Lie group classifications and exact solutions for time-fractional Burgers equation

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Abstract

Lie group method provides an efficient tool to solve nonlinear partial differential equations. This paper suggests a fractional Lie group method for fractional partial differential equations. A time-fractional Burgers equation is used as an example to illustrate the effectiveness of the Lie group method and some classes of exact solutions are obtained.

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Key words Lie group method; Fractional Burgers equation; Fractional characteristic method

1 Introduction

Many methods of mathematical physics have been developed to solve differential equations, among which Lie group method is an efficient approach to derive the exact solution of nonlinear partial differential equations.

Since Sophus Lie's group analysis work more than 100 years ago, Lie group theory has become more and more pervasive in its influence on other mathematical disciplines [1, 2]. There are, however, there few applications of Lie method in fractional calculus. Then a question may naturally arise: is there a fractional Lie group method for fractional differential equations?

Some researchers investigated Lie group method for fractional differential equations in sense of the Caputo derivative and derived scaling transformation and similarity solutions [3–5]. Considering the classical Lie group method, method of characteristic is used to solve symmetry equations. Recently, with the modified Riemann-Liouville derivative [6–8], we first propose a

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more generalized fractional characteristic method [9] than Jumarie's Lagrange method [10]. Using our fractional characteristic method, the generalized symmetry equations generating by the prolongation technique can be solved, and a fractional Lie group method was presented for an anomalous diffusion equation [9].

In this study, we investigate a simplified version of the fractional Burgers equations [5]

$$u_t^{(\alpha)} = u_{xx} + u_x^2,\ x \in (0,\ \infty),\ \ 0<t,\ 0<\alpha < 1, \tag{1}$$

and derive its group classifications. In order to investigate the local behaviors of the above equation, the fractional derivative is in the sense of the modified Riemann-Liouville [6–8].

2 Fractional Calculus and Some Properties

From the viewpoint of Brown motion, Jumarie proposed the modified Riemann-Liouville derivative [6–8],

$$D_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_0^x (x-\xi)^{n-\alpha-1}(f(\xi)-f(0))\,d\xi,\quad n-1<\alpha<n, \tag{2}$$

where the derivative on the right-hand side is the Riemann-Liouville fractional derivative and $n \in Z^+$.

(a) Fractional Taylor series

Recently, Jumarie-Taylor series [11] was proposed

$$df(x) = \sum_{i=1}^{\infty} \frac{h^{k\alpha}}{(k\alpha)!} f^{(k\alpha)}(x),\quad 0<\alpha < 1. \tag{3}$$

Here $f(x)$ is a $k\alpha$-differentiable function and k is an arbitrary positive integer.

Taking $k = 1$, $f(x)$ is a α-differentiable function. We can derive that

$$df(x) = \frac{D_x^\alpha f(x)(dx)^\alpha}{\Gamma(1+\alpha)}. \tag{4}$$

(b) Fractional Leibniz product law

If we set $D_x^\alpha u(x)$ and $D_x^\alpha v(x)$ exist, we can readily find that

$$D_x^\alpha(uv) = u^{(\alpha)}v + uv^{(\alpha)}. \tag{5}$$

The properties of Jumarie's derivative were summarized in [11]. The extension of Jumarie's fractional derivative and integral to variations approach by Almeida et al. [12, 13]. Fractional variational interactional method and Adomian decomposition method are proposed for fractional differential equations [14, 15].

(c) Integration with respect to $(dx)^\alpha$

$$ {}_0I_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\xi)^{\alpha-1} f(\xi) d\xi = \frac{1}{\Gamma(\alpha+1)} \int_0^x f(\xi)(d\xi)^{\alpha}, 0 < \alpha \leq 1. \tag{6} $$

(d) Generalized Newton-Leibniz Law

Assume $D_x^{\alpha} f(x)$ is an integrable function in the interval $[0, a]$. Obviously,

$$ \frac{1}{\Gamma(1+\alpha)} \int_0^a D_x^{\alpha} f(x)(dx)^{\alpha} = f(a) - f(0), 0 < \alpha < 1, \tag{7} $$

$$ \frac{1}{\Gamma(1+\alpha)} \int_0^x D_{\xi}^{\alpha} f(\xi)(d\xi)^{\alpha} = f(x) - f(a), \tag{8} $$

and

$$ \frac{D_x^{\alpha}}{\Gamma(1+\alpha)} \int_0^x f(\xi)(d\xi)^{\alpha} = f(x), \quad 0 < \alpha < 1. \tag{9} $$

(e) Some other useful properties

$$ f^{(\alpha)}([x(t)]) = \frac{df}{dx} x^{(\alpha)}(t), \quad 0 < \alpha < 1, \tag{10} $$

$$ D_x^{\alpha} x^{\beta} = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} x^{\beta-\alpha}, \quad 0 < \beta < 1, \tag{11} $$

$$ \int (dx)^{\beta} = x^{\beta}. \tag{12} $$

The above properties (a)–(d) can be found in Ref. [11]. We must point out that $f(x)$ should be differentiable w.r.t x in Eq. (10), and x^{β} is an α order function in Eq. (11).

3 A Characteristic Method for Fractional Differential Equations

It is well known that the method of characteristics has played a very important role in mathematical physics. Preciously, the method of characteristics is used to solve the initial value problem for general first order. With the modified Riemann-Liouville derivative, Jumaire ever gave a Lagrange characteristic method [10], in which the time-fractional order equals to the space-fractional order. We present a more generalized fractional method of characteristics and use it to solve linear fractional partial equations.

Consider the following first order equation,

$$ a(x,t)\frac{\partial u(x,t)}{\partial x} + b(x,t)\frac{\partial u(x,t)}{\partial t} = c(x,t). \tag{13} $$

The goal of the method of characteristics is to change coordinates from $(x,\ t)$ to a new coordinate system $(x_0,\ s)$ in which the partial differential equation becomes an ordinary differential equation along certain curves in the $x-t$ plane. The curves are called the characteristic curves.

More generally, we consider to extend this method to linear space-time fractional differential equations

$$a(x,t)\frac{\partial^{\beta} u(x,t)}{\partial x^{\beta}} + b(x,t)\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = c(x,t), \quad 0 < \alpha, \beta < 1. \tag{14}$$

With the fractional Taylor's series in two variables [11]

$$du = \frac{\partial^{\beta} u(x,t)}{\Gamma(1+\beta)\partial x^{\beta}}(dx)^{\beta} + \frac{\partial^{\alpha} u(x,t)}{\Gamma(1+\alpha)\partial t^{\alpha}}(dt)^{\alpha}, \quad 0 < \alpha, \ \beta < 1, \tag{15}$$

similarly, we derive the generalized characteristic curves

$$\frac{du}{ds} = c(x,t), \tag{16}$$

$$\frac{(dx)^{\beta}}{\Gamma(1+\beta)ds} = a(x,t), \tag{17}$$

$$\frac{(dt)^{\alpha}}{\Gamma(1+\alpha)ds} = b(x,t). \tag{18}$$

Eqs. (16)–(18) can be reduced as Jumaire's Lagrange method of characteristic if $\alpha = \beta$ in [10].

4 A Fractional Lie Group Method

In the classical Lie method for partial differential equations, the one-parameter Lie group of transformations in $(x, \ t, \ u)$ is given by

$$\begin{aligned} \tilde{x} &= x + \varepsilon\xi(x,t,u) + O(\varepsilon^2), \\ \tilde{t} &= t + \varepsilon\tau(x,t,u) + O(\varepsilon^2), \\ \tilde{u} &= u + \varepsilon\phi(x,t,u) + O(\varepsilon^2), \end{aligned}$$

where ε is the group parameter.

Use the set of fractional vector fields instead of the one of integer order

$$V = \xi(x,t,u)D_x^{\beta} + \tau(x,t,u)D_t^{\alpha} + \phi(x,t,u)D_u, \quad 0 < \alpha < 1, \quad 0 < \beta < 1. \tag{19}$$

For the fractional second order prolongation $Pr^{(2\beta)}V$ of the infinitesimal generators, we proposed [9]

$$Pr^{(2\beta)}V = V + \phi^{[t]}\frac{\partial\phi}{\partial D_t^{\alpha}u} + \phi^{[x]}\frac{\partial\phi}{\partial D_x^{\beta}u} + \phi^{[tt]}\frac{\partial\phi}{\partial D_t^{2\alpha}u} + \phi^{[xx]}\frac{\partial\phi}{\partial D_x^{2\beta}u} + \phi^{[xt]}\frac{\partial\phi}{\partial D_x^{\beta}D_t^{\alpha}u}. \tag{20}$$

As a result, we can have

$$Pr^{(2\beta)}V(\Delta[u]) = 0, \tag{21}$$

on $\Delta[u] = 0$.

In the time-fractional Burgers equation, Eq. (1), we only need to consider the case of the fractional order of space $\beta = 1$. Thus, the corresponding Lie algebra of infinitesimal symmetries is the set of fractional vector fields in the form

$$V = \xi(x,t,u)D_x + \tau(x,t,u)D_t^\alpha + \phi(x,t,u)D_u. \tag{22}$$

We assume the one-parameter Lie group of transformations in $(x,\ t,\ u)$ given by

$$\begin{aligned} &\tilde{x} = x + \varepsilon\xi(x,t,u) + O(\varepsilon), \\ &\tfrac{\tilde{t}^\alpha}{\Gamma(1+\alpha)} = \tfrac{t^\alpha}{\Gamma(1+\alpha)} + \varepsilon\tau(x,t,u) + O(\varepsilon), \\ &\tilde{u} = u + \varepsilon\phi(x,t,u) + O(\varepsilon), \end{aligned} \tag{23}$$

where ε is the group parameter.

The generalized second prolongation satisfies

$$Pr^{(2)}V = V + \phi^t \frac{\partial\phi}{\partial D_t^\alpha u} + \phi^x \frac{\partial\phi}{\partial D_x u} + \phi^{tt}\frac{\partial\phi}{\partial D_t^{2\alpha} u} + \phi^{xx}\frac{\partial\phi}{\partial u_{xx}} + \phi^{xt}\frac{\partial\phi}{\partial D_t^\alpha u_x}. \tag{24}$$

Using the following condition

$$Pr^{(2)}V(\Delta[u]) = 0, \quad \Delta[u] = 0, \tag{25}$$

we can have

$$(\phi^t - \phi^{xx} - 2u_x\phi^x)\big|_{\Delta[u]=0} = 0. \tag{26}$$

The generalized prolongation vector fields are reduced as

$$\begin{aligned} &\phi^t = D_t^\alpha\phi - (D_t^\alpha\xi)D_x u - (D_t^\alpha\tau)D_t^\alpha u, \\ &\phi^x = D_x\phi - (D_x\xi)D_x u - (D_x\tau)D_t^\alpha u, \\ &\phi^{xx} = D_x^2\phi - 2(D_x\xi)D_x^2 u - (D_x^2\xi)D_x u - 2(D_x\tau)D_x D_t^\alpha u - (D_x^2\tau)D_t^\alpha u. \end{aligned} \tag{27}$$

Substituting Eq. (27) into Eq. (26) and setting the coefficients of $u_x u_{xt}^{(\alpha)}$, $u_{xt}^{(\alpha)}$, $u_{xx}u_x$, u_x and 1 to zero. Solve the equations with maple software, we can have

$$\begin{aligned} &\xi(x,t,u) = c_1 + c_4 x + 2c_5\tfrac{t^\alpha}{\Gamma(1+\alpha)} + 4c_6\tfrac{xt^\alpha}{\Gamma(1+\alpha)}, \\ &\tau(x,t,u) = c_2 + 2c_4\tfrac{t^\alpha}{\Gamma(1+\alpha)} + 4c_6\tfrac{t^{2\alpha}}{\Gamma^2(1+\alpha)}, \\ &\phi(x,t,u) = c_3 - c_5 x + \tfrac{2c_6 t^\alpha}{\Gamma(1+\alpha)} - c_6 x^2 + a(x,t)e^u, \end{aligned} \tag{28}$$

where $k_t^{(\alpha)} = k_{xx}$.

The Lie algebra of infinitesimal symmetries of Eq. (1) is spanned by the vector field

$$\begin{aligned} &V_1 = \tfrac{\partial}{\partial x},\ \ V_2 = \tfrac{\partial^\alpha}{\partial t^\alpha},\ \ V_3 = \tfrac{\partial}{\partial u},\ \ V_4 = x\tfrac{\partial}{\partial u} + \tfrac{2t^\alpha}{\Gamma(1+\alpha)}\tfrac{\partial^\alpha}{\partial t^\alpha}, \\ &V_5 = -x\tfrac{\partial}{\partial u} + \tfrac{2t^\alpha}{\Gamma(1+\alpha)}\tfrac{\partial}{\partial x}, \\ &V_6 = \tfrac{4xt^\alpha}{\Gamma(1+\alpha)}\tfrac{\partial}{\partial x} + \tfrac{4t^{2\alpha}}{\Gamma^2(1+\alpha)}\tfrac{\partial^\alpha}{\partial t^\alpha} - (x^2 + \tfrac{2t^\alpha}{\Gamma(1+\alpha)})\tfrac{\partial}{\partial u}, \end{aligned} \tag{29}$$

and the infinite-dimensional subalgebra

$$V_k = k(x,t)e^{-u}\frac{\partial}{\partial u}.$$

It is easy to check the two vector fields $\{V_1, V_2, V_3, V_4, V_5, V_6\}$ are closed under the Lie bracket $[a,b] = ab - ba$. In fact, we have

$[V_i, V_i] = 0$ $(i = 0, ..., 6)$, $[V_1, V_2] = [V_1, V_3] = 0$, $[V_1, V_4] = -V_1$, $[V_1, V_5] = V_3$,

$[V_1, V_6] = -2V_5$, $[V_2, V_3] = 0$, $[V_2, V_4] = -2V_2$, $[V_2, V_5] = -2V_1$, $[V_2, V_6] = 2V_3 - 4V_4$,

$[V_3, V_4] = [V_3, V_5] = [V_3, V_6] = 0$, $[V_4, V_5] = -V_5$, $[V_4, V_6] = -2V_6$, $[V_5, V_6] = 0$.

$[V_1, V_k] = -V_{k_x}$, $[V_2, V_k] = -V_{k_t}$, $[V_3, V_k] = -V_k$, $[V_4, V_k] = -V_{k'}$, $[V_5, V_k] = -V_{k''}$,

$[V_6, V_k] = -V_{k'''}$,

where $k' = xk_x + \frac{2t^\alpha}{\Gamma(1+\alpha)}k_t^{(\alpha)}$, $k'' = \frac{2t^\alpha}{\Gamma(1+\alpha)}k_x + xk$ and $k''' = \frac{4xt^\alpha}{\Gamma(1+\alpha)}k_x + \frac{4t^{2\alpha}}{\Gamma^2(1+\alpha)}k_t^{(\alpha)} + (x^2 + \frac{2t^\alpha}{\Gamma(1+\alpha)})k$.

Take the characteristic equation V_5 as an example. The characteristic curve of V_5 can be given

$$\frac{du}{d\varepsilon} = -x, \tag{30}$$

$$\frac{dx}{d\varepsilon} = \frac{2t^\alpha}{\Gamma(1+\alpha)}, \tag{31}$$

$$\frac{(dt)^\alpha}{\Gamma(1+\alpha)d\varepsilon} = 0. \tag{32}$$

Solve the above ordinary equations with the initial value $u = u(x,t,\varepsilon)|_{\varepsilon=0}$, $x = x(\varepsilon)|_{\varepsilon=0}$ and $t = t(\varepsilon)|_{\varepsilon=0}$. The one-parameter group G_i generated by the $V_i (i = 1, ..., 6, \alpha)$ are given as

$$\begin{aligned}
&g_1\colon\ (x,\ \tfrac{t^\alpha}{\Gamma(1+\alpha)},\ u) \to (x+\varepsilon,\ \tfrac{t^\alpha}{\Gamma(1+\alpha)},\ u),\\
&g_2\colon\ (x,\ \tfrac{t^\alpha}{\Gamma(1+\alpha)},\ u) \to (x,\ \tfrac{t^\alpha}{\Gamma(1+\alpha)}+\varepsilon,\ u),\\
&g_3\colon\ (x,\ \tfrac{t^\alpha}{\Gamma(1+\alpha)},\ u) \to (x,\ \tfrac{t^\alpha}{\Gamma(1+\alpha)},\ u+\varepsilon),\\
&g_4\colon\ (x,\ \tfrac{t^\alpha}{\Gamma(1+\alpha)},\ u) \to (xe^\varepsilon,\ \tfrac{t^\alpha e^{2\varepsilon}}{\Gamma(1+\alpha)},\ u),\\
&g_5\colon\ (x,\ \tfrac{t^\alpha}{\Gamma(1+\alpha)},\ u) \to (x+\tfrac{2\varepsilon t^\alpha}{\Gamma(1+\alpha)},\ \tfrac{t^\alpha}{\Gamma(1+\alpha)},\ u-\tfrac{\varepsilon^2 t^\alpha}{\Gamma(1+\alpha)}-x\varepsilon),\\
&g_6\colon\ (x,\ \tfrac{t^\alpha}{\Gamma(1+\alpha)},\ u) \to (\tfrac{x}{1-4\varepsilon\frac{t^\alpha}{\Gamma(1+\alpha)}},\ \tfrac{t^\alpha}{1-4\varepsilon\frac{t^\alpha}{\Gamma(1+\alpha)}},\ u-\tfrac{x^2\varepsilon}{1-4\varepsilon\frac{t^\alpha}{\Gamma(1+\alpha)}}+\log\sqrt{1-4\varepsilon\tfrac{t^\alpha}{\Gamma(1+\alpha)}}),\\
&g_\alpha\colon\ (x,\ \tfrac{t^\alpha}{\Gamma(1+\alpha)},\ u) \to (x,\ \tfrac{t^\alpha}{\Gamma(1+\alpha)},\ \log(e^u+\varepsilon k)).
\end{aligned} \tag{33}$$

Take $\alpha = 1$ in the above classifications. We can derive the results for the case of integer order. Since g_i is a symmetry, if $u = f(x, \frac{t^\alpha}{\Gamma(1+\alpha)})$ is a solution of Eq. (1), then the following u_i are also the solutions of Eq. (1)

$$
\begin{aligned}
&u_1 = f(x-\varepsilon,\ \tfrac{t^\alpha}{\Gamma(1+\alpha)}),\\
&u_2 = f(x,\ \tfrac{t^\alpha}{\Gamma(1+\alpha)}-\varepsilon),\\
&u_3 = f(x,\ \tfrac{t^\alpha}{\Gamma(1+\alpha)})+\varepsilon,\\
&u_4 = f(xe^{-\varepsilon},\ \tfrac{t^\alpha e^{-\varepsilon}}{\Gamma(1+\alpha)}),\\
&u_5 = f(x-\tfrac{2\varepsilon t^\alpha}{\Gamma(1+\alpha)}, \tfrac{t^\alpha}{\Gamma(1+\alpha)})+\tfrac{\varepsilon^2 t^\alpha}{\Gamma(1+\alpha)}-x\varepsilon,\\
&u_6 = f(\tfrac{x}{1+4\varepsilon\frac{t^\alpha}{\Gamma(1+\alpha)}},\ \tfrac{t^\alpha}{1+4\varepsilon\frac{t^\alpha}{\Gamma(1+\alpha)}})-\tfrac{x^2\varepsilon}{1-4\varepsilon\frac{t^\alpha}{\Gamma(1+\alpha)}}-\log\sqrt{1-4\varepsilon\tfrac{t^\alpha}{\Gamma(1+\alpha)}},\\
&u_\alpha = \log(e^{f(x,\frac{t^\alpha}{\Gamma(1+\alpha)})}+\varepsilon k).
\end{aligned}
\tag{34}
$$

Now we consider the applications of the above transformations. From u_1 to u_4, we can only obtain trivial solutions. Therefore, we start from the use of u_5

$$
g_5\colon\ (x,\ \frac{t^\alpha}{\Gamma(1+\alpha)},\ u)\to(x+\frac{2\varepsilon t^\alpha}{\Gamma(1+\alpha)},\frac{t^\alpha}{\Gamma(1+\alpha)},\ u-\frac{\varepsilon^2 t^\alpha}{\Gamma(1+\alpha)}-x\varepsilon), \tag{35}
$$

Assume $u_{5,0}=u_{5,0}(x,\ \frac{t^\alpha}{\Gamma(1+\alpha)})=f(x,\ \frac{t^\alpha}{\Gamma(1+\alpha)})$ is one exact solution of Eq. (1). Take $u_{5,0}=c$, where c is a arbitrary constant and also a trivial solution. We can get a new nontrivial exact solution as

$$
u_{5,1}=c+\frac{\varepsilon^2 t^\alpha}{\Gamma(1+\alpha)}-x\varepsilon. \tag{36}
$$

Further more, continue this iteration process, we can derive a new exact solution of Eq. (1)

$$
u_{5,2}=u_{5,1}(x-\frac{2\varepsilon t^\alpha}{\Gamma(1+\alpha)},\ \frac{t^\alpha}{\Gamma(1+\alpha)})+\frac{\varepsilon^2 t^\alpha}{\Gamma(1+\alpha)}-x\varepsilon=c-2x\varepsilon+\frac{4\varepsilon^2 t^\alpha}{\Gamma(1+\alpha)}. \tag{37}
$$

Similarly, take $u_{6,0}=c$, then we can have

$$
u_{6,1}=c-\frac{x^2\varepsilon}{1+4\varepsilon\frac{t^\alpha}{\Gamma(1+\alpha)}}-\log\sqrt{1+4\varepsilon\frac{t^\alpha}{\Gamma(1+\alpha)}}, \tag{38}
$$

and

$$
\begin{aligned}
u_{6,2}&=u_{6,1}(\frac{x}{1+4\varepsilon\frac{t^\alpha}{\Gamma(1+\alpha)}},\ \frac{t^\alpha}{1+4\varepsilon\frac{t^\alpha}{\Gamma(1+\alpha)}})-\frac{x^2\varepsilon}{1+4\varepsilon\frac{t^\alpha}{\Gamma(1+\alpha)}}-\log\sqrt{1+4\varepsilon\frac{t^\alpha}{\Gamma(1+\alpha)}}\\
&=c-\frac{2\varepsilon x^2}{1+\frac{8\varepsilon t^\alpha}{\Gamma(1+\alpha)}}-\log\sqrt{1+\frac{8\varepsilon t^\alpha}{\Gamma(1+\alpha)}}.
\end{aligned}
\tag{39}
$$

We can readily verify that $u_{5,1}$, $u_{5,2}$, $u_{6,1}$ and $u_{6,2}$ are four exact solutions of Eq. (1). Assuming the fractional order $\alpha=1$, the exact solutions we give here can be reduced as the exact iteration solution in Ref. [16] if we set the coefficients $a=b=1$.

5 Conclusions

Fractional differential equations have caught considerable attention due to their various applications in real physical problems. However, there is no a systematic method to derive the exact solutions. Now, the problem is partly solved in this paper. The presented paper can be applied to other fractional partial differential equations and investigate their non-smooth properties.

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