Special and exceptional Jordan dialgebras

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Abstract

In this paper, we study the class of Jordan dialgebras. We develop an approach for reducing problems on dialgebras to the case of ordinary algebras. It is shown that straightforward generalizations of the classical Cohn's, Shirshov's, and Macdonald's Theorems do not hold for dialgebras. However, we prove dialgebraic analogues of these statements. Also, we study polylinear special identities which hold in all special Jordan algebras and do not hold in all Jordan algebras. We find a natural correspondence between special identities for ordinary algebras and dialgebras.

INTRODUCTION

One of the most important classes of nonassociative algebras is the class of Lie algebras defined by the anti-commutativity and Jacobi identities $x^2 = 0$, $(xy)z + (zx)y + (yz)x = 0$. This is well-known that every associative algebra A turns into a Lie algebra with respect to the new product $[a, b] = ab - ba$, $a, b \in A$. The Lie algebra obtained is denoted by $A^{(-)}$. The classical Poincaré—Birkhoff—Witt Theorem implies every Lie algebra to be embedded into $A^{(-)}$ for an appropriate associative algebra A.

Leibniz algebras introduced in [1] are the most popular non-commutative analogues of Lie algebras. An algebra $(L, [\cdot, \cdot])$ is said to be a (right) Leibniz algebra if the product $[\cdot, \cdot]\colon L\times L \to L$ satisfies the following (right) Leibniz identity:

$$[[x, y], z] = [[x, z], y] + [x, [y, z]]. \qquad (1)$$

To get an analogue of the Poincaré—Birkhoff—Witt Theorem for Leibniz algebras, the notion of an associative dialgebra was introduced in [2]. Namely, an associative dialgebra is a linear space D with two bilinear operations $\vdash, \dashv\colon D \times D \to D$ satisfying certain axioms. The new product $[x, y] = x \dashv y - y \vdash x$, $x, y \in D$, satisfies (1), so D is a Leibniz algebra with respect to this new product. The Leibniz algebra obtained is denoted by $D^{(-)}$. As it was shown in [3, 4], every Leibniz algebra can be embeddable into $D^{(-)}$ for an appropriate associative dialgebra D.

Another important class of nonassociative algebras is the class of Jordan algebras defined by the commutativity and Jordan identity $(x^2y)x = x^2(yx)$. This is well-known that if A is an associative algebra over a field on characteristic $\neq 2$ then A with respect to the new product $a \circ b = \frac{1}{2}(ab + ba)$ is a Jordan algebra denoted by $A^{(+)}$. For Jordan algebras, the analogue of the Poincaré—Birkhoff—Witt theorem is not true: There exist Jordan algebras that can not be embedded into $A^{(+)}$ for any associative algebra A. Therefore, the following notion makes sense: If a Jordan algebra J is a subalgebra of $A^{(+)}$ for some associative algebra A then it is said to be a special Jordan algebra.

The notion of a Jordan dialgebra was introduced in [5] as a particular example of a general algebraic definition of what is a variety of dialgebras. This general operadic approach leads

*The author was partially supported by ADTP (Grant 2.1.1.419), RFBR (Grant 09-01-00157-A), SSc-3669.2010.1, and Federal Aim Program (contracts N. 02.740.11.5191, N. 14.740.11.0346).

to three identities defining the variety of Jordan dialgebras. Independently, the notion of quasi-Jordan algebra emerged in [6] as the variety of non-commutative analogues of Jordan algebras. Namely, if one considers an associative dialgebra D with respect to a new product $x \circ y = \frac{1}{2}(x \dashv y + y \vdash x)$, $x, y \in D$, then the algebra obtained is a quasi-Jordan algebra. In [6], two identities were stated to define the variety of quasi-Jordan algebras. Later in [7], the third (missing) one was noticed, so the notions of quasi-Jordan algebras and Jordan dialgebras went to coincidence.

In [8], the natural notions of a special Jordan dialgebra and of a special identity (s-identity, for sort) were introduced. An s-identity of Jordan dialgebras is an identity which holds in all special Jordan dialgebras but does not hold in some Jordan dialgebra. In this note, we show the correspondence between polylinear s-identities of Jordan algebras and Jordan dialgebras (Theorem 20). In particular, one of the main results of [8] follows from this theorem.

Also, several natural problems were posed in [8]: How to generalize the classical statements known for Jordan algebras to the case of dialgebras. This paper is devoted to the solution of all these problems. We prove the analogues of the following theorems:

- The Cohn's Theorem [9] on the characterization of elements of free special Jordan algebra as symmetric elements of free associative algebra.
- The Cohn's example [9] of the exceptional Jordan algebra which is the homomorphic image of the two-generated special Jordan algebra. It follows from this example that the class of special Jordan algebras is not a variety.
- The Shirshov's Theorem [10] on the speciality of two-generated Jordan algebra.
- The Macdonald's Theorem [10] on special identities in three variables.

The main method of study is the following. Given a Jordan dialgebra J, we build two Jordan algebras $\bar{J}$ and $\widehat{J}$ as described in [11]. The classical theorems hold for these Jordan algebras, and their properties allow to make conclusions about the dialgebra J. Moreover, the theory of conformal algebras [12] is deeply involved into considerations.

1 PRELIMINARIES

1.1 Dialgebras

A linear space D with two bilinear operations of a product $\vdash, \dashv \colon D \times D \to D$ is called a *dialgebra*. The base field is denoted by $\Bbbk$. A dialgebra is *associative* if it satisfies the identities

$$(x \dashv y) \vdash z = (x \vdash y) \vdash z, \quad x \dashv (y \vdash z) = x \dashv (y \dashv z) \tag{2}$$

and

$$\begin{aligned} (x, y, z)_{\vdash} &:= (x \vdash y) \vdash z - x \vdash (y \vdash z) = 0, \\ (x, y, z)_{\dashv} &:= (x \dashv y) \dashv z - x \dashv (y \dashv z) = 0, \\ (x, y, z)_{\times} &:= (x \vdash y) \dashv z - x \vdash (y \dashv z) = 0. \end{aligned} \tag{3}$$

This class of dialgebras is well investigated in [3].

A dialgebra that meets the identities (2), is called a *0-dialgebra.* If D is a 0-dialgebra then the subspace $D_0 = \mathrm{Span}\{a \vdash b - a \dashv b \mid a, b \in D\}$ is an ideal of D and the quotient dialgebra $\bar{D} = D/D_0$ can be identified with an ordinary algebra. The space D may be endowed with left and right actions of $\bar{D}$:

$$\bar{a} \cdot x = a \vdash x, \quad x \cdot \bar{a} = x \dashv a, \quad x, a \in D,$$

where $\bar{a}$ denotes the image of a in $\bar{D}$.

Let A be an algebra that acts on a linear space M via some operations $\circ\colon A\times M\to M$ and $\circ\colon M\times A\to M$. In this case, we can define the algebra $(A\oplus M,\circ)$, where the product $\circ$ is given by the formula $(a+m)\circ(b+n)=ab+(a\circ n+m\circ b)$, that is, $M\circ M=0$. The algebra obtained is called the split null extension of A by means of M.

We have seen before that we can define actions of the algebra $\bar{D}$ on the dialgebra D, so the split null extension $\bar{D}\oplus D$ is defined. We will denote it by $\widehat{D}$.

In any dialgebra D a *dimonomial* is an expression of the form $w=(a_1\ldots a_n)$, where $a_1,\ldots,a_n\in D$ and parentheses indicate some placement of parentheses with some choice of operations. By induction we can define the *central letter* $c(w)$ of a dimonomial: if $w\in D$, then $c(w)=w$, otherwise $c(w_1\vdash w_2)=c(w_2)$ and $c(w_1\dashv w_2)=c(w_1)$. Let $c(w)=a_k$. If D is 0-dialgebra, then $w=(a_1\vdash\ldots\vdash a_{k-1}\vdash a_k\dashv a_{k+1}\dashv\ldots\dashv a_n)$ for the same parenthesizing that in $(a_1\ldots a_n)$. We will denote this w by $(a_1\ldots a_{k-1}\dot{a}_k a_{k+1}\ldots a_n)$. In an associative dialgebra parenthesizing does not matter, so it is reasonable to use the notation $w=a_1\ldots a_{k-1}\dot{a}_k a_{k+1}\ldots a_n$, where the dot indicates the central letter.

Let X be a set of generators. Obviously, that the basis of the free dialgebra $\mathrm{DiAlg}\langle X\rangle$ generated by X consists of dimonomials with a free placement of parentheses and a free choice of operations. It is clear that the basis of the free 0-dialgebra $\mathrm{DiAlg0}\langle X\rangle$ is the set of dimonomials $(a_1\ldots a_{k-1}\dot{a}_k a_{k+1}\ldots a_n)$ where $k=1,\ldots,n$ and $a_1,\ldots,a_n\in X$. At last, the basis of the free associative dialgebra $\mathrm{DiAs}\langle X\rangle$ consists of dimonomials $a_1\ldots a_{k-1}\dot{a}_k a_{k+1}\ldots a_n$ (see [3]).

If $X=\{x_1,\ldots,x_n\}$ then every dipolynomial $f\in\mathrm{DiAs}\langle X\rangle$ can be presented as a sum $f=f_1+\ldots+f_n$, where each f_i collects all those dimonomials with central letter x_i, $i=1,\ldots,n$.

1.2 Jordan dialgebras

Let us consider the class of Jordan dialgebras over a field $\Bbbk$ such that $\mathrm{char}\,\Bbbk\neq 2,3$. In this case, the variety of Jordan algebras Jord over the field $\Bbbk$ is defined by the following polylinear identities

$$x_1x_2=x_2x_1,\ \ J(x_1,x_2,x_3,x_4)=0,$$

where

$$\begin{aligned}J(x_1,x_2,x_3,x_4)&=x_1(x_2(x_3x_4))+(x_2(x_1x_3))x_4+x_3(x_2(x_1x_4))\\&-(x_1x_2)(x_3x_4)-(x_1x_3)(x_2x_4)-(x_3x_2)(x_1x_4)\end{aligned}$$

is the Jordan identity in a polylinear form [10].

Hence using the general definition of a variety of dialgebras [5] we obtain that the class of Jordan dialgebras is defined by two 0-identities (2) and the following identities

$$\begin{gathered}x_1\vdash x_2=x_2\dashv x_1,\\ J(\dot{x}_1,x_2,x_3,x_4)=0,\quad J(x_1,\dot{x}_2,x_3,x_4)=0,\\ J(x_1,x_2,\dot{x}_3,x_4)=0,\quad J(x_1,x_2,x_3,\dot{x}_4)=0.\end{gathered}\tag{4}$$

The variety of Jordan dialgebras is denoted DiJord. We can express both operations in a Jordan dialgebra through one operation: $a\vdash b=ab$, $a\dashv b=ba$. Then an ordinary algebra arises that is a noncommutative analogue of a Jordan algebra. The corresponding variety is defined by the system of identities

$$[x_1x_2]x_3=0,\quad (x_1^2,x_2,x_3)=2(x_1,x_2,x_1x_3),\quad x_1(x_1^2x_2)=x_1^2(x_1x_2),$$

that is equivalent to identities (4).

Such algebras are investigated in [7, 8, 13].

1.3 Conformal algebras

The notion of a conformal algebra over a field of zero characteristic was introduced by V. G. Kac [12] as a tool of the conformal field theory in mathematical physics. Over a field of an arbitrary characteristic, it is reasonable to use the following equivalent definition [5]: a *conformal algebra* is a linear space C endowed with a linear mapping $T\colon C \to C$ and a set of bilinear operations (n-products) $(\cdot\,_{(n)}\,\cdot)\colon C \times C \to C$. For all $a, b \in C$ there exist just a finite number of elements $n \in \mathbb{Z}^+$ such that $a\,_{(n)}\,b \neq 0$ (locality property). In addition, these operations satisfy the following properties:

$$Ta\,_{(n)}\,b = a\,_{(n-1)}\,b,\ n \geq 1, \quad Ta\,_{(0)}\,b = 0,$$
$$T(a\,_{(n)}\,b) = a\,_{(n)}\,Tb + Ta\,_{(n)}\,b,\ n \geq 0,$$

for all $a, b \in C$.

Let Var be a variety of ordinary algebras. It was defined in [14] what is the corresponding variety of conformal algebras. In [15] this notion was rephrased in terms of pseudo-algebras, that works for nonzero characteristic of $\Bbbk$. Since we use the term "conformal algebra" for a pseudo-algebra over $\Bbbk[T]$ in this paper, it is possible to define the class of these objects corresponding to the variety Var of ordinary algebras. This class is not a real variety of algebras, but we will also denote it by Var.

It was also observed in [5] that if C is a conformal algebra of a variety Var, then the space C can be endowed with a structure of a dialgebra by means of

$$a \vdash b = a\,_{(0)}\,b, \quad a \dashv b = \sum_{s \geq 0} T^s(a\,_{(s)}\,b).$$

The dialgebra obtained is denoted by $C^{(0)}$, it belongs to the variety DiVar.

The simplest example of a conformal algebra can be constructed as follows. Let A be an ordinary algebra, then a conformal product is uniquely defined on $\Bbbk[T] \otimes A$ by the following formulas for $a, b \in A$:

$$a\,_{(n)}\,b = \begin{cases} ab, & n = 0, \\ 0, & n > 0. \end{cases}$$

The conformal algebra obtained is denoted by $\operatorname{Cur} A$ and is called a current conformal algebra. If an algebra A belongs to a variety Var, then $\operatorname{Cur} A$ is a conformal algebra of the same variety. In the language of category theory, we can say that Cur is a functor from the category of algebras to the category of conformal algebras. If $\varphi\colon A \to B$ is a homomorphism of algebras, then the mapping $\operatorname{Cur}\varphi\colon \operatorname{Cur} A \to \operatorname{Cur} B$ acting by the rule $\operatorname{Cur}\varphi(f(T) \otimes a) = f(T) \otimes \varphi(a)$ is a morphism of conformal algebras.

In [13] it was proved that an arbitrary dialgebra D is embedded into the dialgebra $(\operatorname{Cur} \widehat{D})^{(0)}$.

1.4 Notation for varieties of algebras and dialgebras

An arbitrary variety of ordinary algebras we denote Var, the free algebra in this variety generated by a set X is denoted by $\operatorname{Var}\langle X\rangle$. The corresponding variety of dialgebras is denoted by DiVar, the free dialgebra is denoted by $\operatorname{DiVar}\langle X\rangle$. The denotation for concrete varieties is analogous, for example, Jord is the variety of Jordan algebras, $\operatorname{DiJord}\langle X\rangle$ is the free Jordan dialgebra.

2 SPECIAL JORDAN DIALGEBRAS

In this section $\operatorname{char}\Bbbk \neq 2$. This is necessary to define the Jordan product correctly.

2.1 Special and exceptional Jordan dialgebras

Let D be an associative dialgebra. If we define on the set D new operations

$$a_{(\vdash)} b = \frac{1}{2}(a \vdash b + b \dashv a),\ a_{(\dashv)} b = \frac{1}{2}(a \dashv b + b \vdash a) \tag{5}$$

then we obtain a new dialgebra which is denoted by $D^{(+)}$. It is easy to check that this dialgebra is Jordan [7].

A dialgebra J is called *special*, if $J \hookrightarrow D^{(+)}$ for some associative dialgebra D. Jordan dialgebras that are not special we call *exceptional*. Further, we will denote the operations in a special Jordan dialgebra through $_{(\vdash)}$ and $_{(\dashv)}$. These operations are expressed through associative operations by the formula (5).

The definition of special Jordan dialgebras has been introduced by the analogy with ordinary algebras, where a Jordan algebra J is called special, if $J \hookrightarrow A^{(+)}$ for some associative algebra A and the product in $A^{(+)}$ is given by the formula

$$a \circ b = \frac{1}{2}(ab + ba). \tag{6}$$

Let now D be an associative dialgebra. The mapping $*\colon D \to D$ is called an *involution* (involution of the second type [11]) of the dialgebra D, if $*$ is linear and

$$(a^*)^* = a, \quad (a \vdash b)^* = b^* \dashv a^*, \quad (a \dashv b)^* = b^* \vdash a^* \tag{7}$$

for all a, $b \in D$.

The set $H(D, *) = \{x \in D \mid x = x^*\}$ of symmetric elements with respect to $*$ is closed relative to operations (5). This set is a subalgebra of the algebra $D^{(+)}$. So, $H(D, *)$ is a special Jordan dialgebra.

Construct the example of an exceptional Jordan dialgebra. Prove the next

Proposition 1. *Let $(J, \circ)$ be an exceptional Jordan algebra and suppose the condition $x \circ J = 0$, $x \in J$, implies $x = 0$. Then J as a dialgebra with equal operations $x_{(\vdash)} y := x \circ y$ and $x_{(\dashv)} y := x \circ y$ is an exceptional Jordan dialgebra.*

Proof. Assume the opposite. Let $J \hookrightarrow D^{(+)}$ where $(D, \vdash, \dashv)$ is an associative dialgebra and the product in $D^{(+)}$ is given by the formula (5). Consider $I = \mathrm{Span}\{a \vdash b - a \dashv b \mid a,\, b \in D\}$ that is an ideal of D. Then $\bar{D} = D/I$ is an ordinary associative algebra and $\varphi\colon D^{(+)} \to \bar{D}^{(+)}$ is the natural homomorphism of a Jordan dialgebra on its quotient algebra. The composition of the embedding $\hookrightarrow$ and φ is a homomorphism too, we denote this homomorphism through ψ. It is clear that $K := \ker \psi$ is an ideal of J. Since ψ is a restriction φ on J so $K = \ker \psi \subseteq \ker \varphi = I$. We have $I \vdash J = J \dashv I = 0$, this is a consequence of the 0-identity. Hence $I \circ J = I_{(\vdash)} J = \frac{1}{2}(I \vdash J + J \dashv I) = 0$, from conditions of the proposition we obtain $I = 0$ therefore and $K = 0$. So ψ is an embedding and $J \hookrightarrow \bar{D}^{(+)}$, i. e., J is exceptional. □

Let $\mathbf{C}$ be the Cayley-Dickson algebra over the field $\Bbbk$, $\mathrm{char}\,\Bbbk \neq 2$. Consider an algebra $H(\mathbf{C}_3)$ of those 3×3 matrices over $\mathbf{C}$ that are symmetric relative the involution in $\mathbf{C}$. This is so called Albert algebra. It is well-known that $J = H(\mathbf{C}_3)$ is a simple exceptional Jordan algebra, so J satisfies the conditions of Proposition 1. Therefore, $H(\mathbf{C}_3)$ is an exceptional Jordan dialgebra.

2.2 Analogue of the Cohn's Theorem for dialgebras

Let $\mathrm{Alg}\,\langle X\rangle$ be a free non-associative algebra generated by X, $\mathrm{As}\,\langle X\rangle$ be a free associative algebra, $\mathrm{DiAlg}\,\langle X\rangle$ be a free non-associative dialgebra, $\mathrm{DiAs}\,\langle X\rangle$ be a free associative dialgebra

[3]. Products in $\mathrm{Alg}\langle X\rangle$ and $\mathrm{As}\langle X\rangle$, also in $\mathrm{DiAlg}\langle X\rangle$ and $\mathrm{DiAs}\langle X\rangle$ are denoted identically. There is no confusion because by the origin of elements it is clear which the product we mean. Fix $z \in X$ and introduce the following mappings.

A mapping $\mathcal{J}\colon \mathrm{Alg}\langle X\rangle \to \mathrm{As}\langle X\rangle$ is defined by linearity, on non-associative words it is defined by induction on a length: if $x \in X$ then $\mathcal{J}(x) = x$; if $uv \in \mathrm{Alg}\langle X\rangle$ then $\mathcal{J}(uv) = \frac{1}{2}(\mathcal{J}(u)\mathcal{J}(v) + \mathcal{J}(v)\mathcal{J}(u))$. So, the value of $\mathcal{J}$ on a non-associative polynomial f is equal to an associative polynomial obtained from f by means of rewriting all products in f as Jordan ones by the formula (6). By analogy, in the case of dialgebras a mapping $\mathcal{J}_{\mathrm{Di}}\colon \mathrm{DiAlg}\langle X\rangle \to \mathrm{DiAs}\langle X\rangle$ is defined. It is linear, it acts identically on $x \in X$ and $\mathcal{J}_{\mathrm{Di}}(u \vdash v) = \frac{1}{2}(\mathcal{J}_{\mathrm{Di}}(u) \vdash \mathcal{J}_{\mathrm{Di}}(v) + \mathcal{J}_{\mathrm{Di}}(v) \dashv \mathcal{J}_{\mathrm{Di}}(u))$, $\mathcal{J}_{\mathrm{Di}}(u \dashv v) = \frac{1}{2}(\mathcal{J}_{\mathrm{Di}}(u) \dashv \mathcal{J}_{\mathrm{Di}}(v) + \mathcal{J}_{\mathrm{Di}}(v) \vdash \mathcal{J}_{\mathrm{Di}}(u))$.

Introduce the following notation

$$\mathrm{Alg}_z\langle X\rangle = \{\Phi \in \mathrm{Alg}\langle X\rangle \mid \Phi = \sum f_i,\ f_i \text{ — monomials},\ \deg_z f_i = 1\},$$

$$\mathrm{DiAlg}_z\langle X\rangle = \{\Phi \in \mathrm{DiAlg}\langle X\rangle \mid \Phi = \sum f_i,\ f_i \text{ — dimonomials},\ \deg_z f_i = 1,\ c(f_i) = z\},$$

where $c(f_i)$ stands for the central letter of a dimonomial f_i. A mapping $\Psi^z_{\mathrm{Alg}}\colon \mathrm{Alg}_z\langle X\rangle \to \mathrm{DiAlg}_z\langle X\rangle$ places signs of dialgebraic operations in a non-associative polynomial in such a way that every sign points to z. By induction it can be defined as follows: $\Psi^z_{\mathrm{Alg}}(z) = z$; if z is contained by u then $\Psi^z_{\mathrm{Alg}}(uv) = \Psi^z_{\mathrm{Alg}}(u) \dashv v^{\dashv}$; if z is contained by v then $\Psi^z_{\mathrm{Alg}}(uv) = u^{\vdash} \vdash \Psi^z_{\mathrm{Alg}}(v)$. There we introduce two mappings ${}^{\vdash}, {}^{\dashv}\colon \mathrm{Alg}\langle X\rangle \to \mathrm{DiAlg}\langle X\rangle$. The first mapping maps a word u to $u^{\vdash}$ where the word $u^{\vdash}$ has the same multipliers as u and all signs of operations point to the right. In $v^{\dashv}$ all signs of operations point to the left respectively. Further in Lemmas 2 and 3 we use mappings ${}^{\vdash}, {}^{\dashv}\colon \mathrm{As}\langle X\rangle \to \mathrm{DiAs}\langle X\rangle$ which are defined and denoted in a similar way.

Analogously, we may define the sets $\mathrm{As}_z\langle X\rangle$, $\mathrm{DiAs}_z\langle X\rangle$ and a mapping $\Psi^z_{\mathrm{As}}\colon \mathrm{As}_z\langle X\rangle \to \mathrm{DiAs}_z\langle X\rangle$.

Define the following sets:

$$\mathrm{SJ}\langle X\rangle = \mathcal{J}(\mathrm{Alg}\langle X\rangle),$$
$$\mathrm{DiSJ}\langle X\rangle = \mathcal{J}_{\mathrm{Di}}(\mathrm{DiAlg}\langle X\rangle).$$

From the definition of the mapping $\mathcal{J}$ it is clear that $\mathrm{SJ}\langle X\rangle$ is a subalgebra in $\mathrm{As}\langle X\rangle^{(+)}$ generated by the set X. Similarly, $\mathrm{DiSJ}\langle X\rangle \hookrightarrow \mathrm{DiAs}\langle X\rangle^{(+)}$.

An element from $\mathrm{As}\langle X\rangle$ is called a *Jordan polynomial* if it belongs to $\mathrm{SJ}\langle X\rangle$. By analogy, an element from $\mathrm{DiAs}\langle X\rangle$ is called a *Jordan dipolynomial* if it belongs to $\mathrm{DiSJ}\langle X\rangle$.

Lemma 2. *For arbitrary $u \in \mathrm{DiAs}\langle X\rangle$, $v \in \mathrm{Alg}\langle X\rangle$ we have*

$$u \dashv \mathcal{J}(v)^{\dashv} = u \dashv \mathcal{J}_{\mathrm{Di}}(v^{\dashv}) = u \dashv \mathcal{J}_{\mathrm{Di}}(v^{\vdash}),$$

$$\mathcal{J}(v)^{\vdash} \vdash u = \mathcal{J}_{\mathrm{Di}}(v^{\vdash}) \vdash u = \mathcal{J}_{\mathrm{Di}}(v^{\dashv}) \vdash u.$$

Proof. Use an induction on the length of the word v. A base is evident. Let $v = v_1v_2$. Then

$$u \dashv \mathcal{J}(v)^{\dashv} = u \dashv \mathcal{J}(v_1v_2)^{\dashv} = \frac{1}{2}u \dashv (\mathcal{J}(v_1)^{\dashv} \dashv \mathcal{J}(v_2)^{\dashv} + \mathcal{J}(v_2)^{\dashv} \dashv \mathcal{J}(v_1)^{\dashv})$$
$$= \frac{1}{2}u \dashv (\mathcal{J}_{\mathrm{Di}}(v_1^{\dashv}) \dashv \mathcal{J}_{\mathrm{Di}}(v_2^{\dashv}) + \mathcal{J}_{\mathrm{Di}}(v_2^{\dashv}) \vdash \mathcal{J}_{\mathrm{Di}}(v_1^{\dashv})) = u \dashv \mathcal{J}_{\mathrm{Di}}(v_1^{\dashv} \dashv v_2^{\dashv}) = u \dashv \mathcal{J}_{\mathrm{Di}}(v^{\dashv}).$$

All remaining equalities are proved in the same way. □

Lemma 3. *For all $\Phi \in \mathrm{Alg}_z\langle X\rangle$ we have*

$$\Psi^z_{\mathrm{As}}(\mathcal{J}(\Phi)) = \mathcal{J}_{\mathrm{Di}}(\Psi^z_{\mathrm{Alg}}(\Phi)).$$

Proof. Since all mappings are linear, it is enough to prove the statement for the case when Φ is a word. If $\Phi = z$ then the claim is evident. If $\Phi = uv$ then z can belong to either u or v. Let z belongs to u. Then using Lemma 2 we obtain

$$\begin{aligned}\Psi^z_{\mathrm{As}}(\mathcal{J}(\Phi)) &= \Psi^z_{\mathrm{As}}(\frac{1}{2}[\mathcal{J}(u)\mathcal{J}(v) + \mathcal{J}(v)\mathcal{J}(u)]) \\ &= \frac{1}{2}[\Psi^z_{\mathrm{As}}(\mathcal{J}(u)) \dashv \mathcal{J}(v)^{\dashv} + \mathcal{J}(v)^{\vdash} \vdash \Psi^z_{\mathrm{As}}(\mathcal{J}(u))] \\ &= \frac{1}{2}[\mathcal{J}_{\mathrm{Di}}(\Psi^z_{\mathrm{Alg}}(u)) \dashv \mathcal{J}_{\mathrm{Di}}(v^{\dashv}) + \mathcal{J}_{\mathrm{Di}}(v^{\dashv}) \vdash \mathcal{J}_{\mathrm{Di}}(\Psi^z_{\mathrm{Alg}}(u))] \\ &= \mathcal{J}_{\mathrm{Di}}(\Psi^z_{\mathrm{Alg}}(u) \dashv v^{\dashv}) = \mathcal{J}_{\mathrm{Di}}(\Psi^z_{\mathrm{Alg}}(\Phi)).\end{aligned}$$

The case when z belongs to v is proved similarly. □

Remind about the quotient that has been defined in Section 1.1. This quotient compares every 0-dialgebra with an ordinary algebra. The quotient of a dialgebra generated by a set $X = \{x_i \mid i \in I\}$ is an algebra generated by the set $\bar{X} = \{\bar{x}_i \mid i \in I\}$. Further we will identify elements from X with elements from $\bar{X}$. Following this agreement we obtain, for example, $\overline{\mathrm{DiAs}\langle X\rangle} = \mathrm{As}\langle X\rangle$.

Proposition 4. *Let* $f \in \mathrm{DiAs}_z\langle X\rangle$*. Then*

$$f \in \mathrm{DiSJ}\langle X\rangle \Leftrightarrow \bar{f} \in \mathrm{SJ}\langle X\rangle.$$

Proof. "$\Rightarrow$". Let $f \in \mathrm{DiSJ}\langle X\rangle$ that is $f = \mathcal{J}_{\mathrm{Di}}(\Phi)$ for some $\Phi \in \mathrm{DiAlg}\langle X\rangle$. Then $\bar{f} = \overline{\mathcal{J}_{\mathrm{Di}}(\Phi)} = \mathcal{J}(\bar{\Phi})$, so $\bar{f} \in \mathrm{SJ}\langle X\rangle$. There we have used the equality $\overline{\mathcal{J}_{\mathrm{Di}}(\Phi)} = \mathcal{J}(\bar{\Phi})$ which is easy to prove by induction on the length of Φ.

"$\Leftarrow$". Let $\bar{f} \in \mathrm{SJ}\langle X\rangle$ that is $\bar{f} = \mathcal{J}(\Phi)$ for some $\Phi \in \mathrm{Alg}\langle X\rangle$. Since the degrees on variables do not change when we apply $\mathcal{J}$, we obtain $\Phi \in \mathrm{Alg}_z\langle X\rangle$. Thereby, $\Phi \in \mathrm{Alg}_z\langle X\rangle$. By Lemma 3 we obtain $\mathcal{J}_{\mathrm{Di}}(\Psi^z_{\mathrm{Alg}}(\Phi)) = \Psi^z_{\mathrm{As}}(\mathcal{J}(\Phi)) = \Psi^z_{\mathrm{As}}(\bar{f}) = f$, the last equality in the sequence is true because $f \in \mathrm{DiAs}_z\langle X\rangle$. So, $f \in \mathrm{DiSJ}\langle X\rangle$. □

Consider the dialgebra

$$\Lambda_X = \mathrm{DiAs}\langle X\rangle / I,$$

where I is the ideal of $\mathrm{DiAs}\langle X\rangle$ generated by the set $\{f_{x,y} = x \dashv y + y \vdash x \mid x, y \in X\}$. This dialgebra is the analogue of the exterior algebra (Grassmann algebra). Further we will identify the set X and its image $\bar{X} \subseteq \Lambda_X$. Following this agreement we suppose that Λ_X is generated by the set X.

Theorem 5. *Let X be a linearly ordered set. Then the basis of the algebra Λ_X consists of words $\dot{x}_1 x_2 \ldots x_k$, $k \geq 1$, $x_i \in X$, $x_2 < x_3 < \cdots < x_k$.*

Proof. Use the theory of Gröbner-Shirshov bases for associative dialgebras developed in [16]. Let $S_0 = \{f_{x,y} \mid x, y \in X\}$ be the initial set of defining relations. Compositions of left product $z \dashv f_{x,y}$ belong to the ideal I as well as compositions of right product $f_{x,y} \vdash z$, $x, y, z \in X$. The set of defining relations obtained

$$x \dashv y + y \vdash x; \quad x \dashv y \dashv z + x \dashv z \dashv y,\ y > z; \quad x \vdash y \vdash z + y \vdash x \vdash z,\ x > y; \quad x \vdash x \vdash y; \quad x \dashv y \dashv y$$

is a Gröbner-Shirshov basis. Reduced words are of the form

$$\dot{x}_1 x_2 \ldots x_k,\ k \geq 1,\ x_2 < x_3 < \cdots < x_k,$$

and the set of all reduced words by [16] is a linear basis of the algebra Λ_X. □

Define an involution $*$ on $\mathrm{DiAs}\langle X\rangle$ in the following way:

$$(x_{i_1}\ldots\dot{x}_{i_k}\ldots x_{i_n})^* = x_{i_n}\ldots\dot{x}_{i_k}\ldots x_{i_1},$$

and extend to dipolynomials by linearity. This mapping reverses words and signs of dialgebraic operations. It is easy to check that the mapping $*$ satisfies properties of an involution (7). Through $\mathrm{DiH}\langle X\rangle$ we denote the Jordan dialgebra $H(\mathrm{DiAs}\langle X\rangle, *)$ of symmetric elements from $\mathrm{DiAs}\langle X\rangle$ with respect to $*$ with the product (5).

Further $\{u\}$ denotes $u+u^*$ where u is a basic word from $\mathrm{DiAs}\langle X\rangle$. Note that $\{u\}=\{u^*\}$.

An analogous involution on $\mathrm{As}\langle X\rangle$ we denote by $*$ too. It acts like as

$$(x_{i_1}\ldots x_{i_k}\ldots x_{i_n})^* = x_{i_n}\ldots x_{i_k}\ldots x_{i_1},$$

on monomials and extends to polynomials by linearity.

The next theorem is an analogue of the classical Cohn's Theorem [10] that describes Jordan polynomials from ≤ 3 variables as symmetric elements of the free associative algebra.

Theorem 6. *For any set X we have $\mathrm{DiSJ}\langle X\rangle \subseteq \mathrm{DiH}\langle X\rangle$. If $|X|\leq 2$ then there is an equality, if $|X|>2$ then there is a strict inclusion.*

Proof. "$\subseteq$" follows from the equality $\mathcal{J}_{\mathrm{Di}}(\Phi)^* = \mathcal{J}_{\mathrm{Di}}(\Phi)$ which holds for all $\Phi\in\mathrm{DiAlg}\langle X\rangle$. As before, this equality can be proved by induction on the length of Φ considering cases $\Phi = u\vdash v$ and $\Phi = u\dashv v$.

Let $|X|=2$. In order to prove the equality, consider an arbitrary $f\in\mathrm{DiH}\langle x,y\rangle$, i. e., $f\in\mathrm{DiAs}\langle x,y\rangle$ and $f=f^*$. We need to show that $f\in\mathrm{DiSJ}\langle x,y\rangle$. The dipolynomial f is equal to a sum of dimonomials $f=\sum f_i$. Further, $f=\frac{1}{2}(f+f^*)=\frac{1}{2}\sum(f_i+f_i^*)$. Without loss of generality we may assume $f=a+a^*$ where a is a dimonomial. Suppose x is the central letter on a. So f can be written in a form $f=u\dot{x}v+v^*\dot{x}u^*$ where $u,\, v\in\mathrm{DiAs}\langle x,y\rangle$ or equal to empty words. Consider $g(x,y,z)=u\dot{z}v+v^*\dot{z}u^*\in\mathrm{DiAs}\langle x,y,z\rangle$. Since $\bar{g}=\bar{g}^*$ then $\bar{g}\in\mathrm{SJ}\langle x,y,z\rangle$ by the classical Cohn's Theorem. In addition, $g\in\mathrm{DiAs}_z\langle x,y,z\rangle$ hence Proposition 4 implies $g\in\mathrm{DiSJ}\langle x,y,z\rangle$. It means that there exists a dipolynomial $\Phi(x,y,z)$ such that $g=\mathcal{J}_{\mathrm{Di}}(\Phi(x,y,z))$. Substituting $z:=x$ in the last equality we obtain $f=\mathcal{J}_{\mathrm{Di}}(\Phi(x,y,x))$. Therefore, $f\in\mathrm{DiSJ}\langle x,y\rangle$. We have proved the equality for $|X|=2$ and thus for $|X|=1$.

Let $|X|>2$. In order to prove the strict inclusion consider the dipolynomial $\{y\dot{x}xz\} = y\dot{x}xz+zx\dot{x}y\in\mathrm{DiH}\langle X\rangle$ where $x,y,z\in X$. There exists a homomorphism $\sigma\colon\mathrm{DiAs}\langle x,y,z\rangle\to\mathrm{Di}\Lambda\langle x,y,z\rangle$ such that $\sigma(x)=x$, $\sigma(y)=y$, $\sigma(z)=z$. All Jordan dipolynomials of degree greater that 1 map to zero by this homomorphism. Using the basis of $\mathrm{Di}\Lambda\langle x,y,z\rangle$ from Theorem 5 we obtain

$$\sigma(\{y\dot{x}xz\}) = 2\dot{x}xyz\neq 0.$$

(When we use Theorem 5 we suppose that $x<y<z$.) So, the dipolynomial $\{y\dot{x}xz\}$ does not belong to $\mathrm{DiSJ}\langle X\rangle$. $\square$

2.3 Homomorphic images of special Jordan dialgebras

In this section we construct the example of an exceptional two-generated Jordan dialgebra which is a homomorphic image of a special Jordan dialgebra.

Theorem 7. *Consider the special Jordan dialgebra $\mathrm{DiSJ}\langle x,y\rangle$, and let I be its ideal generated by the element $k=\frac{1}{2}(\dot{x}x+x\dot{x})-\frac{1}{2}(\dot{y}y+y\dot{y})$. Then the quotient dialgebra $J=\mathrm{DiSJ}\langle x,y\rangle/I$ is exceptional.*

Proof. Denote by $\widehat{I}$ the ideal of $\mathrm{DiAs}\langle x,y\rangle$ generated by the set I.

It is evident that J is special if and only if $\widehat{I}\cap\mathrm{DiSJ}\langle x,y\rangle = I$ (this is an analogue of the Cohn's Lemma, see [10]).

Consider $f = kx\dot{x}y + y\dot{x}xk \in \widehat{I}$. Since

$$f = \{x^3\dot{x}y\} - \{y^2x\dot{x}y\} \in \mathrm{DiH}\,\langle x, y\rangle,$$

then $f \in \mathrm{DiSJ}\,\langle x, y\rangle$ by Theorem 6. It remains to show that $f \notin I$.

Assume $f \in I$. Then there exists a dipolynomial

$$\varphi(x, y, z) \in \mathrm{DiSJ}\,\langle x, y, z\rangle \subset \mathrm{DiH}\,\langle x, y, z\rangle$$

such that $\varphi(x, y, k) = f$. In addition, every summand from φ contains at most one entry of z.

Write

$$\varphi(x, y, z) = \varphi_1(x, y, z) + \varphi_2(x, y, z) + \ldots, \quad \deg_z \varphi_n = n.$$

The total degree of f (with respect to all variables) is equal to 5, hence $\varphi_n = 0$ when $n \geq 3$. Therefore $\varphi(x, y, z) = \varphi_1(x, y, z) + \varphi_2(x, y, z)$.

Suppose $\varphi_1 := \varphi_{1,0} + \varphi_{1,1} + \varphi_{1,2} + \varphi_{1,3}$, where $\deg_x \varphi_{1,0} = 0$, $\deg_x \varphi_{1,1} = 1$, $\deg_x \varphi_{1,2} = 2$, $\deg_x \varphi_{1,3} = 3$; $\varphi_2 := \varphi_{2,0} + \varphi_{2,1}$, where $\deg_x \varphi_{2,0} = 0$, $\deg_x \varphi_{2,1} = 1$.

After the substitution $z = k$ all summands in $\varphi_{1,1}$, $\varphi_{1,3}$ and $\varphi_{2,1}$ have degree 1, 3 or 5 in x. All summands from $\varphi_{1,0}$, $\varphi_{1,2}$ and $\varphi_{2,0}$ have degree 0, 2 or 4 in x. Since f contains summands of only 2-nd and 4-th degree in x, we have $\varphi_{1,1} + \varphi_{1,3} + \varphi_{2,1} = 0$.

Therefore, $\varphi = \varphi_{1,0} + \varphi_{1,2} + \varphi_{2,0}$.

Since x is the central letter of the dipolynomial f, central letters of dimonomials from φ can be variables x and z. Every dipolynomial from $\mathrm{DiH}\,\langle x, y, z\rangle$ with this property is equal to a linear combination of the next dipolynomials:

$$\begin{gathered}
\{\dot{x}yxz\},\ \{xy\dot{x}z\},\ \{xyx\dot{z}\}, \quad \{y\dot{x}xz\},\ \{yx\dot{x}z\},\ \{yxx\dot{z}\},\\
\{\dot{x}xyz\},\ \{x\dot{x}yz\},\ \{xxy\dot{z}\}, \quad \{\dot{x}yzx\},\ \{xyz\dot{x}\},\ \{xy\dot{z}x\},\\
\{yz\dot{x}x\},\ \{yzx\dot{x}\},\ \{y\dot{z}xx\}, \quad \{y\dot{x}zx\},\ \{yxz\dot{x}\},\ \{yx\dot{z}x\},\\
\{\dot{z}yyy\},\ \{y\dot{z}yy\}, \quad \{\dot{z}zy\},\ \{z\dot{z}y\}, \{\dot{z}yz\}.
\end{gathered}$$

Consequently $\varphi(x, y, z)$ has the form

$$\begin{gathered}
\alpha_1\{\dot{x}yxz\} + \alpha_2\{y\dot{x}xz\} + \alpha_3\{\dot{x}xyz\} + \alpha_4\{\dot{x}yzx\} + \alpha_5\{yz\dot{x}x\} + \alpha_6\{y\dot{x}zx\}\\
+\beta_1\{xy\dot{x}z\} + \beta_2\{yx\dot{x}z\} + \beta_3\{x\dot{x}yz\} + \beta_4\{xyz\dot{x}\} + \beta_5\{yzx\dot{x}\} + \beta_6\{yxz\dot{x}\}\\
+2\gamma_1\{xyx\dot{z}\} + 2\gamma_2\{yxx\dot{z}\} + 2\gamma_3\{xxy\dot{z}\} + 2\gamma_4\{xy\dot{z}x\} + 2\gamma_5\{y\dot{z}xx\} + 2\gamma_6\{yx\dot{z}x\}\\
+2\delta_1\{\dot{z}yyy\} + 2\delta_2\{y\dot{z}yy\} + 2\delta_3\{\dot{z}zy\} + 2\delta_4\{z\dot{z}y\} + 2\delta_5\{\dot{z}yz\}.
\end{gathered}$$

Substituting $z = k$ and using the equalities

$$\begin{aligned}
2\dot{z}z &= (\dot{x}x + x\dot{x} - \dot{y}y - y\dot{y}) \dashv (xx - yy)\\
&= \dot{x}x^3 + x\dot{x}x^2 - \dot{y}yx^2 - y\dot{y}x^2 - \dot{x}xy^2 - x\dot{x}y^2 + \dot{y}y^3 + y\dot{y}y^2,\\
2z\dot{z} &= (xx - yy) \vdash (\dot{x}x + x\dot{x} - \dot{y}y - y\dot{y})\\
&= x^2\dot{x}x + x^3\dot{x} - x^2\dot{y}y - x^2y\dot{y} - y^2\dot{x}x - y^2x\dot{x} + y^2\dot{y}y + y^3\dot{y},
\end{aligned}$$

we obtain $\varphi(x, y, k)$ is equal to

$$\begin{gathered}
\alpha_1\{\dot{x}yx^3\} + \alpha_2\{y\dot{x}x^3\} + \alpha_3\{\dot{x}xyx^2\} + \alpha_4\{\dot{x}yx^3\} + \alpha_5\{yx^2\dot{x}x\} + \alpha_6\{y\dot{x}x^3\}\\
-\alpha_1\{\dot{x}yxy^2\} - \alpha_2\{y\dot{x}xy^2\} - \alpha_3\{\dot{x}xy^3\} - \alpha_4\{\dot{x}y^3x\} - \alpha_5\{y^3\dot{x}x\} - \alpha_6\{y\dot{x}y^2x\}\\
+\beta_1\{xy\dot{x}x^2\} + \beta_2\{yx\dot{x}x^2\} + \beta_3\{x\dot{x}yx^2\} + \beta_4\{xyx^2\dot{x}\} + \beta_5\{yx^3\dot{x}\} + \beta_6\{yx^3\dot{x}\}\\
-\beta_1\{xy\dot{x}y^2\} - \beta_2\{yx\dot{x}y^2\} - \beta_3\{x\dot{x}y^3\} - \beta_4\{xy^3\dot{x}\} - \beta_5\{y^3x\dot{x}\} - \beta_6\{yxy^2\dot{x}\}\\
+\gamma_1\{xyx\dot{x}x\} + \gamma_2\{yx^2\dot{x}x\} + \gamma_3\{x^2y\dot{x}x\} + \gamma_4\{xy\dot{x}x^2\} + \gamma_5\{y\dot{x}x^3\} + \gamma_6\{yx\dot{x}x^2\}\\
+\gamma_1\{xyx^2\dot{x}\} + \gamma_2\{yx^3\dot{x}\} + \gamma_3\{x^2yx\dot{x}\} + \gamma_4\{xyx\dot{x}x\} + \gamma_5\{yx\dot{x}x^2\} + \gamma_6\{yx^2\dot{x}x\}\\
-\gamma_1\{xyx\dot{y}y\} - \gamma_2\{yx^2\dot{y}y\} - \gamma_3\{x^2y\dot{y}y\} - \gamma_4\{xy\dot{y}yx\} - \gamma_5\{y\dot{y}yx^2\} - \gamma_6\{yx\dot{y}yx\}\\
-\gamma_1\{xyxy\dot{y}\} - \gamma_2\{yx^2y\dot{y}\} - \gamma_3\{x^2y^2\dot{y}\} - \gamma_4\{xy^2\dot{y}x\} - \gamma_5\{y^2\dot{y}x^2\} - \gamma_6\{yxy\dot{y}x\}
\end{gathered}$$

$$+\delta_1\{\dot{x}xy^3\} + \delta_1\{x\dot{x}y^3\} - \delta_1\{\dot{y}y^4\} - \delta_1\{y\dot{y}y^3\}$$
$$+\delta_2\{y\dot{x}xy^2\} + \delta_2\{yx\dot{x}y^2\} - \delta_2\{y\dot{y}y^3\} - \delta_2\{y^2\dot{y}y^2\}$$
$$+\delta_3\{\dot{x}x^3y\} + \delta_3\{x\dot{x}x^2y\} - \delta_3\{\dot{y}yx^2y\} - \delta_3\{y\dot{y}x^2y\}$$
$$-\delta_3\{\dot{x}xy^3\} - \delta_3\{x\dot{x}y^3\} + \delta_3\{\dot{y}y^4\} + \delta_3\{y\dot{y}y^3\}$$
$$+\delta_4\{x^2\dot{x}xy\} + \delta_4\{x^3\dot{x}y\} - \delta_4\{x^2\dot{y}y^2\} - \delta_4\{x^2y\dot{y}y\}$$
$$-\delta_4\{y^2\dot{x}xy\} - \delta_4\{y^2x\dot{x}y\} + \delta_4\{y^2\dot{y}y^2\} + \delta_4\{y^3\dot{y}y\}$$
$$+\delta_5\{\dot{x}xyx^2\} + \delta_5\{x\dot{x}yx^2\} - \delta_5\{\dot{y}y^2x^2\} - \delta_5\{y\dot{y}yx^2\}$$
$$-\delta_5\{\dot{x}xy^3\} - \delta_5\{x\dot{x}y^3\} + \delta_5\{\dot{y}y^4\} + \delta_5\{y\dot{y}y^3\}.$$

This expression must coincide with $f = \{x^3\dot{x}y\} - \{y^2x\dot{x}y\}$. In particular, a sum of all dimonomials with the central letter y must be equal to zero:

$$0 = \gamma_1\{y\dot{y}xyx\} + (\gamma_2 + \delta_3)\{y\dot{y}x^2y\} + (\gamma_3 + \gamma_5 + \delta_4 + \delta_5)\{y\dot{y}yx^2\}$$
$$+\gamma_4\{xy\dot{y}yx\} + \gamma_6\{xy\dot{y}xy\} + \gamma_1\{\dot{y}yxyx\} + (\gamma_2 + \delta_3)\{\dot{y}yx^2y\} + (\gamma_3 + \delta_5)\{\dot{y}y^2x^2\}$$
$$+\gamma_4\{x\dot{y}y^2x\} + (\gamma_5 + \delta_4)\{x^2\dot{y}y^2\} + \gamma_6\{x\dot{y}yxy\} + (\delta_1 - \delta_3 - \delta_5)\{\dot{y}y^4\}$$
$$+(\delta_1 + \delta_2 - \delta_3 - \delta_4 - \delta_5)\{y\dot{y}y^3\} + (\delta_2 - \delta_4)\{y^2\dot{y}y^2\}.$$

All coefficients in this sum have to be zero. Solving the obtained system we have $\gamma_2 = -\delta_3$, $\gamma_3 = -\delta_5$, $\gamma_5 = -\delta_4$, $\delta_1 = \delta_3 + \delta_5$, $\delta_2 = \delta_4$, $\gamma_1 = \gamma_4 = \gamma_6 = 0$.

Substitute the obtained relations to $\varphi(x, y, k)$ we get that all summands with coefficients γ and δ are eliminated.

Further, consider the remaining summands (we divide them into two groups by $\deg_y$):

$$(\alpha_1 + \alpha_4)\{\dot{x}yx^3\} + (\alpha_2 + \alpha_6)\{y\dot{x}x^3\} + \alpha_3\{\dot{x}xyx^2\} + \alpha_5\{yx^2\dot{x}x\}$$
$$+\beta_1\{xy\dot{x}x^2\} + \beta_2\{yx\dot{x}x^2\} + \beta_3\{x\dot{x}yx^2\} + \beta_4\{xyx^2\dot{x}\} + (\beta_5 + \beta_6)\{yx^3\dot{x}\}$$
$$= \{x^3\dot{x}y\},$$

$$\alpha_1\{\dot{x}yxy^2\} + \alpha_2\{y\dot{x}xy^2\} + \alpha_3\{\dot{x}xy^3\} + \alpha_5\{y^3\dot{x}x\} + \alpha_6\{y\dot{x}y^2x\}$$
$$+\beta_1\{xy\dot{x}y^2\} + \beta_2\{yx\dot{x}y^2\} + \beta_3\{x\dot{x}y^3\} + (\alpha_4 + \beta_4)\{\dot{x}y^3x\} + \beta_5\{y^3x\dot{x}\} + \beta_6\{yxy^2\dot{x}\}$$
$$= \{y^2x\dot{x}y\}.$$

The last two equalities imply $\alpha_2 = 1$ and other coefficients are equal to zero. Therefore,

$$\varphi(x, y, z) = \{y\dot{x}xz\} - 2\delta_3\{yxx\dot{z}\} - 2\delta_5\{xxy\dot{z}\} - 2\delta_4\{y\dot{z}xx\}$$
$$+2(\delta_3 + \delta_5)\{\dot{z}yyy\} + 2\delta_4\{y\dot{z}yy\} + 2\delta_3\{\dot{z}zy\} + 2\delta_4\{z\dot{z}y\} + 2\delta_5\{\dot{z}yz\}.$$

By assumption this dipolynomial is Jordan. When we expand Jordan products then the central letter is preserved, hence the dipolynomials consisting of dimonomials from $\varphi(x, y, z)$ with the fixed central letter must be Jordan. In particular, if we choose the central letter x then the dipolynomial $\{y\dot{x}xz\}$ must be Jordan, but this is not true by the proof of Theorem 6.

The contradiction obtained proves that $f \notin I$. □

3 S-IDENTITIES

In this section $\operatorname{char} \Bbbk = 0$, so we can perform the process of full linearization of identities and varieties of algebras are always defined by polylinear identities.

3.1 Equality of varieties $\mathcal{H}$DiSJ and Di$\mathcal{H}$SJ

Consider a class of special Jordan dialgebras SJ. The class SJ is not a variety because it is not close relative the taking of homomorphic images. Consider the operator $\mathcal{H}$ acting on classes of algebraic systems

$$\mathcal{H}(K) = \{A \mid A = \varphi(B) \text{ for } B \in K, \varphi\colon B \to A \text{ is a homomorphism}\}.$$

It is well-known that $\mathcal{H}(\mathrm{SJ})$ is a variety of algebras which we denote $\mathcal{H}$SJ.

Remind (see Section 1.1) that if $D \in \mathrm{DiAlg0}$ then D can be endowed with left and right actions of the algebra $\bar{D}$ by the rules $\bar{x}y = x \vdash y$, $y\bar{x} = y \dashv x$, where x, $y \in D$. Let Var be a variety of ordinary algebras. In the paper [11] it is shown that $D \in \mathrm{DiVar}$ if and only if $\bar{D} \in \mathrm{Var}$ and D is a Var-bimodule over $\bar{D}$ in the sense of Eilenberg, i. e., the split null extension $\widehat{D} = \bar{D} \oplus D$ belongs to the variety Var.

In this way we can define a variety of dialgebras Di$\mathcal{H}$SJ by a variety $\mathcal{H}$SJ.

Let DiSJ be the class of special Jordan dialgebras. Consider the closure $\mathcal{H}(\mathrm{DiSJ})$ of this class relative to the operator $\mathcal{H}$. The variety obtained we denote by $\mathcal{H}$DiSJ.

The purpose of this section is to show that $\mathcal{H}\mathrm{DiSJ} = \mathrm{Di}\mathcal{H}\mathrm{SJ}$.

Lemma 8. $\mathrm{DiSJ}\,\langle X\rangle$ *is a free algebra in the variety* $\mathcal{H}$DiSJ.

Proof. Let $J' \in \mathcal{H}\mathrm{DiSJ}$ be a homomorphic image of $J \in \mathrm{DiSJ}$, D be an associative dialgebra such that $J \hookrightarrow D^{(+)}$. We have the following commutative diagram

$$\begin{array}{ccccc}
J' & \longleftarrow & J & \xrightarrow{\subseteq} & D \\
\uparrow & & \uparrow & & \uparrow \\
X & \xrightarrow{\subseteq} & \mathrm{DiSJ}\langle X\rangle & \xrightarrow{\subseteq} & \mathrm{DiAs}\langle X\rangle
\end{array}$$

By the universal property of $\mathrm{DiAs}\langle X\rangle$ there exists an unique homomorphism $\mathrm{DiAs}\langle X\rangle \to D$ such that its restriction to $\mathrm{DiSJ}\langle X\rangle$ is the homomorphism $\mathrm{DiSJ}\langle X\rangle \to J$. The last homomorphism in a composition with the mapping $J \to J'$ gives the required homomorphism $\mathrm{DiSJ}\langle X\rangle \to J'$. ☐

A *bar-unit* of a 0-dialgebra D is an element $e \in D$ such that $x \dashv e = e \vdash x = x$ for every $x \in D$ and e belongs to the associative center of D that is

$$(x, e, y)_\times = (e, x, y)_\dashv = (x, y, e)_\vdash = 0$$

for all x, $y \in D$.

Proposition 9 (Pozhidaev [11, Theorem 2.2])**.** *For every associative dialgebra D there exists an associative dialgebra D_e with the bar-unit e such that $D \hookrightarrow D_e$.*

Lemma 10. *Let J be a special Jordan dialgebra. Then there exists a special Jordan dialgebra J_e such that $J \hookrightarrow J_e$ and $\bar{e}$ is a unit in the algebra $\bar{J}_e$.*

Proof. By the defintion of a special Jordan dialgebra it follows that $J = (J, _{(\vdash)}, _{(\dashv)})$ is embedded into $D^{(+)}$ for some associative dialgebra $D = (D, \vdash, \dashv)$. By Proposition 9 we have an embedding $D^{(+)} \hookrightarrow D_e^{(+)}$ where e is a bar-unit in D_e. Therefore, $J_e = D_e^{(+)}$ is the required dialgebra.

Further, $e \vdash x = x \dashv e = x$ holds for every $x \in D_e$, so in J_e we have $e_{(\vdash)}\, x = \frac{1}{2}(e \vdash x + x \dashv e) = x$, $x\,_{(\dashv)}\, e = x$. Hence $\bar{e}\bar{x} = \bar{x}\bar{e} = \bar{x}$ in the quotient algebra $\bar{J}_e$, so $\bar{e}$ is a unit in $\bar{J}_e$. ☐

Lemma 11. *Let J be a special Jordan dialgebra and such that $\bar{J}$ contains a unit. Then $\bar{J}$ is special.*

Proof. Let D be an associative dialgebra such that $J \hookrightarrow D^{(+)}$. Denote $\langle D, D\rangle := \mathrm{Span}\{a \vdash b - a \dashv b \mid a, b \in D\}$, $[J, J] := \mathrm{Span}\{a \,{}_{(\vdash)}\, b - a \,{}_{(\dashv)}\, b \mid a, b \in J\}$. As before $\bar{D} = D/\langle D, D\rangle$ is an associative algebra. Since $J \subseteq D$ we have $[J, J] \subseteq \langle D, D\rangle$. Then the homomorphism $\varphi\colon \bar{J} \to \bar{D}^{(+)}$ is well-defined by the rule $x + [J, J] \mapsto x + \langle D, D\rangle$.

$$\begin{array}{ccc} J & \xrightarrow{\ \subseteq\ } & D \\ \downarrow & & \downarrow \\ \bar{J} & \xrightarrow{\ \varphi\ } & \bar{D} \end{array}$$

It is evident that φ is injective if and only if $\langle D, D\rangle \cap J = [J, J]$.

Let $x \in \langle D, D\rangle \cap J$. Then $x \vdash y = y \dashv x = 0$ for every $y \in D$, hence $x \,{}_{(\vdash)}\, y = \frac{1}{2}(x \vdash y + y \dashv x) = 0$ in J and $\bar{x}\bar{y} = \bar{0}$ in $\bar{J}$. Take $\bar{y} = 1 \in \bar{J}$ and obtain $\bar{x} = \bar{0}$, i. e., $x \in [J, J]$. So, φ is injective and $\bar{J}$ is special. □

Let J be a Jordan algebra, A be an associative algebra with a unit, then a homomorphism from J to $A^{(+)}$ is called an *associative specialization* $\sigma\colon J \to A$. This is a linear mapping such that

$$\sigma(ab) = \frac{1}{2}(\sigma(a)\sigma(b) + \sigma(b)\sigma(a))$$

for all $a, b \in J$.

Two associative specializations are called *commuting* if $[\sigma_1(a), \sigma_2(b)] = 0$ for all $a, b \in J$.

A bimodule M over J is *special* if there exists an embedding of M into a bimodule N such that if $v \in N$, $a \in J$ then

$$a \cdot v = \frac{1}{2}(\sigma_1(a)v + \sigma_2(a)v), \tag{8}$$

where σ_1, σ_2 are commuting associative specializations of J into $\mathrm{Hom}(N, N)$.

Theorem 12 (Jacobson [17, theorem II.17])**.** *Let J be a special Jordan algebra, M be a bimodule over J. Then the bimodule M is special if and only if the split null extension $J \oplus M$ is a special Jordan algebra.*

Lemma 13. *Let J be a special Jordan dialgebra and $\bar{J}$ be a special Jordan algebra. Then $\widehat{J} = \bar{J} \oplus J$ is special too.*

Proof. Since $J = (J, {}_{(\vdash)}, {}_{(\dashv)})$ is special, we have $J \hookrightarrow D^{(+)}$ where $D = (D, \vdash, \dashv)$ is an associative dialgebra. The dialgebra J is a $\bar{J}$-bimodule: $\bar{a} \cdot v = a \,{}_{(\vdash)}\, v = v \,{}_{(\dashv)}\, a = v \cdot \bar{a}$, where $\bar{a} \in \bar{J}$, $v \in J$.

Prove that the bimodule J over the special Jordan algebra $\bar{J}$ is special. The bimodule J is embedded into D and D is a $\bar{J}$-bimodule too. Consider mappings $\sigma_1, \sigma_2\colon \bar{J} \to \mathrm{Hom}(D, D)$ defined by the rule

$$\sigma_1(\bar{a})\colon d \mapsto a \vdash d \in D,\ \sigma_2(\bar{a})\colon d \mapsto d \dashv a \in D, \quad d \in D,\ a \in J \subseteq D.$$

These mappings are defined correctly. Show that they are associative specializations. Indeed for every $\bar{a}$, $\bar{b} \in \bar{J}$, $d \in D$

$$\sigma_1(\bar{a}\bar{b})(d) = \sigma_1(\overline{a \,{}_{(\vdash)}\, b})(d) = \frac{1}{2}(a \vdash b + b \dashv a) \vdash d =$$
$$\frac{1}{2}(b \vdash a \vdash d + a \vdash b \vdash d) = \frac{1}{2}(\sigma_1(\bar{a})\sigma_1(\bar{b}) + \sigma_1(\bar{b})\sigma_1(\bar{a}))(d).$$

(We write a composition of mappings as $fg(x) = g(f(x))$.) Analogously, one may check that σ_2 is an associative specialization.

The relation (8) follows from the definition of the operation in our bimodule.

Moreover, σ_1 and σ_2 are commuting because

$$[\sigma_1(\bar{a}), \sigma_2(\bar{b})](d) = (\sigma_1(\bar{a})\sigma_2(\bar{b}) + \sigma_2(\bar{b})\sigma_1(\bar{a}))(d) = (a \vdash d) \dashv b - a \vdash (d \dashv b) = 0.$$

So, J is a special $\bar{J}$-bimodule and by Theorem 12 we obtain that $\widehat{J}$ is special. □

In papers [5, 15] conformal algebras were investigated and the following fact was proved.

Proposition 14. *If an algebra A belongs to a variety* Var *then a dialgebra* $(\mathrm{Cur}\, A)^{(0)}$ *belongs to a variety* DiVar.

Prove an auxiliary statement.

Lemma 15. *If* $\widehat{J} \in \mathcal{H}\mathrm{SJ}$ *then* $J \in \mathcal{H}\mathrm{DiSJ}$.

Proof. Use conformal algebras. Let the algebra $\widehat{J}$ generated by a set X belongs to the variety $\mathcal{H}\mathrm{SJ}$. Since $\mathrm{SJ}\langle X\rangle$ is a free algebra of the variety $\mathcal{H}\mathrm{SJ}$, there exists a surjective homomorphism $\varphi\colon \mathrm{SJ}\langle X\rangle \to \widehat{J}$. Then $\mathrm{Cur}\,\varphi\colon \mathrm{Cur}\,\mathrm{SJ}\langle X\rangle \to \mathrm{Cur}\,\widehat{J}$ is a morphism of conformal algebras and particulary dialgebras. It is known [13] that $J \hookrightarrow (\mathrm{Cur}\,\widehat{J})^{(0)}$. So $(\mathrm{Cur}\,\varphi)^{-1}[J]$ is a subdialgebra in $(\mathrm{Cur}\,\mathrm{SJ}\langle X\rangle)^{(0)}$. The algebra $\mathrm{SJ}\langle X\rangle \in \mathrm{SJ}$ so by the definition of SJ there exists an associative algebra A such that $\mathrm{SJ}\langle X\rangle \hookrightarrow A^{(+)}$, hence $\mathrm{Cur}\,\mathrm{SJ}\langle X\rangle \hookrightarrow \mathrm{Cur}\,A^{(+)}$ and $(\mathrm{Cur}\,\mathrm{SJ}\langle X\rangle)^{(0)} \in \mathrm{DiSJ}$. To complete the proof we need to note that $J = \mathrm{Cur}\,\varphi((\mathrm{Cur}\,\varphi)^{-1}[J])$, where $(\mathrm{Cur}\,\varphi)^{-1}[J] \hookrightarrow (\mathrm{Cur}\,\mathrm{SJ}\langle X\rangle)^{0} \in \mathrm{DiSJ}$ and so $J \in \mathcal{H}\mathrm{DiSJ}$. □

Now we can prove the following theorem.

Theorem 16. $\mathcal{H}\mathrm{DiSJ} = \mathrm{Di}\mathcal{H}\mathrm{SJ}$.

Proof. To prove the inclusion "$\subseteq$" consider a free algebra $\mathrm{DiSJ}\langle X\rangle$ in the variety $\mathcal{H}\mathrm{DiSJ}$. By Lemma 10 we have $\mathrm{DiSJ}\langle X\rangle \hookrightarrow J_e$, J_e is special and $1 \in \bar{J}_e$. Then by Lemma 11 $\bar{J}_e$ is special, hence by Lemma 13 $\widehat{J}_e$ is a special Jordan algebra and $J_e \in \mathrm{Di}\mathcal{H}\mathrm{SJ}$. Therefore, $\mathrm{DiSJ}\langle X\rangle \in \mathrm{Di}\mathcal{H}\mathrm{SJ}$. Since the free algebra of the variety $\mathcal{H}\mathrm{DiSJ}$ belongs to the variety $\mathrm{Di}\mathcal{H}\mathrm{SJ}$, the variety $\mathcal{H}\mathrm{DiSJ}$ is embedded into $\mathrm{Di}\mathcal{H}\mathrm{SJ}$.

Prove the inclusion "$\supseteq$". Let $J \in \mathrm{Di}\mathcal{H}\mathrm{SJ}$. By the definition of a variety of dialgebras in the sense of Eilenberg it means that $\widehat{J} \in \mathcal{H}\mathrm{SJ}$, hence by Lemma 15 we obtain $J \in \mathcal{H}\mathrm{DiSJ}$. □

3.2 s-identities in dialgebras

Let Var be a variety of algebras, $X = \{x_1, x_2, \ldots\}$ be a countable set. Consider a mapping $\varphi_{\mathrm{Var}}\colon \mathrm{Alg}\langle X\rangle \to \mathrm{Var}\langle X\rangle$ which maps $x_i \mapsto x_i$. Let $T_0(\mathrm{Var})$ be a set of polylinear polynomials from $\ker\varphi_{\mathrm{Var}}$, these are exactly all polylinear identities of Var. We suppose that the variety is defined by polylinear identities that is $\mathrm{Var} = \{A \mid A \vDash T_0(\mathrm{Var})\}$. There we use the denotation $A \vDash f$ which means that the identity $f(x_1, \ldots, x_n) = 0$ holds on the algebra A.

Further, let $\mathrm{DiAlg0}\langle X\rangle$ be a free 0-dialgebra, $\varphi_{\mathrm{DiVar}}\colon \mathrm{DiAlg0}\langle X\rangle \to \mathrm{DiVar}\langle X\rangle$, $T_0(\mathrm{DiVar})$ be a set of polylinear dipolynomials from $\ker\varphi_{\mathrm{DiVar}}$, i. e., all polylinear identities from DiVar.

In paper [11] the following theorem was proved.

Theorem 17 (Pozhidaev [11, Theorem 3.2])**.** *Let* $D \in \mathrm{DiAlg0}$. *Then the following conditions are equivalent:*

1. $D \in \mathrm{DiVar}$;
2. $\widehat{D} = \bar{D} \oplus D \in \mathrm{Var}$ (*the definition in the sense of Eilenberg*);
3. $D \vDash \Psi^{x_i}_{\mathrm{Alg}} f$ *for every* $f \in T_0(\mathrm{Var})$, $\deg f = n$, $i = 1, \ldots, n$ (*the definition in the sense of* [5]).

Prove the following

Proposition 18. *Let* $f = f(x_1, \ldots, x_n) \in \mathrm{DiAlg0}\langle X\rangle$ *be polylinear,* $f = \Psi^{x_j}_{\mathrm{Alg}} \bar{f}$ *for some* j. *Then*

$$f \in T_0(\mathrm{DiVar}) \Leftrightarrow \bar{f} \in T_0(\mathrm{Var}).$$

Proof. Since evidently $\mathrm{Var} \subseteq \mathrm{DiVar}$, the statement "$\Rightarrow$" is trivial.

To prove "$\Leftarrow$" consider an identity $\bar{f} \in T_0(\mathrm{Var})$. By Theorem 17 for arbitrary $D \in \mathrm{DiVar}$ we have $D \models \Psi^{x_i}_{\mathrm{Alg}}\, \bar{f}$ for all $i = 1, \ldots, n$, but $\Psi^{x_j}_{\mathrm{Alg}}\, \bar{f} = f$ and so $f \in T_0(\mathrm{DiVar})$. □

Proposition 19. *Let* $f = f(x_1, \ldots, x_n) \in \mathrm{DiAlg0}\langle X \rangle$ *be polylinear,* $f = f_1 + \ldots + f_n$ *where* f_i *consists of all dimonomials in* f *with a central letter* x_i*. Then*

$$f \in T_0(\mathrm{DiVar}) \Leftrightarrow f_i \in T_0(\mathrm{DiVar}) \textit{ for all } i = 1, \ldots, n.$$

Proof. "$\Leftarrow$" is evident.

Prove "$\Rightarrow$". Let $f \in T_0(\mathrm{DiVar})$, consider an arbitrary algebra $A \in \mathrm{Var}$. Then by Proposition 14 we obtain $(\mathrm{Cur}\, A)^{(0)} \in \mathrm{DiVar}$, hence $(\mathrm{Cur}\, A)^{(0)} \models f$, where $f = f(x_1, \ldots, x_n)$. Fix $i = 1, \ldots, n$ and assign the following values to variables: $x_i := Ta_i$, $a_i \in A$, $x_j := a_j$ for all $j \neq i$, $a_j \in A$. The properties of a conformal product imply

$$0 = f(a_1, \ldots, Ta_i, \ldots, a_n) = T\bar{f}_i(a_1, \ldots, a_n).$$

From the last equality we obtain $\bar{f}_i(a_1, \ldots, a_n) = 0$, so $A \models \bar{f}_i$ and $\bar{f}_i \in T_0(\mathrm{Var})$. By the previous proposition $f_i \in T_0(\mathrm{DiVar})$. □

Remind that f is called a polylinear s-identity (in the case of ordinary algebras) if

$$f \in T_0(\mathcal{H}\mathrm{SJ}) \setminus T_0(\mathrm{Jord}) := \mathrm{SId}.$$

A similar notion can be introduced for dialgebras [8]

$$f \in T_0(\mathcal{H}\mathrm{DiSJ}) \setminus T_0(\mathrm{DiJord}) := \mathrm{DiSId}.$$

Theorem 20 (about the correspondence of polylinear s-identities)**.**

1. *Let* $g = g(x_1, \ldots, x_n) \in \mathrm{SId}$*. Then* $\Psi^{x_i}_{\mathrm{Alg}}\, g \in \mathrm{DiSId}$ *for all* $i = 1, \ldots, n$.
2. *Let* $f = f(x_1, \ldots, x_n) \in \mathrm{DiSId}$*,* $f = f_1 + \ldots + f_n$ *(by a central letter). Then there exists* $j \in \{1, \ldots, n\}$ *such that* $\bar{f}_j \in \mathrm{SId}$.

Proof. Prove the statement 1. Let $g \in \mathrm{SId}$, hence by the definition SId we have $g \in T_0(\mathcal{H}\mathrm{SJ})$ and $g \notin T_0(\mathrm{Jord})$. Proposition 18 implies $\Psi^{x_i}_{\mathrm{Alg}}\, g \in T_0(\mathrm{Di}\mathcal{H}\mathrm{SJ})$, $\Psi^{x_i}_{\mathrm{Alg}}\, g \notin T_0(\mathrm{DiJord})$. It follows from the equality of varieties $\mathcal{H}\mathrm{DiSJ} = \mathrm{Di}\mathcal{H}\mathrm{SJ}$ that $\Psi^{x_i}_{\mathrm{Alg}}\, g \in \mathrm{DiSId}$.

For proving the statement 2 consider $f \in \mathrm{DiSId}$. By the definition of DiSId and Theorem 16 we have $f \in T_0(\mathcal{H}\mathrm{DiSJ}) = T_0(\mathrm{Di}\mathcal{H}\mathrm{SJ})$ and $f \notin T_0(\mathrm{DiJord})$. It follows from $f \in T_0(\mathrm{Di}\mathcal{H}\mathrm{SJ})$ by Proposition 19 that $f_i \in T_0(\mathrm{Di}\mathcal{H}\mathrm{SJ})$ for all i. It follows from $f \notin T_0(\mathrm{DiJord})$ that j exists such that $f_j \notin T_0(\mathrm{DiJord})$. Further, by Proposition 18, $\bar{f}_i \in T_0(\mathcal{H}\mathrm{SJ})$ and $\bar{f}_j \notin T_0(\mathrm{DiJord})$, hence by the definition SId we obtain $\bar{f}_j \in \mathrm{SId}$. □

Now we can easily prove the following corollary which was proved in [8] by computer algebra methods.

Corollary 21. *There are no s-identities for Jordan dialgebras of degree* ≤ 7 *and there exists a polylinear s-identity of a degree 8.*

Proof. Let f be a s-identity for Jordan dialgebras, $\deg f = k \leq 7$. After a full linearization of f we can suppose that f is polylinear that is $f \in \mathrm{DiSId}$ and $f = f_1 + \ldots + f_k$ by central letters. It follows from Theorem 20 about the corresponding of polylinear s-identities that $\bar{f}_i \in \mathrm{SId}$ for some i, $\deg \bar{f}_i \leq k$, but Glennie proved [18] that such an identity does not exist.

It is known [19] that there exists f which is a s-identity for Jordan algebras, $\deg f = 8$. Again we can suppose that f is polylinear. Then Theorem 20 implies $\Psi^{x_i}_{\mathrm{Alg}} f$ is a required polylinear s-identity for all $i = 1, \ldots, 8$. □

3.3 Analogues for dialgebras of Shirshov's and Macdonald's Theorems

After the generalization of the Cohn's Theorem to the case of dialgebras a question appears about a generalization of the Shirshov's Theorem for special Jordan algebras, notably is it true that every Jordan dialgebra with two generators is special? The answer to this questions is negative, it follows from Theorem 7. However, the following analogue of the Shirshov's Theorem holds for dialgebras.

Theorem 22. *Let J be a one-generated dialgebra. Then J is special.*

Proof. We have $J \in \mathrm{DiJord}$. Then by the definition a variety of dialgebras in the sense of Eilenberg $\bar{J} \in \mathrm{Jord}$, $\widehat{J} = \bar{J} \oplus J \in \mathrm{Jord}$. Let x be the generative element of J. Then $\widehat{J} = \langle \bar{x}, x \rangle$, so $\widehat{J}$ is a two-generated Jordan algebra. By the Shirshov's Theorem we obtain that $\widehat{J}$ is special. We have $J \hookrightarrow (\mathrm{Cur}\,\widehat{J})^{(0)}$ and so J is special too. □

Consider the particular case when two-generated dialgebra is free.

Theorem 23. *Let $J = \mathrm{DiJord}\,\langle x, y \rangle$ be the free Jordan dialgebra generated by x, y. Then J is special.*

Proof. We need to show that $J \in \mathrm{DiSJ}$. First, prove $J \in \mathcal{H}\mathrm{DiSJ}$. Assume the converse, i. e., $J \notin \mathcal{H}\mathrm{DiSJ}$. By Lemma 15 we obtain $\widehat{J} = \bar{J} \oplus J \notin \mathcal{H}\mathrm{SJ}$. Since $\widehat{J} \in \mathrm{Jord} \setminus \mathcal{H}\mathrm{SJ}$, there exists a polylinear s-identity $f(x_1, \ldots, x_n)$ of Jordan algebras such that $\mathrm{SJ} \models f$ but $\widehat{J} \not\models f$. Therefore, we may find $u_1, \ldots, u_n \in \widehat{J}$ such that $f(u_1, \ldots, u_n) \neq 0$. Since the polynomial f is polylinear, we can suppose that either $u_i \in \bar{J}$ or $u_i \in J$ for all i. A quantity of elements $u_i \in J$ does not exceed one otherwise, $f(u_1, \ldots, u_n) = 0$ because $J \cdot J = 0$. Consider two possible cases. The first case is when all $u_i \in \bar{J}$. Then $\bar{J} \not\models f$, which is impossible since $\bar{J} \in \mathrm{SJ}$ and f is an s-identity. The second case is when $u_1, \ldots, u_{n-1} \in \bar{J}$, $u_n \in J$. The algebra $\bar{J}$ is generated by $\bar{x}$ and $\bar{y}$, so $u_i = u_i(\bar{x}, \bar{y})$, $i = 1, \ldots, n$. Then denote $g(\bar{x}, \bar{y}, u_n) := f(u_1(\bar{x}, \bar{y}), \ldots, u_{n-1}(\bar{x}, \bar{y}), u_n) \neq 0$. Note that g does not hold on $\widehat{J}$. The polynomial $g(x, y, z)$ vanishes in SJ, $\deg_z g = 1$, hence by the Macdonald's Theorem we obtain $g = 0$ in Jord. The contradiction obtained proves that $J \in \mathcal{H}\mathrm{DiSJ}$.

Prove that $J \in \mathrm{DiSJ}$. We know that J is a homomorphic image of some special Jordan algebra J_0 under some mapping $\varphi\colon J_0 \to J$. Let x_0 and y_0 are preimages of x and y with respect to φ. Consider a subdialgebra U in J_0 generated by x_0 and y_0. Since the dialgebra J_0 is special, subdialgebra U is special too. The dialgebra $J = \mathrm{DiJord}\,\langle x, y \rangle$ is free in the variety of Jordan dialgebras, hence every mapping of x and y to U extends to a homomorphism. Map x and y to x_0 and y_0 respectively. Since x_0 and y_0 generate U, we obtain a surjective homomorphism inverse to a homomorphism $\varphi|_U$. Therefore, $J \backsimeq U$ is a special Jordan dialgebra. □

Corollary 24. *If an identity $f(x, y)$ in two variables holds in all special Jordan dialgebras then it holds in all Jordan dialgebras.*

Proof. Consider $f(x, y)$ as an element of the free Jordan dialgebra $\mathrm{DiJord}\,\langle x, y \rangle$. By the previous theorem $\mathrm{DiJord}\,\langle x, y \rangle$ is a special Jordan dialgebra, therefore $\mathrm{DiJord} \models f$. □

In the paper [8] the s-identity of dialgebras was found which depends on three variables and is linear by one of variables. So the naive generalization of the Macdonald's Theorem to the case of dialgebras is not true. But if an identity is linear in the central letter then the following theorem is true which is an analogue of the Macdonald's Theorem.

Theorem 25. *Let $f = f(x, y, \dot{z})$ be a dipolynomial which is linear in z. If* $\mathrm{DiSJ} \models f$ *then* $\mathrm{DiJord} \models f$.

Proof. Let DiSJ $\models f$, then $\mathcal{H}$DiSJ $\models f$. Consider a Jordan algebra $\bar{J} \in \mathcal{H}$SJ as a dialgebra J with equal left and right products. Then $\bar{J} \in \mathcal{H}$SJ and $\widehat{J} = \bar{J} \oplus J = \bar{J} \oplus \bar{J} \in \mathcal{H}$SJ, so $J \in$ Di$\mathcal{H}$SJ $= \mathcal{H}$DiSJ. We obtain $J \models f$, hence $\bar{J} \models \bar{f}$. Therefore, $\mathcal{H}$SJ $\models \bar{f} = f(x, y, z)$, so by the classical Macdonald's Theorem we have Jord $\models \bar{f}$.

It remains to note that if $f = f(x, y, \dot{z})$ is a polylinear dipolynomial such that Jord $\models \bar{f} = f(x, y, z)$ then DiJord $\models f$: It follows immediately from the definition [5] of what is a variety of dialgebras. The polynomial $f(x, y, z)$ can be nonlinear in x and y. Suppose $\deg_x f = n$, $\deg_y f = m$. Consider the full linearization

$$g(x_1, \ldots, x_n, y_1, \ldots, y_m, z) = L_x^n L_y^m f(x, y, z)$$

of the identity $f(x, y, z)$ (notations from [10, ch. 1]). Then Jord $\models g(x_1, \ldots, x_n, y_1, \ldots, y_m, z)$ and so DiJord $\models g(x_1, \ldots, x_n, y_1, \ldots, y_m, \dot{z})$.

If we now identify variables, then

$$g(x, \ldots, x, y, \ldots, y, \dot{z}) = n!m!f(x, y, \dot{z}).$$

In this section the characteristic of the basic field is equal to zero, so we can divide by $n!m!$ and hence $f(x, y, \dot{z})$ is an identity on DiJord. □

In the end of paper the author thanks Kolesnikov P. S., Pozhidaev A. P. and Gubarev V. Yu. for helpful discussions and valuable comments.

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