VALUATIONS AND ASYMPTOTIC INVARIANTS FOR SEQUENCES OF IDEALS

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Abstract. We study asymptotic jumping numbers for graded sequences of ideals, and show that every such invariant is computed by a suitable real valuation of the function field. We conjecture that every valuation that computes an asymptotic jumping number is necessarily quasi-monomial. This conjecture holds in dimension two. In general, we reduce it to the case of affine space and to graded sequences of valuation ideals. Along the way, we study the structure of a suitable valuation space.

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Introduction

Given a nonzero ideal $\mathfrak{a}$ on a smooth complex variety X, the log canonical threshold $\mathrm{lct}(\mathfrak{a})$ is a fundamental invariant in both singularity theory and birational geometry (see, for example, [Laz], [EM] or [Kol]). Analytically, it can be described as follows: if $\mathfrak{a}$ is generated by $f_1, \ldots, f_m$, then

$$\mathrm{lct}(\mathfrak{a}) = \sup\{s > 0 \mid (\sum_i |f_i|^2)^{-s} \text{ is locally integrable}\}.$$

Alternatively, the invariant admits the following description in terms of valuations:

$$\mathrm{lct}(\mathfrak{a}) = \inf_E \frac{A(\mathrm{ord}_E)}{\mathrm{ord}_E(\mathfrak{a})}, \tag{0.1}$$

where E varies over the prime divisors over X, and where $A(\mathrm{ord}_E) - 1$ is the coefficient of the divisor E on Y in the relative canonical class $K_{Y/X}$. In fact, in the above formula one can take the infimum over all real valuations of $K(X)$ with center on X. A key fact for the study of the log canonical threshold is that if $\mu\colon Y \to X$ is a log resolution of the ideal $\mathfrak{a}$, that is, μ is proper and birational, Y is nonsingular, and $\mathfrak{a} \cdot \mathcal{O}_Y$ is the ideal of a divisor D such that $D + K_{Y/X}$ has simple normal crossings, then there is a prime divisor E on Y that achieves the infimum in (0.1). These divisors play an important role in understanding the singularities of $\mathfrak{a}$.

In this paper we undertake the systematic study of similar invariants in the case of sequences of ideals. We focus on *graded sequences of ideals* $\mathfrak{a}_\bullet$: these are sequences of ideals $(\mathfrak{a}_m)_{m\geq 1}$ on X such that $\mathfrak{a}_p \cdot \mathfrak{a}_q \subseteq \mathfrak{a}_{p+q}$ for all p and q. In order to simplify the statements, in this introduction we also assume that all $\mathfrak{a}_m$ are nonzero. The main geometric example of a graded sequence is given by the ideals defining the base locus of $|L^m|$, where L is an effective line bundle on the smooth projective variety X. Note that the interesting behavior of this sequence takes place when the section $\mathbf{C}$-algebra $\oplus_{m\geq 0}\Gamma(X, L^m)$ is not finitely generated.

Given a graded sequence $\mathfrak{a}_\bullet$, one can define an *asymptotic log canonical threshold* $\mathrm{lct}(\mathfrak{a}_\bullet)$ as the limit

$$\mathrm{lct}(\mathfrak{a}_\bullet) := \lim_{m\to\infty} m \cdot \mathrm{lct}(\mathfrak{a}_m) = \sup_m m \cdot \mathrm{lct}(\mathfrak{a}_m) \in \mathbf{R}_{\geq 0} \cup \{\infty\}.$$

We show that as above, we have

$$\mathrm{lct}(\mathfrak{a}_\bullet) = \inf_E \frac{A(\mathrm{ord}_E)}{\mathrm{ord}_E(\mathfrak{a}_\bullet)}, \tag{0.2}$$

where $\mathrm{ord}_E(\mathfrak{a}_\bullet) = \lim_{m\to\infty} \frac{\mathrm{ord}_E(\mathfrak{a}_m)}{m} = \inf_m \frac{\mathrm{ord}_E(\mathfrak{a}_m)}{m}$. More generally, one can define $v(\mathfrak{a}_\bullet)$ and $A(v)$ for every valuation v of $K(X)$ with center on X, and in (0.2) we may take the infimum over all such valuations different from the trivial one. It is easy to see that in this setting there might be no divisor E such that the infimum in (0.2) is achieved by ord_E (we give such an example with $\mathfrak{a}_\bullet$ a graded sequence of monomial ideals in §8). The following is (a special case of) our first main result:

Theorem A. *For every graded sequence of ideals $\mathfrak{a}_\bullet$, there is a real valuation v of $K(X)$ with center on X that computes $\mathrm{lct}(\mathfrak{a}_\bullet)$, that is, such that $\mathrm{lct}(\mathfrak{a}_\bullet) = \frac{A(v)}{v(\mathfrak{a}_\bullet)}$.*

We make the following conjecture.

Conjecture B. *Let $\mathfrak{a}_\bullet$ be a graded sequence of ideals on X such that* $\mathrm{lct}(\mathfrak{a}_\bullet) < \infty$.

- ***Weak version****: there exists a quasi-monomial valuation v that computes* $\mathrm{lct}(\mathfrak{a}_\bullet)$.
- ***Strong version****: any valuation v that computes* $\mathrm{lct}(\mathfrak{a}_\bullet)$ *must be quasi-monomial*[1].

Recall that a quasi-monomial valuation is a valuation v of $K(X)$ with the following property. There is a proper birational morphism $\mu\colon Y \to X$, with Y nonsingular, and coordinates $y_1, \ldots, y_r$ at a point $p \in Y$, as well as $\alpha_1, \ldots, \alpha_r \in \mathbf{R}_{\geq 0}$ such that if f can be written at p as $f = \sum_{u \in \mathbf{Z}^r_{\geq 0}} c_u y^u$, then

$$v(f) = \min\{\sum_i \alpha_i u_i \mid u = (u_1, \ldots, u_r) \in \mathbf{Z}^r_{\geq 0}, c_u \neq 0\}.$$

Equivalently, such valuations are known as Abhyankar valuations (see §3.2). Note that a positive answer to the above conjecture would give a strong finiteness property of any graded sequence, even when the sequence is not finitely generated, that is, when the $\mathcal{O}_X$-algebra $\bigoplus_{m \geq 0} \mathfrak{a}_m$ is not finitely generated.

As a consequence of Theorem A, we show that in order to prove Conjecture B it suffices to consider certain special graded sequences $\mathfrak{a}_\bullet$, namely those attached to an arbitrary real valuation w of $K(X)$, by taking $\mathfrak{a}_m = \{f \mid w(f) \geq m\}$. See Theorem 7.7.

Our second main result reduces Conjecture B to the case of affine space over an algebraically closed field. Furthermore, in the strong version we may assume that ν is a valuation of transcendence degree zero. In order to get such a statement, we need to work in a slightly more general setting that we now explain. To a nonzero ideal $\mathfrak{a}$, one associates its multiplier ideals $\mathcal{J}(\mathfrak{a}^t)$, where $t \in \mathbf{R}_{\geq 0}$. These are ideals on X with $\mathcal{J}(\mathfrak{a}^{t_1}) \subseteq \mathcal{J}(\mathfrak{a}^{t_2})$ if $t_1 > t_2$, and $\mathcal{J}(\mathfrak{a}^t) = \mathcal{O}_X$ for $0 \leq t \ll 1$. One knows that there is an unbounded sequence of positive rational numbers $0 < t_1 < t_2 < \ldots$ such that $\mathcal{J}(\mathfrak{a}^t)$ is constant for $t \in [t_{i-1}, t_i)$ and $\mathcal{J}(\mathfrak{a}^{t_i}) \neq \mathcal{J}(\mathfrak{a}^{t_{i-1}})$ for all $i \geq 1$ (with the convention that $t_0 = 0$). These t_i are the *jumping numbers* of $\mathfrak{a}$, introduced and studied in [ELSV]. From this point of view, the log canonical threshold $\mathrm{lct}(\mathfrak{a})$ is simply the smallest jumping number t_1.

We index the jumping numbers of $\mathfrak{a}$ as follows. Given a nonzero ideal $\mathfrak{q}$ on X, let $\mathrm{lct}^{\mathfrak{q}}(\mathfrak{a})$ be the smallest t such that $\mathfrak{q} \not\subseteq \mathcal{J}(\mathfrak{a}^t)$. In particular, we recover $\mathrm{lct}(\mathfrak{a})$ as $\mathrm{lct}^{\mathcal{O}_X}(\mathfrak{a})$. The advantage of considering higher jumping numbers comes from the fact that it allows replacing X by any smooth X', where X' is proper and birational over X: in this case $\mathrm{lct}^{\mathfrak{q}}(\mathfrak{a}) = \mathrm{lct}^{\mathfrak{q}'}(\mathfrak{a}')$, where $\mathfrak{a}' = \mathfrak{a} \cdot \mathcal{O}_{X'}$ and $\mathfrak{q}' = \mathfrak{q} \cdot \mathcal{O}_{X'}(-K_{X'/X})$. Many of the subtle properties of the log canonical threshold are not shared by the higher jumping numbers. However, for our purposes, considering also $\mathrm{lct}^{\mathfrak{q}}(\mathfrak{a})$ does not create any additional difficulties.

In particular, given a graded sequence of ideals $\mathfrak{a}_\bullet$, one defines as above

$$\mathrm{lct}^{\mathfrak{q}}(\mathfrak{a}_\bullet) := \lim_{m\to\infty} m \cdot \mathrm{lct}^{\mathfrak{q}}(\mathfrak{a}_m) = \sup_m m \cdot \mathrm{lct}^{\mathfrak{q}}(\mathfrak{a}_m) \in \mathbf{R}_{\geq 0} \cup \{\infty\},$$

[1]Given v, the existence of $\mathfrak{a}_\bullet$ such that v computes $\mathrm{lct}(\mathfrak{a}_\bullet) < \infty$ is equivalent to the following two properties: $A(v) < \infty$, and for every valuation w of $K(X)$ with center on X such that $w(\mathfrak{a}) \geq v(\mathfrak{a})$ for every ideal $\mathfrak{a}$ on X, we have $A(w) \geq A(v)$; see Theorem 7.8.

and we have

$$\mathrm{lct}^{\mathfrak{q}}(\mathfrak{a}_\bullet) = \inf_v \frac{A(v) + v(\mathfrak{q})}{v(\mathfrak{a}_\bullet)}, \tag{0.3}$$

where the supremum is over all real valuations of $K(X)$ with center on X, and different from the trivial one. With this notation we have versions of Theorem A and Conjecture B for $\mathrm{lct}^{\mathfrak{q}}(\mathfrak{a}_\bullet)$. Furthermore, we reduce the general version of Conjecture B to the following conjecture about valuations.

Conjecture C. *Let $X = \mathbf{A}^n_k$, where k is an algebraically closed field of characteristic zero and where $n \geq 1$. Let $\mathfrak{a}_\bullet$ be a graded sequence of ideals on X and $\mathfrak{q}$ a nonzero ideal on X such that $\mathrm{lct}^{\mathfrak{q}}(\mathfrak{a}_\bullet) < \infty$ and such that $\mathfrak{a}_1 \supseteq \mathfrak{m}^p$, where $p \geq 1$ and $\mathfrak{m} = \mathfrak{m}_\xi$ is the ideal defining a closed point $\xi \in X$.*

- ***Weak version****: there exists a quasi-monomial valuation v computing $\mathrm{lct}^{\mathfrak{q}}(\mathfrak{a}_\bullet)$ and having center ξ on X.*
- ***Strong version****: any valuation of transcendence degree 0 computing $\mathrm{lct}^{\mathfrak{q}}(\mathfrak{a}_\bullet)$ and having center ξ on X, must be quasi-monomial.*

Theorem D. *If Conjecture C holds for all $n \leq d$, then Conjecture B holds for all X with $\dim(X) \leq d$.*

We give a proof of the strong version of Conjecture C in dimension ≤ 2. The argument is similar to the one used in [FJ3], where a version of Conjecture B is proved. However, as opposed to [FJ3], the proof given here does not use the detailed tree structure of the valuation space at a point.

As it is always the case when dealing with graded sequences of ideals (see [Laz], [ELMNP], [Mus] and [FJ3]), a key tool is provided by the corresponding system of asymptotic multiplier ideals $\mathfrak{b}_\bullet = (\mathfrak{b}_t)_{t \in \mathbf{R}_{>0}}$. These are defined by $\mathfrak{b}_t = \mathcal{J}(\mathfrak{a}_m^{t/m})$ for m divisible enough. The invariant $\mathrm{lct}^{\mathfrak{q}}(\mathfrak{a}_\bullet)$ can be recovered as the smallest λ such that $\mathfrak{q} \not\subseteq \mathfrak{b}_\lambda$. A fundamental property of $\mathfrak{b}_\bullet$ is provided by the Subadditivity Theorem [DEL], which says that $\mathfrak{b}_{s+t} \subseteq \mathfrak{b}_s \mathfrak{b}_t$ for every $s, t \geq 0$. To a general such *subadditive* systems of ideals $\mathfrak{b}_\bullet$ (not necessarily associated to a graded system) we introduce and study asymptotic invariants. A key property for us is that a graded sequence $\mathfrak{a}_\bullet$ has, roughly speaking, the same asymptotic invariants as its system $\mathfrak{b}_\bullet$ of multiplier ideals.

We now describe the key idea in the proof of Theorems A and D. Given a graded sequence $\mathfrak{a}_\bullet$ and a nonzero ideal $\mathfrak{q}$ with $\lambda = \mathrm{lct}^{\mathfrak{q}}(\mathfrak{a}_\bullet) < \infty$, let ξ be the generic point of an irreducible component of the subscheme defined by $(\mathfrak{b}_\lambda \colon \mathfrak{q})$. After localizing and completing at ξ, we may assume that $X = \mathrm{Spec}\ k[\![x_1, \ldots, x_n]\!]$ for a characteristic zero field k, and that ξ is the closed point. We show that if $\mathfrak{m}$ is the ideal defining ξ, and $p \gg 0$, then $\mathrm{lct}^{\mathfrak{q}}(\mathfrak{a}_\bullet) = \mathrm{lct}^{\mathfrak{q}}(\mathfrak{c}_\bullet)$, where $\mathfrak{c}_\ell = \sum_{i=0}^{\ell} \mathfrak{a}_i \cdot \mathfrak{m}^{p(\ell - i)}$. Using a compactness argument for the space of normalized valuations with center at ξ, we construct a valuation v with center at ξ, which computes $\mathrm{lct}^{\mathfrak{q}}(\mathfrak{c}_\bullet)$. It is now easy to see that v also computes $\mathrm{lct}^{\mathfrak{q}}(\mathfrak{a}_\bullet)$. This proves the general version of Theorem A. In order to prove Theorem D for the weak versions of the conjectures, we need two extra steps: we show that after replacing X by a higher model, the valuation v that we construct has transcendence degree zero, and then we show that we may replace k by its algebraic closure $\overline{k}$, and Spec $\overline{k}[\![x_1,\ldots,x_n]\!]$ by $\mathbf{A}^n_{\overline{k}}$. In this case, assuming Conjecture C, we can choose v to be quasi-monomial.

A general principle in our work is to study a graded system $\mathfrak{a}_\bullet$ of ideals on X through the induced function $v \mapsto v(\mathfrak{a}_\bullet)$ on the space Val_X of real-valued valuations on $K(X)$ admitting a center on X. We show in Theorem 4.9 that Val_X can be viewed as a projective limit of simplicial cone complexes equipped with an integral affine structure, a description which leads us to extend the log discrepancy from divisorial to arbitrary valuations. In fact, the precise understanding of the log discrepancy plays a key role in the proof of Theorem D.

Spaces of valuations, such as Berkovich spaces [Ber1], are fundamental objects in non-archimedean geometry. More surprisingly, they have recently seen a number of applications to problems over the complex numbers [Ber2, KS, FJ2, FJ3, BFJ1, BdFF, Ked1, Ked2]. The space Val_X is a dense subset of the Berkovich analytic space X^{an} and has the advantage of being birationally invariant (as a set). It is also closely related to the valuation space considered in [BFJ1]. See §6.3 for more details. Expecting the space Val_X to be useful for further studies, we spend some time analyzing it in detail. However, on a first reading, the reader may want to skim through §§3-5.

We mention that part of our motivation comes from the Openness Conjecture of Demailly and Kollár [DK] for plurisubharmonic (psh) functions. The connection between valuation theory and this conjecture has been highlighted by the two-dimensional result in [FJ3], and by the higher-dimensional framework in [BFJ1]. In the setting of psh functions, one can define analogues of the invariant $\mathrm{lct}(\mathfrak{a}_\bullet)$, and one can formulate an analogue of Conjecture B, which would imply in particular the Openness Conjecture. While in general there is no graded sequence associated to a psh function φ, Demailly's approximation technique (see [DK]) allows one to get a subadditive system of ideals $\mathfrak{b}_\bullet$. We expect that methods similar to the ones used in this paper should give analogues of Theorems A and D for psh functions (in particular, this would reduce an analytic statement, the Openness Conjecture, to the valuation-theoretic Conjecture C above). We hope to treat the case of psh functions in future work.

As explained above, we make use of localization and completion. In order to do this, we work from the beginning with regular schemes of finite type over a formal power series ring over a field of characteristic zero. The basic results about log canonical thresholds and multiplier ideals carry over to this setting. Some of the more subtle results, whose proofs use vanishing theorems, are reduced to the familiar setting in the appendix.

The paper in structured as follows. In §1 we set up some notation and definitions, and in §2 we introduce the asymptotic invariants for graded sequences and subadditive systems of ideals. We prove here their basic properties, and in particular, we relate the invariants of a graded sequence and those of the corresponding subadditive system of asymptotic multiplier ideals. In §3 we introduce the quasi-monomial valuations and prove some general properties that will be needed later. Section 4 contains some results concerning the structure of the valuation space, while in §5 we use this framework to extend the log discrepancy function to the whole valuation space. In Sections 4 and 5 we follow the approach in [BFJ1], with some modifications due to the fact that we do not restrict to valuations centered at one point. In §6 we return to subadditive and graded sequences, and extend some results that we proved for divisorial valuations to arbitrary valuations. Section 7 is the central section of the paper, in which we prove our main results. In §8 we consider a special case, that of graded sequences of monomial ideals. In this case the picture can be completely described, and in particular, we see that Conjecture B has a positive answer. We give a proof of Conjecture C in the two-dimensional case in §9. The appendix shows how to extend some basic results about multiplier ideals, the Restriction and the Subadditivity Theorems, from the case of varieties over a field to our more general setting.

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1. Preliminaries

Our main interest is in nonsingular algebraic varieties. However, it is sometimes convenient to allow localization and completion at a not necessarily closed point. In order to cover both these cases, and also have log resolutions of ideals, we work in the following setting: the ambient scheme X is separated, nonsingular, irreducible, of finite type over a ring $R = k[\![t_1, \ldots, t_n]\!]$, where k is a characteristic zero field (not necessarily algebraically closed). However, most of the time the reader will not lose much by assuming that we deal with separated, nonsingular algebraic varieties over k, with k algebraically closed.

The main tool in our study is provided by multiplier ideals. For the theory of multiplier ideals in the case of varieties over a field k we refer to [Laz]. The definition and basic properties carry over easily to our framework, see [dFM] and [dFEM]. The key fact that we have log resolutions in this setting follows from [Tem1]. The only subtlety is in extending the results that rely on vanishing theorems, since such results are not known in our framework. We explain in the appendix how the extension of some basic results, the Restriction and the Subadditivity Theorems, can be carried out. From now on, without further discussion, we will not distinguish between the classical setting and ours when dealing with multiplier ideals.

1.1. Valuations.

We will consider the set Val_X of all real valuations of the function field $K(X)$ of X that have center on X. The former condition means that if $\mathcal{O}_v$ is the valuation ring of v, then there is a point $\xi = c_X(v) \in X$ (the *center* of v) such that we have a local inclusion of local rings $\mathcal{O}_{X,\xi} \hookrightarrow \mathcal{O}_v$. Note that since X is separated, the center is unique. We sometimes call the closure of $c_X(v)$ the center of v, too. The *trivial* valuation is the valuation with center at the generic point of X, or equivalently, the valuation whose only finite value is zero. We denote by $\mathrm{Val}_X^* \subseteq \mathrm{Val}_X$ the subset of nontrivial valuations. Notice that if X is a variety over a field k, then the restriction of any $v \in \mathrm{Val}_X$ to k is the trivial valuation.

It is clear that for every v as above, since the ring $\mathcal{O}_{X,\xi}$ is Noetherian, there is no infinite decreasing sequence $v(f_1) > v(f_2) > \dots$, with all f_i in $\mathcal{O}_{X,\xi}$. Indeed, the sequence of ideals $\mathfrak{a}_i = \{f \in \mathcal{O}_{X,\xi} \mid v(f) \geq v(f_i)\}$ would be strictly increasing. In particular, we see that there is a minimal $v(f)$, where f varies over the maximal ideal of $\mathcal{O}_{X,\xi}$.

If $v \in \mathrm{Val}_X$, and if $\mathfrak{a}$ is an ideal[2] on X, then we put $v(\mathfrak{a}) := \min_f v(f)$, where the minimum is over local sections of $\mathfrak{a}$ that are defined in a neighborhood of $c_X(v)$. If Z is the subscheme defined by $\mathfrak{a}$, we also write this as $v(Z)$. In fact, it turns out to be natural to instead view a valuation as taking values on ideals rather than rational functions. Let $\mathcal{I}$ be the set of nonzero ideals on X. It has the structure of an ordered semiring, with the order given by inclusion, and the operations given by addition and multiplication. The set $\mathbf{R}_{\geq 0}$ also has an ordered semiring structure, with operations given by minimum and addition. As above, a valuation $v \in \mathrm{Val}_X$ induces a function $v\colon \mathcal{I} \to \mathbf{R}_{\geq 0}$ by $v(\mathfrak{a}) := \min\{v(f) \mid f \in \mathfrak{a} \cdot \mathcal{O}_{X,\xi}\}$, where $\xi = c_X(v)$, and this function is easily seen to be a homomorphism of semirings:

$$v(\mathfrak{a} \cdot \mathfrak{b}) = v(\mathfrak{a}) + v(\mathfrak{b}) \quad \text{and} \quad v(\mathfrak{a} + \mathfrak{b}) = \min\{v(\mathfrak{a}), v(\mathfrak{b})\}. \tag{1.1}$$

Note that such a homomorphism is automatically order-preserving in the sense $v(\mathfrak{a}) \geq v(\mathfrak{b})$ if $\mathfrak{a} \subseteq \mathfrak{b}$. Indeed, $\mathfrak{a} \subseteq \mathfrak{b}$ implies $\mathfrak{a} + \mathfrak{b} = \mathfrak{b}$. Moreover, the above homomorphism has ξ as a center on X in the sense that $v(\mathfrak{a}) > 0$ if and only if $\xi \in V(\mathfrak{a})$. Conversely, if $v\colon \mathcal{I} \to \mathbf{R}_{\geq 0}$ is a semiring homomorphism admitting $\xi \in X$ as a center, then v induces a valuation in Val_X centered at ξ. Indeed, if $f \in \mathcal{O}_{X,\xi}$, then we define $v(f) := v(\mathfrak{a})$ for any ideal $\mathfrak{a}$ on X such that $\mathfrak{a} \cdot \mathcal{O}_{X,\xi}$ is principal, generated by f. One can check that this is well-defined, and it extends to a valuation of $K(X)$ having center at ξ. It is clear that these two maps between Val_X and semiring homomorphisms $\mathcal{I} \to \mathbf{R}_{\geq 0}$ with center on X are mutual inverses.

1.2. **Divisorial valuations and log discrepancy.** A distinguished role is played by the *divisorial* valuations ord_E, where E is a *divisor over* X, that is, a prime divisor on a normal variety Y, having a proper birational morphism $\pi\colon Y \to X$. It follows from results on resolution of singularities in this setting (see [Tem1]) that we may always choose Y nonsingular, with E a nonsingular divisor. The *log discrepancy* $A(\mathrm{ord}_E)$ is defined as $\mathrm{ord}_E(K_{Y/X}) + 1$, where $K_{Y/X}$ is the *relative canonical divisor*[3]. Note that the log discrepancy depends on the variety X: whenever there is some ambiguity, we denote it by $A_X(\mathrm{ord}_E)$.

There is some subtlety in computing the log discrepancy in our setting, so we discuss this briefly, and refer for details to [dFEM]. The difficulty comes from the fact that our schemes are not of finite type over a field. However, recall that all our schemes are of finite type over a formal power series ring $R = k[\![t_1, \dots, t_n]\!]$. For each such scheme Y one introduced in [dFEM] a coherent *sheaf of special differentials* $\Omega'_{Y/k}$, together with a (special) k-derivation $d'\colon \mathcal{O}_Y \to \Omega'_{Y/k}$. If $\xi \in Y$ is a nonsingular point, then $\Omega'_{Y/k,\xi}$ is a free $\mathcal{O}_{Y,\xi}$-module of rank $\dim(\mathcal{O}_{Y,\xi}) + \dim_{k(\xi)} \Omega'_{k(\xi)/k}$, where $k(\xi)$ is the residue field of ξ. Furthermore,

[2] By "ideal on X" we shall mean "coherent ideal sheaf on X" throughout the paper.

[3] In our context, this can be defined as the 0^{th} Fitting ideal of $\Omega_{Y/X}$, see [dFEM]. As usual, this is a divisor supported on the exceptional locus of π.

if $y_1,\ldots,y_r$ form a regular system of parameters at ξ, then the $d'(u_1)_\xi,\ldots,d'(u_r)_\xi$ are part of a basis of $\Omega'_{Y/k,\xi}$.

Suppose now that $\varphi\colon Y'\to Y$ is a proper birational morphism of nonsingular schemes as above. If $\xi'\in Y'$ and $\xi=\varphi(\xi')\in Y$, then the Dimension Formula (see [Mat, Theorem 15.6]) gives $\dim(\mathcal{O}_{Y',\xi'})=\dim(\mathcal{O}_{Y,\xi})-\mathrm{trdeg}(k(\xi')/k(\xi))$. On the other hand, there are sequences

$$\Omega'_{Y/k,\xi}\otimes_{\mathcal{O}_{Y,\xi}}\mathcal{O}_{Y',\xi'}\xrightarrow{T}\Omega'_{Y'/k,\xi'}\to\Omega_{Y'/Y}\to 0, \tag{1.2}$$

$$0\to\Omega'_{k(\xi)/k}\otimes_{k(\xi)}k(\xi')\to\Omega_{k(\xi')/k}\to\Omega_{k(\xi')/k(\xi)}\to 0. \tag{1.3}$$

It follows that as the 0^{th} Fitting ideal of $\Omega_{Y'/Y}$, the ideal defining the divisor $K_{Y'/Y}$ is given at ξ' by $\det(T)$. Note also that by definition $T(d'(u)\otimes 1)=d'(\varphi^*(u))$.

Lemma 1.1. *Let $\varphi\colon Y'\to Y$ be a proper birational morphism between nonsingular schemes as above. Consider $\xi'\in Y'$ and $\xi=\varphi(\xi')\in Y$, and let us choose regular systems of parameters $\underline{y}=(y_1,\ldots,y_r)$ and $\underline{y'}=(y'_1,\ldots,y'_s)$ at ξ and ξ', respectively. Suppose that*

$$\varphi^*(y_i)=u_i\cdot\prod_{j=1}^{s}(y'_j)^{b_{i,j}},$$

for every $1\leq i\leq r$, and suitable $u_i\in\mathcal{O}_{Y',\xi'}$. If D'_j denotes the closure of $V(y'_j)$, then

(i) *We have $A_Y(\mathrm{ord}_{D'_j})\geq\sum_{i=1}^r b_{i,j}$.*
(ii) *If $r=s$ and if the image of each u_i in $k(\xi')$ is nonzero, then we have equality in* (i) *if and only if $\det(b_{i,j})\neq 0$.*

Proof. With the notation in (1.2), we see that

$$T(d'(y_\ell))\in B\cdot\Omega'_{Y'/k,\xi'}+\sum_j\frac{B}{y'_j}d'(y'_j),$$

where $B=\prod_{j=1}^s(y'_j)^{b_{i,j}}$, and the sum is over those j with $b_{i,j}>0$. The assertion in (i) follows from this and from our description of $\Omega_{Y/k,\xi}$ and $\Omega_{Y'/k,\xi'}$. Furthermore, an easy (and well-known) computation shows that if $r=s$, and if we write $\det(T)=\prod_{j=1}^s(y'_j)^{b_j}\cdot g$, where $b_j+1=\sum_{i=1}^r b_{i,j}$ for every j, then the image of g in $k(\xi')$ is equal to $\det(b_{i,j})\cdot\prod_{i=1}^s\overline{u_i}$, where $\overline{u_i}$ denotes the image of u_i, which is nonzero. This gives the assertion in (ii). □

A result of [Tem1], generalizing Hironaka's theorem for varieties over a field, guarantees existence of log resolutions in our setting: given an ideal $\mathfrak{a}$ on X, there is a projective birational morphism $\pi\colon Y\to X$ such that Y is nonsingular, $\mathfrak{a}\cdot\mathcal{O}_Y$ is the ideal of a divisor D, and $D+K_{Y/X}$ is a divisor with simple normal crossings.

Recall that given a nonzero ideal $\mathfrak{a}$ and $\lambda\in\mathbf{R}_{\geq 0}$, the multiplier ideal $\mathcal{J}(\mathfrak{a}^\lambda)$ is the ideal on X consisting of those local sections f of $\mathcal{O}_X$ such that

$$\mathrm{ord}_E(f)+A(\mathrm{ord}_E)>\lambda\cdot\mathrm{ord}_E(\mathfrak{a})$$

for all divisors E over X such that f is defined at $c_X(\mathrm{ord}_E)$. In fact, it is enough to only consider those divisors E that appear on any given log resolution of $\mathfrak{a}$. This follows as in the case of schemes of finite type over a field once we have the inequality in Lemma 1.1 (i).

We make the convention that if $\mathfrak{a} = (0)$, then $\mathcal{J}(\mathfrak{a}^\lambda) = \mathcal{O}_X$ if $\lambda = 0$, and it is the zero ideal, otherwise.

1.3. **Jumping numbers.** For every ideal $\mathfrak{a}$ on X, we index the jumping numbers of $\mathfrak{a}$, as follows. Given a nonzero ideal $\mathfrak{q}$ on X, we consider the *log canonical threshold of* $\mathfrak{a}$ *with respect to* $\mathfrak{q}$

$$\mathrm{lct}^{\mathfrak{q}}(\mathfrak{a}) := \min\{\lambda \geq 0 \mid \mathfrak{q} \not\subseteq \mathcal{J}(\mathfrak{a}^\lambda)\}$$

(with the convention $\mathrm{lct}^{\mathfrak{q}}(\mathfrak{a}) = \infty$ if $\mathfrak{a} = \mathcal{O}_X$). Note that when $\mathfrak{q} = \mathcal{O}_X$, this is simply the *log canonical threshold* $\mathrm{lct}(\mathfrak{a})$ and as we vary $\mathfrak{q}$, we recover in this way all the jumping numbers of $\mathfrak{a}$, in the sense of [ELSV]. It is convenient to also consider the reciprocals of these numbers. We define the *Arnold multiplicity* of $\mathfrak{a}$ with respect to $\mathfrak{q}$ to be $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}) := \mathrm{lct}^{\mathfrak{q}}(\mathfrak{a})^{-1}$ (if $\mathfrak{q} = \mathcal{O}_X$, we simply write $\mathrm{Arn}(\mathfrak{a})$). If Z is the subscheme defined by $\mathfrak{a}$ we sometimes write $\mathrm{Arn}^{\mathfrak{q}}(Z)$ for $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a})$. Note that $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}) = 0$ if and only if $\mathfrak{a} = \mathcal{O}_X$, and $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}) = \infty$ if and only if $\mathfrak{a} = (0)$.

Lemma 1.2. *If* $\pi\colon Y \to X$ *is a log resolution of the nonzero ideal* $\mathfrak{a}$*, and if* $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-\sum_i \alpha_i E_i)$ *and* $K_{Y/X} = \sum_i \kappa_i E_i$*, then for every nonzero ideal* $\mathfrak{q}$

$$\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}) = \max_i \frac{\alpha_i}{\kappa_i + 1 + \mathrm{ord}_{E_i}(\mathfrak{q})} = \max_i \frac{\mathrm{ord}_{E_i}(\mathfrak{a})}{A(\mathrm{ord}_{E_i}) + \mathrm{ord}_{E_i}(\mathfrak{q})}. \tag{1.4}$$

Moreover, we have

(i) *If* $\mathfrak{a} \subseteq \mathfrak{b}$*, then* $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}) \geq \mathrm{Arn}^{\mathfrak{q}}(\mathfrak{b})$*;*
(ii) $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}^m) = m \cdot \mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a})$ *for every* $m \geq 1$*;*
(iii) $\mathrm{Arn}^{\mathfrak{q}_1+\mathfrak{q}_2}(\mathfrak{a}) = \max_{i=1,2} \mathrm{Arn}^{\mathfrak{q}_i}(\mathfrak{a})$*;*
(iv) $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a} \cdot \mathfrak{b}) \leq \mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}) + \mathrm{Arn}^{\mathfrak{q}}(\mathfrak{b})$ *for every ideals* $\mathfrak{a}$ *and* $\mathfrak{b}$*.*

Proof. Equation (1.4) is a consequence of the description of multiplier ideals in terms of a log resolution. Properties (i)–(iii) follow from the definition whereas (iv) is a consequence of (1.4). □

If the maximum in (1.4) is achieved for E_i, we say that ord_{E_i} *computes* $\mathrm{lct}^{\mathfrak{q}}(\mathfrak{a})$ (or $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a})$). It is natural to consider the invariants $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a})$ also for $\mathfrak{q} \neq \mathcal{O}_X$, since this case naturally appears when considering pull-backs by birational morphisms, as in

Corollary 1.3. *If* $\mu\colon X' \to X$ *is a proper birational morphism, with both* X *and* X' *nonsingular, then for every ideals* $\mathfrak{a}$*,* $\mathfrak{q}$ *on* X*, with* $\mathfrak{q}$ *nonzero, we have*

$$\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}) = \mathrm{Arn}^{\mathfrak{q}'}(\mathfrak{a}'),$$

where $\mathfrak{a}' = \mathfrak{a} \cdot \mathcal{O}_{X'}$ *and* $\mathfrak{q}' = \mathfrak{q} \cdot \mathcal{O}_{X'}(-K_{X'/X})$*.*

Proof. If $\mathfrak{a} = (0)$, then the assertion is clear. If this is not the case, let $\mu'\colon X'' \to X'$ be a log resolution of $\mathfrak{a}' \cdot \mathcal{O}_{X'}(-K_{X'/X})$, so $\mu \circ \mu'$ is a log resolution of $\mathfrak{a}$. The assertion in the corollary follows from (1.4), using the fact that for every divisor E on X'', we have $\mathrm{ord}_E(K_{X''/X}) = \mathrm{ord}_E(K_{X''/X'}) + \mathrm{ord}_E((\mu')^*(K_{X'/X}))$. □

1.4. **Regular morphisms.** In the proof of our main results, we make use of localization followed by completion, as well as field extensions. Both of these operations are special cases of regular morphisms.

Recall that a morphism $\varphi\colon X' \to X$ is *regular* if it is flat and all its fibers are geometrically regular (since we work in characteristic zero, this simply means regular). An immediate consequence of the definition is that if φ is regular and $Y \to X$ is any morphism, then $Y \times_X X' \to Y$ is regular. In particular, if Y is a regular scheme, then $Y \times_X X'$ is regular; similarly, if D is a divisor on Y having simple normal crossings, then its inverse image on $Y \times_X X'$ has simple normal crossings.

We will assume that both X and X' satisfy our usual assumptions; however, φ is not assumed to be of finite type. For an introduction to regular morphisms, see [Mat, Chapter 32].

Example 1.4. Let K/k be an extension of fields of characteristic zero. Then the induced morphism $\varphi : \mathbf{A}^n_K \to \mathbf{A}^n_k$ is regular and faithfully flat.

Example 1.5. Given a point $\xi \in X$, the canonical morphism $\varphi\colon \operatorname{Spec}\widehat{\mathcal{O}_{X,\xi}} \to X$ is regular. Indeed, since X is of finite type over a complete local Noetherian ring, X is excellent, hence φ is regular (see [Mat, Chapter 32]).

Proposition 1.6. *Let $\mathfrak{a}$ and $\mathfrak{q}$ be nonzero ideals on X. Let $\varphi\colon X' \to X$ be a regular morphism and write $\mathfrak{a}' := \mathfrak{a}\cdot\mathcal{O}_{X'}$, $\mathfrak{q}' := \mathfrak{q}\cdot\mathcal{O}_{X'}$. Then $\mathcal{J}(\mathfrak{a}'^t) = \mathcal{J}(\mathfrak{a}^t)\cdot\mathcal{O}_{X'}$ for every $t \geq 0$. In particular, $\operatorname{lct}^{\mathfrak{q}'}(\mathfrak{a}') \geq \operatorname{lct}^{\mathfrak{q}}(\mathfrak{a}) =: \lambda$ with equality if $V(\mathcal{J}(\mathfrak{a}^\lambda)\colon \mathfrak{q}) \cap \varphi(X') \neq \emptyset$. Further, the latter condition holds if φ is faithfully flat.*

Proof. Let $\mu\colon Y \to X$ be a log resolution of $\mathfrak{a}$, with $\mathfrak{a}\cdot\mathcal{O}_Y = \mathcal{O}_Y(-D)$. Note that $\mu'\colon Y' = Y \times_X X' \to X'$ is a log resolution of $\mathfrak{a}'$. Indeed, Y' is regular since Y is, and it is connected since μ' is proper, has connected fibers, and gives an isomorphism over an open subset of X'. In addition, if $p\colon Y' \to Y$ is the projection, then $\mathfrak{a}'\cdot\mathcal{O}_{Y'} = \mathcal{O}_{Y'}(-p^*(D))$, and $p^*(D) + K_{Y'/X'} = p^*(D + K_{Y/X})$ has simple normal crossings. It now follows from base-change with respect to flat morphisms that $\mathcal{J}(\mathfrak{a}'^t) = \mathcal{J}(\mathfrak{a}^t)\cdot\mathcal{O}_{X'}$ for every $t \in \mathbf{R}_{\geq 0}$.

If $0 \leq t < \lambda$, then $\mathfrak{q} \subseteq \mathcal{J}(\mathfrak{a}^t_\bullet)$, hence $\mathfrak{q}' \subseteq \mathcal{J}(\mathfrak{a}^t)\cdot\mathcal{O}_{X'}$. Therefore, $\operatorname{lct}^{\mathfrak{q}'}(\mathfrak{a}'_\bullet) \geq \operatorname{lct}^{\mathfrak{q}}(\mathfrak{a}_\bullet)$. Now suppose $\varphi^{-1}(V(\mathcal{J}(\mathfrak{a}^\lambda_\bullet)\colon \mathfrak{q})) \neq \emptyset$. In this case, since φ is flat we have $(\mathcal{J}(\mathfrak{a}'^t)\colon \mathfrak{q}') = (\mathcal{J}(\mathfrak{a}^t)\colon \mathfrak{q})\cdot\mathcal{O}_{X'} \neq \mathcal{O}_{X'}$, hence $\mathfrak{q}' \not\subseteq \mathcal{J}(\mathfrak{a}'^t)$ and so $\operatorname{lct}^{\mathfrak{q}'}(\mathfrak{a}') = \lambda$. Finally, note that since $(\mathcal{J}(\mathfrak{a}^\lambda_\bullet)\colon \mathfrak{q}) \neq \mathcal{O}_X$, if φ is surjective, then $\varphi^{-1}(V(\mathcal{J}(\mathfrak{a}^\lambda_\bullet)\colon \mathfrak{q}))$ is clearly nonempty. □

2. Graded and subadditive systems of ideals

We now introduce the main objects that we wish to study.

2.1. **Graded sequences.** A *graded sequence* of ideals $\mathfrak{a}_\bullet = (\mathfrak{a}_m)_{m\in\mathbf{Z}_{>0}}$ is a sequence of ideals on X that satisfies $\mathfrak{a}_p\cdot\mathfrak{a}_q \subseteq \mathfrak{a}_{p+q}$ for every p, $q \geq 1$. We always assume that such a sequence is *nonzero*, in the sense that $\mathfrak{a}_m \neq (0)$ for some m. Then $S = S(\mathfrak{a}_\bullet) := \{m \in \mathbf{Z}_{>0} \mid \mathfrak{a}_m \neq (0)\}$ is a subsemigroup of the positive integers (with respect to addition). By convention we put $\mathfrak{a}_0 = \mathcal{O}_X$.

Example 2.1. The most interesting geometric examples arise as follows: suppose that L is a line bundle of nonnegative Kodaira dimension on a smooth projective variety X. If $\mathfrak{a}_m$ is the ideal defining the base locus of L^m, then $(\mathfrak{a}_m)_{m\geq 1}$ is a graded sequence of ideals.

Example 2.2. To any valuation $v \in \mathrm{Val}_X^*$, we can associate a graded sequence $\mathfrak{a}_\bullet = \mathfrak{a}_\bullet(v)$ of *valuation ideals* given by $\mathfrak{a}_m(v) = \{v \geq m\}$. More precisely, for an affine open subset U of X we have $\Gamma(U, \mathfrak{a}_m) = \{f \in \mathcal{O}_X(U) \mid v(f) \geq m\}$ if $c_X(v) \in U$, and $\Gamma(U, \mathfrak{a}_m) = \mathcal{O}_X(U)$, otherwise. Note that $\mathfrak{a}_m$ is nonzero since v is nontrivial.

Following [ELMNP], one can attach asymptotic invariants to graded sequences of ideals, via the following well-known result. We include a proof, for the reader's convenience.

Lemma 2.3. *Let $(\alpha_m)_{m\geq 1}$ be a sequence of elements in $\mathbf{R}_{\geq 0} \cup \{\infty\}$, that satisfies $\alpha_{p+q} \leq \alpha_p + \alpha_q$ for all p and q. If the set $S := \{m \mid \alpha_m < \infty\}$ is nonempty, then S is a subsemigroup of $\mathbf{Z}_{>0}$, and*

$$\lim_{m\to\infty, m\in S} \frac{\alpha_m}{m} = \inf_{m\geq 1} \frac{\alpha_m}{m}.$$

Proof. Let $T := \inf_{m\geq 1} \alpha_m/m$. We need to show that for every $\tau > T$, we have $\alpha_p/p < \tau$ if $p \gg 0$, with $p \in S$. Let m be such that $\alpha_m/m < \tau$. It is enough to show that for every integer q with $0 \leq q < m$, if $p = m\ell + q \in S$ with $\ell \gg 0$, then $\alpha_p/p < \tau$.

If there is no ℓ such that $m\ell + q \in S$, then there is nothing to prove. Otherwise, let us choose ℓ_0 with $m\ell_0 + q \in S$. For $\ell \geq \ell_0$ we have

$$\frac{\alpha_{m\ell+q}}{m\ell+q} \leq \frac{\alpha_{m\ell_0+q} + (\ell-\ell_0)\alpha_m}{m\ell+q}.$$

Since the right-hand side converges to $\alpha_m/m < \tau$ for $\ell \to \infty$, it follows that

$$\frac{\alpha_{m\ell+q}}{m\ell+q} < \tau \text{ for } \ell \gg 0,$$

which completes the proof. □

Suppose now that $\mathfrak{a}_\bullet$ is a graded sequence of ideals, and $v \in \mathrm{Val}_X$. By taking $\alpha_m = v(\mathfrak{a}_m)$, we define as in [ELMNP]

$$v(\mathfrak{a}_\bullet) := \inf_{m\geq 1} \frac{v(\mathfrak{a}_m)}{m} = \lim_{m\to\infty, m\in S(\mathfrak{a}_\bullet)} \frac{v(\mathfrak{a}_m)}{m}.$$

By the definition of a graded sequence we have $v(\mathfrak{a}_{p+q}) \leq v(\mathfrak{a}_p \cdot \mathfrak{a}_q) = v(\mathfrak{a}_p) + v(\mathfrak{a}_q)$.

Lemma 2.4. *Let $\mathfrak{a}_\bullet(v)$ be the graded sequence of valuation ideals associated to a nontrivial valuation $v \in \mathrm{Val}_X^*$. Then, for any $w \in \mathrm{Val}_X$ we have*

$$w(\mathfrak{a}_\bullet(v)) = \inf \frac{w(\mathfrak{b})}{v(\mathfrak{b})},$$

where $\mathfrak{b}$ ranges over ideals on X for which $v(\mathfrak{b}) > 0$. In particular, $v(\mathfrak{a}_\bullet(v)) = 1$.

Proof. Note first that if $c := \inf\{w(\mathfrak{b})/v(\mathfrak{b}) \mid v(\mathfrak{b}) > 0\}$, then by definition we have $w(\mathfrak{a}_m(v)) \geq c \cdot v(\mathfrak{a}_m(v)) \geq cm$. Dividing by m and letting $m \to \infty$ gives $w(\mathfrak{a}_\bullet(v)) \geq c$. For the reverse inequality, it is enough to show that for every $\varepsilon > 0$, we have $w(\mathfrak{a}_\bullet(v)) < c + \varepsilon$.

By definition of c, there is an ideal $\mathfrak{b}$ on X such that $v(\mathfrak{b}) > 0$ and $w(\mathfrak{b})/v(\mathfrak{b}) < c+\varepsilon$. For every $m \geq 1$, we have $\mathfrak{b}^m \subseteq \mathfrak{a}_{\lfloor m\cdot v(\mathfrak{b})\rfloor}(v)$. Therefore $w(\mathfrak{a}_{\lfloor m\cdot v(\mathfrak{b})\rfloor}(v)) \leq m\cdot w(\mathfrak{b})$, and so

$$\frac{w(\mathfrak{a}_{\lfloor m\cdot v(\mathfrak{b})\rfloor}(v))}{\lfloor m\cdot v(\mathfrak{b})\rfloor} \leq \frac{m\cdot w(\mathfrak{b})}{\lfloor m\cdot v(\mathfrak{b})\rfloor}.$$

As $m\to\infty$ we get $w(\mathfrak{a}_\bullet(v)) \leq \frac{w(\mathfrak{b})}{v(\mathfrak{b})} < c+\varepsilon$. The last assertion about v is clear. □

Similarly, by taking $\alpha_m = \operatorname{Arn}^{\mathfrak{q}}(\mathfrak{a}_m)$, where $\mathfrak{q}$ is a nonzero ideal, we get

$$\operatorname{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet) := \inf_{m\geq 1}\frac{\operatorname{Arn}^{\mathfrak{q}}(\mathfrak{a}_m)}{m} = \lim_{m\to\infty, m\in S(\mathfrak{a}_\bullet)}\frac{\operatorname{Arn}^{\mathfrak{q}}(\mathfrak{a}_m)}{m}.$$

The fact that the conditions in the lemma are satisfied follows from the defining property of a graded sequence, together with Lemma 1.2 (iv). We also write $\operatorname{lct}^{\mathfrak{q}}(\mathfrak{a}_\bullet) = 1/\operatorname{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet)$ (with the convention $\operatorname{lct}^{\mathfrak{q}}(\mathfrak{a}_\bullet)=\infty$ if $\operatorname{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet)=0$). Note that both $v(\mathfrak{a}_\bullet)$ and $\operatorname{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet)$ are finite.

Proposition 2.5. *If $\mu\colon X'\to X$ is a proper birational morphism, with both X and X' nonsingular, then for every graded sequence of ideals $\mathfrak{a}_\bullet$ on X, and every nonzero ideal $\mathfrak{q}$ on X, we have*

$$\operatorname{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet) = \operatorname{Arn}^{\mathfrak{q}'}(\mathfrak{a}'_\bullet),$$

where $\mathfrak{a}'_m = \mathfrak{a}_m\cdot\mathcal{O}_{X'}$ and $\mathfrak{q}' = \mathfrak{q}\cdot\mathcal{O}_{X'}(-K_{X'/X})$.

Proof. The assertion follows by applying Corollary 1.3 to each $\mathfrak{a}_m \neq 0$, and then letting m go to infinity. □

2.2. **Subadditive systems.** A *subadditive system of ideals* $\mathfrak{b}_\bullet$ is a one-parameter family $(\mathfrak{b}_t)_{t\in\mathbf{R}_{>0}}$ of nonzero ideals satisfying $\mathfrak{b}_{s+t}\subseteq\mathfrak{b}_s\cdot\mathfrak{b}_t$ for every $s,t\in\mathbf{R}_{>0}$. Note that this implies $\mathfrak{b}_s\subseteq\mathfrak{b}_t$ for $s\geq t$. By convention we put $\mathfrak{b}_0=\mathcal{O}_X$.

To any such system of ideals, we can associate various invariants via the following lemma (compare with Lemma 2.2 in [Mus]).

Lemma 2.6. *If $\varphi\colon\mathbf{R}_{>0}\to\mathbf{R}_{\geq 0}$ is an increasing function such that $\varphi(mt)\geq m\varphi(t)$ for every $t\in\mathbf{R}_{>0}$ and $m\in\mathbf{Z}_{>0}$, then $\lim_{t\to\infty}\frac{\varphi(t)}{t}$ exists in $\mathbf{R}_{\geq 0}\cup\{\infty\}$, and equals $\sup_{t>0}\frac{\varphi(t)}{t}$.*

Proof. It is enough to show that for every $\tau < \sup_{t>0}\frac{\varphi(t)}{t}$, we have $\frac{\varphi(t)}{t} > \tau$ for $t\gg 0$. Choose $s>0$ such that $\frac{\varphi(s)}{s}>\tau$. We claim that $\frac{\varphi(t)}{t}>\tau$ as long as $1-\frac{s}{t} > \frac{\tau s}{\varphi(s)}$. Indeed, in this case if we choose $m\in\mathbf{Z}_{>0}$ such that $ms\leq t<(m+1)s$, then

$$\frac{\varphi(t)}{t} \geq \frac{\varphi(ms)}{ms}\cdot\frac{ms}{t} \geq \frac{\varphi(s)}{s}\cdot\left(1-\frac{s}{t}\right) > \tau,$$

which completes the proof. □

The lemma applies to $\varphi(t) := v(\mathfrak{b}_t)$ where $v\in\operatorname{Val}_X$ and $\mathfrak{b}_\bullet$ is a subadditive system of ideals. Indeed, if $s\geq t$, then $\mathfrak{b}_s\subseteq\mathfrak{b}_t$ and so $v(\mathfrak{b}_s)\geq v(\mathfrak{b}_t)$. Similarly, $\mathfrak{b}_{mt}\subseteq\mathfrak{b}_t^m$, hence $v(\mathfrak{b}_{mt})\geq v(\mathfrak{b}_t^m) = mv(\mathfrak{b}_t)$. We put $v(\mathfrak{b}_\bullet) := \lim_{t\to\infty}\frac{v(\mathfrak{b}_t)}{t}$.

Example 2.7. We may have $v(\mathfrak{b}_\bullet)=\infty$. For example, if $\mathfrak{a}$ is the ideal defining a closed point $\xi\in X$, then we have a subadditive system of ideals $\mathfrak{b}_\bullet$, where $\mathfrak{b}_t=\mathfrak{a}^{\lfloor t^2\rfloor}$ for all $t>0$. It is clear that for every $v\in \mathrm{Val}_X$ with $c_X(v)=\xi$, we have $v(\mathfrak{b}_\bullet)=\infty$. For more interesting examples, see §6.3.

Similarly, if $\mathfrak{q}$ is a nonzero ideal, it follows from Lemma 1.2 that $\varphi(t):=\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{b}_t)$ satisfies the hypotheses in Lemma 2.6. We define the *asymptotic Arnold multiplicity* of $\mathfrak{b}_\bullet$ with respect to $\mathfrak{q}$ by $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{b}_\bullet):=\lim_{t\to\infty}\frac{\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{b}_t)}{t}$. We also put $\mathrm{lct}^{\mathfrak{q}}(\mathfrak{b}_\bullet)=1/\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{b}_\bullet)$.

One can give an alternative description of the asymptotic Arnold multiplicity:

Proposition 2.8. *If $\mathfrak{b}_\bullet$ is a subadditive system of ideals, and if $\mathfrak{q}$ is a nonzero ideal, then*

$$\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{b}_\bullet)=\sup_E\frac{\mathrm{ord}_E(\mathfrak{b}_\bullet)}{A(\mathrm{ord}_E)+\mathrm{ord}_E(\mathfrak{q})}, \tag{2.1}$$

where the supremum is over all divisors E over X.

Proof. Given any divisor E over X, we have by Lemma 1.2

$$\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{b}_t)\geq\frac{\mathrm{ord}_E(\mathfrak{b}_t)}{A(\mathrm{ord}_E)+\mathrm{ord}_E(\mathfrak{q})}.$$

Dividing by t, and letting t go to infinity gives "$\geq$" in (2.1).

For the reverse inequality, fix $\tau<\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{b}_\bullet)$, pick t such that $\tau<\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{b}_t)/t$, and choose a divisor E over X such that $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{b}_t)=\frac{\mathrm{ord}_E(\mathfrak{b}_t)}{A(\mathrm{ord}_E)+\mathrm{ord}_E(\mathfrak{q})}$. Then

$$\tau<\frac{\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{b}_t)}{t}=\frac{\mathrm{ord}_E(\mathfrak{b}_t)}{t(A(\mathrm{ord}_E)+\mathrm{ord}_E(\mathfrak{q}))}\leq\frac{\mathrm{ord}_E(\mathfrak{b}_\bullet)}{A(\mathrm{ord}_E)+\mathrm{ord}_E(\mathfrak{q})},$$

which proves "$\leq$" in (2.1). □

As we will see in the next subsection, the subadditive systems that come from graded sequences of ideals are of *controlled growth* in the sense that

$$\frac{\mathrm{ord}_E(\mathfrak{b}_t)}{t}>\mathrm{ord}_E(\mathfrak{b}_\bullet)-\frac{A(\mathrm{ord}_E)}{t} \tag{2.2}$$

for every divisor E over X and every $t>0$. In particular, for such a system $\mathrm{ord}_E(\mathfrak{b}_\bullet)$ is finite for all E. The following lemma shows that if condition (2.2) holds, then we can similarly control the convergence of $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{b}_t)/t$.

Lemma 2.9. *If $\mathfrak{b}_\bullet$ is a subadditive system of controlled growth, then for every nonzero ideal $\mathfrak{q}$ and every $t>0$, we have*

$$\frac{\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{b}_t)}{t}\geq\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{b}_\bullet)-\frac{1}{t}.$$

Proof. By the definition of $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{b}_\bullet)$ it is enough to show that

$$\frac{\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{b}_t)}{t}>\frac{\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{b}_s)}{s}-\frac{1}{t}$$

for every $s > 0$. Choose a divisor E over X such that $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{b}_s) = \frac{\mathrm{ord}_E(\mathfrak{b}_s)}{A(\mathrm{ord}_E)+\mathrm{ord}_E(\mathfrak{q})}$. Using Lemma 1.2 and condition (2.2), we deduce

$$\frac{\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{b}_t)}{t} \geq \frac{\mathrm{ord}_E(\mathfrak{b}_t)}{t(A(\mathrm{ord}_E)+\mathrm{ord}_E(\mathfrak{q}))} > \left(\frac{\mathrm{ord}_E(\mathfrak{b}_s)}{s} - \frac{A(\mathrm{ord}_E)}{t}\right) \cdot \frac{1}{A(\mathrm{ord}_E)+\mathrm{ord}_E(\mathfrak{q})}$$
$$= \frac{\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{b}_s)}{s} - \frac{1}{t} \cdot \frac{A(\mathrm{ord}_E)}{A(\mathrm{ord}_E)+\mathrm{ord}_E(\mathfrak{q})} \geq \frac{\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{b}_s)}{s} - \frac{1}{t},$$

concluding the proof. □

Corollary 2.10. *If $\mathfrak{b}_\bullet$ is a subadditive system of ideals of controlled growth, then $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{b}_\bullet)$ is finite for every nonzero ideal $\mathfrak{q}$.*

2.3. **Asymptotic multiplier ideals.** Recall that the *asymptotic multiplier ideals* of a graded sequence $\mathfrak{a}_\bullet$ are defined by $\mathfrak{b}_t := \mathcal{J}(\mathfrak{a}_\bullet^t) := \mathcal{J}(\mathfrak{a}_m^{t/m})$, where m is divisible enough (depending on $t > 0$). We have $\mathfrak{a}_m \subseteq \mathfrak{b}_m$ for all m and it follows from the Subadditivity Theorem that $(\mathfrak{b}_t)_{t>0}$ is a subadditive system of ideals. As above, we set $\mathfrak{b}_0 := \mathcal{O}_X$. For more about graded sequences and their corresponding asymptotic multiplier ideals we refer to [Laz].

Next we show that the asymptotic invariants $\mathrm{lct}^{\mathfrak{q}}(\mathfrak{a}_\bullet)$ can be described in terms of the jumps of the system $\mathfrak{b}_\bullet$:

Proposition 2.11. *If $\mathfrak{a}_\bullet$ is a graded sequence of ideals, and $\mathfrak{b}_\bullet$ is the system of asymptotic multiplier ideals of $\mathfrak{a}_\bullet$, then*

$$\mathrm{lct}^{\mathfrak{q}}(\mathfrak{a}_\bullet) = \min\{\lambda \geq 0 \mid \mathfrak{q} \not\subseteq \mathfrak{b}_\lambda\}$$

for every nonzero ideal $\mathfrak{q}$.

Proof. By definition, $\mathrm{lct}^{\mathfrak{q}}(\mathfrak{a}_\bullet) = \sup_{m\geq 1} m \cdot \mathrm{lct}^{\mathfrak{q}}(\mathfrak{a}_m)$. Hence $t \geq \mathrm{lct}^{\mathfrak{q}}(\mathfrak{a}_\bullet)$ if and only if $t/m \geq \mathrm{lct}^{\mathfrak{q}}(\mathfrak{a}_m)$, or, equivalently, $\mathfrak{q} \not\subseteq \mathcal{J}(\mathfrak{a}_m^{t/m})$ for all m. On the other hand, we have $\mathcal{J}(\mathfrak{a}_m^{t/m}) \subseteq \mathfrak{b}_t$, with equality if m is divisible enough. The result follows. □

We now compare the invariants defined for $\mathfrak{a}_\bullet$ and for the corresponding system of asymptotic multiplier ideals $\mathfrak{b}_\bullet$. In the process we will see that $\mathfrak{b}_\bullet$ has controlled growth.

Proposition 2.12. *If $\mathfrak{a}_\bullet$ is a graded sequence of ideals, and $\mathfrak{b}_\bullet$ is the corresponding subadditive system given by the asymptotic multiplier ideals of $\mathfrak{a}_\bullet$, then:*

(i) *the system $\mathfrak{b}_\bullet$ has controlled growth;*
(ii) *we have $\mathrm{ord}_E(\mathfrak{a}_\bullet) = \mathrm{ord}_E(\mathfrak{b}_\bullet)$ for every divisor E over X.*

We shall later extend (ii) and show that $v(\mathfrak{a}_\bullet) = v(\mathfrak{b}_\bullet)$ for many non-divisorial valuations $v \in \mathrm{Val}_X$. See Proposition 6.2.

Proof. Given $t > 0$, consider m such that $\mathfrak{b}_t = \mathcal{J}(\mathfrak{a}_m^{t/m})$. By the definition of multiplier ideals, we have $\mathrm{ord}_E(\mathcal{J}(\mathfrak{a}_m^{t/m})) > t \cdot \frac{\mathrm{ord}_E(\mathfrak{a}_m)}{m} - A(\mathrm{ord}_E)$, hence

$$\frac{\mathrm{ord}_E(\mathfrak{b}_t)}{t} > \frac{\mathrm{ord}_E(\mathfrak{a}_m)}{m} - \frac{A(\mathrm{ord}_E)}{t} \geq \mathrm{ord}_E(\mathfrak{a}_\bullet) - \frac{A(\mathrm{ord}_E)}{t}. \tag{2.3}$$

By letting t go to infinity in (2.3) we get $\mathrm{ord}_E(\mathfrak{b}_\bullet) \geq \mathrm{ord}_E(\mathfrak{a}_\bullet)$. On the other hand, since $\mathfrak{a}_m \subseteq \mathfrak{b}_m$ for every m, we deduce $\mathrm{ord}_E(\mathfrak{b}_m) \leq \mathrm{ord}_E(\mathfrak{a}_m)$. Dividing by m and letting m go to infinity gives $\mathrm{ord}_E(\mathfrak{b}_\bullet) \leq \mathrm{ord}_E(\mathfrak{a}_\bullet)$. Therefore we have (ii), and now the assertion in (i) follows from (2.3). □

Proposition 2.13. *If $\mathfrak{a}_\bullet$ is a graded sequence of ideals, and $\mathfrak{q}$ is a nonzero ideal, then $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet) = \mathrm{Arn}^{\mathfrak{q}}(\mathfrak{b}_\bullet)$, where $\mathfrak{b}_\bullet$ is the subadditive system given by the asymptotic multiplier ideals of $\mathfrak{a}_\bullet$.*

The case $\mathfrak{q} = \mathcal{O}_X$ is Theorem 3.6 in [Mus]. We include the proof of the general case for the convenience of the reader. The key ingredient is the lemma below, which corresponds to Lemma 3.7 in [Mus].

Lemma 2.14. *If $\mathfrak{a}$ and $\mathfrak{q}$ are nonzero ideals on X, then $\mathrm{Arn}^{\mathfrak{q}}(\mathcal{J}(\mathfrak{a}^\lambda)) \geq \lambda \cdot \mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}) - 1$ for every $\lambda \in \mathbf{R}_{\geq 0}$.*

Proof. Write $J = \mathcal{J}(\mathfrak{a}^\lambda)$. It follows from the definition of the multiplier ideal that $\mathrm{ord}_E(J) > \lambda \cdot \mathrm{ord}_E(\mathfrak{a}) - A(\mathrm{ord}_E)$ for any divisor E above X. Since $\mathrm{ord}_E(\mathfrak{q}) \geq 0$, this implies

$$\mathrm{Arn}^{\mathfrak{q}}(J) \geq \frac{\mathrm{ord}_E(J)}{A(\mathrm{ord}_E) + \mathrm{ord}_E(\mathfrak{q})} \geq \lambda \cdot \frac{\mathrm{ord}_E(\mathfrak{a})}{A(\mathrm{ord}_E) + \mathrm{ord}_E(\mathfrak{q})} - 1.$$

We obtain the desired inequality by picking E that computes $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a})$. □

Proof of Proposition 2.13. Given $t > 0$, let us choose m such that $\mathfrak{b}_t = \mathcal{J}(\mathfrak{a}_m^{t/m})$. We deduce from Lemma 2.14 that

$$\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{b}_t) \geq t \cdot \frac{\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}_m)}{m} - 1 \geq t \cdot \mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet) - 1.$$

Dividing by t and letting t go to infinity gives $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{b}_\bullet) \geq \mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet)$. The opposite inequality follows from the definition of asymptotic Arnold multiplicities and the inclusions $\mathfrak{a}_m \subseteq \mathfrak{b}_m$ for all m. □

Corollary 2.15. *If $\mathfrak{a}_\bullet$ is a graded sequence of ideals and $\mathfrak{q}$ is a nonzero ideal, then*

$$\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet) = \sup_E \frac{\mathrm{ord}_E(\mathfrak{a}_\bullet)}{A(\mathrm{ord}_E) + \mathrm{ord}_E(\mathfrak{q})},$$

where the supremum is over all divisors E over X.

Proof. The assertion follows by combining Propositions 2.8, 2.12 and 2.13. □

3. Quasi-monomial valuations

We now want to extend the considerations in §1–§2 from divisorial to general real valuations. As an important intermediate step, we first study quasi-monomial valuations.

3.1. **Quasi-monomial valuations.** Let X be a scheme as before. Suppose that $\pi\colon Y \to X$ is a proper birational morphism, with Y nonsingular and irreducible, and $\underline{y} = (y_1, \dots, y_r)$ is a system of algebraic coordinates at a point $\xi \in Y$ (that is, a regular system of parameters of $\mathcal{O}_{Y,\xi}$). We use the notation $y^\beta = \prod_{i=1}^r y_i^{\beta_i}$ when $\beta = (\beta_1, \dots, \beta_r) \in \mathbf{Z}^r_{\geq 0}$, and $\langle \alpha, \beta \rangle := \sum_{i=1}^r \alpha_i \beta_i$ when $\alpha, \beta \in \mathbf{R}^r$. Let $k(\xi)$ be the residue field of $\mathcal{O}_{Y,\xi}$.

Lemma 3.1. *To every $\alpha = (\alpha_1, \dots, \alpha_r) \in \mathbf{R}^r_{\geq 0}$ one can associate a valuation $\mathrm{val}_\alpha \in \mathrm{Val}_X$ with the following property: if $f \in \mathcal{O}_{Y,\xi}$ is written in $\widehat{\mathcal{O}_{Y,\xi}}$ as $f = \sum_{\beta \in \mathbf{Z}^r_{\geq 0}} c_\beta y^\beta$, with each c_β either zero or not in the maximal ideal, then*

$$\mathrm{val}_\alpha(f) = \min\{\langle \alpha, \beta \rangle \mid c_\beta \neq 0\}. \tag{3.1}$$

The center of val_α on Y is the generic point of $\bigcap_{\alpha_i > 0} V(y_i)$. In particular, val_α is the trivial valuation if and only if $\alpha = 0$.

A valuation as in the lemma is called a *quasi-monomial* valuation. Note that the trivial valuation on X is considered quasi-monomial.

Proof. Uniqueness is clear since every element of $\widehat{\mathcal{O}_{Y,\xi}}$ can be written as in the lemma. In order to prove existence, let us choose an isomorphism $\widehat{\mathcal{O}_{Y,\xi}} \simeq k(\xi)[\![y_1, \dots, y_r]\!]$. For $f = \sum_{\beta \in \mathbf{Z}^r_{\geq 0}} c_\beta y^\beta$, with $c_\beta \in k(\xi)$, we define $\mathrm{val}_\alpha(f)$ by (3.1). It is easy to see that val_α induces a valuation of $K(Y) = K(X)$ that has the required property, and with the center as described in the lemma. □

Of course, the definition of val_α depends on the morphism π and the divisors $V(y_i)$ corresponding to the system of coordinates $\underline{y}$ at the point $\xi \in Y$. Note that the point ξ is not uniquely determined by the valuation, unless we require all entries of α be positive, in which case $\xi = c_Y(\mathrm{val}_\alpha)$. In general, ξ only lies in the closure of $c_Y(\mathrm{val}_\alpha)$; conversely, given any such ξ, the valuation admits a description as above at ξ.

In practice, instead of considering systems of coordinates, it is more convenient to consider simple normal crossings divisors. Borrowing terminology from the Minimal Model Program, we introduce

Definition 3.2. A *smooth log pair over* X is a pair (Y, D) with Y nonsingular and D a reduced effective simple normal crossings divisor, together with a proper birational morphism $\pi : Y \to X$ which is an isomorphism outside the support of D.

The set of (isomorphism classes of) smooth log pairs over X admits a partial ordering: we say that $(Y', D') \succeq (Y, D)$ if there exists a morphism $\varphi\colon Y' \to Y$ over X with $\mathrm{Supp}(D') \supseteq \mathrm{Supp}(\varphi^*(D))$. Under this ordering, any two smooth log pairs can be dominated by a third, and any smooth log pair dominates $(X, \emptyset)$.

Remark 3.3. Suppose that we have a birational morphism $W \to X$, with W regular, and D_W is a reduced, simple normal crossings divisor on W. By Nagata's compactification theorem (see [Con]), there is a proper birational morphism $\pi\colon Y \to X$ such that W is isomorphic over X to an open subset of Y. By [Tem2], we may resolve the singularities of Y by a resolution that is an isomorphism over W, and therefore assume that Y is regular. Given any such Y, we can find a reduced divisor D on Y whose restriction to W is D_W, and whose support contains the exceptional locus $\mathrm{Exc}(\pi)$ of π. Since D_W has simple normal crossings, it follows from [Tem2] that there is a proper birational morphism $\varphi\colon \widetilde{Y} \to Y$ that is an isomorphism over $W \cup (Y \smallsetminus \mathrm{Supp}(D))$ such that $\widetilde{Y}$ is nonsingular and $\widetilde{D} := \varphi^*(D)_{\mathrm{red}}$ has simple normal crossings[4]. In this case $(\widetilde{Y}, \widetilde{D})$ is a smooth log pair over X, extending (W, D_W).

We denote by $\mathrm{QM}_\xi(Y, D)$ the set of all quasi-monomial valuations v that can be described at the point $\xi \in Y$ with respect to coordinates $y_1, \ldots, y_r$ such that each y_i defines at ξ an irreducible component of D (hence ξ is the generic point of a connected component of the intersection of some of the D_i). We put $\mathrm{QM}(Y, D) = \bigcup_\xi \mathrm{QM}_\xi(Y, D)$.

Remark 3.4. Every quasi-monomial valuation belongs to some $\mathrm{QM}(Y, D)$. Indeed, suppose the valuation v is defined in coordinates $y_1, \ldots, y_r$ at ξ and let D_i be the closure of the divisor defined by (y_i). Since $D = \sum_i D_i$ has simple normal crossings in a neighborhood W of ξ, it follows from Remark 3.3 that there is a smooth log pair $(\widetilde{Y}, \widetilde{D})$ over X extending $(W, D|_W)$. Then $v \in \mathrm{QM}(\widetilde{Y}, \widetilde{D})$.

Definition 3.5. A smooth log pair (Y, D) is *adapted* to a quasi-monomial valuation v if $v \in \mathrm{QM}(Y, D)$. It is a *good* pair adapted to v if the values $v(D_i)$ that are strictly positive are also rationally independent.

The following technical lemma ensures the existence of good log pairs.

Lemma 3.6. *Let $v \in \mathrm{Val}_X$ be quasi-monomial and consider a pair (Y, D) adapted to v. Let $D_1, \ldots, D_r$ be the irreducible components of D containing $\xi = c_Y(v)$.*

(i) *If (Y, D) is a good pair adapted to v and $(Y', D') \succeq (Y, D)$, then (Y', D') is also a good pair adapted to v. Further, $\xi' = c_{Y'}(v)$ is the generic point of a connected component of the intersection of exactly r irreducible components D'_j, $1 \le j \le r$ of D', and if $\varphi\colon Y' \to Y$ is the corresponding morphism, then we can write*

$$\varphi^*(D_i) = \sum_{j=1}^{r} b_{ij} D'_j + E'_i, \quad i = 1, \ldots, r. \tag{3.2}$$

Here E'_i is an effective divisor on Y' whose support does not contain ξ' and the $r \times r$ matrix (b_{ij}) has nonnegative integer entries and nonzero determinant.

(ii) *There always exist a good pair $(Y', D') \succeq (Y, D)$ adapted to v and irreducible components $D'_1, \ldots, D'_r$ of D' such that the representation (3.2) holds. More precisely, $v \in \mathrm{QM}_{\xi'}(Y', D')$, where ξ' lies over ξ and it is the generic point of a connected component of $D'_1 \cap \cdots \cap D'_r$, each E'_i is an effective divisor whose support does not contain ξ', and the $r \times r$ matrix (b_{ij}) has nonnegative integer entries and nonzero determinant. Further, there exists $s \le r$ such that $c_{Y'}(v)$ is the generic point of a connected component of $D'_1 \cap \cdots \cap D'_s$.*

[4] Actually, the statement in [Tem2] only asserts that $\varphi^*(D)$ has normal crossings. However, resolving a normal crossings divisor is standard, so one can obtain the statement that we need.

The construction of the morphism $\varphi : Y' \to Y$ in (ii) is toric in nature. The number s is the rational rank of v; see §3.2.

Proof. In (i), let D_i, $1 \le i \le M$ and D'_j, $1 \le j \le N$ be all the irreducible components of D and D', respectively. We have $\varphi^* D_i = \sum_j b_{ij} D'_j$ for nonnegative integers b_{ij}. After re-indexing, we may suppose that $v(D_i) > 0$ and $v(D'_j) > 0$ if and only if $i \le r$ and $j \le s$, respectively. Note that $c_Y(v)$ is the generic point of a component of $\bigcap_{i \le r} D_i$ and $c_{Y'}(v) \in \bigcap_{j \le s} D'_j$. Since $\varphi(c_{Y'}(v)) = c_Y(v)$, we have $\dim(\mathcal{O}_{Y,c_Y(v)}) \ge \dim(\mathcal{O}_{Y',c_{Y'}(v)})$ by the Dimension Formula (see [Mat, Theorem 15.6]), hence $s \le r$. But, by assumption, the values $v(D_i) = \sum_{j=1}^s b_{ij} v(D'_j)$, $i \le r$ are rationally independent. This implies that $s = r$, that $c_{Y'}(v)$ is the generic point of a component of $\bigcap_{j \le r} D'_j$, that the matrix $(b_{ij})_{i,j=1}^r$ has maximal rank r, and that the values $v(D'_j)$, $1 \le j \le r$ are rationally independent. This completes the proof of (i).

We now turn to (ii). Given a system of coordinates $\underline{y} = (y_1, \dots, y_r)$ at $\xi = c_Y(v)$ such that $D_i = V(y_i)$, we get a morphism $h\colon \operatorname{Spec}(\mathcal{O}_{Y,\xi}) \to \operatorname{Spec}(\mathcal{O}_{\mathbf{A}^r_{\mathbf{Q}},0})$. Note that h is formally smooth, and since $\mathcal{O}_{\mathbf{A}^r_{\mathbf{Q}},0}$ is excellent, it follows by the main theorem in [And] that h is a regular morphism. We call a proper birational morphism $\varphi\colon Y' \to Y$ *toroidal* (with respect to $\underline{y}$)[5] if there is a proper birational morphism of toric varieties $\psi\colon Z = Z(\Delta) \to \mathbf{A}^r_{\mathbf{Q}}$, with Z nonsingular, such that φ and ψ induce isomorphic schemes over $\operatorname{Spec}(\mathcal{O}_{Y,\xi})$ via base-change. The morphism ψ is defined by a fan Δ refining the standard cone defining $\mathbf{A}^r_{\mathbf{Q}}$, and the fact that Z is nonsingular is equivalent with Δ being regular, which means that each cone of Δ is generated by part of a basis for $\mathbf{Z}^r$ (we refer to [Ful] for basic facts on toric varieties and toric morphisms). Note that since h is regular and Z is nonsingular, Y' is nonsingular in a neighborhood of $\varphi^{-1}(\xi)$. On Y' we have finitely many distinguished points lying over ξ (corresponding to the torus-fixed closed points on Z). At each of these points we have a system of toroidal coordinates $\underline{y}' = (y'_1, \dots, y'_r)$ induced by the toric coordinates at the corresponding point on Z (we use again the fact that h is regular). These are uniquely determined up to reordering. One can write $y_i = \prod_j (y'_j)^{b_{i,j}}$, with $b_{i,j} \in \mathbf{Z}_{\ge 0}$, and $\det(b_{i,j}) = \pm 1$.

Starting with a toric proper birational morphism $Z \to \mathbf{A}^r_{\mathbf{Q}}$, with Z nonsingular, there exists a smooth log pair (Y', D') dominating (Y, D), such that $Y' \times_Y \operatorname{Spec} \mathcal{O}_{Y,\xi} \simeq Z \times_{\mathbf{A}^r_{\mathbf{Q}}} \operatorname{Spec} \mathcal{O}_{Y,\xi}$, and such that the toroidal coordinates on Y' define irreducible components of D'. This is a consequence of Remark 3.3.

Given a toroidal morphism $\varphi\colon Y' \to Y$ corresponding to $Z = Z(\Delta) \to \mathbf{A}^r_{\mathbf{Q}}$, we have an affine open cover of $Y'_\xi = \operatorname{Spec} \mathcal{O}_{Y,\xi} \times_Y Y'$ by subsets U_i, induced by the toric affine open subsets on Z. If $\eta' = c_{Y'}(v)$, then $\varphi(\eta') = \xi$, hence there is i such that $\eta' \in U_i$. We have toroidal coordinates $\underline{y}' = (y'_1, \dots, y'_r)$ on U_i such that $y_i = \prod_j (y'_j)^{b_{i,j}}$, with $b_{i,j} \in \mathbf{Z}_{\ge 0}$, and $\det(b_{i,j}) = \pm 1$. Since $\eta' \in U_i$, it follows that $\alpha'_i := v(y'_i) \ge 0$, and we have $\alpha_i = \sum_j b_{i,j} \alpha'_i$. Since the matrix $(b_{i,j})_{i,j=1}^r$ induces a bijection between the monomials in $\underline{y}'$ and the monomials in $\underline{y}$, it is clear that in terms of the coordinates on Y' we have

[5]This is an ad-hoc definition, although related to the usual notion of *toroidal morphism*, see [KKMS].

$v = \mathrm{val}_{\alpha'}$. In particular, if (Y', D') is a smooth log pair such that the closure of each $V(y'_i)$ is a component of D', then (Y', D') is adapted to v.

To complete the proof of (ii) it therefore suffices to prove the following statement. Let $\alpha = (\alpha_1, \dots, \alpha_r) \in \mathbf{R}^r_{>0}$ be any vector and set $s := \dim_{\mathbf{Q}} \sum_i \mathbf{Q}\alpha_i$. Then there exists a regular fan Δ in $\mathbf{Z}^r$ refining the standard fan Δ_1 defining $\mathbf{A}^r_{\mathbf{Q}}$ such that α belongs to the relative interior of a cone of dimension s. To construct Δ, first pick a vector space $W_{\mathbf{Q}} \subseteq \mathbf{Q}^r$ of dimension s such that $\alpha \in W := W_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathbf{R}$. Let σ_1 be any rational simplicial s-dimensional cone $\sigma_1 \subseteq \mathbf{R}^r_{\geq 0} \cap W$ containing α in its interior. Let Δ_1 be any simplicial fan refining Δ_0 and having σ_1 as one of its cones. Now refine Δ_1 to a regular fan Δ using barycentric subdivision as in [Ful, §2.6]. Then Δ will contain a cone $\sigma \subseteq \sigma_1$ containing α in its interior.

Alternatively, the toric birational morphism $Z(\Delta) \to \mathbf{A}^r_{\mathbf{Q}}$ can be constructed explicitly using Perron transformations as in [Zar, Theorem 1]. □

It follows from Lemma 3.6 that given finitely many quasi-monomial valuations $v_1, \dots, v_m$ in Val_X, there exists a log pair (Y, D) which is good and adapted to all the v_i. Furthermore, given finitely many ideals $\mathfrak{a}_1, \dots, \mathfrak{a}_p$ on X, we may assume that (Y, D) gives a log resolution of the product $\mathfrak{a} = \mathfrak{a}_1 \cdot \ldots \cdot \mathfrak{a}_p$: this means that $Y \to X$ is a log resolution of $\mathfrak{a}$ with the inverse image of $V(\mathfrak{a})$ being contained in the support on D.

3.2. **Abhyankar valuations.** Next we recall how to recognize a quasi-monomial valuation algebraically, in terms of its numerical invariants. This will be very useful in the sequel. The *rational rank* $\mathrm{ratrk}(v)$ of a valuation $v \in \mathrm{Val}_X$ is equal to $\dim_{\mathbf{Q}}(\Gamma_v \otimes_{\mathbf{Z}} \mathbf{Q})$, where $\Gamma_v := v(K(X)^*)$ is the value group of v. If k_v and $k(\xi)$ are the residue fields of the valuation ring $\mathcal{O}_v$ and of $\mathcal{O}_{X,\xi}$, respectively, where $\xi = c_X(v)$, then the *transcendence degree* of v is defined as $\mathrm{trdeg}_X(v) = \mathrm{trdeg}(k_v/k(\xi))$. Note that if $\pi\colon Y \to X$ is proper and birational, with Y nonsingular, and $\eta = c_Y(v)$, then $\dim(\mathcal{O}_{Y,\eta}) = \dim(\mathcal{O}_{X,\xi}) - \mathrm{trdeg}(k(\eta)/k(\xi))$ (this follows from the Dimension Formula since π is birational, see [Mat, Theorem 15.6]). This formula can be used to deduce that $\mathrm{trdeg}_X(v)$ is the maximum of $\dim(\mathcal{O}_{X,c_X(v)}) - \dim(\mathcal{O}_{Y,c_Y(v)})$, where the maximum is over all morphisms $Y \to X$ as above.

In this setting, the Abhyankar inequality holds (see [Vaq]):

$$\mathrm{ratrk}(v) + \mathrm{trdeg}_X(v) \leq \dim(\mathcal{O}_{X,\xi}). \tag{3.3}$$

A valuation for which equality is achieved is an *Abhyankar valuation*. Another application of the Dimension formula implies that if $\pi\colon Y \to X$ is proper and birational, with Y nonsingular, then v is an Abhyankar valuation over X if and only if it is an Abhyankar valuation over Y.

Proposition 3.7. *A valuation $v \in \mathrm{Val}_X$ is an Abhyankar valuation if and only if it is quasi-monomial. Moreover, in this case there exists a good smooth log pair (Y, D) adapted to v such that $\varphi(D) \subseteq \overline{c_X(v)}$, where $\varphi\colon Y \to X$ is the associated birational morphism.*

Proof. Let $\xi = c_X(v)$. First suppose v is quasi-monomial and pick a good pair (Y, D) adapted to v. Let $D_1, \dots, D_r$ be the irreducible components of D containing the center

ξ. Then $\dim \mathcal{O}_{Y,\xi} = r$. By assumption, the values $v(D_i)$, $1 \leq i \leq r$, are rationally independent, so $\mathrm{ratrk}(v) = r$. On the other hand, $\mathrm{trdeg}_Y(v) = 0$. Thus v is an Abhyankar valuation.

Conversely, it was shown in [ELS] that every Abhyankar valuation v is quasi-monomial.[6] We sketch the main idea in the proof, slightly modified in order to guarantee $\varphi(D) \subseteq \overline{\xi}$. Note that we may blow-up any closed subset of $\overline{\xi}$: the resulting W over X might be singular, but we may replace W by $W' \to W$ that is an isomorphism over $X \smallsetminus \overline{\xi}$, with W' nonsingular.

Let J denote the ideal defining $\overline{\xi}$. One knows that if v is an Abhyankar valuation of $K(X)$, then the value group Γ_v is a finitely generated free abelian group. We first find a proper morphism $Y \to X$ that is an isomorphism over $X \smallsetminus \overline{\xi}$, with Y nonsingular, such that $\dim \mathcal{O}_{X,\xi} - \dim \mathcal{O}_{Y,\eta} = \mathrm{trdeg}_X(v)$, where $\eta = c_Y(v)$, and there are $f_1, \ldots, f_r \in \mathcal{O}_{Y,\eta}$ such that $v(f_1), \ldots, v(f_r)$ give a basis of Γ_v. Indeed, in order to obtain both conditions, it is enough to perform finitely many times the following operation: given $g, h \in \mathcal{O}_{X,\xi}$, we blow-up a closed subset in $\overline{\xi}$ to get $W \to X$ such that there is $Q \in \mathcal{O}_{W,c_W(v)}$ with $v(Q - \frac{g}{h}) > 0$ or $v(Q - \frac{h}{g}) > 0$. For this it is enough to blow-up the subscheme defined by $(g,h) + J^N$, where $N \cdot v(J) > \max\{v(g), v(h)\}$.

Suppose now that Y is as above, and the f_i are defined in a neighborhood U of ξ, and consider any nonsingular Y' with $\varphi\colon Y' \to Y$ proper and birational such that $\varphi^{-1}(U) \to U$ is a log resolution of $\prod_{i=1}^r ((f_i) + J^N)$, where $N \cdot v(J) > \max_i\{v(f_i)\}$. One can easily see that if $\xi' = c_{Y'}(v)$, then we have coordinates $y'_1, \ldots, y'_r$ at ξ' such that

$$\big((f_i) + J^N\big) \cdot \mathcal{O}_{Y',\xi'} = \left(\prod_{j=1}^r (y'_j)^{b_{ij}}\right), \text{ with } b_{ij} \in \mathbf{Z}_{\geq 0} \text{ and } \det(b_{ij}) = \pm 1, \tag{3.4}$$

and $v(y'_1), \ldots, v(y'_s)$ are linearly independent over $\mathbf{Q}$. It is then clear that v is equal to the quasi-monomial valuation attached to $(v(y'_1), \ldots, v(y'_r))$ in this system of coordinates. One more application of Remark 3.3 gives the conclusion of the proposition. □

Remark 3.8. The trivial valuation is quasi-monomial with rational rank zero. A valuation $v \in \mathrm{Val}_X$ is *divisorial*, that is, a positive multiple of a valuation ord_E, if and only if it is a quasi-monomial valuation with rational rank one.

3.3. **Completion and field extension.** Using the numerical invariants, we now show that the set of quasi-monomial valuations is preserved under two important operations: localization followed by completion; and algebraic field extensions.

Lemma 3.9. *Let ξ be a point on X, and consider the canonical morphism $\varphi\colon X' = \mathrm{Spec}\, R \to X$, where $R = \widehat{\mathcal{O}_{X,\xi}}$. If $v' \in \mathrm{Val}_{X'}$ has center the closed point, and if $v \in \mathrm{Val}_X$ is induced from v' by restriction, then $\mathrm{trdeg}_{X'}(v') = \mathrm{trdeg}_X(v)$ and $\mathrm{ratrk}(v') = \mathrm{ratrk}(v)$. In particular, v is quasi-monomial if and only if v' is quasi-monomial.*

Proof. If $\mathfrak{m}$ is the maximal ideal in R, then $\alpha := v'(\mathfrak{m}) > 0$. Given $f \in R$, let $g \in \mathcal{O}_{X,\xi}$ be such that $(f - g) \in \mathfrak{m}^n$, where $n\alpha > v'(f)$. In this case $v'(f - g) > v'(f)$,

[6] While in [ELS] one considers an algebraic variety over a field, the proof therein also works in our more general framework.

hence $v'(f)=v'(g)$. This shows that v' and v have the same value groups. In particular, $\mathrm{ratrk}(v')=\mathrm{ratrk}(v)$.

Denote by $(\mathcal{O}_{v'},\mathfrak{m}_{v'})$ and $(\mathcal{O}_v,\mathfrak{m}_v)$ the valuation rings corresponding to v' and v, respectively. The equality $\mathrm{trdeg}_{X'}(v')=\mathrm{trdeg}_X(v)$ is equivalent to the field extension $\mathcal{O}_v/\mathfrak{m}_v\hookrightarrow\mathcal{O}_{v'}/\mathfrak{m}_{v'}$ being algebraic. In fact, we will show that $\mathcal{O}_v/\mathfrak{m}_v=\mathcal{O}_{v'}/\mathfrak{m}_{v'}$. Given a nonzero $u\in\mathcal{O}_{v'}$, write $u=\frac{f}{f_1}$, with f, $f_1\in R$. As above, let us consider g, $g_1\in\mathcal{O}_{X,\xi}$ with $v'(f-g)>v'(f)$ and $v'(f_1-g_1)>v'(f_1)$. In particular, we have $v'(f)=v'(g)$ and $v'(f_1)=v'(g_1)$. Since $\frac{f}{f_1}-\frac{g}{g_1}=\frac{fg_1-f_1g}{f_1g_1}$ and $v'(fg_1-f_1g)=v'((f-g)g_1+g(g_1-f_1))>v'(f_1g_1)$, it follows that the class of $\frac{f}{f_1}$ in $\mathcal{O}_{v'}/\mathfrak{m}_{v'}$ lies in $\mathcal{O}_v/\mathfrak{m}_v$. This completes the proof. □

Lemma 3.10. *Let $k\subset K$ be an algebraic field extension, and $\varphi\colon\mathbf{A}^n_K\to\mathbf{A}^n_k$ the corresponding morphism of affine spaces. Suppose that v' is a valuation of $K(x_1,\dots,x_n)$ with center on $\mathbf{A}^n_K$, and let v be its restriction to $k(x_1,\dots,x_n)$. Then* $\mathrm{trdeg}_{\mathbf{A}^n_k}(v)=\mathrm{trdeg}_{\mathbf{A}^n_K}(v')$ *and* $\mathrm{ratrk}(v)=\mathrm{ratrk}(v')$. *In particular, v is quasi-monomial if and only if v' is quasi-monomial.*

Proof. Let $(\mathcal{O}_v,\mathfrak{m}_v)$ and $(\mathcal{O}_{v'},\mathfrak{m}_{v'})$ be the valuation rings of v and v', respectively. Note that we have a local homomorphism $\mathcal{O}_v\hookrightarrow\mathcal{O}_{v'}$. Since the extension $k[x_1,\dots,x_n]\hookrightarrow K[x_1,\dots,x_n]$ is integral, in order to show that $\mathrm{trdeg}_{\mathbf{A}^n_k}(v)=\mathrm{trdeg}_{\mathbf{A}^n_K}(v')$ it is enough to show that the field extension $\mathcal{O}_v/\mathfrak{m}_v\hookrightarrow\mathcal{O}_{v'}/\mathfrak{m}_{v'}$ is algebraic. Given $f\in\mathcal{O}_{v'}$, there is an equation

$$\sum_{i=0}^m c_if^i=0, \tag{3.5}$$

with $c_i\in k(x_1,\dots,x_n)$ not all zero. If $v(c_j)=\min_i v(c_i)$, then $c_i/c_j\in\mathcal{O}_v$ for all i. Dividing by c_j in (3.5), we see that $\overline{f}\in\mathcal{O}_{v'}/\mathfrak{m}_{v'}$ is algebraic over $\mathcal{O}_v/\mathfrak{m}_v$.

Since $k(x_1,\dots,x_n)\subseteq K(x_1,\dots,x_n)$, in order to show that $\mathrm{ratrk}(v)=\mathrm{ratrk}(v')$ it is enough to show that for every $f\in K[x_1,\dots,x_n]$, some integer multiple of $v'(f)$ lies in the value group of v. Consider an equation (3.5) satisfied by f. We can find $i\neq j$ such that $v'(c_if^i)=v'(c_jf^j)$. Hence $(j-i)v'(f)=v(c_i)-v(c_j)$ lies in the value group of v. □

4. Structure of valuation space

Next we investigate the structure of the valuation space Val_X. We show that it is a projective limit of simplicial cone complexes and endowed with a natural integral affine structure. This gives a way of approximating a valuation by quasi-monomial valuations. Our discussion largely follows [BFJ1], with some details added and some modifications made due to the fact that our setting here is slightly different. The exposition also draws on [KS] and bears some resemblance to [Pay].

4.1. **Topology and ordering.** Recall from §1 that we can view the elements of Val_X as either real valuations of the function field of X or as $\mathbf{R}_{\geq 0}$-valued homomorphisms of the semiring of ideals on X. This leads to two natural topologies τ and σ on Val_X. Namely, σ is the weakest topology for which the evaluation map $\mathrm{Val}_X\ni v\to\varphi_f(v):=v(f)$ is continuous for all nonzero rational functions f on X. Similarly, τ is the weakest topology for which the evaluation map $\mathrm{Val}_X \ni v \to \varphi_{\mathfrak{a}}(v) := v(\mathfrak{a})$ is continuous for all nonzero ideals $\mathfrak{a}$ on X.

Lemma 4.1. *The two topologies σ and τ defined above coincide.*

Proof. First suppose that X is affine. Since $v(f/g) = v(f) - v(g)$, we see that σ is the weakest topology that makes all maps φ_f, with $f \in \mathcal{O}(X)$, continuous. In particular, τ is finer than σ. On the other hand, if an ideal $\mathfrak{a}$ is generated by $f_1, \ldots, f_r$, then $\varphi_{\mathfrak{a}} = \min_i \varphi_{f_i}$. Therefore σ is finer than τ, which completes the proof in the affine case.

Next, note that if U is an open subset of X, then the two topologies on $\mathrm{Val}_U \subseteq \mathrm{Val}_X$ are just the subspace topologies with respect to σ and τ on Val_X. For σ this is clear, while for τ this follows from the fact that every coherent ideal sheaf on U is the restriction of a coherent ideal sheaf on X.

Now, if $U \subseteq X$ is open and affine, $\mathrm{Val}_U \subseteq \mathrm{Val}_X$ is closed in Val_X in both the σ and τ topologies. Indeed, if J is the ideal defining $X \setminus U$, with the reduced scheme structure, then $\mathrm{Val}_U = \{v \in \mathrm{Val}_X \mid v(J) = 0\}$, hence it is τ-closed. On the other hand, we also have $\mathrm{Val}_U = \bigcap_{h \in \mathcal{O}(U)} \{v \in \mathrm{Val}_X \mid v(h) \geq 0\}$, hence Val_U is also σ-closed. If we cover X by finitely many affine open subsets U_i, we now deduce the assertion in the lemma for X from the assertion for the U_i. □

Remark 4.2. It follows from the above proof that the map

$$\mathrm{Val}_X \ni v \stackrel{c_X}{\to} c_X(v) \in X$$

is "anticontinuous" in the sense that the inverse image of any open subset is closed.

Definition 4.3. If $v, w \in \mathrm{Val}_X$, then we say that $v \leq w$ if $v(\mathfrak{a}) \leq w(\mathfrak{a})$ for all (nonzero) ideals $\mathfrak{a}$ on X.

This clearly defines a partial ordering under which the trivial valuation is the unique minimal element. Note that this order relation depends on the model X.

Lemma 4.4. *We have $v \leq w$ if and only if $\eta := c_X(w) \in \overline{c_X(v)}$ and $w(f) \geq v(f)$ for any $f \in \mathcal{O}_{X,\eta}$.*

Proof. Let $\xi := c_X(v)$. First suppose $v \leq w$. If J is the ideal defining $\overline{\xi}$ with the reduced scheme structure, then $w(J) \geq v(J) > 0$, so $\eta \in \overline{\xi}$. Pick $f \in \mathcal{O}_{X,\eta} \subseteq \mathcal{O}_{X,\xi}$ and let $\mathfrak{a}$ be an ideal on X for which $\mathfrak{a} \cdot \mathcal{O}_{X,\eta}$ is principal and generated by f. Then $v(f) = v(\mathfrak{a}) \leq w(\mathfrak{a}) = w(f)$.

Conversely, suppose $\eta \in \overline{\xi}$ and that $v(f) \leq w(f)$ for $f \in \mathcal{O}_{X,\eta}$. For any ideal $\mathfrak{a}$ we then have $v(\mathfrak{a}) = \min_{f \in \mathfrak{a} \cdot \mathcal{O}_{X,\xi}} v(f) \leq \min_{f \in \mathfrak{a} \cdot \mathcal{O}_{X,\eta}} v(f) \leq \min_{f \in \mathfrak{a} \cdot \mathcal{O}_{X,\eta}} w(f) = w(\mathfrak{a})$. □

4.2. **Simplicial cone complexes and integral affine structure.** Next we investigate the structure of the subset $\mathrm{QM}(Y, D) \subseteq \mathrm{Val}_X$ for a given smooth log pair (Y, D) over X.

Lemma 4.5. *If (Y, D) is a smooth log pair over X, and if ξ is the generic point of a connected component of the intersection of r irreducible components $D_1, \ldots, D_r$ of D, then the map $\mathrm{QM}_{\xi}(Y, D) \to \mathbf{R}^r$ defined by $v \to (v(D_1), \ldots, v(D_r))$ gives a homeomorphism onto the cone $\mathbf{R}^r_{\geq 0}$.*

Proof. It is clear that this map gives a bijection of $\mathrm{QM}_\xi(Y,D)$ onto $\mathbf{R}^r_{\geq 0}$. The map is continuous since by definition of the topology, $v \to v(D_i)$ is continuous for each i. After unwinding definitions, the continuity of the inverse map boils down to the fact that for any subset $B \subseteq \mathbf{Z}^r_{\geq 0}$, the function $\alpha \to \min_{\beta\in B}\langle \alpha, \beta\rangle$ on $\mathbf{R}^r_{\geq 0}$ is continuous. □

Thus $\mathrm{QM}(Y,D)$ is the union of finitely many simplicial cones $\mathrm{QM}_\xi(Y,D)$. Each of these cones is closed in $\mathrm{QM}(Y,D)$. Indeed, $\mathrm{QM}_\xi(Y,D)$ consists of those $v \in \mathrm{QM}(Y,D)$ such that $v(D_j) = 0$ for $D_j \not\ni \xi$, and such that $c_X(v)$ does not lie on any of the connected components of $\bigcap_{D_j \ni \xi} D_j$ not containing ξ (for the fact that these are closed conditions, see Lemma 4.1 and Remark 4.2). This allows us to view $\mathrm{QM}(Y,D)$ as a *simplicial cone complex*.

Following [KKMS] one can equip $\mathrm{QM}(Y,D)$ with an integral affine structure. We shall not discuss this in detail here, but simply define an *integral linear function* on $\mathrm{QM}(Y,D)$ to be a map $\mathrm{QM}(Y,D) \to \mathbf{R}$ whose restriction to each $\mathrm{QM}_\xi(Y,D)$ is integral linear under the homeomorphism in Lemma 4.5. We can similarly define *integral linear maps* $\mathrm{QM}(Y',D') \to \mathrm{QM}(Y,D)$ (in this case we require that each $\mathrm{QM}_{\xi'}(Y',D')$ is mapped to some $\mathrm{QM}_\xi(Y,D)$). Every such map is continuous.

4.3. **Retraction.** Given a smooth log pair (Y,D) over X, we define a *retraction* map

$$r_{Y,D} : \mathrm{Val}_X \to \mathrm{QM}(Y,D).$$

This maps a valuation v to the unique quasi-monomial valuation $w := r_{Y,D}(v) \in \mathrm{QM}(Y,D)$ such that $w(D_i) = v(D_i)$ for every irreducible component D_i of D. Note that $c_Y(v) \in \overline{\{c_Y(w)\}}$. Clearly $r_{Y,D}$ is the identity on $\mathrm{QM}(Y,D)$ and it is not hard to see that $r_{Y,D}$ is continuous. This justifies the terminology "retraction".

Lemma 4.6. *If $(Y',D') \succeq (Y,D)$ are smooth log pairs, then $r_{Y,D} \circ r_{Y',D'} = r_{Y,D}$. Furthermore, $r_{Y,D}\colon \mathrm{QM}(Y',D') \to \mathrm{QM}(Y,D)$ is integral linear.*

Proof. Let $D_1,\ldots,D_M$ and $D'_1,\ldots,D'_N$ be the irreducible components of D and D', respectively. For the first assertion, it suffices to show that v and $v' := r_{Y',D'}(v)$ take the same values on D_i, $1 \leq i \leq M$. By assumption we have a birational morphism $\varphi : Y' \to Y$ over X and $\varphi^*(D_i) = \sum_{j=1}^N b_{ij} D'_j$ for $1 \leq i \leq M$, where $b_{ij} \geq 0$. Thus

$$v(D_i) = \sum_j b_{ij} v(D'_j) = \sum_j b_{ij} v'(D'_j) = v'(D_i).$$

For the second assertion, let ξ' be the generic point of a connected component of s of the D'_j, say $D'_1,\ldots,D'_s$. Suppose that $D_1,\ldots,D_r$ are the irreducible components of D that contain $\varphi(\xi')$, and let ξ be the generic point of the connected component of $D_1 \cap \cdots \cap D_r$ that contains $\varphi(\xi')$. In this case $r_{Y,D}$ induces a map $\mathrm{QM}_{\xi'}(Y',D') \to \mathrm{QM}_\xi(Y,D)$, that under the identifications $\mathrm{QM}_{\xi'}(Y',D') \simeq \mathbf{R}^s_{\geq 0}$ and $\mathrm{QM}_\xi(Y,D) \simeq \mathbf{R}^r_{\geq 0}$ provided by Lemma 4.5 is given by the matrix $(b_{i,j})$, with $1 \leq i \leq r$ and $1 \leq j \leq s$. □

Lemma 4.7. *Let (Y,D) be a smooth log pair and $v \in \mathrm{Val}_X$. If $w := r_{Y,D}(v)$ and $\xi = c_Y(v)$, then $w(f) \leq v(f)$ for any $f \in \mathcal{O}_{Y,\xi}$. Equality holds if the support of $V(f)$ is locally contained in the support of D at ξ.*

Using Lemma 4.4 we obtain

Corollary 4.8. *For every $v \in \mathrm{Val}_X$, we have $r_{Y,D}(v) \le v$ in the sense of Definition 4.3. More precisely, for any ideal $\mathfrak{a}$ on X we have $r_{Y,D}(v)(\mathfrak{a}) \le v(\mathfrak{a})$, with equality if (Y,D) gives a log resolution of $\mathfrak{a}$.*

Proof of Lemma 4.7. Let $y_1,\dots,y_r$ be a regular system of parameters at ξ such that every component of D passing through ξ is defined by some (y_i). We have $v(y_j) \ge w(y_j)$ for every j, with equality if (y_j) defines a component of D. Write $f \in \widehat{\mathcal{O}_{Y,\xi}}$ as $f = \sum_\beta c_\beta y^\beta$, with c_β either zero, or not in the maximal ideal. Then $w(c_\beta) = v(c_\beta) = 0$ whenever c_β is nonzero, and $v(y^\beta) \ge w(y^\beta)$, with equality if $\beta_i = 0$ whenever (y_i) does not define a component of D. Therefore

$$v(f) \ge \min\{\langle \alpha, \beta\rangle \mid c_\beta \ne 0\} = w(f),$$

with equality if $V(f)$ is supported on D around ξ. □

4.4. **Structure theorem.** We are now in position to exhibit Val_X as a projective limit of simplicial cone complexes.

Theorem 4.9. *The retraction maps induce a homeomorphism*

$$r : \mathrm{Val}_X \to \varprojlim_{(Y,D)} \mathrm{QM}(Y,D).$$

Proof. The map r is continuous since each $r_{Y,D}$ is. Let us construct its inverse. An element of the projective limit is a compatible family of valuations $(v_{Y,D})$. To such a family we associate the function v that on an ideal $\mathfrak{a}$ on X takes the value $v(\mathfrak{a}) := \sup_{(Y,D)} v_{Y,D}(\mathfrak{a})$. By Corollary 4.8 the supremum is attained whenever (Y,D) defines a log resolution of $\mathfrak{a}$. It is easy to check that v defines a valuation in Val_X whose center on X is the unique minimal element among the centers of all the $v_{Y,D}$. We see that r is a continuous bijection. The continuity of r^{-1} follows from Lemma 4.1 and Corollary 4.8. □

Corollary 4.10. *The set of quasimonomial valuations is dense in Val_X. Moreover, if $v \in \mathrm{Val}_X$, then given any neighborhood U of v in Val_X there exists a smooth log pair (Y,D) adapted to v such that $r_{Y,D}(v) \in U$ and such that $\varphi(D) \subseteq \overline{c_X(v)}$, where $\varphi\colon Y \to X$ is the induced morphism.*

Proof. The result is an immediate consequence of Theorem 4.9, except for the requirement that $\varphi(D) \subseteq c_X(v)$. To have this last property, it suffices to show that for any ideal $\mathfrak{a}$ on X there exists a smooth log pair (Y,D) above X such that $\varphi(D) \subseteq c_X(v)$ and $r_{Y,D}(v)(\mathfrak{a}) = v(\mathfrak{a})$. Let $\mathfrak{m}$ be the ideal defining $\overline{c_X(v)}$ with the reduced structure, and pick $n > v(\mathfrak{a})/v(\mathfrak{m})$. Then $v(\mathfrak{a}+\mathfrak{m}^n) = v(\mathfrak{a})$. Now $V(\mathfrak{a}+\mathfrak{m}^n) \subseteq \overline{c_X(v)}$, so there exists a log resolution (Y,D) of $\mathfrak{a}+\mathfrak{m}^n$ such that $\varphi(D) \subseteq \overline{c_X(v)}$. Then

$$v(\mathfrak{a}) = v(\mathfrak{a}+\mathfrak{m}^n) = r_{Y,D}(v)(\mathfrak{a}+\mathfrak{m}^n) \le r_{Y,D}(v)(\mathfrak{a}),$$

hence $v(\mathfrak{a}) = r_{Y,D}(\mathfrak{a})$ by Corollary 4.8. □

Remark 4.11. The set of divisorial valuations is also dense in Val_X, as it is easy to see they are dense in every $\mathrm{QM}(Y,D)$.

5. Log discrepancy

Our next goal is to define the log discrepancy of quasi-monomial, and more general valuations.

5.1. Log discrepancy of quasi-monomial valuations.

Proposition 5.1. *One can associate to every quasi-monomial valuation $v \in \mathrm{Val}_X$ a nonnegative real number $A_X(v)$, its* log discrepancy*, such that*

(i) *A_X coincides with our old definition for divisorial valuations;*
(ii) *for any smooth log pair (Y, D) over X, A_X is integral linear on $\mathrm{QM}(Y, D)$;*
(iii) *for any proper birational morphism $\varphi\colon X' \to X$, with X' nonsingular, and any quasi-monomial valuation $v \in \mathrm{Val}_X$, we have $A_X(v) = A_{X'}(v) + v(K_{X'/X})$.*

Conditions (i) and (ii) together can be rephrased by saying that if $v \in \mathrm{QM}(Y, D)$, then

$$A_X(v) = \sum_{i=1}^{N} v(D_i) \cdot A_X(\mathrm{ord}_{D_i}) = \sum_{i=1}^{N} v(D_i) \cdot (1 + \mathrm{ord}_{D_i}(K_{Y/X})), \tag{5.1}$$

where $D_1, \ldots, D_N$ are the irreducible components of D. Whenever X is understood, we write $A(v)$ instead of $A_X(v)$.

Proof. It is clear that the formula (5.1) uniquely determines $A_X(v)$ for $v \in \mathrm{QM}(Y, D)$. Let us temporarily denote the expression in (5.1) by $A_{X,Y,D}(v)$. We need to show that this is independent of the choice of log pair (Y, D).

Let us first reduce to the case when (Y, D) is a good log pair adapted to v. To do so, we use Lemma 3.6 (ii) to find a good log pair $(Y', D') \succeq (Y, D)$ adapted to v. Furthermore, we may assume that $v \in \mathrm{QM}_{\xi'}(Y', D')$, where ξ' lies over $\xi = c_X(v)$, and that the components of D (resp. D') through ξ (resp. ξ') are $D_1, \ldots, D_r$ (resp. $D'_1, \ldots, D'_r$). We can also assume that we have the formulas (3.2), where the E'_i do not contain ξ', and $\det(b_{i,j}) \neq 0$.

We claim that $A_{X,Y,D}(v) = A_{X,Y',D'}(v)$. Note first that Lemma 1.1 (ii) gives $1 + \mathrm{ord}_{D'_j}(K_{Y'/Y}) = \sum_{i=1}^{r} b_{ij}$. On the other hand, we have $v(E'_i) = 0$ for every i, hence (3.2) implies $v(D_i) = \sum_{j=1}^{r} b_{ij} v(D'_j)$ for $i = 1, \ldots, r$. We also have $\mathrm{ord}_{D'_j}(K_{Y/X}) = \sum_{i=1}^{r} b_{ij}\, \mathrm{ord}_{D_i}(K_{Y/X})$ for $j = 1, \ldots, r$. Putting these together, we get

$$\begin{aligned} A_{X,Y,D}(v) = \sum_i v(D_i)(1 + \mathrm{ord}_{D_i}(K_{Y/X})) &= \sum_{i,j} b_{ij} v(D'_j)(1 + \mathrm{ord}_{D_i}(K_{Y/X}) \\ &= \sum_j v(D'_j)(1 + \mathrm{ord}_{D'_j}(K_{Y'/Y}) + \mathrm{ord}_{D'_j}(K_{Y/X})) \\ &= \sum_j v(D'_j)(1 + \mathrm{ord}_{D'_j}(K_{Y'/X})) = A_{X,Y',D'}(v). \end{aligned}$$

After this reduction, it suffices to show that $A_{X,Y,D}(v)$ is independent of (Y, D) as long as (Y, D) is good for v. Since any two such pairs can be dominated by a third, it suffices to prove that $A_{X,Y,D}(v) = A_{X,Y',D'}(v)$ whenever (Y, D) is good for v and $(Y', D') \succeq (Y, D)$. By Lemma 3.6 (i), (Y', D') is automatically good for v. We can now proceed exactly as above, using Lemma 3.6 (i) and Lemma 1.1 (ii), to show that $A_{X,Y,D}(v) = A_{X,Y',D'}(v)$.

It remains to prove assertion (iii). Pick any smooth log pair (Y, D) over X'. The function $v \to A_{X'}(v) + v(K_{X'/X}) - A_X(v)$ is integral linear on $\mathrm{QM}(Y, D)$ and vanishes when $v = \mathrm{ord}_{D_i}$ for any irreducible component D_i of D. Thus this function vanishes identically, which proves (iii) since (Y, D) was arbitrary. □

Remark 5.2. It is clear from definition that if $v \in \mathrm{Val}_X$ is a quasi-monomial valuation, and if $U \subseteq X$ is an open subset such that $c_X(v) \in U$, then v is quasi-monomial also as an element in Val_U, and $A_U(v) = A_X(v)$.

Lemma 5.3. *Let $(Y', D') \succeq (Y, D)$ be smooth log pairs over X with associated retractions $r = r_{Y,D}$ and $r' = r_{Y',D'}$, respectively. Then $A(r(v)) \le A(r'(v))$ for all $v \in \mathrm{Val}(X)$, with equality if and only if $r'(v) \in \mathrm{QM}(Y, D)$.*

Proof. By Lemma 4.6 we have $r(v) = r(r'(v))$, hence after replacing v by $r'(v)$, we may assume that $v = r'(v) \in \mathrm{QM}(Y', D')$. Write $w := r(v)$.

We first prove that $A(w) \le A(v)$. This inequality follows from Lemma 1.1 (i) when $v = \mathrm{ord}_{D'_j}$ for some irreducible component D'_j of D'. It must then hold on all of $\mathrm{QM}(Y', D')$. Indeed, by Lemma 4.6 and by Proposition 5.1, the function $A(v) - A(r_{Y,D}(v))$ is (integral) linear on $\mathrm{QM}(Y', D')$.

Now suppose that $v \notin \mathrm{QM}(Y, D)$, that is, $v \ne w$. We will show that $A(v) > A(w)$. By condition (iii) in Proposition 5.1, it is enough to show that $A_Y(v) > A_Y(w)$, and therefore we may and will assume that $X = Y$. Furthermore, by Remark 5.2 we may replace Y by an open neighborhood of $c_Y(v)$. If $c_Y(v) \ne c_Y(w)$, then the inequality $A(v) > A(w)$ follows from the first part: after replacing Y by an open neighborhood of $c_Y(v)$, we can find a prime divisor E containing $c_Y(v)$ such that $(Y, D+E)$ is a smooth log pair. Hence $A(v) \ge A(r_{Y,D+E}(v)) > A(w)$. Therefore we may assume that $c_Y(v) = c_Y(w)$. Let $(Y', D') \succeq (Y, D)$ be induced by a suitable toroidal blowup as in Lemma 3.6 (ii), centered at $c_Y(v)$, such that (Y', D') is a good pair adapted to w. Note that $r_{Y',D'}(v) = w$. Since (X, D) is a good pair adapted to w, and $v \ne w$, we have $c_{Y'}(v) \ne c_{Y'}(w)$. As we have seen, this implies $A_{Y'}(v) > A_{Y'}(w)$, hence $A_Y(v) > A_Y(w)$. □

5.2. **Log discrepancy of general valuations.** We now extend the log discrepancy to arbitrary valuations in Val_X. If v is any valuation in Val_X, then we set

$$A(v) = A_X(v) := \sup_{(Y,D)} A(r_{Y,D}(v)) \in \mathbf{R}_{\ge 0} \cup \{\infty\}, \tag{5.2}$$

where the supremum is over all smooth log pairs (Y, D) over X. As a consequence of Lemma 4.6 and Lemma 5.3 we may, in the definition of $A(v)$, take the supremum over sufficiently high pairs (Y, D). This in particular implies that for a quasi-monomial valuation v, the new definition of $A(v)$ is equivalent to the old one. Note that $A(v) > 0$ when v is nontrivial. We also obtain

Corollary 5.4. *For any smooth log pair (Y, D) over X and any valuation $v \in \mathrm{Val}_X$ we have $A(r(v)) \le A(v)$ with equality if and only if $v \in \mathrm{QM}(Y, D)$.*

Remark 5.5. Suppose that $v \in \mathrm{Val}_X$ is an arbitrary valuation, and U is an open subset of X containing $c_X(v)$. It follows from Remarks 3.3 and 5.2 that $A_X(v) = A_U(v)$.

Remark 5.6. Let $\mu\colon X' \to X$ be a proper birational morphism, with X' nonsingular. It follows from Proposition 5.1 (iii) that $A_X(v) = A_{X'}(v) + v(K_{X'/X})$ for any valuation $v \in \mathrm{Val}_X = \mathrm{Val}_{X'}$.

Lemma 5.7. *The log discrepancy function* $A\colon \mathrm{Val}_X \to \mathbf{R}_{\geq 0} \cup \{\infty\}$ *is lower semicontinuous.*

Proof. Since A is continuous on each $\mathrm{QM}(Y, D)$ and $r_{Y,D}\colon \mathrm{Val}_X \to \mathrm{QM}(Y, D)$ is continuous, A is a supremum of continuous functions, hence lower semicontinuous. □

Corollary 5.8. *Given* $v \in \mathrm{Val}_X$ *we have* $A(v) = \sup_{Y,D} A(r_{Y,D}(v))$, *where the supremum is taken over smooth log pairs* (Y, D) *over* X *such that* $\varphi(D) \subseteq \overline{c_X(v)}$, *where* $\varphi\colon Y \to X$ *is the associated morphism.*

Proof. The inequality $A(v) \geq \sup_{Y,D} A(r_{Y,D}(v))$ is definitional. For the reverse inequality, fix $\varepsilon > 0$ and first assume $A(v) < \infty$. By the lower semicontinuity of A, the subset $\{A > A(v) - \varepsilon\} \subseteq \mathrm{Val}_X$ is open and hence contains a valuation of the desired form $r_{Y,D}(v)$ by Corollary 4.10. When $A(v) = \infty$, we instead look at the open set $\{A > \varepsilon^{-1}\}$. □

We will make later use of the following compactness result.

Proposition 5.9. *Let* $\xi \in X$ *be a point and* $\mathfrak{m}$ *the ideal defining* $\overline{\xi}$, *with the reduced scheme structure. For every* $M \in \mathbf{R}_{\geq 0}$, *the set*

$$V_M := \{v \in \mathrm{Val}_X \mid c_X(v) = \xi, v(\mathfrak{m}) = 1, A(v) \leq M\}$$

is a compact subspace of Val_X.

Proof. We consider an element of Val_X as a morphism of semirings $\mathcal{I} \to \mathbf{R}_{\geq 0}$, where $\mathcal{I}$ is the semiring of nonzero ideals on X (see §1.1). Recall that the condition $c_X(v) = \xi$ simply says that $v(\mathfrak{a}) = 0$ if $\mathfrak{a} \not\subseteq \mathfrak{m}$, and $v(\mathfrak{a}) > 0$, otherwise. Of course, in the presence of $v(\mathfrak{m}) = 1$, the second condition is automatically fulfilled.

Note that if $A(v) \leq M$, then for every nonzero $\mathfrak{a} \in \mathcal{I}$ we have $v(\mathfrak{a}) \leq M \cdot \mathrm{Arn}(\mathfrak{a})$. It follows from the definition of the topology on Val_X that

$$W_M := \{v \in \mathrm{Val}_X \mid c_X(v) = \xi, v(\mathfrak{m}) = 1, v(\mathfrak{a}) \leq M \cdot \mathrm{Arn}(\mathfrak{a}) \text{ for all } \mathfrak{a} \in \mathcal{I}\}$$

is a closed subset of $\prod_{\mathfrak{a} \in \mathcal{I}, \mathfrak{a} \subseteq \mathfrak{m}} [1, M \cdot \mathrm{Arn}(\mathfrak{a})]$, hence a compact topological space by Tychonoff's theorem. Moreover, V_M is a closed subset of W_M, since A is lower semicontinuous on Val_X by Proposition 5.7. Thus V_M is compact, □

When $M = +\infty$, the space V_M above is not compact but can be compactified as follows. For simplicity assume that $\xi \in X$ is a closed point. Let $\mathcal{V}_{X,\xi}$ denote the set of all semivaluations v on $\mathcal{O}_{X,\xi}$ for which $v(\mathfrak{m}) = 1$. Such a semivaluation is a map $v\colon \mathcal{O}_{X,\xi} \to \mathbf{R}_{\geq 0} \cup \{\infty\}$ satisfying the usual valuation conditions and such that $\min\{v(f) \mid f \in \mathfrak{m} \cdot \mathcal{O}_{X,\xi}\} = 1$. Note that we allow $v(f) = +\infty$ for nonzero f. This space $\mathcal{V}_{X,\xi}$ is the valuation space considered in [BFJ1], see §6.3 below. As with Val_X, we equip it with the topology of pointwise convergence, turning it into a closed subset of $\prod_{f\in\mathfrak{m}\cdot\mathcal{O}_{X\xi}}[1,\infty]$ and hence compact by Tychonoff's Theorem. One can show that $\mathrm{Val}_X\cap\mathcal{V}_{X,\xi}$ is dense in $\mathcal{V}_{X,\xi}$, see §6.3. We will use $\mathcal{V}_{X,\xi}$ in the proof of Proposition 5.13.

5.3. **Izumi's inequality.** If ξ is a point on X, we denote by $\mathfrak{m}_\xi$ the ideal defining the closure of ξ. Let ord_ξ be the valuation with center ξ, such that $\mathrm{ord}_\xi(f)=\sup\{r\mid f\in\mathfrak{m}^r\cdot\mathcal{O}_{X,\xi}\}$ for every $f\in\mathcal{O}_{X,\xi}$. Note that this is a divisorial valuation: it is equal to ord_{E_ξ}, where E_ξ is the component of the exceptional divisor on $\mathrm{Bl}_{\mathfrak{m}_\xi}(X)$ whose image contains ξ. We shall later make use of the following Izumi-type estimate [Izu, ELS, BFJ2]:

Proposition 5.10. *For any $v\in\mathrm{Val}_X$ we have*

$$v(\mathfrak{m}_\xi)\,\mathrm{ord}_\xi\le v\le A(v)\,\mathrm{ord}_\xi, \tag{5.3}$$

in the sense of Definition 4.3, where $\xi=c_X(v)$.

Proof. The first inequality follows from the definitions. By approximation, the second inequality can be reduced to the case when v is divisorial, and then it goes back at least to Tougeron [Tou, Lemma 1.3, p.178]. Alternatively, it comes from the fact that for $f\in\mathcal{O}_{X,\xi}$ with $\mathrm{ord}_\xi(f)=m$, after replacing X by some open neighborhood of ξ we have $\frac{A(v)}{v(f)}\ge\mathrm{lct}(f)\ge\frac{1}{m}$. For the second inequality, see [Kol, Lemma 8.10] (this treats the case when X is of finite type over a field, but the general case can be easily reduced to this one, arguing as in [dFM, Corollary 2.10]). □

Corollary 5.11. *If $v\in\mathrm{Val}_X$ satisfies $A(v)<\infty$, then v has a unique extension as a valuation $v'\in\mathrm{Val}_{X'}$, where $X'=\mathrm{Spec}\,\widehat{\mathcal{O}_{X,\xi}}$, $\xi=c_X(v)$.*

Proof. Let $\mathfrak{m}=\mathfrak{m}_\xi$ be as above and set $\mathfrak{m}'=\mathfrak{m}\cdot\widehat{\mathcal{O}_{X,\xi}}$. One can always uniquely extend $v\colon\mathcal{O}_{X,\xi}\smallsetminus\{0\}\to\mathbf{R}_{\ge0}$ by $\mathfrak{m}_\xi$-adic continuity to $v'\colon\widehat{\mathcal{O}_{X,\xi}}\smallsetminus\{0\}\to\mathbf{R}_{\ge0}\cup\{\infty\}$. Indeed, for $f\in\widehat{\mathcal{O}_{X,\xi}}\smallsetminus\{0\}$ and $n\ge1$, we can write $\mathfrak{a}'_n:=(f)+\mathfrak{m}'^n=\mathfrak{a}_n\cdot\widehat{\mathcal{O}_{X,\xi}}$ for some ideal $\mathfrak{a}_n$ on X. Then $v'(f)=\lim_{n\to\infty}v(\mathfrak{a}_n)$, where the limit is increasing.

If ξ' denotes the closed point of X', then $\mathrm{ord}_{\xi'}(\mathfrak{a}'_n)=\mathrm{ord}_{\xi'}(f)$ for $n>\mathrm{ord}_{\xi'}(f)$. Now, if $A(v)<\infty$, then (5.3) shows that $v(\mathfrak{a}_n)\le A(v)\,\mathrm{ord}_\xi(\mathfrak{a}_n)=A(v)\,\mathrm{ord}_{\xi'}(f)$ for $n>\mathrm{ord}_{\xi'}(f)$. This shows that $v'(f)\le A(v)\,\mathrm{ord}_{\xi'}(f)<\infty$. □

Remark 5.12. When $A(v)=\infty$, the extension v' of v to $\widehat{\mathcal{O}_{X,\xi}}$ may not be a valuation. For example, if v is the valuation on $\mathbf{A}^2_k$ given by $v(f(x,y))=\mathrm{ord}_t\,f(t,g(t))$, where $g(t)=\sum_{m\ge1}t^m/m!$, then v is a valuation with center at the origin, but $v'(y-g(x))=\infty$.

5.4. **Completion and field extension.** The following technical proposition will play an important role in the proofs of our main results. Note that if $\varphi\colon X'\to X$ is a flat morphism of integral schemes, then φ is dominant, hence it induces a field extension $K(X)\hookrightarrow K(X')$, that in turn induces a continuous map $\mathrm{Val}_{X'}\to\mathrm{Val}_X$ given by restriction of valuations.

Proposition 5.13. *Let $\varphi\colon X'\to X$ be a regular morphism, with X and X' schemes as before. Consider $v'\in\mathrm{Val}^*_{X'}$ and let $v\in\mathrm{Val}^*_X$ be the valuation induced by restriction of v'. Then $A(v')\ge A(v)$. Furthermore, equality holds in the following cases:*

(i) $X' = \operatorname{Spec} \widehat{\mathcal{O}_{X,\xi}} \to X$, *where* $\xi \in X$ *and* v' *is centered at the closed point on* X'*;*
(ii) $X' = \mathbf{A}^n_K$ *and* $X = \mathbf{A}^n_k$*, where* K/k *is an algebraic field extension, and* v' *is centered at* $0 \in X'$*.*

Recall that we have seen in Remark 5.5 that we have the equality $A(v') = A(v)$ also in the case when φ is an open immersion.

Proof. To prove the inequality $A(v') \geq A(v)$ we argue as in the proof of Proposition 1.6. Given any smooth log pair (Y, D) over X, we get a smooth log pair (Y', D') over X', where $Y' = Y \times_X X' \overset{p}{\to} Y$, with $D' = p^*(D)$. Let $\eta = c_Y(v)$ and $\eta' = c_{Y'}(v')$. If E is an irreducible component of D containing η, and if E' is the connected component of $p^*(E)$ containing η', then $v'(E') = v(E)$, hence $A(v') \geq A(r_{Y',D'}(v')) = A(r_{Y,D}(v))$. Therefore $A(v') \geq A(v)$.

We note that in case (ii), in order to prove that $A(v') = A(v)$, it is enough to consider the case when K/k is a finite Galois extension. Indeed, note first that v' can be extended to an element $\overline{v}$ of $\mathrm{Val}_{\mathbf{A}^n_{\overline{k}}}$, where $\overline{k}$ is an algebraic closure of k containing K. By what we have seen so far, $A(\overline{v}) \geq A(v') \geq A(v)$, hence it is enough to show that $A(\overline{v}) = A(v)$. On the other hand, it is easy to see that the inequality we have already proved gives $A(\overline{v}) = \sup_{L/k} A(v_L)$, where L varies over the finite Galois extensions of k contained in $\overline{k}$, and v_L is the restriction of $\overline{v}$ to $\mathbf{A}^n_L$ (this follows from the fact that every smooth log pair over $\mathbf{A}^n_{\overline{k}}$ is defined over some L as above). Therefore it is enough to show that $A(v_L) = A(v)$ for all such L, hence whenever considering case (ii), we will assume that K/k is finite and Galois, with Galois group G.

Recall that by Lemmas 3.9 and 3.10, v is quasi-monomial if and only if v' is quasi-monomial. We first prove the equality in cases (i) and (ii) under this assumption. Let (Y, D) be a good pair adapted to v. If (Y', D') is defined as before, then the same argument as above shows that it is enough to prove that (Y', D') is a good pair adapted to v'. Recall that we put $\eta = c_Y(v)$ and $\eta' = c_{Y'}(v')$. By assumption, we have $\dim(\mathcal{O}_{Y,\eta}) = \mathrm{ratrk}(v)$, and it is enough to show that $\dim(\mathcal{O}_{Y',\eta'}) = \mathrm{ratrk}(v')$. By Lemmas 3.9 and 3.10 we have $\mathrm{ratrk}(v) = \mathrm{ratrk}(v')$, hence it suffices to show that $\dim(\mathcal{O}_{Y',\eta'}) = \dim(\mathcal{O}_{Y,\eta})$. In case (ii) this follows from the fact that the morphism $X' \to X$, hence also $Y' \to Y$, is finite. In case (i), note that since $\mathcal{O}_{Y,\eta} \to \mathcal{O}_{Y',\eta'}$ is flat, we have $\dim(\mathcal{O}_{Y',\eta'}) = \dim(\mathcal{O}_{Y,\eta}) + \dim(\mathcal{O}_{Y',\eta'} \otimes k(\eta))$. However, since η' lies over the closed point of X', we see that $\mathcal{O}_{Y',\eta'} \otimes k(\eta) \simeq k(\eta)$. This completes the proof of $A(v') = A(v)$ when v is quasi-monomial.

Before proving the general case, let us make some preparations. Let $\xi = c_X(v)$ and $\xi' = c_X(v')$, and let $\mathfrak{m}$ and $\mathfrak{m}'$ denote the ideals on X (resp. X') defining $\overline{\xi}$ (resp. $\overline{\xi'}$), so that $\mathfrak{m}' = \mathfrak{m} \cdot \mathcal{O}_{X'}$. Let $V \subseteq \mathrm{Val}_X$ (resp. $V' \subseteq \mathrm{Val}_{X'}$) be the subset of valuations μ (resp. μ') for which $\mu(\mathfrak{m}) = 1$ (resp. $\mu'(\mathfrak{m}') = 1$). The restriction map $\rho\colon V' \to V$ is continuous but not surjective in general. We claim that ρ is *open onto its image*, that is, for any open subset $U' \subseteq V'$ there exists an open subset $U \subseteq V$ such that $\rho(U') = \rho(V') \cap U$. First consider case (i), when ρ is injective. We may assume U' is of the form

$$U' = \{\mu' \in V' \mid s_j < \mu'(\mathfrak{a}'_j) < t_j,\ j = 1, 2, \ldots, n\},$$

where $\mathfrak{a}'_j$ is an ideal on X' and $s_j < t_j \leq +\infty$ for $1 \leq j \leq n$. Pick p large enough so that $p > t_j$ whenever $t_j < \infty$. As $\mu'(\mathfrak{m}') = 1$, replacing $\mathfrak{a}'_j$ by $\mathfrak{a}'_j + \mathfrak{m}'^p$ does not change U'. We may therefore assume that $\mathfrak{a}'_j = \mathfrak{a}_j \cdot \mathcal{O}_{X'}$, for some ideal $\mathfrak{a}_j$ on X. But then $\rho(U') = \rho(V') \cap U$, where $U = \bigcap_j \{\mu \in V \mid s_j < \mu(\mathfrak{a}_j) < t_j\}$ is open.

Suppose now that we are in case (ii), when ρ is surjective. It is convenient to consider the extension of ρ to the spaces of semivaluations introduced at the end of §5.2. More precisely, we have a continuous surjective map induced by restriction $\widetilde{\rho}\colon \mathcal{V}_{X',\xi'} \to \mathcal{V}_{X,\xi}$. We can identify V' and V with subspaces of $\mathcal{V}_{X',\xi}$ and $\mathcal{V}_{X,x}$, respectively, such that ρ is the restriction of $\widetilde{\rho}$ over V. Therefore it is enough to show that $\widetilde{\rho}$ is open.

Note that the Galois group $G = G(K/k)$ has a natural action on X', which induces an action on $\mathcal{V}_{X',\xi'}$ such that the fibers of $\widetilde{\rho}$ coincide with the G-orbits of this action. If U' is an open subset of $\mathcal{V}_{X',\xi'}$, then $\widetilde{\rho}^{-1}(\widetilde{\rho}(U')) = \bigcup_{g\in G} gU'$ is open in $\mathcal{V}_{X',\xi'}$, hence its complement is closed. Since $\mathcal{V}_{X',\xi'}$ is compact, it follows that $\widetilde{\rho}(F)$ is compact, hence closed in $\mathcal{V}_{X,\xi}$. Therefore $\mathcal{V}_{X,\xi} \smallsetminus \widetilde{\rho}(F) = \widetilde{\rho}(U')$ is open in $\mathcal{V}_{X,\xi}$.

We can now prove that $A(v) \geq A(v')$. First suppose $A(v') < \infty$.

Given $\varepsilon > 0$, the set $U' = U'_\varepsilon := \{w' \in V' \mid A(w') > A(v') - \varepsilon\}$ is an open subset of V' by the lower semicontinuity of A. By what precedes, there exists an open subset $U \subseteq V$ such that $\rho(U') = \rho(V') \cap U$. Clearly $v \in U$, so by Corollary 4.10 we can find a smooth log pair (Y, D) above X such that $w := r(v) \in U$, where $r = r_{Y,D}\colon \mathrm{Val}_X \to \mathrm{QM}(Y, D)$ is the corresponding retraction. Since w is quasi-monomial, w is in the image of ρ, so there exists $w' \in U'$ quasi-monomial such that $\rho(w') = w$. We have seen that $A(w') = A(w)$. Thus $A(v) \geq A(w) = A(w') > A(v') - \varepsilon$. As $\varepsilon \to 0$ we get $A(v) \geq A(v')$. The case when $A(v') = \infty$ is treated similarly, by setting $U' = \{w' \in V' \mid A(w') > \varepsilon^{-1}\}$. □

6. Graded and subadditive systems revisited

We now extend to arbitrary valuations some of the results proved in §1–§2 for divisorial valuations.

6.1. **Induced functions on valuation space.**

Lemma 6.1. *If $\mathfrak{a}_\bullet$ is a graded sequence of ideals, then the function $v \mapsto v(\mathfrak{a}_\bullet)$ is upper semicontinuous on Val_X. Similarly, if $\mathfrak{b}_\bullet$ is any subadditive system of ideals, then the function $v \mapsto v(\mathfrak{b}_\bullet)$ is lower semicontinuous.*

Proof. For each t, the function $v \mapsto v(\mathfrak{b}_t)$ is continuous on Val_X by Lemma 4.1. Hence $v \mapsto v(\mathfrak{b}_\bullet) = \sup_t \frac{1}{t} v(\mathfrak{b}_t)$ is lower semicontinuous. The argument for $\mathfrak{a}_\bullet$ is analogous. □

Proposition 6.2. *If $\mathfrak{a}_\bullet$ is a graded sequence of ideals, and $\mathfrak{b}_\bullet$ is the corresponding system of asymptotic multiplier ideals, then for every m such that $\mathfrak{a}_m$ is nonzero we have*

$$v(\mathfrak{a}_\bullet) - \frac{A(v)}{m} \leq \frac{v(\mathfrak{a}_m)}{m} - \frac{A(v)}{m} \leq \frac{v(\mathfrak{b}_m)}{m} \leq \frac{v(\mathfrak{a}_m)}{m} \tag{6.1}$$

for all $v \in \mathrm{Val}_X$, with the second inequality being strict when v is nontrivial. In particular, $v(\mathfrak{a}_\bullet) = v(\mathfrak{b}_\bullet)$ whenever $A(v) < \infty$.

Proof. The first inequality is definitional and the last inequality follows from the inclusion $\mathfrak{a}_m \subseteq \mathfrak{b}_m$. To prove (6.1) it therefore suffices to show that the function $h_m(v) := v(\mathfrak{b}_m) - v(\mathfrak{a}_m) + A(v)$ is positive on Val_X^*. Let (Y, D) be a smooth log pair that defines a log resolution of $\mathfrak{a}_m \cdot \mathfrak{b}_m$. Then $v(\mathfrak{a}_m) = r_{Y,D}(v)(\mathfrak{a}_m)$, $v(\mathfrak{b}_m) = r_{Y,D}(v)(\mathfrak{b}_m)$ and $A(v) \geq A(r_{Y,D}(v))$, so it suffices to show that h_m is positive on the nontrivial valuations in $\mathrm{QM}(Y, D)$. But h_m is linear on $\mathrm{QM}(Y, D)$ and we already know that $h_m(\mathrm{ord}_{D_i}) > 0$ for any irreducible component D_i of D. Hence $h_m > 0$ on Val_X^*, as claimed.

The last assertion follows by letting $m \to \infty$ along a suitable subsequence. □

Remark 6.3. The same argument shows that if $\mathfrak{a}_\bullet$ and $\mathfrak{b}_\bullet$ are as above, then $\frac{v(\mathfrak{b}_t)}{t} > v(\mathfrak{a}_\bullet) - \frac{A(v)}{t}$ for every $v \in \mathrm{Val}_X^*$ with $A(v) < \infty$, and for every $t > 0$.

Corollary 6.4. *If $\mathfrak{a}_\bullet$ is a graded sequence of ideals on X, then the function $v \mapsto v(\mathfrak{a}_\bullet)$ is continuous on $\{v \in \mathrm{Val}_X \mid A(v) < \infty\}$.*

Proof. Let $W = \{v \in \mathrm{Val}_X \mid A(v) < \infty\}$. Proposition 6.2 gives $v(\mathfrak{a}_\bullet) = v(\mathfrak{b}_\bullet)$ for $v \in W$. Therefore Lemma 6.1 implies that the map $v \to v(\mathfrak{a}_\bullet)$ is both lower and upper semicontinuous, hence continuous, on W. □

Proposition 6.5. *If $\mathfrak{b}_\bullet$ is a subadditive sequence of ideals on X, of controlled growth, then*

$$v(\mathfrak{b}_\bullet) - \frac{A(v)}{t} \leq \frac{v(\mathfrak{b}_t)}{t} \leq v(\mathfrak{b}_\bullet) \tag{6.2}$$

for every t and every $v \in \mathrm{Val}_X$.

Proof. The second inequality is definitional. For the first inequality, it is enough to show that for every s and every $v \in \mathrm{Val}_X$, we have

$$h(v) := \frac{v(\mathfrak{b}_t)}{t} - \frac{v(\mathfrak{b}_s)}{s} + \frac{A(v)}{t} \geq 0. \tag{6.3}$$

Pick a smooth log pair (Y, D) that defines a log resolution of $\mathfrak{b}_s \cdot \mathfrak{b}_t$. Then $h \geq h \circ r_{Y,D}$ so it suffices to prove $h(v) \geq 0$ when $v \in \mathrm{QM}(Y, D)$. But this follows since h is linear on $\mathrm{QM}(Y, D)$ and $h(\mathrm{ord}_{D_i}) > 0$ for every irreducible component D_i of D. □

Corollary 6.6. *If $\mathfrak{b}_\bullet$ is a subadditive sequence of ideals on X, of controlled growth, then the function $v \mapsto v(\mathfrak{b}_\bullet)$ is continuous on any subset of Val_X on which A is bounded. In particular, $v \mapsto v(\mathfrak{b}_\bullet)$ is continuous on $\mathrm{QM}(Y, D)$ for any smooth log pair (Y, D).*

Proof. The function $v \mapsto h(v) := v(\mathfrak{b}_\bullet)$ is the pointwise limit of the continuous functions $v \mapsto \frac{1}{m} v(\mathfrak{b}_m)$ and by Proposition 6.5 the convergence is uniform on subsets where A is bounded. This proves the first assertion. For the second assertion, note that h is continuous on $\mathrm{QM}(Y, D) \cap \{A \leq 1\}$, hence on all of $\mathrm{QM}(Y, D)$ by homogeneity. □

6.2. **Jumping numbers.**

Lemma 6.7. *If $\mathfrak{a}$ and $\mathfrak{q}$ are nonzero ideals on X, then*

$$\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}) = \max_{v \in \mathrm{Val}_X^*} \frac{v(\mathfrak{a})}{A(v) + v(\mathfrak{q})}. \tag{6.4}$$

Suppose that $\mathfrak{a} \neq \mathcal{O}_X$ *and* (Y, D) *is a smooth log pair over* X *giving a log resolution of* $\mathfrak{a}\cdot\mathfrak{q}$. *Then equality in* (6.4) *is achieved for* v *if and only if* $v \in \mathrm{QM}(Y, D)$ *and* ord_{D_i} *computes* $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a})$ *for every irreducible component* D_i *of* D *for which* $v(D_i) > 0$. *In particular,* v *must be quasi-monomial.*

Proof. Let $\chi(v) = \frac{v(\mathfrak{a})}{A(v)+v(\mathfrak{q})}$. By Corollary 4.8 and Corollary 5.4, $\chi \circ r_{Y,D} \geq \chi$ with strict inequality if $v \notin \mathrm{QM}(Y, D)$ and $v(\mathfrak{a}) > 0$. Thus v achieves the maximum in (6.4) if and only if $v \in \mathrm{QM}(Y, D)$ and v belongs to the zero locus of the function $v \mapsto v(\mathfrak{a}) - \mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a})(A(v) + v(\mathfrak{q}))$. But this function is linear on $\mathrm{QM}(Y, D)$. The result follows. □

Corollary 6.8. *If* $\mathfrak{b}_\bullet$ *is a subadditive system of ideals, and* $\mathfrak{q}$ *is a nonzero ideal, then*

$$\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{b}_\bullet) = \sup_{v\in \mathrm{Val}_X^*, A(v)<\infty} \frac{v(\mathfrak{b}_\bullet)}{A(v) + v(\mathfrak{q})}. \tag{6.5}$$

Proof. By Proposition 2.8, we only need to show that

$$\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{b}_\bullet) \geq \frac{v(\mathfrak{b}_\bullet)}{A(v) + v(\mathfrak{q})} \tag{6.6}$$

when $A(v) < \infty$. But Lemma 6.7 gives $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{b}_t) \cdot (A(v) + v(\mathfrak{q})) \geq v(\mathfrak{b}_t)$ for every $t > 0$. Dividing by t and letting $t \to \infty$ gives (6.6). □

Corollary 6.9. *If* $\mathfrak{a}_\bullet$ *is a graded sequence of ideals, then for every nonzero ideal* $\mathfrak{q}$ *we have*

$$\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet) = \sup_{v\in \mathrm{Val}_X^*} \frac{v(\mathfrak{a}_\bullet)}{A(v) + v(\mathfrak{q})}. \tag{6.7}$$

Proof. This follows by combining Corollary 6.8 and Propositions 6.2 and 2.13. □

As a consequence of Corollary 2.15 and Corollary 6.9 we get

Corollary 6.10. *For graded sequence* $\mathfrak{a}_\bullet$ *of ideals, the following conditions are equivalent:*

(i) $\mathrm{Arn}(\mathfrak{a}_\bullet) = 0$;
(ii) $\mathrm{ord}_E(\mathfrak{a}_\bullet) = 0$ *for all divisors* E *over* X;
(iii) $v(\mathfrak{a}_\bullet) = 0$ *for all* $v \in \mathrm{Val}_X$ *with* $A(v) < \infty$;
(iv) $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet) = 0$ *for every nonzero ideal* $\mathfrak{q}$ *on* X.

Remark 6.11. The right-hand side of (6.5) is a priori undefined when $A(v) = \infty$ as in this case we could also have $v(\mathfrak{b}_\bullet) = \infty$. On the other hand, for a graded sequence $\mathfrak{a}_\bullet$ we always have $v(\mathfrak{a}_\bullet) < \infty$, so the right-hand side of (6.7) is well-defined for any nontrivial valuation $v \in \mathrm{Val}_X^*$.

6.3. **Comparison with other valuation spaces.** While our usage of the valuation space Val_X is, to our knowledge, new, it is certainly related to other approaches. For simplicity, suppose that X is a smooth variety over an algebraically closed field k of characteristic zero and equip k with the trivial valuation.

In this context, the Berkovich space X^{an} is defined (as a topological space) as follows [Ber1]. When $X = \mathrm{Spec}\, A$ is affine, X^{an} is the set of semivaluations $v : A \to [0, +\infty]$

whose restriction to k is trivial. In general, X^{an} is obtained by glueing the (closed) subsets U^{an}, where U ranges over an open affine covering of X. Just as for Val_X, the topology on X^{an} is defined in terms of pointwise convergence. Thus Val_X embeds in X^{an}.

In fact, Val_X is dense in X^{an}. Let us sketch a proof for completeness. We may assume $X = \operatorname{Spec} A$ is affine. Consider any $v \in X^{\mathrm{an}}$. If $v \notin \mathrm{Val}_X$, then the prime ideal $I := \{v = \infty\} \subseteq A$ is nonzero. Now look at the prime ideal $J := \{v > 0\} \supseteq I$ with associated point $\xi \in X$. If $I = J$, then the semivaluation v satisfies $v(f) = \infty$ if $f \in J$ and $v(f) = 0$ otherwise. Hence the divisorial valuation $n \operatorname{ord}_\xi \in \mathrm{Val}_X$ tends to v as $n \to \infty$. Now suppose $J \supsetneq I$ so that $0 < v(J) < \infty$. If (Y, D) is a smooth log pair above X for which the associated birational morphism $\varphi : Y \to X$ satisfies $\varphi(D) \subset \overline{\xi}$, then we can define the retraction $r_{Y,D}(v) \in \mathrm{Val}_X$ as in §4.3. We claim that every neighborhood of v in X^{an} contains an element of the form $r_{Y,D}(v)$. To see this, fix $f \in A$. It suffices to find a sequence (Y_n, D_n) such that $r_{Y_n,D_n}(v)(f) \to v(f)$ as $n \to \infty$, but for this we may take (Y_n, D_n) to be a log resolution of $(f) + J^n$.

When X is projective, X^{an} is compact [Ber1, Theorem 3.5.3], hence defines a compactification of Val_X. Note that while Val_X is a birational invariant of X, X^{an} is not.

Given any closed point $\xi \in X$ we can also, as in §5.2, consider the compact subset $\mathcal{V}_{X,\xi} \subseteq X^{\mathrm{an}}$ consisting of semivaluations for which $v(\mathfrak{m}_\xi) = 1$. This is the valuation space studied in [BFJ1], By [BFJ1, Theorem 1.16] (see also [Ber1, Thu2]), $\mathcal{V}_{X,\xi}$ is contractible. The argument above shows that $\mathrm{Val}_X \cap \mathcal{V}_{X,\xi}$ is dense in $\mathcal{V}_{X,\xi}$. In fact, for each smooth log pair (Y, D) as above, $\mathcal{V}_{X,\xi} \cap \mathrm{QM}(Y, D)$ is a simplicial complex and by [BFJ1, Theorem 1.13], $\mathcal{V}_{X,\xi}$ is homeomorphic to the projective limit of these complexes. It follows from Corollary 5.8 that the log discrepancy defined in [BFJ1, Definition 3.4] (called thinness there) coincides with the one defined in this paper.

In [BFJ1], a class of plurisubharmonic (psh) functions on $\mathcal{V}_{X,\xi}$ was defined. A posteriori, a function on $\mathcal{V}_{X,\xi}$ is plurisubharmonic if and only if it is of the form $v \to -v(\mathfrak{b}_\bullet)$ where $\mathfrak{b}_\bullet$ is a subadditive system of controlled growth satisfying $\mathfrak{b}_t \supseteq \mathfrak{m}_\xi^{p_t}$, for each t, where $p_t \geq 1$. This cone of psh functions has good compactness properties and is studied in detail in [BFJ2, BFJ3].

In dimension $\dim X = 2$, $\mathcal{V}_{X,\xi}$ is naturally an $\mathbf{R}$-tree, being a contractible projective limit of one-dimensional simplicial complexes. A function on $\mathcal{V}_{X,\xi}$ is psh if and only if it satisfies certain convexity conditions [FJ1, BR, Thu1].[7] This allows us to construct graded and subadditive sequences with interesting behavior. For example, given coordinates (x, y) at ξ and a strictly increasing sequence $1 \leq \beta_1 < \beta_2 < \dots$ of rational numbers with unbounded denominators we can define a valuation $v \in \mathrm{Val}_X \cap \mathcal{V}_{X,\xi}$ by $v(f) = \operatorname{ord}_{x=0} f(x, \sum_j x^{\beta_j})$. Such a valuation satisfies $\operatorname{trdeg}(v) = 0$, $\operatorname{ratrk}(v) = 1$ and is called *infinitely singular* in [FJ1]. If the β_j grow sufficiently fast, then there exists a psh function φ on $\mathcal{V}_{X,\xi}$ for which $\varphi(\operatorname{ord}_0) = -1$ and $\varphi(v) = -\infty$. This translates into the existence of a subadditive sequence $\mathfrak{b}_\bullet$ of controlled growth such that $\operatorname{ord}_0(\mathfrak{b}_\bullet) = 1$ but $v(\mathfrak{b}_\bullet) = \infty$. In particular, $A(v) = \infty$. One can also show that the associated graded system $\mathfrak{a}_\bullet = \mathfrak{a}_\bullet(v)$ satisfies $v(\mathfrak{a}_\bullet) = 1$ but $w(\mathfrak{a}_\bullet) = 0$ for all $w \in \mathcal{V}_{X,\xi} \setminus \{v\}$.

[7] In [FJ1, FJ2, FJ3], the negative of a psh function is called a tree potential.

7. Valuations computing asymptotic invariants

Now we are ready to formulate our main results and conjectures. We keep our previous setup.

7.1. Results and conjectures.

Definition 7.1. A valuation $v \in \mathrm{Val}_X^*$ *computes* $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet)$, for a nonzero ideal $\mathfrak{q}$ and a graded sequence of ideals $\mathfrak{a}_\bullet$ if $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet) = \frac{v(\mathfrak{a}_\bullet)}{A(v)+v(\mathfrak{q})}$.

Equivalently, v then computes $\mathrm{lct}^{\mathfrak{q}}(\mathfrak{a}_\bullet)$. Of course, if $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet) = \infty$, any valuation computes $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet)$, so we shall focus on the case $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet) > 0$ in the sequel. In this case any v computing $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet)$ satisfies $A(v) < \infty$.

Remark 7.2. If $v \in \mathrm{Val}_X^*$ computes $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet) > 0$, then $c_X(v)$ lies in the zero-locus of $(\mathfrak{b}_\lambda\colon \mathfrak{q})$, where $\lambda = \mathrm{lct}^{\mathfrak{q}}(\mathfrak{a}_\bullet)$ and $\mathfrak{b}_\lambda = \mathcal{J}(\mathfrak{a}_\bullet^\lambda)$. Indeed, if f is a local section of $(\mathfrak{b}_\lambda\colon \mathfrak{q})$ defined in a neighborhood of $c_X(v)$, then $f\cdot\mathfrak{q} \subseteq \mathfrak{b}_\lambda$, hence

$$v(f) + v(\mathfrak{q}) \geq v(\mathfrak{b}_\lambda) > \lambda\cdot v(\mathfrak{a}_\bullet) - A(v) = v(\mathfrak{q}),$$

in view of Remark 6.3. It follows that $v(f) > 0$, so f vanishes at $c_X(v)$.

The following result generalizes Theorem A from the introduction.

Theorem 7.3. *Let $\mathfrak{a}_\bullet$ be a graded sequence of ideals on X, and $\mathfrak{q}$ a nonzero ideal. If $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet) = \lambda^{-1} > 0$, then for every generic point ξ of an irreducible component of $V(\mathcal{J}(\mathfrak{a}_\bullet^\lambda)\colon \mathfrak{q})$, there is a valuation $v \in \mathrm{Val}_X^*$ that computes $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet)$, with $c_X(v) = \xi$.*

As we will see in Remark 8.5 below, the valuation v cannot always be taken divisorial. However, we state

Conjecture 7.4. *Let $\mathfrak{a}_\bullet$ be a graded sequence of ideals on X and $\mathfrak{q}$ a nonzero ideal on X such that $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet) = \lambda^{-1} > 0$.*

- ***Weak version**: for any generic point ξ of an irreducible component of the subscheme defined by $(\mathcal{J}(\mathfrak{a}_\bullet^\lambda)\colon \mathfrak{q})$, there exists a quasi-monomial valuation $v \in \mathrm{Val}_X^*$ that computes $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet)$ and with $c_X(v) = \xi$.*
- ***Strong version**: any valuation $v \in \mathrm{Val}_X^*$ that computes $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet)$ must be quasi-monomial.*

While we are unable to prove either version of Conjecture 7.4, we shall reduce them, in two ways, to statements that hopefully are easier to prove. First, we reduce to the case of an affine space over an algebraically closed field.

Conjecture 7.5. *Let $X = \mathbf{A}_k^n$, where k is an algebraically closed field of characteristic zero and where $n \geq 1$. Let $\mathfrak{a}_\bullet$ be a graded sequence of ideals on X and $\mathfrak{q}$ a nonzero ideal on X such that $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet) > 0$ and such that $\mathfrak{a}_1 \supseteq \mathfrak{m}^p$, where $p \geq 1$ and $\mathfrak{m} = \mathfrak{m}_\xi$ is the ideal defining a closed point $\xi \in X$.*

- ***Weak version**: there exists a quasi-monomial valuation $v \in \mathrm{Val}_X^*$ computing $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet)$ and with $c_X(v) = \xi$.*

- ***Strong version***: *any valuation* $v \in \mathrm{Val}_X^*$ *of transcendence degree 0 computing* $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet)$ *must be quasi-monomial.*

The strong version of Conjecture 7.5 is trivially true in dimension one. A proof in dimension two will be given in §9 and the monomial case is treated in §8. The following result strengthens Theorem D in the introduction.

Theorem 7.6. *If the weak (resp. strong) version of Conjecture 7.5 holds for every* $n \le N$, *then the weak (resp. strong) version of Conjecture 7.4 holds for all* X *with* $\dim(X) \le N$.

Second, we reduce to the case of a graded sequence of valuation ideals.

Theorem 7.7. *In Conjecture 7.4 (weak and strong version) we may assume that* $\mathfrak{a}_\bullet$ *is a graded sequence of valuation ideals, that is,* $\mathfrak{a}_m = \{f \mid w(f) \ge m\}$ *for some* $w \in \mathrm{Val}_X^*$.

We also have a related result.

Theorem 7.8. *Let* $v \in \mathrm{Val}_X^*$ *be a nontrivial valuation with* $A(v) < \infty$ *and* $\mathfrak{q}$ *a nonzero ideal on* X. *Then the following assertions are equivalent:*

(i) *There is a graded sequence of ideals* $\mathfrak{a}_\bullet$ *on* X *such that* v *computes* $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet) > 0$.
(ii) *There is a subadditive system of ideals* $\mathfrak{b}_\bullet$ *(which can be assumed of controlled growth) such that* v *computes* $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{b}_\bullet) > 0$.
(iii) *For every* $w \in \mathrm{Val}_X$ *such that* $w \ge v$ *in the sense of Definition 4.3, we have* $A(w) + w(\mathfrak{q}) \ge A(v) + v(\mathfrak{q})$.
(iv) *If* $\mathfrak{a}'_m = \{f \mid v(f) \ge m\}$, *then* v *computes* $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}'_\bullet)$.

In (ii), by a valuation $v \in \mathrm{Val}_X^*$ computing $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{b}_\bullet)$ we mean that $A(v) < \infty$ and $v(\mathfrak{b}_\bullet)/(A(v) + v(\mathfrak{q})) = \mathrm{Arn}^{\mathfrak{q}}(\mathfrak{b}_\bullet)$.

From the equivalence of (i) and (iii) we obtain

Corollary 7.9. *If* $\mathfrak{q}$ *is a nonzero ideal on* X *and* v *computes* $\mathrm{Arn}(\mathfrak{a}_\bullet) > 0$ *for some graded sequence* $\mathfrak{a}_\bullet$, *then* v *also computes* $\mathrm{Arn}^{\mathfrak{q}}(\widetilde{\mathfrak{a}}_\bullet) > 0$ *for some (other) graded sequence* $\widetilde{\mathfrak{a}}_\bullet$.

7.2. **Valuation ideals.** Now we give the proofs of the results in Section 7. We start by the reductions to the case of graded sequences of valuation ideals, specifically Theorems 7.7 and 7.8.

Proof of Theorem 7.8. We will show that (i)⇒(ii)⇒(iii)⇒(iv)⇒(i)

The implication (i)⇒(ii) follows from Proposition 2.13 and Proposition 6.2: it is enough to take $\mathfrak{b}_\bullet$ to be given by the asymptotic multiplier ideals of $\mathfrak{a}_\bullet$.

In order to show (ii)⇒(iii), suppose that v computes $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{b}_\bullet) > 0$. If $w \ge v$, then clearly $w(\mathfrak{b}_\bullet) \ge v(\mathfrak{b}_\bullet)$. Now Corollary 6.8 gives $\frac{w(\mathfrak{b}_\bullet)}{A(w)+w(\mathfrak{q})} \le \frac{v(\mathfrak{b}_\bullet)}{A(v)+v(\mathfrak{q})}$, hence $\frac{A(w)+w(\mathfrak{q})}{A(v)+v(\mathfrak{q})} \ge \frac{w(\mathfrak{b}_\bullet)}{v(\mathfrak{b}_\bullet)} \ge 1$. Therefore we have (iii).

Now suppose (iii) holds. By Lemma 2.4, $v(\mathfrak{a}'_\bullet) = 1$. To prove (iv) it therefore suffices, by Corollary 6.9, to show that for every $w \in \mathrm{Val}^*_X$ we have

$$\frac{w(\mathfrak{a}'_\bullet)}{A(w) + w(\mathfrak{q})} \leq \frac{1}{A(v) + v(\mathfrak{q})}. \tag{7.1}$$

If $w(\mathfrak{a}'_\bullet) = 0$, then (7.1) is trivial, so suppose $w(\mathfrak{a}'_\bullet) > 0$. Since the left hand side is invariant under scaling of w, we may assume $w(\mathfrak{a}'_\bullet) = 1$. By Lemma 2.4 this implies $w \geq v$. The assumption (iii) now gives $A(w) + w(\mathfrak{q}) \geq A(v) + v(\mathfrak{q})$, so that (7.1) holds.

Finally, the implication (iv)⇒(i) is trivial: if v computes $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}'_\bullet)$, then $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}'_\bullet) = (A(v) + v(\mathfrak{q}))^{-1} > 0$. This completes the proof. □

Now we turn to Theorem 7.7. The assertion corresponding to the strong versions of the conjectures follows from the implication (i)⇒(iv) in Theorem 7.8. The assertion concerning the weak statements of the conjectures is a consequence of Theorem 7.3 and the following result.

Proposition 7.10. *Assume that $v \in \mathrm{Val}^*_X$ computes $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet) > 0$ and define $\mathfrak{a}'_\bullet$ by $\mathfrak{a}'_m = \{f \mid v(f) \geq m\}$. Then $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}'_\bullet) = \mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet)$ and any $w \in \mathrm{Val}^*_X$ that computes $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}'_\bullet)$ also computes $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet)$.*

Proof. Since $v \in \mathrm{Val}_X$ computes $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet) > 0$ we must have $A(v) < \infty$ and $v(\mathfrak{a}_\bullet) > 0$. After rescaling v, we may assume $v(\mathfrak{a}_\bullet) = 1$. By Lemma 2.4 we also have $v(\mathfrak{a}'_\bullet) = 1$. Since v computes $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet)$, it also computes $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}'_\bullet)$ by Theorem 7.8. This yields

$$\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}'_\bullet) = \frac{v(\mathfrak{a}'_\bullet)}{A(v) + v(\mathfrak{q})} = \frac{1}{A(v) + v(\mathfrak{q})} = \frac{v(\mathfrak{a}_\bullet)}{A(v) + v(\mathfrak{q})} = \mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet).$$

Now $v(\mathfrak{a}_\bullet) = 1$ implies $\mathfrak{a}_m \subseteq \mathfrak{a}'_m$ for every m. In particular, $w(\mathfrak{a}'_m) \leq w(\mathfrak{a}_m)$ for all m and all $w \in \mathrm{Val}^*_X$, hence $w(\mathfrak{a}'_\bullet) \leq w(\mathfrak{a}_\bullet)$. If w computes $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}'_\bullet)$, we therefore get

$$\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet) = \mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}'_\bullet) = \frac{w(\mathfrak{a}'_\bullet)}{A(w) + w(\mathfrak{q})} \leq \frac{w(\mathfrak{a}_\bullet)}{A(w) + w(\mathfrak{q})},$$

so that w computes $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet)$ (and $w(\mathfrak{a}'_\bullet) = v(\mathfrak{a}'_\bullet)$). □

7.3. **Birational and regular morphisms.** Throughout this subsection, $\varphi\colon X' \to X$ is either a birational, or a regular morphism. Let $\mathfrak{a}_\bullet$ be a graded sequence of ideals on X, $\mathfrak{q}$ a nonzero ideal on X and $\mathfrak{a}'_\bullet$, $\mathfrak{q}'$ their transforms to X', defined by $\mathfrak{a}'_m := \mathfrak{a}_m \cdot \mathcal{O}_{X'}$ and $\mathfrak{q}' := \mathfrak{q} \cdot \mathcal{O}_{X'}(-K_{X'/X})$ (in the birational case) or $\mathfrak{q}' = \mathfrak{q} \cdot \mathcal{O}_{X'}$ (in the regular case).

Lemma 7.11. *Suppose that $\varphi\colon X' \to X$ is a proper birational morphism with X' nonsingular. Then $\mathrm{Arn}^{\mathfrak{q}'}(\mathfrak{a}'_\bullet) = \mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet)$. Moreover, $v \in \mathrm{Val}_X = \mathrm{Val}_{X'}$ computes $\mathrm{Arn}^{\mathfrak{q}'}(\mathfrak{a}'_\bullet)$ if and only if it computes $\mathrm{Arn}^{\mathfrak{q}'}(\mathfrak{a}'_\bullet)$.*

Proof. The equality $\mathrm{Arn}^{\mathfrak{q}'}(\mathfrak{a}'_\bullet) = \mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet)$ is exactly Proposition 2.5. The last assertion in the lemma follows from $v(\mathfrak{a}_\bullet) = v(\mathfrak{a}'_\bullet)$, $v(\mathfrak{q}') = v(\mathfrak{q}) + v(K_{X'/X})$ and $A_X(v) = A_{X'}(v) + v(K_{X'/X})$; see Remark 5.6. □

Proposition 7.12. *Suppose that* $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet) = \lambda^{-1} > 0$. *Let* $\xi \in X$ *be a point in the subscheme defined by* $(\mathcal{J}(\mathfrak{a}_\bullet^\lambda) \colon \mathfrak{q})$. *Let* $\varphi \colon \operatorname{Spec} \widehat{\mathcal{O}_{X,\xi}} \to X$ *be the canonical morphism. Then* $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet) = \mathrm{Arn}^{\mathfrak{q}'}(\mathfrak{a}'_\bullet)$. *Moreover, if* $v' \in \mathrm{Val}_{X'}$ *is a valuation centered at the closed point and* $v \in \mathrm{Val}_X$ *denotes its restriction to* X, *then* v *computes* $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet)$ *if and only if* v' *computes* $\mathrm{Arn}^{\mathfrak{q}'}(\mathfrak{a}'_\bullet)$.

Proof. Since φ is regular (see Example 1.5), the equality of Arnold multiplicities follows from Proposition 1.6. Proposition 5.13 implies $A(v') = A(v)$. Since $v'(\mathfrak{q}') = v(\mathfrak{q})$ and $v'(\mathfrak{a}'_\bullet) = v(\mathfrak{a}_\bullet)$, it is now clear that v' computes $\mathrm{Arn}^{\mathfrak{q}'}(\mathfrak{a}'_\bullet)$ if and only if v' computes $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet)$. □

Proposition 7.13. *If* K/k *is an algebraic field extension and* $\varphi \colon X' = \mathbf{A}^n_K \to \mathbf{A}^n_k = X$ *is the canonical map, then* $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet) = \mathrm{Arn}^{\mathfrak{q}'}(\mathfrak{a}'_\bullet)$. *Moreover, for* $v' \in \mathrm{Val}^*_{X'}$ *let* $v \in \mathrm{Val}^*_X$ *be the restriction of* v' *to* X. *Then* v *computes* $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet)$ *if and only if* v' *computes* $\mathrm{Arn}^{\mathfrak{q}'}(\mathfrak{a}'_\bullet)$.

Proof. Since φ is regular and faithfully flat (see Example 1.4), the equality $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet) = \mathrm{Arn}^{\mathfrak{q}'}(\mathfrak{a}'_\bullet)$ follows from Proposition 1.6. Proposition 5.13 implies $A(v') = A(v)$. Since $v'(\mathfrak{q}') = v(\mathfrak{q})$ and $v'(\mathfrak{a}'_\bullet) = v(\mathfrak{a}_\bullet)$, it is now clear that v' computes $\mathrm{Arn}^{\mathfrak{q}'}(\mathfrak{a}'_\bullet)$ if and only if v computes $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet)$. □

7.4. **Enlarging a graded sequence.** Fix a graded sequence $\mathfrak{a}_\bullet$ of ideals, a nonzero ideal $\mathfrak{q}$ on X and a point $\xi \in X$. For the proof of Theorem 7.6 it is useful to enlarge $\mathfrak{q}$ and $\mathfrak{a}_\bullet$ so that they vanish only at ξ. Given an integer $p \geq 1$, define $\mathfrak{c}_\bullet$ by

$$\mathfrak{c}_j = \sum_{i=0}^{j} \mathfrak{a}_i \cdot \mathfrak{m}^{p(j-i)}, j \geq 0 \tag{7.2}$$

where $\mathfrak{m} = \mathfrak{m}_\xi$ is the ideal defining $\overline{\xi}$. Note that $\mathfrak{c}_1 \supseteq \mathfrak{m}^p$.

Proposition 7.14. *Assume* $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet) = \lambda^{-1} > 0$ *and let* ξ *be the generic point of an irreducible component of the subscheme defined by* $(\mathcal{J}(\mathfrak{a}_\bullet^\lambda) \colon \mathfrak{q})$. *Define* $\mathfrak{c}_\bullet$ *using* (7.2). *Then, for* $p \gg 0$, $\mathrm{lct}^{\mathfrak{q}}(\mathfrak{c}_\bullet) = \mathrm{lct}^{\mathfrak{q}}(\mathfrak{a}_\bullet) = \lambda$ *and if* $v \in \mathrm{Val}^*_X$ *computes* $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{c}_\bullet)$, *then* v *also computes* $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet)$.

Proposition 7.15. *Suppose that* $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet) = \lambda^{-1} > 0$ *and that* $\mathfrak{m}^p \subseteq \mathfrak{a}_1$. *If* $N \geq \lambda p$ *and* $\mathfrak{r} = \mathfrak{q} + \mathfrak{m}^N$, *then* $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet) = \mathrm{Arn}^{\mathfrak{r}}(\mathfrak{a}_\bullet)$. *Furthermore, if* $v \in \mathrm{Val}^*_X$, *then* v *computes* $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet)$ *if and only if* v *computes* $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{c}_\bullet)$.

Proof of Proposition 7.14. In order to prove $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{c}_\bullet) = \mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet)$ for $p \gg 0$, let us first consider the special case when $\mathfrak{m} \subseteq \sqrt{(\mathcal{J}(\mathfrak{a}_\bullet^\lambda) \colon \mathfrak{q})}$. Then there exists a positive integer n such that $\mathfrak{m}^n \cdot \mathfrak{q} \subseteq \mathcal{J}(\mathfrak{a}_\bullet^\lambda)$. Set $\lambda' := \mathrm{lct}^{\mathfrak{m}^n\mathfrak{q}}(\mathfrak{a}_\bullet) > \lambda$ and pick $p > n/(\lambda' - \lambda)$. Fix $0 < \varepsilon < 1$ such that $p > n/((1-\varepsilon)\lambda' - \lambda)$.

Note that $v(\mathfrak{c}_\bullet) = \min\{v(\mathfrak{a}_\bullet), pv(\mathfrak{m})\}$ for all $v \in \mathrm{Val}^*_X$. Thus

$$\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{c}_\bullet) = \sup_{v \in \mathrm{Val}^*_X} \frac{\min\{v(\mathfrak{a}_\bullet), pv(\mathfrak{m})\}}{A(v) + v(\mathfrak{q})} \geq \sup_{v \in V_\varepsilon} \frac{\min\{v(\mathfrak{a}_\bullet), pv(\mathfrak{m})\}}{A(v) + v(\mathfrak{q})},$$

where V_ε is the set of $v \in \mathrm{Val}^*_X$ for which $\frac{v(\mathfrak{a}_\bullet)}{A(v)+v(\mathfrak{q})} \geq (1-\varepsilon)/\lambda$.

By the definition of λ' we have

$$\frac{n\cdot v(\mathfrak{m})}{v(\mathfrak{a}_\bullet)} \geq \lambda' - \frac{A(v)+v(\mathfrak{q})}{v(\mathfrak{a}_\bullet)}$$

for all $v \in \mathrm{Val}_X^*$. This implies

$$\begin{aligned}\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{c}_\bullet) &\geq \sup_{v\in V_\varepsilon} \frac{v(\mathfrak{a}_\bullet)}{A(v)+v(\mathfrak{q})} \min\left\{1, \frac{p}{n}\left(\lambda' - \frac{A(v)+v(\mathfrak{q})}{v(\mathfrak{a}_\bullet)}\right)\right\} \\ &\geq \sup_{v\in V_\varepsilon} \frac{v(\mathfrak{a}_\bullet)}{A(v)+v(\mathfrak{q})} \min\left\{1, \frac{p}{n}\left(\lambda' - \frac{\lambda}{1-\varepsilon}\right)\right\} = \sup_{v\in V_\varepsilon} \frac{v(\mathfrak{a}_\bullet)}{A(v)+v(\mathfrak{q})} = \mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet).\end{aligned}$$

Therefore $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{c}_\bullet) \geq \mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet)$, and the reverse inequality is obvious.

We now treat the general case. Consider the natural morphism $\varphi\colon \mathrm{Spec}\, R = X' \to X$, where $R = \widehat{\mathcal{O}_{X,\xi}}$. Let ξ' denote the closed point of X', and let $\mathfrak{m}' = \mathfrak{m}\cdot R$, $\mathfrak{a}'_j = \mathfrak{a}_j \cdot R$, $\mathfrak{q}' = \mathfrak{q}\cdot R$, and $\mathfrak{c}'_j = \mathfrak{c}_j \cdot R = \sum_{i=0}^{j} \mathfrak{a}'_i \cdot \mathfrak{m}'^{p(j-i)}$. Note that $\mathfrak{m}'$ is the ideal defining the closed point of X'. It follows from Proposition 1.6 that $\mathcal{J}(\mathfrak{a}'^{\lambda}_\bullet) = \mathcal{J}(\mathfrak{a}^{\lambda}_\bullet)\cdot R$. By construction, $\sqrt{(\mathcal{J}(\mathfrak{a}'^{\lambda}_\bullet)\colon \mathfrak{q}')} = \sqrt{(\mathcal{J}(\mathfrak{a}^{\lambda}_\bullet)\colon \mathfrak{q})}\cdot R = \mathfrak{m}'$, so by the case already treated, we have $\mathrm{lct}^{\mathfrak{q}'}(\mathfrak{a}'_\bullet) = \mathrm{lct}^{\mathfrak{q}'}(\mathfrak{c}'_\bullet)$ for $p \gg 0$. Therefore

$$\mathrm{lct}^{\mathfrak{q}}(\mathfrak{a}_\bullet) \leq \mathrm{lct}^{\mathfrak{q}}(\mathfrak{c}_\bullet) \leq \mathrm{lct}^{\mathfrak{q}'}(\mathfrak{c}'_\bullet) = \mathrm{lct}^{\mathfrak{q}'}(\mathfrak{a}'_\bullet), \tag{7.3}$$

where the first inequality follows from the inclusions $\mathfrak{a}_j \subseteq \mathfrak{c}_j$, and the second one from Proposition 1.6. Since $\mathrm{lct}^{\mathfrak{q}'}(\mathfrak{a}'_\bullet) = \mathrm{lct}^{\mathfrak{q}}(\mathfrak{a}_\bullet)$ by the same proposition, it follows that all inequalities in (7.3) are equalities. In particular, $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{c}_\bullet) = \mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet)$.

Suppose now that v is a valuation that computes $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{c}_\bullet)$. Since $\mathfrak{a}_j \subseteq \mathfrak{c}_j$ for every j, we have $v(\mathfrak{c}_\bullet) \leq v(\mathfrak{a}_\bullet)$. Therefore

$$\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{c}_\bullet) = \frac{v(\mathfrak{c}_\bullet)}{A(v)+v(\mathfrak{q})} \leq \frac{v(\mathfrak{a}_\bullet)}{A(v)+v(\mathfrak{q})} \leq \mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet) = \mathrm{Arn}^{\mathfrak{q}}(\mathfrak{c}_\bullet). \tag{7.4}$$

All the inequalities in (7.4) have to be equalities, hence v also computes $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet)$. □

Proof of Proposition 7.15. It follows from Proposition 2.11 that $\mathfrak{q} \not\subseteq \mathcal{J}(\mathfrak{a}^{\lambda}_\bullet)$, but $\mathfrak{q} \subseteq \mathcal{J}(\mathfrak{a}^{t}_\bullet)$ for every $t < \lambda$. In order to prove that $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet) = \mathrm{Arn}^{\mathfrak{r}}(\mathfrak{a}_\bullet)$, it is enough to show that under our assumptions, $\mathfrak{m}^N \subseteq \mathcal{J}(\mathfrak{a}^{\lambda}_\bullet)$. This follows since

$$\mathfrak{m}^N \subseteq \mathcal{J}(\mathfrak{m}^N) \subseteq \mathcal{J}(\mathfrak{m}^{\lambda p}) \subseteq \mathcal{J}(\mathfrak{a}_1^{\lambda}) \subseteq \mathcal{J}(\mathfrak{a}^{\lambda}_\bullet).$$

Suppose now that $v \in \mathrm{Val}_X^*$. Since

$$\frac{v(\mathfrak{a}_\bullet)}{A(v)+v(\mathfrak{q})} \leq \frac{v(\mathfrak{a}_\bullet)}{A(v)+v(\mathfrak{r})},$$

it follows that if v computes $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet)$, then v also computes $\mathrm{Arn}^{\mathfrak{r}}(\mathfrak{a}_\bullet)$. For the converse, it is enough to show that if v computes $\mathrm{Arn}^{\mathfrak{r}}(\mathfrak{a}_\bullet)$, then $v(\mathfrak{q}) = v(\mathfrak{r})$. Note that since $\mathfrak{m}^p \subseteq \mathfrak{a}_1$, we have $v(\mathfrak{a}_\bullet) \leq p\cdot v(\mathfrak{m})$. Therefore

$$v(\mathfrak{m}) \geq \frac{v(\mathfrak{a}_\bullet)}{p} = \frac{A(v)+v(\mathfrak{r})}{\lambda p} > \frac{v(\mathfrak{r})}{N},$$

hence $v(\mathfrak{m}^N) > v(\mathfrak{r}) = \min\{v(\mathfrak{q}), v(\mathfrak{m}^N)\}$. This shows that $v(\mathfrak{r}) = v(\mathfrak{q})$, and completes the proof of the proposition. □

7.5. **Proof of Theorem 7.3.** Let $\mathfrak{m} = \mathfrak{m}_\xi$ be the ideal defining ξ. After applying Propositions 7.14 and 7.15 (and increasing p) we may assume that $\mathfrak{m}^p \subseteq \mathfrak{a}_1$ and $\mathfrak{m}^p \subseteq \mathfrak{q}$ for some $p \geq 1$.

Consider the canonical morphism $\varphi\colon X' = \operatorname{Spec} R \to X$, where $R = \widehat{\mathcal{O}_{X,\xi}}$. Since X is nonsingular, Cohen's structure theorem yields an isomorphism $R \simeq k[\![x_1, \ldots, x_d]\!]$ for a field k. We put $\mathfrak{a}'_m = \mathfrak{a}_m \cdot R$, $\mathfrak{q}' = \mathfrak{q} \cdot R$, and $\mathfrak{m}' = \mathfrak{m} \cdot R$, so $\mathfrak{m}'$ is the ideal defining the closed point 0 of X'. By Proposition 7.12 it suffices to find a valuation $v' \in \operatorname{Val}^*_{X'}$ with center at 0 that computes $\operatorname{Arn}^{\mathfrak{q}'}(\mathfrak{a}'_\bullet)$. Indeed, in this case the restriction v of v' to X has center $c_X(v) = \xi$ and computes $\operatorname{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet)$.

Therefore we may assume $X = \operatorname{Spec} k[\![x_1, \ldots, x_d]\!]$, and that $\mathfrak{a}_1$ and $\mathfrak{q}$ contain $\mathfrak{m}^p$ for some p, where $\mathfrak{m}$ is the ideal defining the closed point of X. Fix $0 < \varepsilon < \operatorname{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet)$ and suppose $v \in \operatorname{Val}^*_X$ is such that $\frac{v(\mathfrak{a}_\bullet)}{A(v)+v(\mathfrak{q})} > \varepsilon$. Since $\mathfrak{m}^p \subseteq \mathfrak{a}_1$, we have $v(\mathfrak{a}_\bullet) \leq p v(\mathfrak{m})$. In particular, v has center at the closed point. After rescaling v, we may assume $v(\mathfrak{m}) = 1$, so that $v(\mathfrak{a}_\bullet) \leq p$, and therefore $A(v) \leq A(v) + v(\mathfrak{q}) \leq M$, where $M = p/\varepsilon$. We conclude that $\operatorname{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet) = \sup_{v \in V_M} \frac{v(\mathfrak{a}_\bullet)}{A(v)+v(\mathfrak{q})}$, where

$$V_M = \{v \in \operatorname{Val}_X \mid v(\mathfrak{m}) = 1, A(v) \leq M\}.$$

By Proposition 5.9, V_M is compact. Furthermore, by Proposition 5.7, A is lower semicontinuous and by Corollary 6.4 the functions $v \to v(\mathfrak{q})$ and $v \to v(\mathfrak{a}_\bullet)$ are continuous on V_M The function $v \to v(\mathfrak{a}_\bullet)/(A(v) + v(\mathfrak{q}))$ is therefore upper semicontinuous on V_M, hence achieves its maximum at some $v \in V_M$. This completes the proof of Theorem 7.3.

7.6. **Proof of Theorem 7.6.** Assume $\lambda := \operatorname{lct}^{\mathfrak{q}}(\mathfrak{a}_\bullet) < \infty$. The proof proceeds similarly to the proof of Theorem 7.3, repeatedly using localization, completion, and field extensions.

We start by considering the weak versions of Conjectures 7.4 and 7.5. Let ξ be the generic point of an irreducible component of the subscheme defined by $(\mathcal{J}(\mathfrak{a}_\bullet^\lambda)\colon \mathfrak{q})$. In view of Propositions 7.14 and 7.15, we may assume that $\mathfrak{m}^p \subseteq \mathfrak{a}_1$ and $\mathfrak{m}^p \subseteq \mathfrak{q}$, where $p \gg 0$ and $\mathfrak{m} = \mathfrak{m}_\xi$ is the ideal defining ξ.

After invoking Proposition 7.12 and Lemma 3.9 we may replace X by $\operatorname{Spec} \widehat{\mathcal{O}_{X,\xi}}$. By Cohen's structure theorem, we may therefore assume $X = \operatorname{Spec} k[\![x_1, \ldots, x_d]\!]$ for a field k. We still have that $\mathfrak{m}^p \subseteq \mathfrak{a}_1$ and $\mathfrak{m}^p \subseteq \mathfrak{q}$, where $\mathfrak{m}$ defines the closed point of X. These inclusions allow us to apply Proposition 7.12 and Lemma 3.9 "in reverse", and assume $X = \mathbf{A}^n_k$ and $\xi = 0$. Finally we can use Proposition 7.13 and Lemma 3.10 with $K = \overline{k}$ to reduce to the case when k is algebraically closed. But then we are in the situation of Conjecture 7.5.

Finally we consider the strong versions of Conjectures 7.4 and 7.5. Pick any $v \in \operatorname{Val}^*_X$ computing $\operatorname{lct}^{\mathfrak{q}}(\mathfrak{a}_\bullet)$. We must show that v is quasi-monomial. After replacing X by a higher model and using Lemma 7.11, we may assume $\operatorname{trdeg}_X(v) = 0$. The proof is now almost identical to what we did for the weak version. Let $\xi = c_X(v)$. By Theorem 7.8, we may assume that $\mathfrak{a}_m = \{f \mid v(f) \geq m\}$. In particular, there is $p \geq 1$ such that $\mathfrak{m}^p \subseteq \mathfrak{a}_1$, where

$\mathfrak{m}$ is the ideal defining ξ. By applying Proposition 7.15, we may also assume that $\mathfrak{m}^N \subseteq \mathfrak{q}$ for some $N \geq 1$. Two applications of Proposition 7.12 and Lemma 3.9 reduce us to the case when $X = \mathbf{A}^n_k$, $\operatorname{trdeg}(v) = 0$ and $c_X(v) = \xi$, where $\xi \in \mathbf{A}^n_k$ is a closed point. Invoking Proposition 7.13 and Lemma 3.10 with $K = \overline{k}$ (note that v extends to a valuation in $\operatorname{Val}_{\mathbf{A}^n_K}$), we see that we may assume that k is algebraically closed, and then we are in position to apply Conjecture 7.5. This completes the proof of Theorem 7.6.

8. The monomial case

In this section we assume that X is the n-dimensional affine space $\operatorname{Spec}(k[x_1, \ldots, x_n])$, where k is a field of characteristic zero, and $\mathfrak{a}_\bullet$ is a graded sequence of monomial ideals (that is, each $\mathfrak{a}_m$ is generated by monomials). As we will see, in this case it is natural to focus on *monomial valuations*: these are the quasi-monomial valuations in $\operatorname{QM}(X, H)$, where $H = H_1 + \cdots + H_n$, with $H_i = V(x_i)$. Every such valuation v is of the form val_w, where $w = (w_1, \ldots, w_n) \in \mathbf{R}^n_{\geq 0}$ is given by $w_i = v(x_i)$. Note that the log discrepancy is then given by $A(\operatorname{val}_w) = \langle e, w \rangle$, where $e = (1, \ldots, 1)$, and where we put $\langle u, w \rangle = \sum_{i=1}^n u_i w_i$ whenever $u, w \in \mathbf{R}^n$.

Denote by $r = r_{X,H} : \operatorname{Val}_X \to \operatorname{QM}(X, H)$ the retraction map. Thus $\bar{v} := r(v)$ is the monomial valuation for which $\bar{v}(x_i) = v(x_i)$ for all i. Thus $\bar{v}(\mathfrak{a}_\bullet) = v(\mathfrak{a}_\bullet)$ and $\bar{v}(\mathfrak{q}) \leq v(\mathfrak{q})$ for any ideal $\mathfrak{q}$. Moreover, by Lemma 5.3, we have $A(\bar{v}) \leq A(v)$ with equality if and only if $v = \bar{v}$ is monomial. This immediately implies that if v is not monomial, then

$$\frac{v(\mathfrak{a}_\bullet)}{A(v) + v(\mathfrak{q})} < \frac{\bar{v}(\mathfrak{a}_\bullet)}{A(\bar{v}) + \bar{v}(\mathfrak{q})} \leq \operatorname{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet);$$

hence v does not compute $\operatorname{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet)$.

On the other hand, consider the simplex $\Sigma = \{w \in \mathbf{R}^n_{\geq 0} \mid \langle e, w \rangle = 1\}$. Then $A(\operatorname{val}_w) = 1$ for all $w \in \Sigma$. It is clear that $w \to \operatorname{val}_w(\mathfrak{q})$ is continuous on Σ and by Lemma 6.4 the same is true for $w \to \operatorname{val}_w(\mathfrak{a}_\bullet)$. Thus the 0-homogeneous function

$$w \to \frac{\operatorname{val}_w(\mathfrak{a}_\bullet)}{A(\operatorname{val}_w) + \operatorname{val}_w(\mathfrak{q})}$$

attains its supremum on Σ. We have proved the following version of Conjecture 7.4:

Proposition 8.1. *If $\mathfrak{a}_\bullet$ is a graded sequence of monomial ideals and $\mathfrak{q}$ is any ideal, then $\operatorname{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet)$ is computed by some monomial valuation. Furthermore, any valuation computing $\operatorname{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet)$ is monomial.*

We shall now use this proposition to recover a formula by Howald [How] for the multiplier ideal $\mathcal{J}(\mathfrak{a}_\bullet^\lambda)$. Let us first observe that $\mathcal{J}(\mathfrak{a}_\bullet^\lambda)$ is a monomial ideal. To see this, let $f \in k[x_1, \ldots, x_n]$ be any polynomial and let $\mathfrak{q} = \mathfrak{q}_f$ be the monomial ideal generated by the monomials that appear in f with nonzero coefficient. It suffices to show that $\operatorname{Arn}^{(f)}(\mathfrak{a}_\bullet) = \operatorname{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet)$. But this is clear by Proposition 8.1 since $v(f) = v(\mathfrak{q})$ for any monomial valuation v.

To describe Howald's formula, we recall from [Mus] (see also [Wol]) how to associate a convex region $P(\mathfrak{a}_\bullet)$ to $\mathfrak{a}_\bullet$. For every $m \geq 1$, consider the Newton polyhedron of $\mathfrak{a}_m$

$$P(\mathfrak{a}_m) = \text{convex hull of } \{u \in \mathbf{Z}^n_{\geq 0} \mid x^u \in \mathfrak{a}_m\}.$$

Our assumption that $\mathfrak{a}_m \neq (0)$ for some m implies that some $P(\mathfrak{a}_m)$ is nonempty. The fact that $\mathfrak{a}_\bullet$ is a graded sequence of ideals gives $P(\mathfrak{a}_m) + P(\mathfrak{a}_\ell) \subseteq P(\mathfrak{a}_{m+\ell})$ for all m and ℓ. In particular, we have $\frac{1}{m}P(\mathfrak{a}_m) \subseteq \frac{1}{pm}P(\mathfrak{a}_{mp})$. We put

$$P(\mathfrak{a}_\bullet) := \overline{\bigcup_m \frac{1}{m}P(\mathfrak{a}_m)}.$$

This is a nonempty closed convex subset of $\mathbf{R}^n_{\geq 0}$, with the property that

$$P(\mathfrak{a}_\bullet) + \mathbf{R}^n_{\geq 0} \subseteq P(\mathfrak{a}_\bullet). \tag{8.1}$$

Indeed, each $P(\mathfrak{a}_m)$ satisfies the same property.

Remark 8.2. Given any nonempty closed convex subset $P \subseteq \mathbf{R}^n_{\geq 0}$ with the property (8.1) there exists a graded sequence $\mathfrak{a}_\bullet$ of monomial ideals such that $P(\mathfrak{a}_\bullet) = P$. Indeed, we can take $\mathfrak{a}_m = (x^u \mid u \in \mathbf{Z}^n_{\geq 0} \cap mP)$ for all $m \geq 1$. In general, the subset $P(\mathfrak{a}_\bullet)$ does not determine $\mathfrak{a}_\bullet$ uniquely. However, as the results below show, if $P(\mathfrak{a}_\bullet) = P(\mathfrak{a}'_\bullet)$, then $\mathfrak{a}_\bullet$ and $\mathfrak{a}'_\bullet$ should be regarded as equisingular.

As an instance of basic convex analysis we next show that the convex set $P = P(\mathfrak{a}_\bullet)$ determines, and is determined by, the concave function $w \to \mathrm{val}_w(\mathfrak{a}_\bullet)$ on $\mathbf{R}^n_{\geq 0}$.

Lemma 8.3. *If $\mathfrak{a}_\bullet$ is a sequence of monomial ideals on $\mathbf{A}^n_k$, then*

$$\mathrm{val}_w(\mathfrak{a}_\bullet) = \inf\{\langle u, w\rangle \mid u \in P(\mathfrak{a}_\bullet)\} \quad \textit{for } w \in \mathbf{R}^n_{\geq 0}. \tag{8.2}$$

Conversely, we have

$$P(\mathfrak{a}_\bullet) = \{u \in \mathbf{R}^n_{\geq 0} \mid \langle u, w\rangle \geq \mathrm{val}_w(\mathfrak{a}_\bullet) \textit{ for all } w \in \mathbf{R}^n_{\geq 0}\}. \tag{8.3}$$

Proof. It is immediate from the definition that $\mathrm{val}_w(\mathfrak{a}_m) = \min\{\langle u, w\rangle \mid u \in P(\mathfrak{a}_m)\}$. It follows that

$$\mathrm{val}_w(\mathfrak{a}_\bullet) = \inf_m \frac{w(\mathfrak{a}_m)}{m} = \inf_m \inf_{u \in \frac{1}{m}P(\mathfrak{a}_m)} \langle u, w\rangle = \inf_{u \in P(\mathfrak{a}_\bullet)} \langle u, w\rangle.$$

The inclusion "$\subseteq$" in (8.3) from from the description of $\mathrm{val}_w(\mathfrak{a}_\bullet)$. On the other hand, if $u_0 \notin P(\mathfrak{a}_\bullet)$, then we can find $v \in \mathbf{R}^n$ and $b \in \mathbf{R}$ such that $\langle u, v\rangle \geq b$ for every $u \in P(\mathfrak{a}_\bullet)$, while $\langle u_0, v\rangle < b$ (this is a general fact about closed convex subsets of $\mathbf{R}^n$, see Theorem 4.5 in [Brø]). It follows from (8.1) that $v \in \mathbf{R}^n_{\geq 0}$, hence $\langle u_0, v\rangle < \mathrm{val}_v(\mathfrak{a}_\bullet)$. □

We can now state and prove Howald's formula.

Proposition 8.4. *If $\mathfrak{a}_\bullet$ is a graded sequence of monomial ideals, then*

$$\mathcal{J}(\mathfrak{a}^\lambda_\bullet) = (x^u \mid u + e \in \mathrm{Int}(\lambda P(\mathfrak{a}_\bullet))). \tag{8.4}$$

Equivalently, $\mathrm{Arn}^{(x^u)}(\mathfrak{a}_\bullet)$ is equal to the unique number $\alpha \geq 0$ such that $\alpha(u+e)$ lies on the boundary of $P = P(\mathfrak{a}_\bullet)$. Moreover, a nontrivial monomial valuation val_w computes

$\mathrm{Arn}^{(x^u)}(\mathfrak{a}_\bullet)$ *if and only if* w *determines a supporting hyperplane of* P *at* $\alpha(u+e)$*, that is,* $\langle \alpha(u+e), w\rangle \leq \langle u', w\rangle$ *for all* $u' \in P(\mathfrak{a}_\bullet)$.

If $\mathfrak{a}_m = \mathfrak{a}^m$ for some monomial ideal $\mathfrak{a}$, then $P(\mathfrak{a}_\bullet) = P(\mathfrak{a})$, $\mathcal{J}(\mathfrak{a}_\bullet^\lambda) = \mathcal{J}(\mathfrak{a}^\lambda)$ and (8.4) becomes Howald's original formula from [How].

Proof. By Proposition 8.1, $\mathrm{Arn}^{(x^u)}(\mathfrak{a}_\bullet)$ is the unique number $\alpha \geq 0$ such that $\mathrm{val}_w(\mathfrak{a}_\bullet) \leq \alpha\langle e+u, w\rangle$ for all $w \in \mathbf{R}^n_{\geq 0}$ with equality for at least one $w \neq 0$. By (8.3) this means exactly that $\alpha(u+e)$ belongs to the boundary of $P(\mathfrak{a}_\bullet)$. Moreover, val_w computes $\mathrm{Arn}^{(x^u)}(\mathfrak{a}_\bullet)$ if and only if $\mathrm{val}_w(\mathfrak{a}_\bullet) = \alpha\langle e+u, w\rangle$, and by (8.2) this means that w defines a supporting hyperplane of P at $\alpha(u+e)$. □

Example 8.5. If we put $P = \{(x,y) \in \mathbf{R}^2_{\geq 0} \mid (x+1)y \geq 1\}$, we get a graded sequence of ideals $\mathfrak{a}_\bullet$ such that $\mathrm{Arn}^{\mathcal{O}_X}(\mathfrak{a}_\bullet) = \frac{-1+\sqrt{5}}{2}$. Furthermore, if $w = \mathrm{val}_{(a,b)}$, then $w(\mathfrak{a}_\bullet) = 2\sqrt{ab} - a$. We see that the nontrivial valuation w computes $\mathrm{Arn}^{\mathcal{O}_X}(\mathfrak{a}_\bullet)$ if and only if $(a,b) = q(1-\alpha, 1)$ for some $q \in \mathbf{R}_{>0}$. In particular, this shows that $\mathrm{Arn}^{\mathcal{O}_X}(\mathfrak{a}_\bullet)$ is not computed by any divisorial monomial valuation.

9. The two-dimensional case

Our goal in this section is to give a proof of the strong version of Conjecture 7.5 in the two-dimensional case. Let k be an algebraically closed field of characteristic zero, $X = \mathbf{A}^2_k$ and $R = k[x,y]$. We put $\mathfrak{m} = (x,y)$. Consider a graded sequence $\mathfrak{a}_\bullet$ of $\mathfrak{m}$-primary ideals and a nonzero ideal $\mathfrak{q}$ on X. Note that there exists $N \geq 1$ such that $\mathfrak{m}^{jN} \subseteq \mathfrak{a}_j$ for all j. We assume that $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet) > 0$, and we have to show that any valuation in Val_X with center at 0 that computes $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet)$ must be quasi-monomial.

For $v \in \mathrm{Val}^*_X$ write

$$\chi(v) = \frac{v(\mathfrak{a}_\bullet)}{A(v) + v(\mathfrak{q})},$$

so that $\mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet)$ is the supremum of χ. As in the proof of Theorem 7.3, it suffices to take the supremum over v centered at the origin, normalized by $v(\mathfrak{m}) = 1$ and satisfying $A(v) \leq M$ for some fixed $M < \infty$. For such valuations, the Izumi-type estimate in (5.3) becomes

$$\mathrm{ord}_0 \leq v \leq A(v) \cdot \mathrm{ord}_0, \tag{9.1}$$

on R, where ord_0 is the divisorial valuation given by the order of vanishing at 0.

Now assume $v_* \in \mathrm{Val}_X$ satisfies $v_*(\mathfrak{m}) = 1$ and $A(v_*) \leq M$ but that v_* is not quasi-monomial. We will show that $\chi(v_*) < \mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet)$. The argument that follows is essentially equivalent to the one in [FJ3], but it avoids appealing to the detailed structure of the valuative tree described in [FJ1]. The key ingredient is a uniform control on strict transforms of curves under birational morphisms, see Lemma 9.2.

Note that $\mathrm{trdeg}(v_*) = 0$ and $\mathrm{ratrk}(v_*) = 1$,[8] or else v_* would be an Abhyankar valuation, hence quasi-monomial. The idea is to find a suitably chosen increasing sequence

[8] Such a valuation is infinitely singular in the terminology of [FJ1].

of smooth log pairs (Y_n, D_n) above $\mathbf{A}^2$ such the corresponding retractions $v_n := r_{Y_n,D_n}(v_*)$ increase to v_*.[9] Furthermore, we will achieve $\chi(v_n) > \chi(v_{n+1})$ for $n \gg 0$ and $\chi(v_n) \to \chi(v_*)$, which in particularly implies that $\chi(v_*) < \mathrm{Arn}^{\mathfrak{q}}(\mathfrak{a}_\bullet)$.

To start the procedure, let $\pi_0 : Y_0 \to \mathbf{A}^2$ be the blowup of $\mathbf{A}^2$ at the origin, with exceptional divisor E_0. Since $\operatorname{trdeg} v_* = 0$, the center of v_* on Y_0 is a closed point $p_0 \in E_0$.

Lemma 9.1. *There exist (algebraic) local coordinates (z_0, w_0) at p_0 on Y_0 such that $E_0 = \{z_0 = 0\}$ and $v_*(z_0) = 1$, $v_*(w_0) = s_0/r_0$ for positive integers r_0, s_0 with $\gcd(r_0, s_0) = 1$ and $r_0 \geq 2$.*

Here the key point is $r_0 \geq 2$. The coordinate w_0 is not unique, but the numbers r_0 and s_0 are.

Proof. Pick any coordinate $z_0 \in \mathcal{O}_{Y_0,p_0}$ such that $E_0 = \{z_0 = 0\}$. Then $v_*(z_0) = v_*(\mathfrak{m}) = 1$. Note that $v_*(\mathcal{O}_{Y_0,p_0} \smallsetminus \{0\})$ is a discrete subsemigroup of $\mathbf{R}_{\geq 0}$. Indeed, if $v_*(f_1) < v_*(f_2) < \dots \leq M$ is a bounded increasing sequence, then we have a decreasing sequence of ideals $\{f \mid v_*(f) \geq v_*(f_i)\}$, all containing the zero-dimensional ideal $\{f \mid v_*(f) \geq M\}$. By the Izumi estimate (9.1) we have $v_*(w) \leq A_{Y_0}(v_*) \operatorname{ord}_{p_0}(w_0)$. Hence we can pick $w_0 \in \mathcal{O}_{Y_0,p_0}$ such that (z_0, w_0) form local coordinates at p_0 and such that $v_*(w_0)$ is maximal. As $\operatorname{ratrk} v_* = 1$, we have $v_*(w_0) \in \mathbf{Q}$ and can write $v_*(w_0) = s_0/r_0$ for positive integers r_0, s_0 with $\gcd(r_0, s_0) = 1$. We have to show that $r_0 \geq 2$.

Suppose to the contrary that $r_0 = 1$. Since $v_*(z_0^{s_0}) = v_*(w_0)$ and $\operatorname{trdeg}(v_*) = 0$, it follows that there is $\vartheta \in k^*$ such that $v_*(w_0 + \vartheta z_0^{s_0}) > v_*(w_0)$. Since $(z_0, w_0 + \vartheta z_0^{s_0})$ is a system of coordinates at p_0, this contradicts the maximality in the choice of $v_*(w_0)$. □

With the notation in the lemma, let v_1 be the monomial valuation in coordinates (z_0, w_0) such that $v_1(z_0) = 1$, $v_1(w_0) = s_0/r_0$. Then v_1 is divisorial and $v_1(\mathfrak{m}) = 1$. Let $\rho_1 : Y_1 \to Y_0$ be a modification above p_0[10] such that the center of v_1 on Y_1 is an exceptional prime divisor E_1. We may and will assume that ρ_1 is a toroidal modification, in the sense that the divisorial valuation ord_E associated to each exceptional prime divisor $E \subseteq Y_1$ is monomial in the coordinates (z_0, w_0) at p_0. (There is a minimal such ρ_1 which can be explicitly described by the continued fractions expansion of s_0/r_0, but we don't need this information.) The center of v_* on Y_1 must be a *free* point $p_1 \in E_1$ (i.e. not belonging to any other exceptional prime divisor) or else v_* would not take the correct value on z_0 or on w_0. Moreover, if D_1 is the reduced exceptional divisor for $\pi_0 \circ \rho_1 : Y_1 \to \mathbf{A}^2$, then v_1 is equal to the retraction $r_{Y_1,D_1}(v_*)$.

Consider now v_* as a valuation on Y_1 with center at p_1. Up to a factor r_0, the situation is then exactly the same as the one we had when considering v_* at (Y_0, p_0): now $v_*(E_1) = r_0^{-1}$, whereas previously $v_*(E_0) = 1$. We can find new coordinates (z_1, w_1) at p_1 such that $E_1 = \{z_1 = 0\}$ and $v_*(w_1)$ is maximal. The proof of Lemma 9.1 gives $v_*(w_1) = \frac{s_1}{r_0 r_1}$ for positive integers $r_1, s_1$with $\gcd(r_1, s_1) = 1$ and $r_1 \geq 2$. Let v_2 be the monomial valuation in coordinates (z_1, w_1) taking the same values as v_* on these coordinates. We

[9]This approach can be used to classify valuations on surfaces and recover the structure of the valuative tree as described in [FJ1]; see also [Spi].

[10]By this, we mean that ρ_1 is proper, and an isomorphism over $Y_0 \smallsetminus \{p_0\}$, with Y_1 nonsingular.

can find a toroidal modification $\rho_2\colon Y_2 \to Y_1$ above p_1 such that the center of v_2 (resp. v_*) on Y_2 is an exceptional prime divisor E_2 (resp. a free point $p_2 \in E_2$).

This procedure can be continued indefinitely, giving rise to sequences $(v_j)_{j\geq 1}$, $(E_j)_{j\geq 0}$, $(p_j)_{j\geq 0}$, $(z_j, w_j)_{j\geq 0}$ and $(r_j, s_j)_{j\geq 0}$. We write $b_n = r_{n-1}r_{n-2}\dots r_0$. One can check that $b_n = \mathrm{ord}_{E_n}(\mathfrak{m})$. Since $r_j \geq 2$ for all j, we have $b_n \geq 2^n$. By Corollary 5.4 we have $A(v_j) < A(v_*)$ for all j.

We have the following estimate, whose proof uses elementary intersection theory.

Lemma 9.2. *Let $\pi_0\colon Y_0 \to \mathbf{A}^2$ be the blowup of the origin with exceptional divisor E_0, and consider a point $p_0 \in E_0$. Further, let $\rho : Y \to Y_0$ be a modification above E_0. Consider an exceptional prime divisor $E \subseteq Y$ mapping to p_0 and a free point p on E. Then, for any effective divisor $H \subseteq \mathbf{A}^2$ we have*

$$\mathrm{ord}_p(\widetilde{H}|_E) \leq b^{-1}\cdot \mathrm{ord}_{p_0}(\widetilde{H}_0|_{E_0}) \leq b^{-1}\cdot \mathrm{ord}_0(H), \tag{9.2}$$

where $\widetilde{H}_0$ and $\widetilde{H}$ are the strict transforms of H by π_0 and $\pi = \pi_0 \circ \rho$, respectively, and where $b = \mathrm{ord}_E(\mathfrak{m})$.

We will apply Lemma 9.2 to $\rho = \rho_n \circ \dots \circ \rho_1$. We then have $b = b_n = \mathrm{ord}_{E_n}(\mathfrak{m}) \geq 2^n$, so $\mathrm{ord}_p(\widetilde{H}) \ll \mathrm{ord}_0(H)$ for $n \gg 0$.

Proof. The second inequality is clear since $E_0 \simeq \mathbf{P}^1$ and the degree of $\widetilde{H}_0|_{E_0}$ equals $\mathrm{ord}_0(H)$. To prove the first inequality we write $\rho^*E_0 = bE + E'$, where E' is a π-exceptional divisor whose support does not contain p. It then follows that

$$\mathrm{ord}_{p_0}(\widetilde{H}_0|_{E_0}) = (\widetilde{H}_0\cdot E_0)_{p_0} = (\rho_*\widetilde{H}\cdot E_0)_{p_0} \geq (\widetilde{H}\cdot \rho^*E_0)_p = b\cdot(\widetilde{H}\cdot E)_p = b\cdot \mathrm{ord}_p(\widetilde{H}|_E).$$

□

Lemma 9.3. *The quasi-monomial valuations v_n satisfy $v_n \leq v_{n+1}$ on $\mathbf{A}^2$. Moreover, $v_n \to v_*$ and $\chi(v_n) \to \chi(v_*)$ as $n \to \infty$.*

Proof. It follows from Lemma 4.6 and Corollary 4.8 that $\nu_n \leq v_{n+1} \leq v_*$ on $\mathbf{A}^2$. We claim that v_n converges to v_* as $n \to \infty$, that is, $v_n(f) \to v_*(f)$ for every $f \in R = k[x,y]$. Now $v_*(f) > v_n(f)$ if and only if the strict transform $\widetilde{H}_n$ of $H := \{f = 0\}$ on Y_n contains p_n, and the latter is equivalent to $\mathrm{ord}_{p_n}(\widetilde{H}_n|_{E_n}) \geq 1$. Thus Lemma 9.2 implies that $v_n(f) = v_*(f)$ as soon as $2^n > \mathrm{ord}_0(f)$.

Let us finally note that $\chi(v_n) \to \chi(v_*)$. Indeed, $v_n(\mathfrak{a}_\bullet)$ and $v_n(\mathfrak{q})$ increase to $v_*(\mathfrak{a}_\bullet)$ and $v_*(\mathfrak{q})$, respectively, by Corollary 6.4. Moreover, since A is lower semicontinuous we have $\liminf_n A(v_n) \geq A(v_*)$. But $A(v_*) \geq A(v_n)$, so $\lim_{n\to\infty} A(v_n) = A(v_*) < \infty$. As $v^*(\mathfrak{q})$ and $v^*(\mathfrak{a}_\bullet)$ are finite, we conclude that $\lim_{n\to\infty}\chi(v_n) = \chi(v_*)$. □

Lemma 9.4. *We have $\chi(v_n) > \chi(v_{n+1})$ for $n \gg 0$.*

Together, Lemmas 9.3 and 9.4 show that $\chi(v_*) < \chi(v_n)$ for n large, and this completes the proof of Conjecture 7.5 in dimension two.

Proof of Lemma 9.4. Pick n_0 such that $2^{n_0} > A(v_*) + v_*(\mathfrak{q})$. In particular, $2^{n_0} > \operatorname{ord}_0(\mathfrak{q})$. By Lemma 9.2, the strict transform of $\mathfrak{q}$ on Y_n does not vanish at p_n for $n \geq n_0$.

Fix $n \geq n_0$ and consider our local coordinates (z_n, w_n) at $p_n \in E_n \subseteq Y_n$. For $t > 0$, let $v_{n,t}$ be the monomial valuation in (z_n, w_n) with $v_{n,t}(z_n) = b_n^{-1}$ and $v_{n,t}(w_n) = b_n^{-1}t$. Thus $v_{n,0} = v_n$ and $v_{n,s_n/r_n} = v_{n+1}$. Note that $v_{n,t}(\mathfrak{m}) = 1$ for all t.

Let us study the function $t \to \chi(v_{n,t})$. First, $A(v_{n,t}) = A(v_n)+b_n^{-1}t$. Second, $v_{n,t}(\mathfrak{q}) = v_n(\mathfrak{q})$ for $n \geq n_0$. For an ideal $\mathfrak{a} \subseteq R$ with $V(\mathfrak{a}) \subseteq \{0\}$, the function, $t \to v_{n,t}(\mathfrak{a})$ is concave (and piecewise linear) for $t \geq 0$. Let $\tilde{\mathfrak{a}}$ be the strict transform of $\mathfrak{a}$ on Y_n. Then, for $0 < t \ll 1$, we deduce using Lemma 9.2:

$$v_{n,t}(\mathfrak{a}) = v_n(\mathfrak{a}) + b_n^{-1}t \cdot \operatorname{ord}_{p_n}(\tilde{\mathfrak{a}}|_{E_n}) \leq v_n(\mathfrak{a}) + 2^{-n}b_n^{-1}t \cdot \operatorname{ord}_0(\mathfrak{a}).$$

By concavity, the same inequality holds for all $t > 0$. Applying this with $\mathfrak{a} = \mathfrak{a}_m$, dividing by m, and then letting m go to infinity, we obtain the inequality

$$v_{n,t}(\mathfrak{a}_\bullet) \leq v_n(\mathfrak{a}_\bullet) + 2^{-n}b_n^{-1}t \cdot \operatorname{ord}_0(\mathfrak{a}_\bullet)$$

for all $t \geq 0$. Hence

$$\chi(v_{n,t}) \leq \frac{v_n(\mathfrak{a}_\bullet) + 2^{-n}b_n^{-1}t \cdot \operatorname{ord}_0(\mathfrak{a}_\bullet)}{A(v_n) + v_n(\mathfrak{q}) + b_n^{-1}t},$$

for all $t \geq 0$, with equality for $t = 0$. Here the right hand side is strictly decreasing in t if and only if

$$2^{-n}b_n^{-1} \operatorname{ord}_0(\mathfrak{a}_\bullet) \cdot (A(v_n) + v_n(\mathfrak{q})) < b_n^{-1}v_n(\mathfrak{a}_\bullet). \tag{9.3}$$

Now $A(v_n) \leq A(v_*) < \infty$, $v_n(\mathfrak{q}) = v_*(\mathfrak{q}) < \infty$ and $v_n(\mathfrak{a}_\bullet) \geq \operatorname{ord}_0(\mathfrak{a}_\bullet)$, so (9.3) holds for $n \geq n_0$ by our choice of n_0. Thus $\chi(v_{n+1}) < \chi(v_n)$ for $n \geq n_0$, completing the proof. □

Appendix A. Multiplier ideals on schemes of finite type over formal power series rings

We explain how to deduce some basic results about multiplier ideals, the Restriction and the Subadditivity Theorems, in our setting from the classical one. Recall that we assume X is a nonsingular scheme of finite type over a ring $R = k[\![x_1, \ldots, x_n]\!]$. Our goal is to prove the following:

Theorem A.1. *If H is a nonsingular closed subscheme of codimension one in X, then $\mathcal{J}((\mathfrak{a} \cdot \mathcal{O}_H)^\lambda) \subseteq \mathcal{J}(\mathfrak{a}^\lambda) \cdot \mathcal{O}_H$ for every ideal $\mathfrak{a}$ on X and every $\lambda \in \mathbf{R}_{\geq 0}$.*

Theorem A.2. *If $\mathfrak{a}$ and $\mathfrak{b}$ are ideals on X, and λ, μ are nonnegative real numbers, then $\mathcal{J}(\mathfrak{a}^\lambda \mathfrak{b}^\mu) \subseteq \mathcal{J}(\mathfrak{a}^\lambda) \cdot \mathcal{J}(\mathfrak{b}^\mu)$.*

For the proofs, it will be convenient to consider the following reindexing of multiplier ideals. If $t > 0$, we put $\mathcal{J}(\mathfrak{a}^{t-}) := \mathcal{J}(\mathfrak{a}^{t-\varepsilon})$ for $0 < \varepsilon \ll 1$. Of course, we have $\mathcal{J}(\mathfrak{a}^t) = \mathcal{J}(\mathfrak{a}^{(t+\varepsilon)-})$ for $0 < \varepsilon \ll 1$. Similarly, if $\mathfrak{a}$ and $\mathfrak{b}$ are two ideals, and s, $t > 0$, then we put $\mathcal{J}(\mathfrak{a}^{s-}\mathfrak{b}^{t-}) := \mathcal{J}(\mathfrak{a}^{s-\varepsilon}\mathfrak{b}^{t-\varepsilon})$ for $0 < \varepsilon \ll 1$. Since $\mathcal{J}(\mathfrak{a}^s\mathfrak{b}^t) = \mathcal{J}(\mathfrak{a}^{(s+\varepsilon)-}\mathfrak{b}^{(t+\varepsilon)-})$ for $0 < \varepsilon \ll 1$, having the statement in Theorem A.2 for all λ, $\mu \geq 0$ is equivalent with having $\mathcal{J}(\mathfrak{a}^{\lambda-}\mathfrak{b}^{\mu-}) \subseteq \mathcal{J}(\mathfrak{a}^{\lambda-}) \cdot \mathcal{J}(\mathfrak{b}^{\mu-})$ for every λ, $\mu > 0$. The same holds for Theorem A.1.

Lemma A.3. *Suppose that $X = \operatorname{Spec} k[\![x_1, \ldots, x_m]\!]$. If $\mathfrak{m}$ is the ideal defining the closed point in X, then*

$$\mathcal{J}(\mathfrak{a}^{t-}) = \bigcap_{N \geq 1} \mathcal{J}((\mathfrak{a} + \mathfrak{m}^N)^{t-})$$

for every $t > 0$.

Proof. We may assume $\mathfrak{a}$ is nonzero: otherwise the assertion follows from $\bigcap_N \mathcal{J}(\mathfrak{m}^{N-}) = \bigcap_N \mathfrak{m}^{N-m} = (0)$. Given $g \in \mathcal{O}(X)$, we have $g \in \mathcal{J}(\mathfrak{a}^{t-})$ if and only if for every divisor E over X

$$\operatorname{ord}_E(g) + A(\operatorname{ord}_E) \geq t \cdot \operatorname{ord}_E(\mathfrak{a}). \tag{A.1}$$

Furthermore, if this is not the case, then one can find a divisor E with center at the closed point such that (A.1) fails (for this, one can argue as in the proof of [dFM, Lemma 2.6]). If $N > \operatorname{ord}_E(\mathfrak{a})$, then $\operatorname{ord}_E(\mathfrak{a}) = \operatorname{ord}_E(\mathfrak{a} + \mathfrak{m}^N)$, and we see that $g \notin \mathcal{J}((\mathfrak{a} + \mathfrak{m}^N)^{t-})$. □

Remark A.4. Using the same proof, one sees that more generally, if $\mathfrak{a}$ and $\mathfrak{b}$ are two ideals as in the lemma, and if $s, t > 0$, then

$$\mathcal{J}(\mathfrak{a}^{s-}\mathfrak{b}^{t-}) = \bigcap_{N \geq 1} \mathcal{J}((\mathfrak{a} + \mathfrak{m}^N)^{s-}(\mathfrak{b} + \mathfrak{m}^N)^{t-})$$

for every s, $t > 0$.

Lemma A.5. *Let $(R, \mathfrak{m})$ be a complete local Noetherian ring, and $(I_N)_{N \geq 1}$ and $(J_N)_{N \geq 1}$ be sequences of ideals in R with $I_{N+1} \subseteq I_N$ and $J_{N+1} \subseteq J_N$ for all N. Write $I = \bigcap_{N \geq 1} I_N$ and $J = \bigcap_{N \geq 1} J_N$.*

(i) *We have $IJ = \bigcap_{N \geq 1} I_N J_N$.*
(ii) *For every ideal I' in R, we have $\bigcap_{N \geq 1}(I' + I_N) = I' + I$.*

Proof. Since R/I is complete in the $\mathfrak{m}$-adic topology, and the filtration given by $(I_N/I)_{N \geq 1}$ is separated, it follows from a theorem of Chevalley (see [ZS, Thm. 13, pp.270–271]) that given any ℓ there is N such that $I_N \subseteq I + \mathfrak{m}^\ell$. Similarly, we see that after possibly increasing N, we may also assume that $J_N \subseteq J + \mathfrak{m}^\ell$. Therefore $I_N J_N \subseteq IJ + \mathfrak{m}^\ell$, so

$$\bigcap_{N \geq 1} I_N J_N \subseteq \bigcap_{\ell \geq 1} (IJ + \mathfrak{m}^\ell) = IJ,$$

where the equality follows from Krull's Intersection Theorem. As the other inclusion is trivial, this proves (i).

The argument for (ii) is similar: we get from Chevalley's theorem that

$$\bigcap_{N \geq 1} (I' + I_N) \subseteq \bigcap_{\ell \geq 1} (I' + I + \mathfrak{m}^\ell) = I' + I,$$

which completes the proof. □

Proof of Theorem A.1. If X is a scheme of finite type over a field k, then the result is well-known see [Laz, Section 9.5.A]. Note that since taking multiplier ideals commutes with passing to the algebraic closure (see Proposition 1.6 and Example 1.4), in this case one can assume that k is algebraically closed.

In the general case, it is enough to prove the two assertions after replacing X by $\mathrm{Spec}(\widehat{\mathcal{O}_{X,\xi}})$, where ξ is any point of X. Indeed, this follows since taking multiplier ideals commutes with this operation, see Proposition 1.6 and Example 1.5. Therefore, by Cohen's Structure Theorem we may assume that $X = \mathrm{Spec}\, k[\![x_1, \ldots, x_m]\!]$, for some m, and that H is defined by the ideal (x_1).

Note that the assertion in the theorem holds for every λ if we replace $\mathfrak{a}$ by $\mathfrak{a} + \mathfrak{m}^N$, where $\mathfrak{m}$ is the ideal defining the closed point of X. Indeed, in this case there is an ideal $\mathfrak{a}_N$ on $\mathbf{A}_k^m$ such that $\mathfrak{a}_N \cdot \mathcal{O}_X = \mathfrak{a} + \mathfrak{m}^N$. In this case, we deduce the assertion on X from the assertion on $\mathbf{A}_k^m$, and the fact that taking multiplier ideals commutes with completion at the origin.

As we have mentioned, this implies that

$$\mathcal{J}(((\mathfrak{a} + \mathfrak{m}^N) \cdot \mathcal{O}_H)^{\lambda-}) \subseteq \mathcal{J}((\mathfrak{a} + \mathfrak{m}^N)^{\lambda-}) \cdot \mathcal{O}_H$$

for all $\lambda > 0$. Intersecting over $N \geq 1$, and using Lemma A.3 and Lemma A.5 (ii) we get

$$\mathcal{J}((\mathfrak{a} \cdot \mathcal{O}_H)^{\lambda-}) \subseteq \mathcal{J}(\mathfrak{a}^{\lambda-}) \cdot \mathcal{O}_H$$

for every $\lambda > 0$. As we have seen, this gives the assertion in the theorem. □

Proof of Theorem A.2. Again, the result is known when X is of finite type over a field (see [Laz, Section 9.5.B]). Arguing as in the proof of Theorem A.1, we see that we may assume $X = \mathrm{Spec}\, k[\![x_1, \ldots, x_m]\!]$, and that we have

$$\mathcal{J}((\mathfrak{a} + \mathfrak{m}^N)^{\lambda-}(\mathfrak{b} + \mathfrak{m}^N)^{\mu-}) \subseteq \mathcal{J}((\mathfrak{a} + \mathfrak{m}^N)^{\lambda-}) \cdot \mathcal{J}((\mathfrak{b} + \mathfrak{m}^N)^{\mu-})$$

for all λ, $\mu > 0$. Taking the intersection over $N \geq 1$ and using Lemma A.3 (see also Remark A.4) and Lemma A.5 (i), we deduce

$$\mathcal{J}(\mathfrak{a}^{\lambda-}\mathfrak{b}^{\mu-}) \subseteq \mathcal{J}(\mathfrak{a}^{\lambda-}) \cdot \mathcal{J}(\mathfrak{b}^{\mu-})$$

for all λ, $\mu > 0$. As we have seen, this implies the assertion in the theorem. □

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