l \leq k-1$:

- If $R_{l+1} = 0$, then $i_{l+1} \geq i_l + 1$ and $j_l \leq j_{l+1} \leq j_l + 1$.
- If $R_{l+1} = 1$, then $i_l \leq i_{l+1} \leq i_l + 1$ and $j_{l+1} \geq j_l + 1$.

We call any sequence of triples satisfying these two properties a *legal* sequence of length k. For any i, j, k, let $\mathcal{S}_k(i,j)$ denote the collection of legal finite sequences of length k whose final triple is

$$(i_k, j_k, R_k) = (i, j, 0).$$

If π is a legal finite sequence of length k, let $V_{z,\pi}$ be the event that $k_\sigma = k, R_\sigma = 0$ and the renewal times up to and including σ give the sequence π. Figure 5 illustrates this definition.

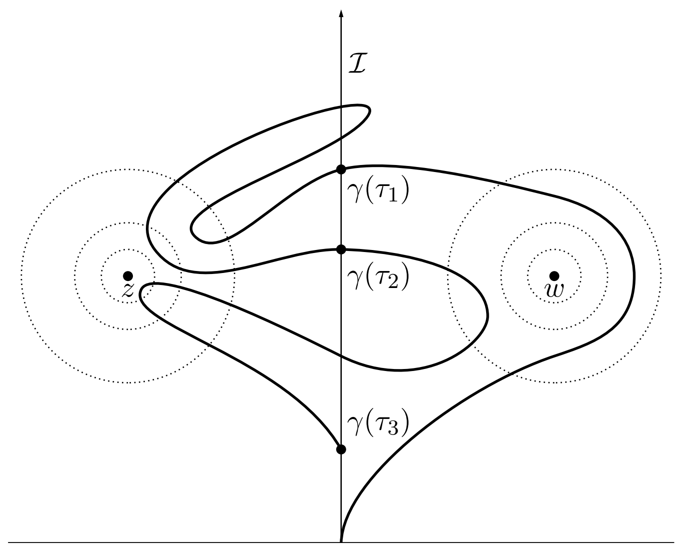

FIGURE 5. An example of a curve γ (shown in bold) in $V_{z,\pi}$ where $\pi = [(0,0,0),(0,1,1),(2,1,0),(3,1,0)]$.

Define K_l for $1 \le l \le k$ by

$$K_l = \begin{cases} i_{l-1} & \text{if } \; R_l = 0, \\ j_{l-1} & \text{if } \; R_l = 1. \end{cases}$$

The next proposition gives the fundamental estimate.

Proposition 4.14. *There exist* c *and an* $\alpha > 0$ *such that the following holds. Let* i, j, k *be integers and let* $\pi \in \mathcal{S}_k(i,j,0)$. *Then*

$$\mathbb{P}\left[V_{z,\pi} \cap \{\xi_w < \infty\}\right] \le c^k \, 2^{(m+n)(d-2)} \, e^{-\alpha(i+j-n)} \prod_{l=1}^{k} e^{-\alpha K_l}.$$

Proof. Note that on the event V_z we may say by Lemma 2.11 that

$$\mathbb{P}\{\xi_w < \infty \mid \mathcal{F}_k\} \le c \left[\frac{2^{-j}}{\Delta_k^*(w)}\right]^{\beta/2} 2^{(m-j)(d-2)}.$$

We will proceed by splitting the event $V_{z,\pi}$ into the case where $\Delta_k^*(w) \geq 2^{-j}$ and the case where it is not.

First note

$$\begin{aligned}
&\mathbb{P}[V_{z,\pi} \cap \{\Delta_k^*(w) \geq 2^{-j/2}\} \cap \{\xi_w < \infty\}] \\
&\quad \leq c\,\mathbb{P}[V_{z,\pi} \cap \{\Delta_k^*(w) \geq 2^{-j}\}]2^{-\beta j/4}2^{(m-j)(d-2)} \\
&\quad \leq c\,\mathbb{P}[V_{z,\pi}]2^{-\beta j/4}2^{(m-j)(d-2)} \\
&\quad \leq c^k\,2^{-\beta j/4}2^{(m-j)(d-2)}2^{(i+j)(d-2)}\prod_{l=1}^{k} e^{-\alpha K_l} \\
&\quad = c^k\,2^{-\beta j/4}2^{(m+n)(d-2)}2^{(i-n)(d-2)}\prod_{l=1}^{k} e^{-\alpha K_l} \\
&\quad \leq c^k\,2^{(m+n)(d-2)}2^{-\mu(i+j-n)}\prod_{l=1}^{k} e^{-\alpha K_l}
\end{aligned}$$

for some choice of μ by repeated applications of Lemma 4.10.

Thus we need only understand the event

$$\mathbb{P}[V_{z,\pi}\cap\{\Delta_k^*(w) < 2^{-j/2}\}\cap\{\xi_w < \infty\}] \leq c\,\mathbb{P}[V_{z,\pi}\cap\{\Delta_k^*(w) < 2^{-j/2}\}]2^{(m-j)(d-2)}.$$

For the event $\{\Delta_k^*(w) < 2^{-j/2}\}$ there must be at least one l such that $\Delta_l^*(w) \geq 2^{-j/2}$ and $\Delta_{l+1}^*(w) < 2^{-j/2}$. By using Lemma 4.12 for that single step if $R_l = 1$ or Lemma 4.13 if $R_l = 0$ and 4.10 for all other steps we have that

$$\begin{aligned}
&\mathbb{P}[V_{z,\pi} \cap \{\Delta_k^*(w) < 2^{-j/2}\}] \\
&\quad \leq \sum_{l=0}^{k-1} \mathbb{P}[V_{z,\pi} \cap \{\Delta_l^*(w) \geq 2^{-j/2}; \Delta_{l+1}^*(w) < 2^{-j/2}\}] \\
&\quad \leq kc^k\,2^{-\alpha j/2}2^{(i+j)(d-2)}\prod_{l=1}^{k} e^{-\alpha K_l}.
\end{aligned}$$

By combining this with the above event we see that

$$\begin{aligned}
&\mathbb{P}[V_{z,\pi} \cap \{\Delta_k^*(w) < 2^{-j/2}\} \cap \{\xi_w < \infty\}] \\
&\quad \leq kc^k\,2^{(m-j)(d-2)}2^{-\alpha j/2}2^{(i+j)(d-2)}\prod_{l=1}^{k} e^{-\alpha K_l} \\
&\quad = c^k\,2^{(m+n)(d-2)}2^{-\mu(j+i-n)}\prod_{l=1}^{k} e^{-\alpha K_l}.
\end{aligned}$$

where c is being used generically and for some choice of μ. Thus by choosing μ and α to be the same (which we can do by taking the minimum for both) we get the desired result. ∎

We will now show how this proposition implies the main theorem. The proof rests upon the following combinatorial lemma.

Lemma 4.15. *For every $\alpha > 0$, there exist c and a $u > 0$ such that for all k*

$$\sum_{\pi \in \mathcal{S}_k(i,j,0)} 1\{K_k \geq r\} \prod_{l=1}^{k} e^{-\alpha K_l} \leq c\, e^{-uk^2}\, e^{-\alpha r}.$$

Proof. We fix α and allow all constants to depend on α. Let

$$\Sigma_k = \sum_{[m]_k} \prod_{l=1}^{k} e^{-\alpha m_l},$$

where the sum is over all strictly increasing finite sequences of positive integers, written as $[m]_k := [m_1, m_2, \ldots, m_k]$. We first claim that

$$\Sigma_k \leq c_1\, e^{-\alpha k^2/4}.$$

Consider the following recursive relation:

$$\begin{aligned}
\Sigma_k &= \sum_{[m]_k} \prod_{l=1}^{k} e^{-\alpha m_l} \\
&\leq \sum_{[m]_{k-1}} \sum_{m_k = k}^{\infty} e^{-\alpha m_k} \prod_{l=1}^{k-1} e^{-\alpha m_l} \\
&= \Sigma_{k-1} \sum_{j=k}^{\infty} e^{-\alpha j} \\
&\leq c_2\, \Sigma_{k-1} e^{-\alpha k}.
\end{aligned}$$

Applying this bound inductively to Σ_k yields

$$\Sigma_k \leq c_2^k \exp\Big\{ -\alpha \sum_{i=1}^{k} i \Big\} \leq c_1\, e^{-\alpha k^2/4}$$

as desired.

To choose a legal sequence in $\mathcal{S}_k(i,j,0)$ there are 2^{k-1} ways to choose the values $R_1, \ldots, R_{k-1}$. Given the values of $R_1, \ldots, R_{k-1}$ we choose the increases of the integers. If $R_l = 0$, then $i_l > i_{l-1}$ and $j_l = j_{l-1}$ or $j_l = j_{l-1} + 1$. The analogous inequalities hold of $R_1 = 1$. There are 2^k ways to choose whether $j_l = j_{l-1}$ or $j_l = j_{l-1} + 1$ (or the corresponding jump for i_l if $R_1 = 1$). In the other components we have to increase by an integer. We therefore get that the sum is bounded above by

$$2^{k-1} \max_{0\leq l\leq k-1} 2^l \Sigma_l \cdot 2^{k-l-1}\Sigma_{k-l-1} \leq c^k \max_{0\leq l\leq k-1} e^{-\alpha l^2/4}\, e^{-\alpha(k-l-1)^2/4}$$
$$\leq ce^{-uk^2}.$$

■

By combining Proposition 4.14 and Lemma 4.15, there exist c such that

$$\sum_{k=1}^{\infty} \sum_{\pi\in\mathcal{S}_k(i,j,0)} \mathbb{P}\left[V_{z,\pi} \cap \{\xi_w < \infty\}\right] \leq c\, 2^{(m+n)(d-2)}\, e^{-\alpha(j+i-n)},$$

and hence by summing over $i \geq n-1, j \geq 0$ we get

$$\begin{aligned}\mathbb{P}\left[V_z \cap \{\xi < \infty\}\right] &\leq \mathbb{P}\left[V_z \cap \{\xi_w < \infty\}\right] \\ &= \sum_{i=n-1}^{\infty}\sum_{j=0}^{\infty}\sum_{k=1}^{\infty}\sum_{\pi\in\mathcal{S}_k(i,j,0)} \mathbb{P}\left[V_{z,\pi}\cap\{\xi_w<\infty\}\right] \\ &\leq c2^{(m+n)(d-2)} = c\,\varepsilon^{2-d}\,\delta^{2-d}.\end{aligned}$$

By the symmetry of z, w we have the bound

$$\mathbb{P}\left[V_w \cap \{\xi < \infty\}\right] \leq c\,\varepsilon^{2-d}\,\delta^{2-d}$$

and hence

$$\mathbb{P}\left\{\Delta_\infty(z) \leq \varepsilon, \Delta_\infty(w) \leq \delta\right\} = \mathbb{P}\{\xi < \infty\} \leq c\,\varepsilon^{2-d}\,\delta^{2-d}$$

as required to complete the proof of Proposition 4.1, and hence the proof of Beffara's estimate.

With the proof set up in this way, we may now rapidly complete our proof of the existence of the multi-point Green's function. By mirroring the proof above, we may conclude that for $\rho = 2^{-\ell}$ (and hence for all ρ) that

$$\begin{aligned}\mathbb{P}\left[V_z \cap \{\xi < \infty, \Delta_\sigma(w) \leq \rho\}\right] &\leq \mathbb{P}\left[V_z \cap \{\xi_w < \infty, \Delta_\sigma(w) \leq \rho\}\right] \\ &= \sum_{i=n-1}^{\infty}\sum_{j=\ell}^{\infty}\sum_{k=1}^{\infty}\sum_{\pi\in\mathcal{S}_k(i,j,0)} \mathbb{P}\left[V_{z,\pi}\cap\{\xi_w<\infty\}\right] \\ &\leq c2^{(m+n)(d-2)}e^{-\alpha\ell} = c\,\varepsilon^{2-d}\,\delta^{2-d}\,\rho^{\alpha}.\end{aligned}$$

This proves the Proposition 4.2, and hence completes the proof of the existence of the multi-point Green's function.

Appendix A. The existence of the I_t

The aim of this appendix is to prove the existence of the separating set I_t desired above.

Definition. Let γ be a curve in the upper half plane and let $z, w, \mathcal{I}$ be as above. Let $\mathcal{I}_t = \mathcal{I} \setminus \gamma(0, t]$. We will denote by I_t the unique open interval contained in $\mathcal{I}$ such that the following four properties hold. For any $t \leq t'$ we have:

- I_t is a connected component of $\mathcal{I}_t$,
- the I_t are decreasing, which is to say $I_{t'} \subseteq I_t$,
- $H_t \setminus I_t$ has exactly two connected components, one containing z and one containing w, and
- $I_t = I_{t'}$ whenever $\gamma(t, t'] \cap \mathcal{I} = \emptyset$.

It may, at first glance, seem simple to define such sets inductively. However, in general the set of times that a curve γ crosses $\mathcal{I}$ may be uncountable and have no well defined notion of "the previous crossing." To avoid this issue and show this notion is well defined, we require a few topological lemmas.

Lemma A.1. *Let U be a connected open set in $\mathbb{C}$ separated by a smooth simple curve $\eta : [0, 1] \to \overline{U}$. Let $V \subset U$ be a connected open subset. Then for any points $z, w \in V$, there exits a curve $\xi : [0, 1] \to V$ from z to w which intersects η a finite number of times.*

Proof. This proof mirrors the classic proof that a connected open set is path connected. Define an equivalence relation on V where points $z, w \in V$ are equivalent if z can be connected to w by a curve ξ which intersects η a finite number of times. This can readily be shown to satisfy the requirements of an equivalence relation. Let V_α denote the open connected components of $V \setminus \eta$.

If z, w are both in the same V_α then they may be connected by a curve which does not intersect η, hence each V_α is contained entirely in a single equivalence class.

Consider a disc, D, contained in V centered on a point $\eta(t_0)$ for some $t_0 \in (0, 1)$ with components V_α and V_β on either side of η near this point. Since η is smooth and simple, by choosing D sufficiently small we may find a diffeomorphism ϕ so that $\phi(D) = \mathbb{D}$ and $\phi(\eta \cap D) = \{it : t \in (-1, 1)\}$. Connect $-1/2$ to $1/2$ by the straight line between them, which only intersects the image of η once. Taking the image of this line under ϕ^{-1} gives a curve ξ satisfying the conditions of the equivalence relation connecting two points, one in V_α and one in V_β. Thus components of $V \setminus \eta$ which are directly separated by η are in the same equivalence class. Since V is connected, the only equivalence class is V itself. ∎

Suppose U is a connected open set in $\mathbb{C}$ separated by a curve $\eta : (0,1) \to U$ into two components U_1, U_2 with points $z \in U_1$ and $w \in U_2$. Let V be a connected subset of U. Define $\mathcal{D}_V(z,w;\eta)$ to be the the set of connected components of $V \cap \eta$ which disconnects z from w in V.

Corollary A.2. *Let U be a connected open set in $\mathbb{C}$ separated by a smooth simple curve $\eta : [0,1] \to \overline{U}$ into two components U_1, U_2 with $z \in U_1$ and $w \in U_2$. Let $V \subset U$ be a connected open subset containing z and w. Then $|\mathcal{D}_V(z,w,\eta)|$ is finite and odd.*

Proof. To see that the number is finite, take the curve ξ between z and w as in the above lemma and note that any η_i which separates z from w must intersect ξ.

To see that it is odd, consider the connected components of $V' := V \setminus \bigcup_{\gamma \in \mathcal{D}_V(z,w;\eta)} \gamma$. There are exactly $|\mathcal{D}_V(z,w;\eta)| + 1$ such components. η separates U into two components, and hence the components of V' are alternately contained in U_1 and U_2. Since the component containing z is in U_1 and the component containing w is in U_2 there must be an even number of components of V', which makes $|\mathcal{D}_V(z,w;\eta)|$ odd. ∎

This general topological lemma has the following consequence in our setting. To simplify notation, we will define $\mathcal{D}_t = \mathcal{D}_{H_t}(z,w,\mathcal{I})$.

Corollary A.3. *Fix $0 \le t' \le t < \infty$. Then a connected component I of $\mathcal{I}_{t'}$ separates z from w in $H_{t'}$ if and only if the number of elements of $\mathcal{D}_t$ contained in I is odd.*

Proof. The 'only if' direction is precisely Corollary A.2. Thus we wish to show that if the number of elements of $\mathcal{D}_t$ contained in I is odd then I separates z from w.

Assume not, so the number of elements of $\mathcal{D}_t$ contained in I is odd but I does not separate z from w. $H_{t'} \setminus I$ has two components, one of which contains both z and w. Consider any curve η connecting z to w. Without loss of generality assume that η crosses each element of $\mathcal{D}_t$ exactly once by simply removing any portion of the curve between the first and last times that it crosses each element of $\mathcal{D}_t$. Since η crosses each element of $\mathcal{D}_t$ contained in I precisely once we know η crosses I an odd number of times and hence it must start and end in different components of $H_{t'} \setminus I$ which contradicts the fact that it connects z to w. ∎

We may now use this to prove that I_t is well defined.

Proof of well-definedness of I_t. For a component I of $\mathcal{I}_t$ and $t' < t$ let $C_{t'}(I)$ denote the component of $\mathcal{I}_{t'}$ which contains I. We claim there exists a unique component of $\mathcal{I}_t$, which we will denote I_t, such that for all $0 \le t' \le t$ we have

$C_{t'}(I_t) \in \mathcal{D}_{t'}$. Note that such an I_t clearly satisfies all the conditions of the definition.

First we prove existence. Let $\{J_i\}_{i=1}^{\infty}$ be the connected components of $\mathcal{I}_t$. Assume that none satisfy the above condition, which is to say that for each i there exists a $t_i \leq t$ so that $C_{t_i}(J_i)$ does not separate z from w in H_{t_i}. Now $\{C_{t_i}(J_i)\}_{i=1}^{\infty}$ covers $\mathcal{I}_t$ since the K_i did as well, and moreover since by construction the $C_{t_i}(J_i)$ are either contained in each other or disjoint we may find a sub-collection $\{C_{t_{i_k}}(J_{i_k})\}_{k=1}^{\infty}$ which covers $\mathcal{I}_t$ with all elements pairwise disjoint. By Corollary A.3 there are an even number of elements of $\mathcal{D}_t$ contained in $C_{t_{i_k}}(J_{i_k})$ for each k. However, since they cover disjointly, this implies that $|\mathcal{D}_t|$ is even, which contradicts Corollary A.2 completing the proof of existence.

Now we establish uniqueness. Let $I_t^{(1)}, I_t^{(2)}, \dots I_t^{(\ell)}$ denote the components of $\mathcal{I}_t$ such that for all $0 \leq t' \leq t$ we have $C_{t'}(I_t^{(i)}) \in \mathcal{D}_{t'}$, and assume that $\ell > 1$. Define

$$t_0 = \sup\{t' \ : \ \exists_{i \neq j} \text{ s.t. } C_{t'}(I_t^{(i)}) = C_{t'}(I_t^{(j)})\}.$$

By this definition, it is clear that $\gamma(t_0) \in \mathcal{I}$. Moreover, there exists a $t_1 < t_0$ such that $\gamma[t_1, t_0) \cap \mathcal{I} = \emptyset$ since if there did not then $\gamma(t_0)$ is a limit point of $\gamma(0, t_0) \cap \mathcal{I}$ which implies that an earlier time would have separated all the $I_t^{(i)}$ from each other contradicting the choice of t_0. The components of $\mathcal{I}_{t_0}$ are precisely those of $\mathcal{I}_{t_1}$ except for a single component, call it J, which is split into J_1, J_2 in $\mathcal{I}_{t_0}$ by $\gamma(t_0)$. By the choice of t_0, J is $C_{t_1}(I_t^{(i)})$ for some i and both of J_1, J_2 are $C_{t_0}(I_t^{(i)})$ for some i. This is a contradiction since by Corollary A.3 each of J, J_1, J_2 must contain an odd number of elements of $\mathcal{D}_t$. ■

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