# The Curse of Dimensionality for Monotone and Convex Functions of Many Variables

Aicke Hinrichs*, Erich Novak†
Mathematisches Institut, Universität Jena
Ernst-Abbe-Platz 2, 07740 Jena, Germany
email: a.hinrichs@uni-jena.de, erich.novak@uni-jena.de

Henryk Woźniakowski‡
Department of Computer Science, Columbia University,
New York, NY 10027, USA, and
Institute of Applied Mathematics, University of Warsaw
ul. Banacha 2, 02-097 Warszawa, Poland
email: henryk@cs.columbia.edu

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**Abstract**

We study the integration and approximation problems for monotone and convex bounded functions that depend on $d$ variables, where $d$ can be arbitrarily large. We consider the worst case error for algorithms that use finitely many function values. We prove that these problems suffer from the curse of dimensionality. That is, one needs exponentially many (in $d$) function values to achieve an error $\varepsilon$.

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# 1 Introduction

Many multivariate problems suffer from the curse of dimensionality. A partial list of such problems can be found in e.g., [6, 7]. The phrase *curse of dimensionality* was coined by Bellman already in 1957 and means that the complexity[1] of a $d$-variate problem is an exponential function in $d$. This is usually proved for multivariate problems defined on the unit balls of normed linear spaces. We stress that the curse of dimensionality may hold independently of the smoothness of functions and may hold even for analytic functions.

The choice of the unit ball as the domain of a multivariate problem is not essential and can be slightly generalized. What is important and heavily used in the proof is that the domain $F_d$ of the $d$ variate problem is balanced ($f \in F_d$ implies $-f \in F_d$) and convex ($f_1, f_2 \in F_d$ and $t \in [0,1]$ imply that $tf_1 + (1-t)f_1 \in F_d$). It is not clear if the curse of dimensionality may hold for domains $F_d$ being not balanced or not convex.

In this paper we study classes of monotone and convex $d$-variate bounded functions. Such classes are obviously *not* balanced and the previous analysis to prove the curse of dimensionality does not apply. We study the integration problem and the approximation problem in the $L_p$ norm with $p \in [1, \infty]$. We consider the worst case setting and algorithms that use finitely many function values. In particular, we ask what is the minimal number of $d$-variate function values that is needed to achieve an error $\varepsilon$.

It turns out that the approximation problem in the $L_p$ norm for both monotone and convex functions is no easier than the integration problem. This means that lower error bounds for integration also hold for approximation. Hence, it is enough to prove the curse of dimensionality for the integration problem.

The integration problem for monotone functions has been studied by Papageorgiou [8], and for convex functions by Katscher, Novak and Petras [4]. They proved the optimal rate of convergence and provided lower and upper bounds on the $n$th minimal error. From these bounds we can conclude the lack of some tractability properties defined later, but cannot conclude whether the curse of dimensionality holds.

In this paper we prove that for both monotone and convex functions, the curse of dimensionality holds for the integration problem and therefore also holds for the approximation problem in the $L_p$ norm. The proof relies on identifying "fooling" functions $f^-$ and $f^+$ which are both monotone or both convex, which share the same $n$ function values used by an algorithm, and whose integrals differ as much as possible. Here "as much as possible" means that the error is at most $\varepsilon$ only if $n$ is exponentially large in $d$. The fooling functions

[1] By complexity we mean the minimal cost of computing an $\varepsilon$-approximation. The complexity is bounded from below by the information complexity which is defined as the minimal number of function values needed to compute an $\varepsilon$-approximation. In this paper we prove that even the information complexity suffers from the curse of dimensionality.

for the monotone class take only values 0 or 1 depending on the points used by an algorithm. The fooling functions for the convex class are $f^- = 0$ and $f^+$ is chosen such that it vanishes at $n$ points used by an algorithm, and its integral is maximized. Using the results of Elekes [1] and Dyer, Füredi and McDiarmid [2] on random volumes of cubes, we prove that the integral of $f^+$ is of order 1 for large $d$, if $n$ is smaller than, say, $(12/11)^d$.

Restricting the algorithms for the integration problem to use only function values is quite natural. However, for the approximation problem it would be also interesting to consider algorithms that use finitely many arbitrary linear functionals. We believe that the $L_p$ approximation problem still suffers from the curse of dimensionality for this general information, and pose this question as an open problem. The paper by Gilewicz, Konovalov and Leviatan [3] may be relevant in this case. This paper presents the order of convergence for the approximation problem for $s$-monotone functions (in one variable).

We finally add a comment on the worst case setting used in this paper. Since integration for monotone and convex classes suffers from the curse of dimensionality in the worst case setting, it seems natural to switch to the randomized setting where algorithms can use function values at randomized sample points. Now we can use the classical Monte Carlo algorithm. Since all monotone and convex integrands are bounded by one, the error bound of Monte Carlo is $n^{-1/2}$, without any additional constant. Hence, $\varepsilon^{-2}$ function values at randomized sample points are enough to guarantee a randomized error $\varepsilon$. This means that the integration problem for both monotone and convex functions is *strongly polynomially tractable*[2] in the randomized setting. The exponent 2 of $\varepsilon^{-1}$ is optimal since the optimal orders of convergence for randomized algorithms are $n^{-1/2-1/d}$ for monotone functions, see [8], and $n^{-1/2-2/d}$ for convex functions, see [4]. Hence, for large $d$ we cannot guarantee a randomized error $\varepsilon$ with $\varepsilon^{-p}$ function values with $p < 2$. This proves that the switch for the worst case setting to the randomized setting breaks the curse of dimensionality for the integration problem defined for monotone and convex functions.

Not much seems to be known about the $L_p$ approximation problem in the randomized setting for monotone or convex functions. It is not clear if we still have the curse of dimensionality in the randomized setting. We pose this as another open problem.

## 2 Integration

We mainly study the integration problem, i.e., we want to approximate

$$\mathrm{INT}_d(f) = \int_{[0,1]^d} f(x)\,\mathrm{d}x,$$

[2]This means that (3) holds with $q = 0$. In this case we can choose $C = 1$ and $p = 2$.

for bounded functions $f : [0,1]^d \to [0,1]$ that are monotone (more precisely, non-decreasing in each variable $x_j$ if the other variables are fixed) or convex. Hence, we consider the classes

$$F_d^{\rm mon} = \{f : [0,1]^d \to [0,1] \mid \ f \text{ is monotone}\}$$

and

$$F_d^{\rm con} = \{f : [0,1]^d \to [0,1] \mid \ f \text{ is convex}\}.$$

We approximate the integral $\mathrm{INT}_d(f)$ by algorithms $A_n$ that use information about $f$ given by $n$ function values. Hence, $A_n$ has the form

$$A_n(f) = \phi_n(f(t_1), f(t_2), \dots, f(t_n)), \tag{1}$$

where $n$ is a nonnegative integer, $\phi_n : \mathbb{R}^n \to \mathbb{R}$ is an arbitrary mapping, and the choice of arbitrary sample points $t_j \in [0,1]^d$ can be adaptive. That is, $t_j$ may depend on the already computed values $f(t_1), f(t_2), \dots, f(t_{j-1})$. For $n = 0$, the mapping $A_n$ is a constant real number. More details can be found in e.g., [5, 6, 7, 9].

We define the $n$th minimal error of such approximations in the worst case setting as

$$e_n^{\rm int}(F_d) = \inf_{A_n} \sup_{f \in F_d} |\mathrm{INT}_d(f) - A_n(f)| \quad \text{for} \quad F_d \in \{F_d^{\rm mon}, F_d^{\rm con}\}.$$

For $n = 0$, it is easy to see that the best algorithm is $A_0 = \frac{1}{2}$ for the two classes considered in this paper, and we obtain

$$e_0^{\rm int}(F_d^{\rm mon}) = e_0^{\rm int}(F_d^{\rm con}) = \tfrac{1}{2} \quad \text{for all} \quad d \in \mathbb{N}.$$

Hence, the integration problems are well scaled and it is enough to study the absolute error. The *information complexity* is the inverse function of $e_n^{\rm int}(F_d)$ given by

$$n^{\rm int}(F_d, \varepsilon) = \min\{n \mid \ e_n^{\rm int}(F_d) \le \varepsilon\} \quad \text{for} \quad F_d \in \{F_d^{\rm mon}, F_d^{\rm con}\}.$$

It is trivial that $n^{\rm int}(F_d^{\rm mon}, \varepsilon) = n^{\rm int}(F_d^{\rm con}, \varepsilon) = 0$ for all $\varepsilon \ge \frac{1}{2}$.

# 3 Known and new results

The integration problems for monotone and for convex functions were studied before, we refer to the paper by Papageorgiou [8] for monotone functions, and to the paper by Katscher, Novak and Petras [4] for convex functions. Here we mention some of the known results and indicate our new results concerning the *curse of dimensionality*.

For the class $F_d^{\rm mon}$ of monotone functions it was proved by Papageorgiou [8] that

$$e_n^{\rm int}(F_d^{\rm mon}) = \Theta(n^{-1/d}).$$

Hence, the *optimal order of convergence* is $n^{-1/d}$. More precisely, it is proved in [8] that there are some positive numbers $c, C$ independent of $n$ and $d$ such that for all $d, n \in \mathbb{N}$ we have

$$c\, d^{-1}\, n^{-1/d} \le e_n^{\rm int}(F_d^{\rm mon}) \le C\, d\, n^{-1/d}. \tag{2}$$

It is interesting to note that the ratio between the upper and the lower bound is of the order $d^2$ , i.e., it is polynomial in $d$, not exponential as it is the case for many other spaces.

The bound (2) yields

$$\left\lceil \left(\frac{c}{d\,\varepsilon}\right)^d \right\rceil \le n^{\rm int}(F_d^{\rm mon}, \varepsilon) \le \left\lceil \left(\frac{C\,d}{\varepsilon}\right)^d \right\rceil.$$

From this we conclude that *polynomial tractability* and even *weak tractability* do not hold. That is, it is *not* true that there are non-negative $C, q, p$ such that for all $d \in \mathbb{N}$ and $\varepsilon \in (0, \frac12)$ we have

$$n^{\rm int}(F_d^{\rm mon}, \varepsilon) \le C\, d^q \varepsilon^{-p} \quad \text{(polynomial tractability)}, \tag{3}$$

as well as it is *not* true that

$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln\, n^{\rm int}(F_d^{\rm mon}, \varepsilon)}{\varepsilon^{-1} + d} = 0 \quad \text{(weak tractability)}.$$

Nevertheless, the lower bound on $n^{\rm int}(F_d^{\rm mon}, \varepsilon)$ is useless for a fixed $\varepsilon > 0$ and large $d$, since for $d \ge c/\varepsilon$ we do not obtain a bound better than $n^{\rm int}(F_d^{\rm mon}, \varepsilon) \ge 1$. Thus, it is not clear whether the information complexity $n^{\rm int}(F_d^{\rm mon}, \varepsilon)$ is exponential in $d$ for a fixed $\varepsilon \in (0, \frac12)$. In this paper we will prove that

$$n^{\rm int}(F_d^{\rm mon}, \varepsilon) \ge 2^d\,(1 - 2\varepsilon) \quad \text{for all} \quad d \in \mathbb{N},\ \varepsilon \in (0, \tfrac12).$$

This means that $n^{\rm int}(F_d^{\rm mon}, \varepsilon)$ is indeed exponential in $d$, that is the integration problem suffers from the *curse of dimensionality*.

We now turn to the class $F_d^{\rm con}$ of convex functions. It was proved by Katscher, Novak and Petras [4] that

$$e_n^{\rm int}(F_d^{\rm con}) = \Theta(n^{-2/d}).$$

Again, the *optimal order of convergence* is known, now it is $n^{-2/d}$. More precisely, it was proved in [4] that there are some positive numbers $c_d, C$, with $c_d$ being exponentially small in $d$ whereas $C$ is independent of $d$, such that we have for all $n \in \mathbb{N}$

$$c_d\, n^{-2/d} \le e_n^{\rm int}(F_d^{\rm con}) \le C\, d\, n^{-2/d}. \tag{4}$$

The bound (4) yields

$$\left\lceil \left(\frac{c_d}{\varepsilon}\right)^{d/2} \right\rceil \le n^{\rm int}(F_d^{\rm con}, \varepsilon) \le \left\lceil \left(\frac{C\, d}{\varepsilon}\right)^{d/2} \right\rceil.$$

From this we conclude that polynomial tractability does not hold. The lower bound in (4) is useless for a fixed $\varepsilon > 0$ and large $d$, and therefore it is not clear if we have weak tractability or the curse of dimensionality. In this paper we will prove that there exists $\varepsilon_0 \in (0, 1/4)$ such that

$$n^{\rm int}(F_d^{\rm con}, \varepsilon) \ge \frac{1}{2(d+1)} \left(\frac{11}{10}\right)^d \quad \text{for all} \quad d \in \mathbb{N},\ \varepsilon \in (0, \varepsilon_0].$$

Hence, the integration problem also suffers from the curse of dimensionality for convex functions.

# 4 The class of monotone functions

We consider integration for monotone functions. Assume that $A_n$ is an arbitrary (possibly adaptive) algorithm for the class $F_d^{\rm mon}$. For $x = [x_1, x_2, \dots, x_d] \in [0,1]^d$, consider the "fooling" function

$$f^*(x) = \begin{cases} 0 & \text{if } \sum_{k=1}^d x_k < d/2, \\ 1 & \text{if } \sum_{k=1}^d x_k \ge d/2. \end{cases}$$

Obviously, $f^* \in F_d^{\rm mon}$ and therefore the algorithm $A_n$ will use function values

$$f^*(t_1), f^*(t_2), \dots, f^*(t_n)$$

for some sample points $t_j \in [0,1]^d$. Since the algorithm $A_n$ can *only* use the computed function values, we obtain

$$A_n(f) = A_n(f^*)$$

for all $f \in F_d^{\rm mon}$ if $f(t_k) = f^*(t_k)$ for $k = 1, 2, \dots, n$.

Take first the case $n = 1$. Suppose first that $f^*(t_1) = 0$, i.e., $\sum_{j=1}^d t_{1,j} < d/2$ for $t_1 = [t_{1,1}, t_{1,2}, \ldots, t_{1,d}]$. Define $f^- = 0$ and the function

$$f^+(x) = \begin{cases} 0 & \text{if } x \le t_1 \text{ (in every coordinate)}, \\ 1 & \text{otherwise.} \end{cases}$$

Then $f^-, f^+ \in F_d^{\text{mon}}$ and they yield the same information as $f^*$, i.e.,

$$f^-(t_1) = f^+(t_1) = f^*(t_1) = 0.$$

Using the standard proof technique it can be checked that

$$\max_{y \in [0,1]^d,\ \sum_{j=1}^d y_j \le d/2} \prod_{j=1}^d y_j = \max_{y \in [0,1]^d,\ \sum_{j=1}^d y_j \ge d/2} \prod_{j=1}^d (1 - y_j) = 2^{-d}.$$

Then

$$\text{INT}_d(f^+) = 1 - \text{INT}_d(1 - f^+) = 1 - \int_{x \le t_1} \mathrm{d}x = 1 - \prod_{j=1}^d t_{1,j}.$$

This implies that

$$\text{INT}_d(f^+) - \text{INT}_d(f^-) \ge 1 - 2^{-d}. \tag{5}$$

The case with $f^*(t_1) = 1$ is similar. Now take $f^+ = 1$ and

$$f^-(x) = \begin{cases} 1 & \text{if } x \ge t_1, \\ 0 & \text{otherwise.} \end{cases}$$

Again $f^+$ and $f^-$ are from $F_d^{\text{mon}}$ and they yield the same information as $f^*$. We also obtain (5). We estimate the error of $A_1$ on the whole class $F_d^{\text{mon}}$ by

$$\begin{aligned} \sup_{f \in F_d^{\text{mon}}} |\text{INT}_d(f) - A_n(f)| &\ge \max\left(|\text{INT}_d(f^+) - A_n(f^*)|, |\text{INT}_d(f^-) - A_n(f^*)|\right) \\ &\ge \tfrac{1}{2}\left(\text{INT}_d(f^+) - \text{INT}_d(f^-)|\right) \ge \tfrac{1}{2}\left(1 - 2^{-d}\right). \end{aligned}$$

Since this holds for all algorithms, we conclude that

$$e_1(F_d^{\text{mon}}, \varepsilon) \ge \tfrac{1}{2}\left(1 - 2^{-d}\right).$$

The general case with $n \in \mathbb{N}$ is similar. Assume that $\ell$ of the function values yield $f^*(t_k) = 0$ while $n - \ell$ function values yield $f^*(t_k) = 1$. Without loss of generality, we may assume that

$$\begin{aligned} f^*(t_j) &= 0 \quad \text{for} \quad j = 1, 2, \ldots, \ell, \\ f^*(t_j) &= 1 \quad \text{for} \quad j = \ell + 1, \ell + 2, \ldots, n. \end{aligned}$$

Define the two functions,

$$f^+(x) = \begin{cases} 0 & \text{if } x \leq t_1 \text{ or } x \leq t_2 \text{ or } \ldots \text{ or } x \leq t_\ell, \\ 1 & \text{otherwise.} \end{cases}$$

and

$$f^-(x) = \begin{cases} 1 & \text{if } x \geq t_{\ell+1} \text{ or } x \geq t_{\ell+2} \text{ or } \ldots \text{ or } x \geq t_n, \\ 0 & \text{otherwise.} \end{cases}$$

Then $f^+, f^- \in F_d^{\text{mon}}$ with

$$f^+(t_k) = f^-(t_k) = f^*(t_k) \quad \text{for all} \quad k = 1, 2, \ldots, n.$$

Furthermore, we have

$$\text{INT}_d(f^-) \leq \sum_{j=1}^{n-\ell} \int_{x \geq t_{\ell+j}} 1 \, \mathrm{d}x \leq (n - \ell) 2^{-d}.$$

Similarly it is easy to show that $\text{INT}_d(f^+) \geq 1 - 2^{-d} \cdot \ell$, so that

$$\text{INT}_d(f^+) - \text{INT}_d(f^-) \geq 1 - 2^{-d} \cdot n.$$

Therefore the worst case error of $A_n$ is at least $\frac{1}{2}(1 - 2^{-d} n)$. Since this holds for an arbitrary $A_n$ we also have

$$e_n(F_d^{\text{mon}}) \geq \tfrac{1}{2} \left(1 - 2^{-d} n\right).$$

This leads to the following theorem.

**Theorem 1.** For each fixed $\varepsilon \in (0, \frac{1}{2})$, the information complexity is at least

$$n^{\text{int}}(F_d^{\text{mon}}, \varepsilon) \geq 2^d \, (1 - 2\varepsilon) \quad \text{for all} \quad d \in \mathbb{N}.$$

Thus, the integration problem for monotone functions suffers from the curse of dimensionality.

# 5 The class of convex functions

We now consider integration for convex function and prove the curse of dimensionality.

**Theorem 2.** There exists $\varepsilon_0 \in (0, \frac{1}{2})$ such that for each fixed $\varepsilon \in (0, \varepsilon_0)$ the information complexity is at least

$$n^{\mathrm{int}}(F_d^{\mathrm{con}}, \varepsilon) \geq \frac{1}{d+1} \left(\frac{11}{10}\right)^d \left(1 - \frac{\varepsilon}{\varepsilon_0}\right) \quad \text{for all} \quad d \in \mathbb{N}.$$

Thus, the integration problem of convex functions suffers from the curse of dimensionality.

The idea of the proof is as follows. Assume again that we have an arbitrary (possibly adaptive) algorithm $A_n$ for the class $F_d^{\mathrm{con}}$. For the zero function $f^- = 0$ the algorithm $A_n$ uses function values at certain sample points $x_1, x_2, \ldots, x_n$. This implies that $A_n$ uses the same sample points $x_1, x_2, \ldots, x_n$ for any function $f$ from $F_d^{\mathrm{con}}$ with

$$f(x_1) = f(x_2) = \cdots = f(x_n) = 0.$$

In particular, let $f^+$ be the largest such function,

$$f^+(x) = \sup\{f(x) \mid \ f(x_j) = 0,\ j = 1, 2, \ldots, n,\ f \in F_d^{\mathrm{con}}\ \}.$$

Clearly, $f^+ \in F_d^{\mathrm{con}}$, $f^+(x_j) = 0$ for $j = 1, 2, \ldots, n$, $f(x) \geq 0$ for all $x \in [0,1]^d$, and $f^+$ has the maximal value of the integral among such functions. The integral $\mathrm{INT}_d(f^+)$ is the volume of the subset under the graph of the function $f^+$. This subset under the graph is the complement in $[0,1]^{d+1}$ of the convex hull of the points $(x_1, 0), (x_2, 0), \ldots, (x_n, 0) \in [0,1]^{d+1}$ and $[0,1]^d \times \{1\} \subset [0,1]^{d+1}$. Denoting this convex hull by $C$, we obtain

$$\mathrm{INT}_d(f^+) = 1 - \mathrm{vol}_{d+1}(C).$$

Since the algorithm $A_n$ computes the same result for the functions $f^-$ and $f^+$ but $\mathrm{INT}_d(f^-) = 0$ we conclude that $A_n$ has error at least

$$\tfrac{1}{2}\big(1 - \mathrm{vol}_{d+1}(C)\big)$$

on one of these functions. Theorem 2 now follows directly from the next theorem which gives an estimate of the volume of the set $C$ by setting $\varepsilon_0 = t_0/2$.

**Theorem 3.** Let $P$ be an $n$-point set in $[0,1]^d \times \{0\}$. Then the $(d+1)$-dimensional volume of the convex hull $C$ of $P \cup \big([0,1]^d \times \{1\}\big)$ is at most

$$\mathrm{vol}_{d+1}(C) \leq (1 - t_0) + (d+1)\, n\, t_0 \left(\frac{10}{11}\right)^d$$

for some $t_0 \in (0,1)$ independent of $d$ and $n$.

*Proof.* Let $Q = [0,1]^d$ and $Q_t = [0,1]^d \times \{t\} \subset \mathbb{R}^{d+1}$ for $t \in [0,1]$. Let $P \subset Q_0$ be an $n$-point set and let $C$ be the convex hull of $P \cup Q_1$. We want to show that

$$\mathrm{vol}_{d+1}(C) \le (1-t_0) + (d+1)\, n\, t_0 \left(\frac{10}{11}\right)^d.$$

Let $C_t = C \cap Q_t$ be the slice of $C$ at height $t$. For a point $z = (z_1, z_2, \ldots, z_d, z_{d+1}) \in \mathbb{R}^{d+1}$ let $\overline{z} = (z_1, z_2, \ldots, z_d)$ be its projection onto the first $d$ coordinates. Similarly, for a set $M \subset \mathbb{R}^{d+1}$, let $\overline{M}$ be the set of all points $\overline{z}$ with $z \in M$.

Since

$$\mathrm{vol}_{d+1}(C) = \int_0^1 \mathrm{vol}_d(C_t)\,\mathrm{d}t = \int_0^1 \mathrm{vol}_d(\overline{C}_t)\,\mathrm{d}t \le (1-t_0) + \int_0^{t_0} \mathrm{vol}_d(\overline{C}_t)\,\mathrm{d}t,$$

it is enough to prove that

$$\mathrm{vol}_d(\overline{C}_t) \le (d+1)\, n \left(\frac{10}{11}\right)^d \quad \text{for all} \quad t \in [0, t_0].$$

Carathéodory's theorem states that any point in the convex hull of a set $M$ in $\mathbb{R}^d$ is already contained in the convex hull of a subset of $M$ consisting of at most $d+1$ points. Hence, every point of $P$ is contained in the convex hull of $d+1$ vertices of $Q_0$. It follows that it is enough to show that

$$\mathrm{vol}_d(\overline{C}_t) \le n \left(\frac{10}{11}\right)^d \tag{6}$$

whenever $P$ is an $n$-point set of such vertices of $Q_0$. So we assume now that $P$ is such a set.

Let

$$w_t = ((1+t)/2, (1+t)/2, \ldots, (1+t)/2, t) \in Q_t.$$

For each vertex $v \in P$, let $B_v \subset Q_0$ be the intersection of the ball with center $\frac{1}{2}(w_0 + v)$ and radius $\frac{1}{2}\|w_0 - v\|$ with $Q_0$. Observe that $C_0$ is the convex hull of $P$. By Elekes' result from [1],

$$C_0 \subset \bigcup_{v \in P} B_v.$$

It follows that

$$C = \mathrm{conv}(P \cup Q_1) \subset \bigcup_{v \in P} \mathrm{conv}(B_v \cup Q_1)$$

since each point in this convex hull lies on a segment between a point in some $B_v$ and a point in $Q_1$. Since all sets $\mathrm{conv}(B_v \cup Q_1)$ are congruent, the inequality (6) immediately follows if we show that

$$\mathrm{vol}_d(\overline{D}_t) \leq \left(\frac{10}{11}\right)^d \quad \text{for all} \quad t \in [0, t_0], \tag{7}$$

where $D_t = \mathrm{conv}(B_v \cup Q_1) \cap Q_t$ is the section of the convex hull at height $t$. We can now restrict ourselves to the case that $v$ is a fixed vertex in $P$, say $v = (0, 0, \ldots, 0, 0)$.

Let $O$ be the origin in $\mathbb{R}^d$. Let $E_t \subset Q$ be the intersection of the ball with center $\frac{1}{2}\overline{w}_t$ and diameter $\|\overline{w}_t\|$ with $Q$. Then $\overline{D}_t \subset E_t$, so (7) is proved once we show

$$\mathrm{vol}_d(E_t) \leq \left(\frac{10}{11}\right)^d \quad \text{for all} \quad t \in [0, t_0]. \tag{8}$$

To this end we follow the approach from [2]. Set $2s = \frac{1}{2}(1 + t)$. Then

$$\mathrm{vol}_d(E_t) = \mathbb{P}\Big(\sum_{j=1}^{d} (X_j - s)^2 \leq ds^2\Big)$$

where $X_1, X_2, \ldots, X_d$ are independent uniformly distributed in $[0, 1]$. We now use Markov's inequality

$$\mathbb{P}(|Y| \geq a) \leq \frac{\mathbb{E}(|Y|)}{a},$$

which holds for all real random variables $Y$ and all $a > 0$. We take $a = 1$ and

$$Y = \exp\left(\alpha\Big(ds^2 - \sum_{j=1}^{d}(X_j - s)^2\Big)\right),$$

and conclude that $\mathrm{vol}_d(E_t)$ is smaller than

$$\mathbb{E}\exp\left(\alpha\Big(ds^2 - \sum_{j=1}^{d}(X_j - s)^2\Big)\right) = \Big(\mathbb{E}\exp\big(\alpha(2sX - X^2)\big)\Big)^d$$

where $X$ is uniformly distributed in $[0, 1]$ and $\alpha > 0$ is arbitrary. This implies

$$\mathrm{vol}_d(E_t) \leq \Big(\inf_{\alpha > 0} g(s, \alpha)\Big)^d$$

where

$$g(s, \alpha) = \int_0^1 \exp(\alpha(2sx - x^2))\,\mathrm{d}x.$$

By continuity and the proof in [2] we find a positive $t_0$, and for each $t \in [0, t_0]$, we find some positive $\alpha$ such that

$$g(s, \alpha) < \frac{10}{11},$$

where $2s = \frac{1}{2}(1+t)$. Now (8) follows and the proof is completed. $\square$

# 6 $L_p$ approximation

The $L_p$ approximation problem is defined by

$$\mathrm{APP}_d \,:\, F_d \to L_p([0,1]^d) \quad \text{with} \quad \mathrm{APP}_d(f) = f$$

for $F_d \in \{F_d^{\mathrm{mon}}, F_d^{\mathrm{con}}\}$ and the standard $L_p([0,1]^d)$ space.

The algorithms $A_n$ are now given by (1) with $\phi_n : \mathbb{R}^n \to L_p([0,1]^d)$. The $n$th minimal error for the $L_p$ approximation problem in the worst case setting is defined by

$$e_n^{\mathrm{app}}(F_d) = \inf_{A_n} \sup_{f \in F_d} \|\mathrm{APP}_d(f) - A_n(f)\|_{L_p([0,1]^d)}.$$

For $n = 0$, the initial error is again $\frac{1}{2}$. The information complexity is now

$$n^{\mathrm{app}}(F_d, \varepsilon) = \min\{n \mid \ e_n^{\mathrm{app}}(F_d) \le \varepsilon\}.$$

Note that lower bounds for integration also hold for $L_p$ approximation. Indeed, take an arbitrary algorithm $A_n$ for the $L_p$ approximation problem, and let

$$A_n^{\mathrm{int}}(f) = \int_{[0,1]^d} A_n(f)(x)\,\mathrm{d}x.$$

Then $A_n^{\mathrm{int}}$ approximates the integral of $f$ and we have

$$\mathrm{INT}_d(f) - A_n^{\mathrm{int}}(f) = \int_{[0,1]^d} \big(f(x) - A_n(f)(x)\big)\,\mathrm{d}x.$$

This yields

$$\big|\mathrm{INT}_d(f) - A_n^{\mathrm{int}}(f)\big| \le \int_{[0,1]^d} |f(x) - A_n(f)(x)|\ \mathrm{d}x \le \left(\int_{[0,1]^d} |f(x) - A_n(f)(x)|^p\ \mathrm{d}x\right)^{1/p}.$$

Since this holds for all algorithms $A_n$, we have

$$e_n^{\mathrm{int}}(F_d) \le e_n^{\mathrm{app}}(F_d) \quad \text{and} \quad n^{\mathrm{int}}(F_d, \varepsilon) \le n^{\mathrm{app}}(F_d, \varepsilon),$$

as claimed. In particular, the curse of dimensionality also holds for the $L_p$ approximation problem for both classes $F_d^{\mathrm{mon}}$ and $F_d^{\mathrm{con}}$.

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