Generalized holomorphic analytic torsion

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Abstract

In this paper we extend the holomorphic analytic torsion classes of Bismut and Köhler to arbitrary projective morphisms between smooth algebraic complex varieties. To this end, we propose an axiomatic definition and give a classification of the theories of generalized holomorphic analytic torsion classes for arbitrary projective morphisms.

For a proper formulation, we first introduce a formalism of hermitian structures on objects of the bounded derived category of coherent sheaves on a smooth complex variety. As a byproduct we build a category $\overline{\mathbf{Sm}}_{*/\mathbb{C}}$ whose objects are smooth complex varieties and whose morphisms are pairs $(f, \overline{T}_f)$ formed by a projective morphism and a hermitian structure on its relative tangent complex T_f.

With this language, we study the theories of analytic torsion classes for closed immersions and for projective spaces. A compatibility criterion enables to combine both theories to produce analytic torsion classes for arbitrary projective morphisms. Our main theorem sets a bijection between theories of generalized analytic torsion classes and real additive genera. The extension of the holomorphic analytic torsion classes of Bismut and Köhler is obtained as the theory of generalized analytic torsion classes associated to $-R/2$, R being the R-genus.

The relation with Grothendieck duality is explored, leading to the notion of the dual analytic torsion theory and self-dual theories. We also study the vanishing of the analytic torsion classes of de Rham complexes. This gives a characterization of Bismut-Köhler higher analytic torsion classes and a conceptual explanation of the R-genus.

We end by applying the theory developed so far to describe the singularities of the analytic torsion for degenerating families of curves. By elementary geometric considerations and computations of Bott-Chern classes, we recover the core result of the work of Bismut-Bost on the singularity of the Quillen metric.

Contents

1 Introduction

The aim of this paper is to extend the classes of analytic torsion forms introduced by Bismut and Köhler to arbitrary projective morphisms between complex algebraic varieties. The main tool for this extension is an axiomatic characterization of all the possible theories of holomorphic analytic torsion classes. Before stating precisely what we mean by a theory of holomorphic analytic torsion classes, we briefly recall the origin of the analytic torsion.

The R-torsion is a topological invariant attached to certain euclidean flat vector bundles on a finite CW-complex. This invariant was introduced by Reidemeister and generalized by Franz in order to distinguish non-homeomorphic lens spaces that have the same homology and homotopy groups. Let W be a connected CW-complex and let K be an orthogonal representation of $\pi_1(W)$. Then K defines a flat vector bundle with an euclidean inner product E_K. Assume that the chain complex of W with values in E_K is acyclic. Then the R-torsion is the determinant of this complex with respect to a preferred basis.

Later, Ray and Singer introduced an analytic analogue of the R-torsion and they conjectured that, for compact riemannian manifolds, this analytic torsion agrees with the R-torsion. This conjecture was proved by Cheeger and Müller. If W is a riemannian manifold and K is as before, then we have the de Rham complex of W with values in E_K at our disposal. The hypothesis on K implies that $(\Omega^*(W, E_K), \mathrm{d})$ is also acyclic. Then the analytic torsion is essentially the determinant of the de Rham complex. Here the difficulty lies in that the vector spaces $\Omega^p(W, E_K)$ are infinite dimensional and therefore the "determinant" has to be defined using a zeta function regularization involving the laplacian. See the paper of Ray and Singer [41] for more details on the construction of R-torsion and analytic torsion.

Ray and Singer observed that, with the help of hermitian metrics, the acyclicity condition can be removed. Moreover, their definition of analytic torsion can be extended to any elliptic complex. In the paper [42], they introduced a holomorphic analogue of the analytic torsion as the determinant of the Dolbeault complex. They also studied some of its properties and computed some examples. In particular, they showed that this invariant depends on the complex structure and they gave a hint that the holomorphic analytic torsion should be interesting in number theory. This holomorphic analytic torsion and its generalizations are the main object of study of the present paper. Since this is the only kind of analytic torsion that we will consider, throughout the paper, by analytic torsion we will mean holomorphic analytic torsion.

In the paper [40], Quillen, using the analytic torsion, associated to each holomorphic hermitian vector bundle on a Riemann surface a hermitian metric on the determinant of its cohomology. Furthermore, he showed that this metric varies smoothly with the holomorphic structure on the vector bundle. He also computed the curvature of the hermitian line bundle on the space of all complex structures obtained in this way.

Subsequently Bismut and Freed [7], [8] generalized the construction of Quillen to families of Dirac operators on the fibers of a smooth fibration. They obtained a smooth metric and a unitary connection on the determinant bundle associated with the family of Dirac operators. Furthermore, they computed the curvature of this connection, which agrees with the degree 2 part of the differential form obtained by Bismut in his proof of the local index theorem [2]. Later, in a series of papers [9], [10], [11], Bismut, Gillet and Soulé considered the case of a holomorphic submersion endowed with a holomorphic hermitian vector bundle. They defined a Quillen type metric on the determinant of the cohomology of the holomorphic vector bundle. In the locally Kähler case, they showed the compatibility with the constructions of Bismut-Freed. In addition they described the variation of the Quillen metric under change of the metric on the vertical tangent bundle and on the hermitian vector bundle. The results of [9], [10], [11] represent a rigidification of [7], [8]. All in all, these works explain the relationship between analytic torsion and the Atiyah-Singer index theorem and, in the algebraic case, with Grothendieck's relative version of the Riemann-Roch theorem.

In [20], Deligne, inspired by the Arakelov formalism, gave a formula for the Quillen metric that can be seen as a very precise version of the degree one case of the Riemann-Roch theorem for families of curves. This result is in the same spirit as the arithmetic Riemann-Roch theorem of Faltings [23].

In the paper [29], Gillet and Soulé conjectured an arithmetic Riemann-Roch formula that generalizes the results of Deligne and Faltings. Besides the analytic torsion or its avatar, the Quillen metric, this Riemann-Roch formula involves a mysterious new odd additive characteristic class, the R-genus, that they computed with the help of Zagier.

In the work [14] Bismut and Lebeau studied the behavior of the analytic torsion with respect to complex immersions. Their compatibility formula also involved the R-genus. Later Bost [15] and Roessler [43] explained, using geometric arguments, why the same genus appears both in the arithmetic Riemann-Roch formula and the Bismut-Lebeau compatibility formula. However these geometric arguments do not characterize the R-genus.

In the article [30] Gillet and Soulé proved the degree one part of the arithmetic Riemann-Roch theorem. A crucial ingredient of the proof is the compatibility formula of Bismut-Lebeau.

In order to establish the arithmetic Riemann-Roch theorem in all degrees it was necessary to generalize the analytic torsion to define higher analytic torsion classes. In fact, it was clear from [30] that, once a suitable theory of higher analytic torsion classes satisfying certain properties were developed, then the arithmetic Riemann-Roch theorem would follow. A first definition of such forms was given by Gillet and Soulé in [29], but they did not prove all the necessary properties. A second equivalent definition was given in [13] by Bismut and Köhler, where some of the needed properties are proved. The compatibility of higher analytic torsion classes with complex immersions, i.e. the generalization of Bismut-Lebeau compatibility formula, was proved in [3]. As a consequence, in [25] Gillet, Soulé and Rössler extended the arithmetic Riemann-Roch theorem to arbitrary degrees.

In the book [24], Faltings followed a similar strategy to define direct images of hermitian vector bundles and proved an arithmetic Riemann-Roch formula up to a unique unknown odd genus.

The arithmetic Riemann-Roch theorems of Gillet-Soulé and Faltings deal only with projective morphisms between arithmetic varieties such that, at the level of complex points, define a submersion. By contrast, in his thesis [51] Zha follows a completely different strategy to establish an arithmetic Riemann-Roch theorem without analytic torsion. His formula does not involve the R-genus. Moreover Zha's theorem is valid for any projective morphism between arithmetic varieties.

In [44], Soulé advocates for an axiomatic characterization of the analytic torsion, similar to the axiomatic characterization of Bott-Chern classes given by Bismut-Gillet-Soulé in [9]. Note that the R-torsion has also been generalized to higher degrees giving rise to different higher torsion classes. In [33], Igusa gives an axiomatic characterization of these higher torsion classes

We now explain more precisely what we mean by a theory of generalized analytic torsion classes. The central point is the relationship between analytic torsion and the Grothendieck-Riemann-Roch theorem.

Let $\pi\colon X \to Y$ be a smooth projective morphism of smooth complex varieties. Let ω be a closed $(1,1)$ form on X that induces a Kähler metric on the fibers of π. Let T_π be the relative tangent bundle. Then ω induces a hermitian metric on T_π. The relative tangent bundle provided with this metric will be denoted $\overline{T}_\pi$.

Let $\overline{F} = (F, h^F)$ be a hermitian vector bundle on X such that for every $i \geq 0$, $R^i\pi_*F$ is locally free. We consider on $R^i\pi_*F$ the L^2 metric obtained using Hodge theory on the fibers of π and denote the corresponding hermitian vector bundle as $\overline{R^i\pi_*F}$. To these data, Bismut and Köhler associate an analytic torsion differential form τ that satisfies the differential equation

$$* \,\partial\bar{\partial}\tau = \sum(-1)^i \operatorname{ch}(\overline{R^i\pi_*F}) - \pi_*(\operatorname{ch}(\overline{F}) \operatorname{Td}(\overline{T}_\pi)), \tag{1.1}$$

where $*$ is a normalization factor that is irrelevant here (see 3.7). Moreover, if we consider the class of τ up to $\operatorname{Im}\partial + \operatorname{Im}\bar{\partial}$, then τ behaves nicely with respect to changes of metrics.

The Grothendieck-Riemann-Roch theorem in de Rham cohomology says that the differential form on the right side of equation (1.1) is exact. Therefore, the existence of the higher analytic torsion classes provides us an analytic proof of this theorem.

Since the Grothendieck-Riemann-Roch theorem is valid with more generality, it is natural to generalize the notion of higher analytic torsion classes as follows. By reasons that will be apparent later (see Remark 2.35) we will restrict ourselves to the algebraic category. Let $f\colon X \to Y$ be a projective morphism between smooth complex algebraic varieties. Let $\overline{F}$ be a hermitian vector bundle on X. Now, the relative tangent complex T_f and the bounded derived direct image f_*F are objects of the derived category of coherent sheaves on X and Y respectively. Since X and Y are smooth, using resolutions by locally free sheaves, we can choose hermitian structures on T_f and f_*F (see Section 2.2) that we denote with an over-line. Hence we have characteristic forms $\operatorname{ch}(\overline{f_*F})$ and $\operatorname{Td}(\overline{T}_f)$. We will denote by $\overline{f}$ the morphism f together with the choice of hermitian structure on T_f. Then the triple $\overline{\xi} = (\overline{f}, \overline{F}, \overline{f_*F})$ will be called a *relative hermitian vector bundle*. This is a particular case of the relative metrized complexes of Section 3.1.

Then, a *generalized analytic torsion class* for $\overline{\xi}$ is the class modulo $\operatorname{Im}\partial + \operatorname{Im}\bar{\partial}$ of a current that satisfies the differential equation

$$* \,\partial\bar{\partial}\tau = \operatorname{ch}(\overline{f_*F}) - f_*(\operatorname{ch}(\overline{F})\operatorname{Td}(\overline{T}_f)). \tag{1.2}$$

Note that such current τ always exists. Again, the Grothendieck-Riemann-Roch theorem in de Rham cohomology implies that the right hand side of equation (1.2) is an exact current. Thus, if Y is proper, the dd^c-lemma implies the existence of such a current. When Y is non-proper, a compactification argument allow us to reduce to the proper case.

Of course, in each particular case, there are many choices for τ. We can add to τ any closed current and obtain a new solution of equation (1.2). By a *theory of generalized analytic torsion classes* we mean a coherent way of choosing a solution of equation (1.2) for all possible relative hermitian vector bundles, satisfying certain natural minimal set of properties.

Each possible theory of generalized analytic torsion classes gives rise to a definition of direct images in arithmetic K-theory and therefore to an arithmetic Riemann-Roch formula. In fact, the arithmetic Riemann-Roch theorems of Gillet-Soulé and of Zha correspond to different choices of a theory of generalized analytic torsion classes. We leave for a subsequent paper the discussion of the relation with the arithmetic Riemann-Roch formula.

Since each projective morphism is the composition of a closed immersion followed by the projection of a projective bundle, it is natural to study first the analytic torsion classes for closed immersions and projective bundles and then combine them in a global theory of analytic torsion classes.

In [18] the authors studied the case of closed immersions (see Section 3.2). The generalized analytic torsion classes for closed immersions are called singular Bott-Chern classes and we will use both terms interchangeably. The definition of a *theory of singular Bott-Chern classes* is obtained by imposing axioms analogous to those defining the classical Bott-Chern classes [26]. Namely, a theory of singular Bott-Chern classes is an assignment that, to each relative hermitian vector bundle $\overline{\xi} = (\overline{f}, \overline{F}, \overline{f_*F})$, with f a closed immersion, assigns the class of a current $T(\overline{\xi})$ on Y, satisfying the following properties:

(i) the differential equation (1.2);

(ii) functoriality for morphisms that are transverse to f;

(iii) a normalization condition.

A crucial observation is that, unlike the classical situation, these axioms do not uniquely characterize the singular Bott-Chern classes. Consequently there are various non-equivalent theories of singular Bott-Chern classes. They are classified by an arbitrary characteristic class of F and T_f. If we further impose the condition that the theory is *transitive* (that is, compatible with composition of closed immersions) and *compatible with the projection formula* then the ambiguity is reduced to an arbitrary additive genus on T_f. The uniqueness can be obtained by adding to the conditions (i)–(iii) an additional homogeneity property. The theory obtained is transitive and compatible with the projection formula and agrees (up to normalization) with the theory introduced in [12].

Similarly, one can define a theory of analytic torsion classes for projective spaces. This is an assignment that, to each relative hermitian vector bundle $\overline{\xi} = (\overline{f}, \overline{F}, \overline{f_* F})$, where $f\colon \mathbb{P}^n_Y \to Y$ is the projection of a trivial projective bundle, assigns the class of a current $T(\overline{\xi})$ satisfying the properties

(i) differential equation (1.2);

(ii) functoriality;

(iii) an additivity and normalization condition;

(iv) compatibility with the projection formula.

The theories of analytic torsion classes for projective spaces are classified by their values in the cases $Y = \operatorname{Spec} \mathbb{C}$, $n \geq 0$, $F = \mathcal{O}(k)$, $0 \leq k \leq n$ and one particular choice of metrics (see Theorem 3.53).

We say that a theory of analytic torsion classes for closed immersions and one for projective spaces are compatible if they satisfy a compatibility equation similar to Bismut-Lebeau compatibility formula for the diagonal immersion $\Delta\colon \mathbb{P}^n_{\mathbb{C}} \to \mathbb{P}^n_{\mathbb{C}} \times \mathbb{P}^n_{\mathbb{C}}$, $n \geq 0$. Given a theory of singular Bott-Chern classes that is transitive and compatible with the projection formula, there exists a unique theory of analytic torsion classes for projective spaces that is compatible with it (Theorem 3.88).

A central result of this paper is that, given a theory of singular Bott-Chern classes and a compatible theory of analytic torsion classes for projective spaces, they can be combined to produce a unique theory of generalized analytic torsion classes (see Definition 3.108 and Theorem 3.114). In particular, the theories of generalized analytic torsion classes are classified by additive genera.

A first consequence of Theorem 3.114 is that the classes of the analytic torsion forms of Bismut-Köhler arise as the restriction to Kähler fibrations of the theory of generalized analytic torsion classes associated to minus one half of the R-genus (Theorem 3.133). In particular, we have succeeded to extend Bismut-Köhler analytic torsion classes to arbitrary projective morphisms in the algebraic category.

As corollaries of the axiomatic characterization of analytic torsion classes, we obtain new proofs of two previously known results about analytic torsion. First we reprove and generalize the theorems of Berthomieu-Bismut [1] and Ma [35], [36] on the compatibility of analytic torsion with the composition of submersions. Second we reprove a weak form of the theorem of Bismut-Bost on the singularity of the Quillen metric for degenerating families of curves, whose singular fibers have at most ordinary double points [6].

From the axiomatic point of view, the role played by the R-genus is mysterious. It would seem more natural to consider the generalized analytic torsion classes associated to the trivial genus 0. This is the choice made implicitly by Zha in his thesis [51]. In fact, with our point of view, one of the main results of Zha's thesis is the existence of a theory of analytic torsion classes associated to the trivial genus. This theory leads to an arithmetic Riemann-Roch formula identical to the classical one without any correction term. Thus, one is tempted to consider the R-genus as an artifact of the analytic definition of the analytic torsion. Nevertheless, by the work of several authors, the R-genus seems to have a deeper meaning. A paradigmatic example is the computation by Bost and Kühn [34] of the arithmetic self-intersection of the line bundle of modular forms on a modular curve, provided with the Petersson metric. This formula gives an arithmetic meaning to the first term of the R-genus. Thus it is important to characterize the R-genus from an axiomatic point of view and to understand its role in the above computations.

From a theorem of Bismut [5] we know that the Bismut-Köhler analytic torsion classes of the relative de Rham complex of a Kähler fibration vanish. This result is important because one of the main difficulties to apply the arithmetic Riemann-Roch theorem is precisely the estimation of the analytic torsion. Moreover, this result explains why the terms of the R-genus appear in different arithmetic computations. For instance, the equivariant version of this result (due to Maillot and Roessler in degree 0 and to Bismut in general) allows Maillot and Roessler [37] to prove some cases of a conjecture of Gross-Deligne.

The above vanishing property characterizes the analytic torsion classes of Bismut and Köhler. Namely, in Theorem 3.162 we show that, if it exists, a theory of analytic torsion classes that vanishes on the relative de Rham complexes of Kähler fibrations is unique, hence it agrees with the one defined by Bismut and Köhler. In fact, to characterize this theory, it is enough to assume the vanishing of the analytic torsion classes for Kähler fibrations of relative dimension one.

When working with generalized analytic torsion classes for projective morphisms one encounters several technical problems. First, the relative tangent complex is not a vector bundle but a complex. Second, the direct images and higher direct images of a vector bundle are not in general locally free. Finally, when considering the composition of two morphism, one has to deal with resolutions of resolutions, that lead to cumbersome notation. This is aggravated by the second mentioned technical problem. All these issues are easily solved using hermitian structures on the bounded derived category of coherent sheaves, and this is the point of view that we follow in this paper. Beyond allowing us to work in complete generality and to simplify the presentation, the use of hermitian structures on the derived category furnishes us a useful formalism to explore the properties of analytic torsion: natural questions arise by analogy with algebraic geometric facts, such as the compatibility of analytic torsion classes with Grothendieck duality. The extension of Bott-Chern classes to the derived category is interesting in its own right and it is outlined below when we detail the contents of the different sections.

A few words about notations. The normalizations of characteristic classes and Bott-Chern classes in this paper differ from the ones used by Bismut, Gillet-Soulé and other authors. The first difference is that they work with real valued characteristic classes, while we use characteristic classes in Deligne cohomology, that naturally include the algebro-geometric twist. The second difference is a factor 1/2 in Bott-Chern classes, that explains the factor 1/2 that appears in the characteristic class associated to the torsion classes of Bismut-Köhler. This change of normalization appears already in [16] and its objective is to avoid the factor 1/2 that appears in the definition of arithmetic degree in [27, §3.4.3] and the factor 2 that appears in [27, Theorem 3.5.4] when relating Green currents with Beilinson regulator. The origin of this factor is that the natural second order differential equation that appears when defining Deligne-Beilinson cohomology is $\mathrm{d}_{\mathcal{D}} = -2\partial\bar{\partial}$, while the operator used when dealing with real valued forms is

$$\mathrm{d}\,\mathrm{d}^c = \frac{1}{2}\frac{1}{2\pi i}\,\mathrm{d}_{\mathcal{D}}\,.$$

Thus the characteristic classes that appear in the present article only agree with the ones in the papers of Bismut, Gillet and Soulé after renormalization. With respect to the work of these authors we have also changed the sign of the differential equation that characterizes singular Bott-Chern classes. In this way, the same differential equation appears when considering both, singular Bott-Chern classes and analytic torsion classes. This change is necessary to combine them.

A convention that we will use throughout the paper and whose objective is to simplify the notation in the differential equations that involve direct images, is the following. If $\overline{f} = (f, \overline{T}_f)$ is a projective morphism together with a choice of hermitian structure on the relative tangent complex T_f, then we will write

$$\overline{f}_\flat(\omega) := f_*(\omega \bullet \mathrm{Td}(\overline{T}_f)),$$

where $\bullet$ denotes the product in the Deligne complex, that, for the particular degrees we are considering agrees with the exterior product of differential forms.

Let us summarize the main contents of this article.

In Section 2 we develop the theory of hermitian structures on objects of the bounded derived category of coherent sheaves on a smooth algebraic variety X over $\mathbb{C}$. First of all, we define and characterize the notion of *meager complex* (Definition 2.8 and Theorem 2.12). Roughly speaking, meager complexes are bounded acyclic complexes of hermitian vector bundles whose Bott-Chern classes vanish for structural reasons. We then introduce the concept of tight morphism (Definition 2.18) and tight equivalence relation (Definition 2.25) between bounded complexes of hermitian vector bundles. We explain a series of useful computational rules on the monoid of hermitian vector bundles modulo tight equivalence relation, that we call *acyclic calculus* (Theorem 2.27). As a byproduct we see that the submonoid of acyclic complexes modulo meager complexes has a natural structure of abelian group, that we denote $\mathbf{KA}(X)$ (Definition 2.31). The group $\mathbf{KA}(X)$ is a universal abelian group for additive Bott-Chern classes (Theorem 2.32). With these tools at hand, we next define hermitian structures on objects of $\mathbf{D}^{\mathrm{b}}(X)$. A hermitian metric on an object $\mathcal{F}$ of $\mathbf{D}^{\mathrm{b}}(X)$ consists in choosing a bounded complex of hermitian vector bundles $\overline{E}$ and a quasi-isomorphism $E \overset{\sim}{\dashrightarrow} \mathcal{F}$. We introduce an equivalence relation on the set of hermitian metrics on $\mathcal{F}$ and we say that two hermitian metrics *fit tight* (Definition 2.42 and Theorem 2.43) when they are equivalent. Then a hermitian structure on $\mathcal{F}$ is a set of equivalence classes of hermitian metrics on $\mathcal{F}$. The objects of the category $\overline{\mathbf{D}}^{\mathrm{b}}(X)$ are objects of $\mathbf{D}^{\mathrm{b}}(X)$ together with a hermitian structure, and the morphisms are just morphisms in $\mathbf{D}^{\mathrm{b}}(X)$. Theorem 2.47 is devoted to describe the structure of the forgetful functor $\overline{\mathbf{D}}^{\mathrm{b}}(X) \to \mathbf{D}^{\mathrm{b}}(X)$. In particular, we show that the group $\mathbf{KA}(X)$ naturally acts on the fibers of the functor, freely and transitively. An important example of use of hermitian structures is the construction of the *hermitian cone* of a morphism in $\overline{\mathbf{D}}^{\mathrm{b}}(X)$ (Definition 2.48), which is well defined only up to tight isomorphism. We also study several elementary constructions in $\overline{\mathbf{D}}^{\mathrm{b}}(X)$. Here we mention the classes of isomorphisms and distinguished triangles in $\overline{\mathbf{D}}^{\mathrm{b}}(X)$. These classes lie in the group $\mathbf{KA}(X)$ and their properties are listed in Theorem 2.67. As an application we show that $\mathbf{KA}(X)$ receives classes from $K_1(X)$ (Proposition 2.69). Then we proceed to construct Bott-Chern classes for isomorphisms and distinguished triangles in $\overline{\mathbf{D}}^{\mathrm{b}}(X)$. We conclude the section with the definition of the category $\overline{\mathbf{Sm}}_{*/\mathbb{C}}$ (Definition 2.111). The objects of this category are smooth complex varieties. The morphisms are pairs $\overline{f} = (f, \overline{T}_f)$ formed by a projective morphism of smooth complex varieties f, together with a hermitian structure on the tangent complex T_f. The remarkable fact is that the hermitian cone construction enables a composition rule for these morphisms. In fact, the construction of this category was our main motivation to define the hermitian cone.

Section 3 is devoted to the theory of generalized analytic torsion classes. First of all we introduce the data on which analytic torsion classes will depend: the *relative metrized complexes* (Definition 3.3). Then we recall the results of [18] on singular Bott-Chern classes, that we translate in the language of derived categories. Singular Bott-Chern classes are seen as analytic torsion classes for closed immersions. We review the classical anomaly formulas, and also prove a version of the anomaly formulas for distinguished triangles. Before studying analytic torsion classes for trivial projective bundles, we elaborate on regular coherent sheaves. The results of this chapter can be summarized in Corollary 3.43, where a generating class (Definition 3.42) of the category $\mathbf{D}^{\mathrm{b}}(\mathbb{P}^n_X)$ is exhibited. This will reveal useful to check several properties of analytic torsion classes for projective spaces. Next we give the definition of a theory of analytic torsion classes for projective spaces of dimension n (Definition 3.49). The construction and classification of these theories (Theorem 3.53) depend on the choice of the *characteristic numbers* (Definition 3.51). The link between analytic torsion classes for closed immersions and for projective spaces is made through the notion of compatibility (Definition 3.86). In Theorem 3.88 we show that for a given real additive genus S, there exists a choice of characteristic numbers such that the theories of analytic torsion classes T_S (for closed immersions) and T_{t} (for projective spaces of dimension n) are compatible. Several other formulas relating these two theories are then established, all of them are consequence of the notion of compatibility. The axiomatic definition of *generalized analytic torsion classes* is furnished by Definition 3.108. With the help of the properties of compatible theories, a classification theorem for the generalized theories is proven (Theorem 3.114). The classification result is complemented by Theorem 3.121, where a concrete bijection between real additive genera and generalized theories of analytic torsion classes is provided. We explain the relationship between our theories of analytic torsion classes and the analytic torsion forms constructed by Bismut-Köhler (Theorem 3.133), and recover a series of properties previously proven by other authors. The use of the general theory developed so far is illustrated with the construction of the dual theory $T^{\vee}$ to a given theory T of generalized analytic torsion classes (Theorem Definition 3.146). A characterization of self-dual theories (i.e. $T^{\vee} = T$) in terms of the coefficients of the attached real additive genus is stated in Corollary 3.152: self-duality is equivalent to the vanishing of the even coefficients of the genus. As an outcome we obtain a conceptual explanation of the vanishing of the even coefficients of the R-genus of Gillet and Soulé (Corollary 3.153). Self-duality can also be characterized in terms of the de Rham complex of smooth morphisms (Theorem 3.156): a theory T is self-dual if its components of bidegree $(2p-1, p)$,

p odd, in the Deligne complex vanish on de Rham complexes. The digression on dual theories ends with a characterization of the theory of analytic torsion classes of Bismut-Köhler: it is the unique theory vanishing in all degrees on de Rham complexes. To establish this characterization we need to appeal to the non-vanishing of the tautological class κ_{g-2} on the moduli stack $\mathcal{M}_g$ of smooth curves of genus $g \geq 2$.

We close the article with an application of the existence of generalized analytic torsion classes for projective morphisms. We explain how to recover the core result of the work of Bismut-Bost [6] on the singularity of Quillen metrics for degenerating families of curves. In contrast with *loc. cit.*, where the spectral definition of the Ray-Singer analytic torsion is required, our arguments rely on the existence of a generalized theory for arbitrary projective morphisms and some elementary computations of Bott-Chern classes.

We like to point out that the construction of generalized analytic torsion classes of Section 3 is influenced by the thesis of Zha [51]. This is in particular true for the use of regular coherent sheaves and the classification of analytic torsion classes for projective spaces. The use of hermitian derived categories is original in our approach, as is the correspondence of generalized theories with real additive genera and the relation with Grothendieck duality.

Further applications of the theory of generalized analytic torsion classes are left for future work. We plan to prove generalizations of the arithmetic Grothendieck-Riemann-Roch theorem of Gillet-Soulé [30] and Gillet-Rössler-Soulé [25] to arbitrary projective morhisms, along the lines of [18].

It is possible to compute explicitly the characteristic numbers of the unique theory of analytic torsion classes for projective spaces compatible with the homogeneous one. This computation makes more precise the characterization of generalized analytic torsion classes. Nevertheless, since this computation is much more transparent when written in terms of properties of arithmetic Chow groups and the Riemann-Roch theorem, we leave it to the paper devoted to the arithmetic Riemann-Roch theorem.

We also plan to study the possible axiomatic characterization of equivariant analytic torsion classes. Note that the characterization of equivariant singular Bott-Chern forms has already been obtained by Tang in [45].

In the unpublished e-print [47], L. Weng gives another approach to axiomatic analytic torsion classes. In his approach, L. Weng only considers smooth morphism between Kähler fibrations. This forces him to include a continuity condition with respect to the deformation to the normal cone as one of the axioms. The remaining axioms he uses are: the differential equation, functoriality with respect to cartesian squares, compatibility with respect to the projection formula and two anomaly formulas. A collection of differential forms satisfying these axioms are called relative Bott-Chern secondary characteristic classes. Relative Bott-Chern secondary characteristic classes are not unique. The main result of Weng's paper is that any two such theories are related by an additive genus. Moreover he is able to obtain a weak form of the existence theorem for relative Bott-Cherns secondary characteristic classes.

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2 Hermitian structures on objects of $\mathbf{D}^{\mathrm{b}}(X)$ and characteristic classes

2.1 Meager complexes and acyclic calculus

The aim of this section is to construct a universal group for additive Bott-Chern classes of acyclic complexes of hermitian vector bundles. To this end we first introduce and study the class of meager complexes. Any Bott-Chern class that is additive for certain short exact sequences of acyclic complexes (see 2.32) and that vanishes on orthogonally split complexes, necessarily vanishes on meager complexes. Then we develop an acyclic calculus that will ease the task to check if a particular complex is meager. Finally we introduce the group **KA**, which is the universal group for additive Bott-Chern classes.

Let X be a complex algebraic variety over $\mathbb{C}$, namely a reduced and separated scheme of finite type over $\mathbb{C}$. We denote by $\mathbf{V}^{\mathrm{b}}(X)$ the exact category of bounded complexes of algebraic vector bundles on X. Assume in addition that X is smooth over $\mathbb{C}$. Then $\overline{\mathbf{V}}^{\mathrm{b}}(X)$ is defined as the category of pairs $\overline{E} = (E, h)$, where $E \in \operatorname{Ob} \mathbf{V}^{\mathrm{b}}(X)$ and h is a smooth hermitian metric on the complex of analytic vector bundle E^{an}. From now on we shall make no distinction between E and E^{an}. The complex E will be called *the underlying complex of* $\overline{E}$. We will denote by the symbol $\sim$ the quasi-isomorphisms in any of the above categories.

A basic construction in $\mathbf{V}^{\mathrm{b}}(X)$ is the cone of a morphism of complexes. Recall that, if $f\colon E \to F$ is such a morphism, then, as a graded vector bundle $\operatorname{cone}(f) = E[1] \oplus F$ and the differential is given by $\mathrm{d}(x, y) = (-\mathrm{d}\, x, f(x) + \mathrm{d}\, y)$. We can extend the cone construction easily to $\overline{\mathbf{V}}^{\mathrm{b}}(X)$ as follows.

Definition 2.1. If $f\colon \overline{E} \to \overline{F}$ is a morphism in $\overline{\mathbf{V}}^{\mathrm{b}}(X)$, *the hermitian cone* of f, denoted by $\overline{\operatorname{cone}}(f)$, is defined as the cone of f provided with the orthogonal sum hermitian metric.

When the morphism is clear from the context we will sometimes denote $\overline{\operatorname{cone}}(f)$ by $\overline{\operatorname{cone}}(\overline{E}, \overline{F})$.

Remark 2.2. Let $f\colon \overline{E} \to \overline{F}$ be a morphism in $\overline{\mathbf{V}}^{\mathrm{b}}(X)$. Then there is an exact sequence of complexes

$$0 \longrightarrow \overline{F} \longrightarrow \overline{\operatorname{cone}}(f) \longrightarrow \overline{E}[1] \longrightarrow 0,$$

whose constituent short exact sequences are orthogonally split. Conversely, if

$$0 \longrightarrow \overline{F} \longrightarrow \overline{G} \longrightarrow \overline{E}[1] \longrightarrow 0$$

is a short exact sequence all whose constituent exact sequences are orthogonally split, then there is a natural section $s\colon E[1] \to G$. The image of $\mathrm{d}\, s - s\, \mathrm{d}$ belongs to F and, in fact, determines a morphism of complexes

$$f_s := \mathrm{d}\, s - s\, \mathrm{d}\colon \overline{E} \longrightarrow \overline{F}.$$

Moreover, there is a natural isometry $\overline{G} \cong \overline{\mathrm{cone}}(f_s)$.

The hermitian cone has the following useful property.

Lemma 2.3. *Consider a diagram in* $\overline{\mathbf{V}}^{\mathrm{b}}(X)$

$$\begin{array}{ccc} \overline{E}' & \xrightarrow{f'} & \overline{F}' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ \overline{E} & \xrightarrow{f} & \overline{F}. \end{array}$$

Assume that the diagram is commutative up to homotopy and fix a homotopy h. The homotopy h induces morphisms of complexes

$$\psi\colon \overline{\mathrm{cone}}(f') \longrightarrow \overline{\mathrm{cone}}(f)$$
$$\phi\colon \overline{\mathrm{cone}}(-g') \longrightarrow \overline{\mathrm{cone}}(g)$$

and there is a natural isometry of complexes

$$\overline{\mathrm{cone}}(\phi) \xrightarrow{\sim} \overline{\mathrm{cone}}(\psi).$$

Morever, let h' be a second homotopy between $g \circ f'$ and $f \circ g'$ and let ψ' be the induced morphism. If there exists a higher homotopy between h and h', then ψ and ψ' are homotopically equivalent.

Proof. Since $h\colon E' \to F[-1]$ is a homotopy between gf' and fg', we have

$$gf' - fg' = \mathrm{d}\, h + h\, \mathrm{d}\,. \tag{2.4}$$

First of all, define the arrow $\psi\colon \overline{\mathrm{cone}}(f') \to \overline{\mathrm{cone}}(f)$ by the following rule:

$$\psi(x', y') = (g'(x'), g(y') + h(x')).$$

From the definition of the differential of a cone and the homotopy relation (2.4), one easily checks that ψ is a morphism of complexes. Now apply the same construction to the diagram

$$\begin{array}{ccc} \overline{E}' & \xrightarrow{-g'} & \overline{E} \\ {\scriptstyle -f'}\downarrow & & \downarrow{\scriptstyle f} \\ \overline{F'} & \xrightarrow{g} & \overline{F}. \end{array} \tag{2.5}$$

The diagram (2.5) is still commutative up to homotopy and h provides such a homotopy. We obtain a morphism of complexes $\phi : \overline{\mathrm{cone}}(-g') \to \overline{\mathrm{cone}}(g)$, defined by the rule

$$\phi(x', x) = (-f'(x'), f(x) + h(x')).$$

One easily checks that a suitable reordering of factors sets an isometry of complexes between $\overline{\mathrm{cone}}(\phi)$ and $\overline{\mathrm{cone}}(\psi)$. Assume now that h' is a second homotopy and that there is a higher homotopy $s\colon \overline{E}' \to \overline{F}[-2]$ such that

$$h' - h = \mathrm{d}\, s - s\, \mathrm{d}\,.$$

Let $H\colon \overline{\operatorname{cone}}(f') \to \overline{\operatorname{cone}}(f)[-1]$ be given by $H(x', y') = (0, s(x'))$. Then

$$\psi' - \psi = \mathrm{d}\, H + H\, \mathrm{d}\,.$$

Hence ψ and ψ' are homotopically equivalent. □

Recall that, given a morphism of complexes $f\colon \overline{E} \to \overline{F}$, we use the abuse of notation $\overline{\operatorname{cone}}(f) = \overline{\operatorname{cone}}(\overline{E}, \overline{F})$. As seen in the previous lemma, sometimes it is natural to consider $\overline{\operatorname{cone}}(-f)$. With the notation above it will be denoted also by $\overline{\operatorname{cone}}(\overline{E}, \overline{F})$. Note that this ambiguity is harmless because there is a natural isometry between $\overline{\operatorname{cone}}(f)$ and $\overline{\operatorname{cone}}(-f)$. Of course, when more than one morphism between $\overline{E}$ and $\overline{F}$ is considered, the above notation should be avoided.

With this convention, Lemma 2.3 can be written as

$$\overline{\operatorname{cone}}(\overline{\operatorname{cone}}(\overline{E}', \overline{E}), \overline{\operatorname{cone}}(\overline{F}', \overline{F})) \cong \overline{\operatorname{cone}}(\overline{\operatorname{cone}}(\overline{E}', \overline{F}'), \overline{\operatorname{cone}}(\overline{E}, \overline{F})). \tag{2.6}$$

Definition 2.7. We will denote by $\mathscr{M}_0 = \mathscr{M}_0(X)$ the subclass of $\overline{\mathbf{V}}^{\mathrm{b}}(X)$ consisting of

(i) the orthogonally split complexes;

(ii) all objects $\overline{E}$ such that there is an acyclic complex $\overline{F}$ of $\overline{\mathbf{V}}^{\mathrm{b}}(X)$, and an isometry $\overline{E} \to \overline{F} \oplus \overline{F}[1]$.

We want to stabilize $\mathscr{M}_0$ with respect to hermitian cones.

Definition 2.8. We will denote by $\mathscr{M} = \mathscr{M}(X)$ the smallest subclass of $\overline{\mathbf{V}}^{\mathrm{b}}(X)$ that satisfies the following properties:

(i) it contains $\mathscr{M}_0$;

(ii) if $f\colon \overline{E} \to \overline{F}$ is a morphism and two of $\overline{E}$, $\overline{F}$ and $\overline{\operatorname{cone}}(f)$ belong to $\mathscr{M}$, then so does the third.

The elements of $\mathscr{M}(X)$ will be called *meager complexes*.

We next give a characterization of meager complexes. For this, we introduce two auxiliary classes.

Definition 2.9. (i) Let $\mathscr{M}_F$ be the subclass of $\overline{\mathbf{V}}^{\mathrm{b}}(X)$ that contains all complexes $\overline{E}$ that have a finite filtration Fil such that

(A) for every $p, n \in \mathbb{Z}$, the exact sequences

$$0 \to \operatorname{Fil}^{p+1} \overline{E}^n \to \operatorname{Fil}^p \overline{E}^n \to \operatorname{Gr}^p_{\operatorname{Fil}} \overline{E}^n \to 0,$$

with the induced metrics, are orthogonally split short exact sequences of vector bundles;

(B) the complexes $\operatorname{Gr}^{\bullet}_{\operatorname{Fil}} \overline{E}$ belong to $\mathscr{M}_0$.

(ii) Let $\mathscr{M}_S$ be the subclass of $\overline{\mathbf{V}}^{\mathrm{b}}(X)$ that contains all complexes $\overline{E}$ such that there is a morphism of complexes $f\colon \overline{E} \to \overline{F}$ and both $\overline{F}$ and $\overline{\operatorname{cone}}(f)$ belong to $\mathscr{M}_F$.

Lemma 2.10. *Let* $0 \to \overline{E} \to \overline{F} \to \overline{G} \to 0$ *be an exact sequence in* $\overline{\mathbf{V}}^{\mathrm{b}}(X)$ *whose constituent rows are orthogonally split. Assume* $\overline{E}$ *and* $\overline{G}$ *are in* $\mathscr{M}_F$ *. Then* $\overline{F} \in \mathscr{M}_F$*. In particular,* $\mathscr{M}_F$ *is closed under cone formation.*

Proof. For the first claim, notice that the filtrations of $\overline{E}$ and $\overline{G}$ induce a filtration on $\overline{F}$ satisfying conditions 2.9 (**A**) and 2.9 (**B**). The second claim then follows by Remark 2.2. □

Example 2.11. Given any complex $\overline{E} \in \operatorname{Ob} \overline{\mathbf{V}}^{\mathrm{b}}(X)$, the complex $\overline{\operatorname{cone}}(\mathrm{id}_{\overline{E}})$ belongs to $\mathscr{M}_F$. This can be seen by induction on the length of $\overline{E}$ using Lemma 2.10 and the bête filtration of $\overline{E}$. For the starting point of the induction one takes into account that, if $\overline{E}$ has only one non zero degree, then $\overline{\operatorname{cone}}(\mathrm{id}_{\overline{E}})$ is orthogonally split.

Theorem 2.12. *The equality*

$$\mathscr{M} = \mathscr{M}_S$$

holds.

Proof. We start by proving that $\mathscr{M}_F \subset \mathscr{M}$. Let $\overline{E} \in \mathscr{M}_F$ and let Fil be any filtration that satisfies conditions 2.9 (**A**) and 2.9 (**B**). We show that $\overline{E} \in \mathscr{M}$ by induction on the length of Fil. If Fil has length one, then $\overline{E}$ belongs to $\mathscr{M}_0 \subset \mathscr{M}$. If the length of Fil is $k > 1$, let p be such that $\operatorname{Fil}^p \overline{E} = \overline{E}$ and $\operatorname{Fil}^{p+1} \overline{E} \neq \overline{E}$. On the one hand, $\operatorname{Gr}^p_{\operatorname{Fil}} \overline{E}[-1] \in \mathscr{M}_0 \subset \mathscr{M}$ and, on the other hand, the filtration Fil induces a filtration on $\operatorname{Fil}^{p+1} \overline{E}$ fulfilling conditions 2.9 (**A**) and 2.9 (**B**) and has length $k-1$. Thus, by induction hypothesis, $\operatorname{Fil}^{p+1} \overline{E} \in \mathscr{M}$. Then, by Lemma 2.10, we deduce that $\overline{E} \in \mathscr{M}$.

Clearly, the fact that $\mathscr{M}_F \subset \mathscr{M}$ implies that $\mathscr{M}_S \subset \mathscr{M}$. Thus, to prove the theorem, it only remains to show that $\mathscr{M}_S$ satisfies the condition 2.8 (ii).

The content of the next result is that the apparent asymmetry in the definition of $\mathscr{M}_S$ is not real.

Lemma 2.13. *Let* $\overline{E} \in \operatorname{Ob} \overline{\mathbf{V}}^{\mathrm{b}}(X)$*. Then there is a morphism* $f\colon \overline{E} \to \overline{F}$ *with* $\overline{F}$ *and* $\overline{\operatorname{cone}}(f)$ *in* $\mathscr{M}_F$ *if and only if there is a morphism* $g\colon \overline{G} \to \overline{E}$ *with* $\overline{G}$ *and* $\overline{\operatorname{cone}}(g)$ *in* $\mathscr{M}_F$*.*

Proof. Assume that there is a morphism $f\colon \overline{E} \to \overline{F}$ with $\overline{F}$ and $\overline{\operatorname{cone}}(f)$ in $\mathscr{M}_F$. Then, write $\overline{G} = \overline{\operatorname{cone}}(f)[-1]$ and let $g\colon \overline{G} \to \overline{E}$ be the natural map. By hypothesis, $\overline{G} \in \mathscr{M}_F$. Moreover, since there is a natural isometry

$$\overline{\operatorname{cone}}(\overline{\operatorname{cone}}(\overline{E},\overline{F})[-1],\overline{E}) \cong \overline{\operatorname{cone}}(\overline{\operatorname{cone}}(\mathrm{id}_{\overline{E}})[-1],\overline{F}),$$

by Example 2.11 and Lemma 2.10 we obtain that $\overline{\operatorname{cone}}(g) \in \mathscr{M}_F$. Thus we have proved one implication. The proof of the other implication is analogous. □

Let now $f\colon \overline{E} \to \overline{F}$ be a morphism of complexes with $\overline{E}, \overline{F} \in \mathscr{M}_S$. We want to show that $\overline{\operatorname{cone}}(f) \in \mathscr{M}_S$. By Lemma 2.13, there are morphisms of complexes $g\colon \overline{G} \to \overline{E}$ and $h\colon \overline{H} \to \overline{F}$ with $\overline{G}$, $\overline{H}$, $\overline{\operatorname{cone}}(g)$, $\overline{\operatorname{cone}}(h) \in \mathscr{M}_F$. We consider the map $\overline{G} \to \overline{\operatorname{cone}}(h)$ induced by $f \circ g$. Then we write

$$\overline{G'} = \overline{\operatorname{cone}}(\overline{G}, \overline{\operatorname{cone}}(h))[-1].$$

By Lemma 2.10, we have that $\overline{G'} \in \mathscr{M}_F$. We denote by $g'\colon G' \to E$ and $k\colon G' \to H$ the maps $g'(a,b,c) = g(a)$ and $k(a,b,c) = -b$.

There is an exact sequence

$$0 \to \overline{\mathrm{cone}}(h) \to \overline{\mathrm{cone}}(g') \to \overline{\mathrm{cone}}(g) \to 0$$

whose constituent short exact sequences are orthogonally split. Since $\overline{\mathrm{cone}}(h)$ and $\overline{\mathrm{cone}}(g)$ belong to $\mathscr{M}_F$, Lemma 2.10 insures that $\overline{\mathrm{cone}}(g')$ belongs to $\mathscr{M}_F$ as well.

There is a diagram

$$\begin{array}{ccc} \overline{G'} & \xrightarrow{k} & \overline{H} \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle h} \\ \overline{E} & \xrightarrow{f} & \overline{F} \end{array} \tag{2.14}$$

that commutes up to homotopy. We fix the homotopy $s\colon \overline{G}' \to F$ given by $s(a, b, c) = c$. By Lemma 2.3 there is a natural isometry

$$\overline{\mathrm{cone}}(\overline{\mathrm{cone}}(g'), \overline{\mathrm{cone}}(h)) \cong \overline{\mathrm{cone}}(\overline{\mathrm{cone}}(-k), \overline{\mathrm{cone}}(f)).$$

Applying Lemma 2.10 again, we have that $\overline{\mathrm{cone}}(-k)$ and $\overline{\mathrm{cone}}(\overline{\mathrm{cone}}(g'), \overline{\mathrm{cone}}(h))$ belong to $\mathscr{M}_F$. Therefore $\overline{\mathrm{cone}}(f)$ belongs to $\mathscr{M}_S$.

Lemma 2.15. *Let $f\colon \overline{E} \to \overline{F}$ be a morphism in $\overline{\mathbf{V}}^{\mathrm{b}}(X)$.*

(i) If $\overline{E} \in \mathscr{M}_S$ and $\overline{\mathrm{cone}}(f) \in \mathscr{M}_F$ then $\overline{F} \in \mathscr{M}_S$.

(ii) If $\overline{F} \in \mathscr{M}_S$ and $\overline{\mathrm{cone}}(f) \in \mathscr{M}_F$ then $\overline{E} \in \mathscr{M}_S$.

Proof. Assume that $\overline{E} \in \mathscr{M}_S$ and $\overline{\mathrm{cone}}(f) \in \mathscr{M}_F$. Let $g\colon \overline{G} \to \overline{E}$ with $\overline{G} \in \mathscr{M}_F$ and $\overline{\mathrm{cone}}(g) \in \mathscr{M}_F$. By Lemma 2.10 and Example 2.11, we know that $\overline{\mathrm{cone}}(\overline{\mathrm{cone}}(\mathrm{id}_{\overline{G}}), \overline{\mathrm{cone}}(f)) \in \mathscr{M}_F$. But there is a natural isometry of complexes

$$\overline{\mathrm{cone}}(\overline{\mathrm{cone}}(\mathrm{id}_{\overline{G}}), \overline{\mathrm{cone}}(f)) \cong \overline{\mathrm{cone}}(\overline{\mathrm{cone}}(\overline{\mathrm{cone}}(g)[-1], \overline{G}), \overline{F}).$$

Since, by Lemma 2.10, $\overline{\mathrm{cone}}(\overline{\mathrm{cone}}(g)[-1], \overline{G}) \in \mathscr{M}_F$, then $\overline{F} \in \mathscr{M}_S$.

The second statement of the lemma is proved using the dual argument. □

Lemma 2.16. *Let $f\colon \overline{E} \to \overline{F}$ be a morphism in $\overline{\mathbf{V}}^{\mathrm{b}}(X)$.*

(i) If $\overline{E} \in \mathscr{M}_F$ and $\overline{\mathrm{cone}}(f) \in \mathscr{M}_S$ then $\overline{F} \in \mathscr{M}_S$.

(ii) If $\overline{F} \in \mathscr{M}_F$ and $\overline{\mathrm{cone}}(f) \in \mathscr{M}_S$ then $\overline{E} \in \mathscr{M}_S$.

Proof. Assume that $\overline{E} \in \mathscr{M}_F$ and $\overline{\mathrm{cone}}(f) \in \mathscr{M}_S$. Let $g\colon \overline{G} \to \overline{\mathrm{cone}}(f)$ with $\overline{G}$ and $\overline{\mathrm{cone}}(\overline{G}, \overline{\mathrm{cone}}(f))$ in $\mathscr{M}_F$. There is a natural isometry of complexes

$$\overline{\mathrm{cone}}(\overline{G}, \overline{\mathrm{cone}}(f))) \cong \overline{\mathrm{cone}}(\overline{\mathrm{cone}}(\overline{G}[-1], \overline{E}), \overline{F})$$

that shows $\overline{F} \in \mathscr{M}_S$.

The second statement of the lemma is proved by a dual argument. □

Assume now that $f\colon \overline{E} \to \overline{F}$ is a morphism in $\overline{\mathbf{V}}^{\mathrm{b}}(X)$ and $\overline{E}$, $\overline{\mathrm{cone}}(f) \in \mathscr{M}_S$. Let $g\colon \overline{G} \to \overline{E}$ with $\overline{G}$, $\overline{\mathrm{cone}}(g) \in \mathscr{M}_F$. There is a natural isometry

$$\overline{\mathrm{cone}}(\overline{\mathrm{cone}}(\overline{G}, \overline{E}), \overline{\mathrm{cone}}(\mathrm{id}_{\overline{F}})) \cong \overline{\mathrm{cone}}(\overline{\mathrm{cone}}(\overline{G}, \overline{F}), \overline{\mathrm{cone}}(\overline{E}, \overline{F})),$$

that implies $\overline{\operatorname{cone}}(\overline{\operatorname{cone}}(\overline{G},\overline{F}),\overline{\operatorname{cone}}(\overline{E},\overline{F}))\in \mathscr{M}_F$. By Lemma 2.15, we deduce that $\overline{\operatorname{cone}}(\overline{G},\overline{F})\in \mathscr{M}_S$. By Lemma 2.16, $\overline{F}\in \mathscr{M}_S$.

With f as above, the fact that, if $\overline{F}$ and $\overline{\operatorname{cone}}(f)$ belong to $\mathscr{M}_S$ so does $\overline{E}$, is proved by a similar argument. In conclusion, $\mathscr{M}_S$ satisfies the condition 2.8 (ii), hence $\mathscr{M}\subset \mathscr{M}_S$, which completes the proof of the theorem. □

The class of meager complexes satisfies the next list of properties, that follow almost directly from Theorem 2.12.

Theorem 2.17. *(i) If $\overline{E}$ is a meager complex and $\overline{F}$ is a hermitian vector bundle, then the complexes $\overline{F}\otimes\overline{E}$, $\operatorname{Hom}(\overline{F},\overline{E})$ and $\operatorname{Hom}(\overline{E},\overline{F})$, with the induced metrics, are meager.*

(ii) If $\overline{E}^{*,*}$ is a bounded double complex of hermitian vector bundles and all rows (or columns) are meager complexes, then the complex $\operatorname{Tot}(\overline{E}^{*,*})$ is meager.

(iii) If $\overline{E}$ is a meager complex and $\overline{F}$ is another complex of hermitian vector bundles, then the complexes

$$\begin{aligned}\overline{E}\otimes\overline{F}&=\operatorname{Tot}((\overline{F}^{i}\otimes\overline{E}^{j})_{i,j}),\\ \underline{\operatorname{Hom}}(\overline{E},\overline{F})&=\operatorname{Tot}(\operatorname{Hom}((\overline{E}^{-i},\overline{F}^{j})_{i,j}))\ \textit{and}\\ \underline{\operatorname{Hom}}(\overline{F},\overline{E})&=\operatorname{Tot}(\operatorname{Hom}((\overline{F}^{-i},\overline{E}^{j})_{i,j})),\end{aligned}$$

are meager.

(iv) If $f\colon X\to Y$ is a morphism of smooth complex varieties and $\overline{E}$ is a meager complex on Y, then $f^{*}\overline{E}$ is a meager complex on X.

□

We now introduce the notion of tight morphism.

Definition 2.18. A morphism $f\colon \overline{E}\to\overline{F}$ in $\overline{\mathbf{V}}^{\mathrm{b}}(X)$ is said to be *tight* if $\overline{\operatorname{cone}}(f)$ is a meager complex.

Proposition 2.19. *(i) Every meager complex is acyclic.*

(ii) Every tight morphism is a quasi-isomorphism.

Proof. Let $\overline{E}\in\mathscr{M}_F(X)$. Let Fil be any filtration that satisfies conditions 2.9 (**A**) and 2.9 (**B**). By definition, the complexes $\operatorname{Gr}^{p}_{\mathrm{Fil}}\overline{E}$ belong to $\mathscr{M}_0$, so they are acyclic. Hence $\overline{E}$ is acyclic.

If $\overline{E}\in\mathscr{M}_S(X)$, let $\overline{F}$ and $\overline{\operatorname{cone}}(f)$ be as in Definition 2.9 (ii). Then, $\overline{F}$ and $\overline{\operatorname{cone}}(f)$ are acyclic, hence $\overline{E}$ is also acyclic. Thus we have proved the first statement. The second statement is a direct consequence of the first one. □

Many arguments to prove that a certain complex is meager or a certain morphism is tight involve cumbersome diagrams. In order to ease these arguments we will develop a calculus of acyclic complexes.

Before starting we need some preliminary lemmas.

Lemma 2.20. *Let $\overline{E}$, $\overline{F}$ be objects of $\overline{\mathbf{V}}^{\mathrm{b}}(X)$. Then the following conditions are equivalent.*

(i) There exists an object $\overline{G}$ *and a diagram*

$$\begin{array}{ccccc} & & \overline{G} & & \\ & {\scriptstyle f}\swarrow\sim & & \searrow{\scriptstyle g} & \\ \overline{E} & & & & \overline{F}, \end{array}$$

such that $\overline{\mathrm{cone}}(g) \oplus \overline{\mathrm{cone}}(f)[1]$ *is meager.*

(ii) There exists an object $\overline{G}$ *and a diagram*

$$\begin{array}{ccccc} & & \overline{G} & & \\ & {\scriptstyle f}\swarrow\sim & & \searrow{\scriptstyle g} & \\ \overline{E} & & & & \overline{F}, \end{array}$$

such that f *and* g *are tight morphisms.*

Proof. Clearly, (ii) implies (i). To prove the converse implication, if $\overline{G}$ satisfies the conditions of (i), we put $G' = G \oplus \overline{\mathrm{cone}}(f)$ and consider the morphisms $f' \colon \overline{G}' \to E$ and $g' \colon G' \to F$ induced by the first projection $G' \to G$. Then

$$\overline{\mathrm{cone}}(f') = \overline{\mathrm{cone}}(f) \oplus \overline{\mathrm{cone}}(f)[1],$$

that is meager because $\overline{\mathrm{cone}}(f)$ is acyclic, and

$$\overline{\mathrm{cone}}(g') = \overline{\mathrm{cone}}(g) \oplus \overline{\mathrm{cone}}(f)[1],$$

that is meager by hypothesis. □

Lemma 2.21. *The following assertions hold:*

(i) any diagram of tight morphisms

$$\begin{array}{ccccc} \overline{E} & & & & \overline{G} \\ & {\scriptstyle f}\searrow & & \swarrow{\scriptstyle g} & \\ & & \overline{F} & & \end{array}$$

can be completed into a diagram of tight morphisms

$$\begin{array}{ccccc} & & \overline{H} & & \\ & \swarrow & & \searrow & \\ \overline{E} & & & & \overline{G} \\ & \searrow & & \swarrow & \\ & & \overline{F} & & \end{array} \tag{2.22}$$

which commutes up to homotopy;

(ii) *any diagram of tight morphisms*

$$\begin{array}{ccc} & \overline{F} & \\ \swarrow & & \searrow \\ \overline{E} & & \overline{G} \end{array}$$

can be completed into a diagram of tight morphisms

$$\begin{array}{ccc} & \overline{F} & \\ \swarrow & & \searrow \\ \overline{E} & & \overline{G} \\ \searrow & & \swarrow \\ & \overline{H} & \end{array} \tag{2.23}$$

which commutes up to homotopy.

Proof. For the first item, note that there is a natural arrow $\overline{G} \to \overline{\mathrm{cone}}(f)$. Define

$$\overline{H} = \overline{\mathrm{cone}}(\overline{G}, \overline{\mathrm{cone}}(f))[-1].$$

With this choice, diagram (2.22) becomes commutative up to homotopy, taking the projection $H \to F[-1]$ as homotopy. We first show that $\overline{\mathrm{cone}}(\overline{H}, \overline{G})$ is meager. Indeed, there is a natural isometry

$$\overline{\mathrm{cone}}(\overline{H}, \overline{G}) \cong \overline{\mathrm{cone}}(\overline{\mathrm{cone}}(\mathrm{id}_{\overline{G}}), \overline{\mathrm{cone}}(\overline{E}, \overline{F})[-1])$$

and the right hand side complex is meager. Now for $\overline{\mathrm{cone}}(\overline{H}, \overline{E})$. By Lemma 2.3, there is an isometry

$$\overline{\mathrm{cone}}(\overline{\mathrm{cone}}(\overline{H}, \overline{E}), \overline{\mathrm{cone}}(\overline{G}, \overline{F})) \cong \overline{\mathrm{cone}}(\overline{\mathrm{cone}}(\overline{H}, \overline{G}), \overline{\mathrm{cone}}(\overline{E}, \overline{F})). \tag{2.24}$$

The right hand side complex is meager, hence the left hand side is meager as well. Since, by hypothesis, $\overline{\mathrm{cone}}(\overline{G}, \overline{F})$ is meager, the same is true for $\overline{\mathrm{cone}}(\overline{H}, \overline{E})$.

The second statement is proved analogously. □

Definition 2.25. We will say that two complexes $\overline{E}$ and $\overline{F}$ are *tightly related* if any of the equivalent conditions of Lemma 2.20 holds.

It is easy to see, using Lemma 2.21, that to be tightly related is an equivalence relation.

Definition 2.26. We denote by $\overline{\mathbf{V}}^{\mathrm{b}}(X)/\mathscr{M}$ the set of classes of tightly related complexes. The class of a complex $\overline{E}$ will be denoted $[\overline{E}]$.

Theorem 2.27 (Acyclic calculus)**.** *(i) For a complex $\overline{E} \in \mathrm{Ob}\, \overline{\mathbf{V}}^{\mathrm{b}}(X)$, the class $[\overline{E}] = 0$ if and only if $\overline{E} \in \mathscr{M}$.*

(ii) The operation $\oplus$ induces an operation, that we denote $+$, in $\overline{\mathbf{V}}^{\mathrm{b}}(X)/\mathscr{M}$. With this operation $\overline{\mathbf{V}}^{\mathrm{b}}(X)/\mathscr{M}$ is an associative abelian semigroup.

(iii) *For a complex $\overline{E}$, there exists a complex $\overline{F}$ such that $[\overline{F}]+[\overline{E}]=0$, if and only if $\overline{E}$ is acyclic. In this case $[\overline{E}[1]]=-[\overline{E}]$.*

(iv) *For every morphism $f\colon \overline{E}\to\overline{F}$, if E is acyclic, then the equality*

$$[\overline{\operatorname{cone}}(\overline{E},\overline{F})]=[\overline{F}]-[\overline{E}]$$

holds.

(v) *For every morphism $f\colon \overline{E}\to\overline{F}$, if F is acyclic, then the equality*

$$[\overline{\operatorname{cone}}(\overline{E},\overline{F})]=[\overline{F}]+[\overline{E}[1]]$$

holds.

(vi) *Given a diagram*

$$\begin{array}{ccc} \overline{E}' & \xrightarrow{f'} & \overline{F}' \\ {\scriptstyle g'}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ \overline{E} & \xrightarrow{f} & \overline{F} \end{array}$$

in $\overline{\mathbf{V}}^{\mathrm{b}}(X)$, that commutes up to homotopy, then for every choice of homotopy we have

$$[\overline{\operatorname{cone}}(\overline{\operatorname{cone}}(f'),\overline{\operatorname{cone}}(f))]=[\overline{\operatorname{cone}}(\overline{\operatorname{cone}}(-g'),\overline{\operatorname{cone}}(g))].$$

(vii) *Let $f\colon \overline{E}\to\overline{F}$, $g\colon \overline{F}\to\overline{G}$ be morphisms of complexes. Then*

$$[\overline{\operatorname{cone}}(\overline{\operatorname{cone}}(g\circ f),\overline{\operatorname{cone}}(g))]=[\overline{\operatorname{cone}}(f)[1]],$$
$$[\overline{\operatorname{cone}}(\overline{\operatorname{cone}}(f),\overline{\operatorname{cone}}(g\circ f))]=[\overline{\operatorname{cone}}(g)].$$

If one of f or g are quasi-isomorphisms, then

$$[\overline{\operatorname{cone}}(g\circ f)]=[\overline{\operatorname{cone}}(g)]+[\overline{\operatorname{cone}}(f)].$$

If $g\circ f$ is a quasi-isomorphism, then

$$[\overline{\operatorname{cone}}(g)]=[\overline{\operatorname{cone}}(f)[1]]+[\overline{\operatorname{cone}}(g\circ f)].$$

Proof. Statements (i) and (ii) are left to the reader. For assertion (iii), observe that, if $\overline{E}$ is acyclic then $\overline{E}\oplus\overline{E}[1]$ is meager. Thus

$$[\overline{E}]+[\overline{E}[1]]=[\overline{E}\oplus\overline{E}[1]]=0.$$

Conversely, if $[\overline{F}]+[\overline{E}]=0$, then $\overline{F}\oplus\overline{E}$ is meager, hence acyclic. Thus $\overline{E}$ is acyclic.

For property (iv) we consider the map $\overline{F}\oplus\overline{E}[1]\to\overline{\operatorname{cone}}(f)$ defined by the map $\overline{F}\to\overline{\operatorname{cone}}(f)$. There is a natural isometry

$$\overline{\operatorname{cone}}(\overline{F}\oplus\overline{E}[1],\overline{\operatorname{cone}}(f))\cong\overline{\operatorname{cone}}(\overline{E}\oplus\overline{E}[1],\overline{\operatorname{cone}}(\mathrm{id}_F)).$$

Since the right hand complex is meager, so is the first. In consequence

$$[\overline{\operatorname{cone}}(f)]=[\overline{F}\oplus\overline{E}[1]]=[\overline{F}]+[\overline{E}[1]]=[\overline{F}]-[\overline{E}].$$

Statement (v) is proved analogously.

Statement (vi) is a direct consequence of Lemma 2.3.

Statement (vii) is an easy consequence of the previous properties. □

Remark 2.28. In $f\colon \overline{E} \to \overline{F}$ is a morphism and neither $\overline{E}$ nor $\overline{F}$ are acyclic, then $[\overline{\text{cone}}(f)]$ depends on the homotopy class of f and not only on $\overline{E}$ and $\overline{F}$. For instance, let $\overline{E}$ be a non-acyclic complex of hermitian bundles. Consider the zero map and the identity map $0, \text{id}\colon \overline{E} \to \overline{E}$. Since, by Example 2.11, we know that $\overline{\text{cone}}(\text{id})$ is meager, then $[\overline{\text{cone}}(\text{id})] = 0$. By contrast,

$$[\overline{\text{cone}}(0)] = [\overline{E}] + [\overline{E}[-1]] \neq 0$$

because $\overline{E}$ is not acyclic. This implies that we can not extend Theorem 2.27 (iv) or (v) to the case when none of the complexes are acyclic.

Corollary 2.29. *(i) Let*

$$0 \longrightarrow \overline{E} \longrightarrow \overline{F} \longrightarrow \overline{G} \longrightarrow 0$$

be a short exact sequence in $\overline{\mathbf{V}}^{\mathrm{b}}(X)$ *all whose constituent short exact sequences are orthogonally split. If either* $\overline{E}$ *or* $\overline{G}$ *is acyclic, then*

$$[\overline{F}] = [\overline{E}] + [\overline{G}].$$

(ii) Let $\overline{E}^{*,*}$ *be a bounded double complex of hermitian vector bundles. If the columns of* $\overline{E}^{*,*}$ *are acyclic, then*

$$[\text{Tot}(\overline{E}^{*,*})] = \sum_k (-1)^k [\overline{E}^{k,*}].$$

If the rows are acyclic, then

$$[\text{Tot}(\overline{E}^{*,*})] = \sum_k (-1)^k [\overline{E}^{*,k}].$$

In particular, if rows and columns are acyclic

$$\sum_k (-1)^k [\overline{E}^{k,*}] = \sum_k (-1)^k [\overline{E}^{*,k}].$$

Proof. The first item follows from Theorem 2.27 (iv) and (v), by using Remark 2.2. The second assertion follows from the first by induction on the size of the complex, by using the usual filtration of $\text{Tot}(E^{*,*})$. □

As an example of the use of the acyclic calculus we prove

Proposition 2.30. *Let* $f\colon \overline{E} \to \overline{F}$ *and* $g\colon \overline{F} \to \overline{G}$ *be morphisms of complexes. If two of* $f,\ g,\ g \circ f$ *are tight, then so is the third.*

Proof. Since tight morphisms are quasi-isomorphisms, by Theorem 2.27 (vii)

$$[\overline{\text{cone}}(g \circ f)] = [\overline{\text{cone}}(f)] + [\overline{\text{cone}}(g)].$$

Hence the result follows from 2.27 (i). □

Definition 2.31. We will denote by $\mathbf{KA}(X)$ the set of invertible elements of $\overline{\mathbf{V}}^{\mathrm{b}}(X)/\mathscr{M}$. This is an abelian subgroup. By Theorem 2.27 (iii) the group $\mathbf{KA}(X)$ agrees with the image in $\overline{\mathbf{V}}^{\mathrm{b}}(X)/\mathscr{M}$ of the class of acyclic complexes.

The group $\mathbf{KA}(X)$ is a universal abelian group for additive Bott-Chern classes. More precisely, let us denote by $\overline{\mathbf{V}}^0(X)$ the full subcategory of $\overline{\mathbf{V}}^{\mathrm{b}}(X)$ of acyclic complexes.

Theorem 2.32. *Let $\mathscr{G}$ be an abelian group and let $\varphi\colon \operatorname{Ob}\overline{\mathbf{V}}^0(X) \to \mathscr{G}$ be an assignment such that*

(i) (Normalization) For every orthogonally split complex $\overline{E}$, we have $\varphi(\overline{E}) = 0$.

(ii) (Additivity for exact sequences) For every short exact sequence in $\overline{\mathbf{V}}^0(X)$

$$0 \longrightarrow \overline{E} \longrightarrow \overline{F} \longrightarrow \overline{G} \longrightarrow 0,$$

all whose constituent short exact sequences are orthogonally split, the equality

$$\varphi(\overline{F}) = \varphi(\overline{E}) + \varphi(\overline{G})$$

holds.

Then φ factorizes through a group homomorphism $\widetilde{\varphi}\colon \mathbf{KA}(X) \to \mathscr{G}$.

Proof. The second condition tells us that φ is a morphism of semigroups. Thus we only need to show that it vanishes on meager complexes. By assumption it vanishes on orthogonally split complexes. By the second condition and the argument of Example 2.11, it vanishes on complexes of the form $\overline{\operatorname{cone}}(\operatorname{id}_E)$. Again by the second condition, if E is acyclic,

$$\varphi(E) + \varphi(E[1]) = \varphi(\overline{\operatorname{cone}}(\operatorname{id}_E)) = 0.$$

Thus φ vanishes on the class $\mathscr{M}_0$. Then, once more by the second condition, φ vanishes on the class $\mathscr{M}$. □

Remark 2.33. The considerations of this section carry over to the category of complex analytic varieties. If M is a complex analytic variety, one thus obtains for instance a group $\mathbf{KA}^{\mathrm{an}}(M)$. Observe that whenever X is a proper smooth algebraic variety over $\mathbb{C}$, the group $\mathbf{KA}^{\mathrm{an}}(X^{\mathrm{an}})$ is canonically isomorphic to $\mathbf{KA}(X)$.

As an example, we consider the simplest case $\operatorname{Spec}\mathbb{C}$ and we compute the group $\mathbf{KA}(\operatorname{Spec}\mathbb{C})$. Given an acyclic complex E of $\mathbb{C}$-vector spaces, there is a canonical isomorphism

$$\alpha : \det E \longrightarrow \mathbb{C}.$$

If we have an acyclic complex of hermitian vector bundles $\overline{E}$, there is an induced metric on $\det E$. If we put on $\mathbb{C}$ the trivial hermitian metric, then there is a well defined positive real number $\|\alpha\|$, namely the norm of the isomorphism α.

Theorem 2.34. *The assignment $\overline{E} \mapsto \log\|\alpha\|$ induces an isomorphism*

$$\widetilde{\tau}\colon \mathbf{KA}(\operatorname{Spec}\mathbb{C}) \xrightarrow{\simeq} \mathbb{R}.$$

Proof. We just give the steps of the proof leaving its verification to the reader. First, we observe that the assignment in the theorem satisfies the hypothesis of Theorem 2.32. Thus, $\widetilde{\tau}$ exists and is a group morphism. Second, an inductive argument shows that the acyclic complexes

$$[a] := 0 \longrightarrow \mathbb{C} \xrightarrow{a\cdot} \mathbb{C} \longrightarrow 0,$$

for $a \in \mathbb{R}_{>0}$, form a system of generators of $\mathbf{KA}(\operatorname{Spec}\mathbb{C})$. Hence $\widetilde{\tau}$ is surjective. Finally, one shows, using suitable exact sequences, that $[a] + [b] = [ab]$. This implies that $\widetilde{\tau}$ is injective. □

2.2 Definition of $\overline{\mathbf{D}}^{\mathrm{b}}(X)$ and basic constructions

Let X be a smooth algebraic variety over $\mathbb{C}$. We denote by $\mathbf{Coh}(X)$ the abelian category of coherent sheaves on X and by $\mathbf{D}^{\mathrm{b}}(X)$ its bounded derived category. The objects of $\mathbf{D}^{\mathrm{b}}(X)$ are complexes of quasi-coherent sheaves with bounded coherent cohomology. For notational convenience, we also introduce $\mathbf{C}^{\mathrm{b}}(X)$, the abelian category of bounded cochain complexes of coherent sheaves on X. Arrows in $\mathbf{D}^{\mathrm{b}}(X)$ will be written as $\dashrightarrow$, while arrows in $\mathbf{C}^{\mathrm{b}}(X)$ will be denoted by $\to$. The symbol $\sim$ will mean either quasi-isomorphism in $\mathbf{C}^{\mathrm{b}}(X)$ or isomorphism in $\mathbf{D}^{\mathrm{b}}(X)$. Every functor from $\mathbf{D}^{\mathrm{b}}(X)$ to another category will tacitly be assumed to be the derived functor. Therefore we will denote just by f_*, f^*, $\otimes$ and $\underline{\mathrm{Hom}}$ the derived direct image, inverse image, tensor product and internal Hom. Finally, we will refer to locally free sheaves by normal upper case letters (such as F) whereas we reserve script upper case letters for quasi-coherent sheaves in general (for instance $\mathcal{F}$).

Remark 2.35. Because X is in particular a smooth noetherian scheme over $\mathbb{C}$, every object $\mathcal{F}$ of $\mathbf{C}^{\mathrm{b}}(X)$ admits a quasi-isomorphism $F \to \mathcal{F}$, with F an object of $\mathbf{V}^{\mathrm{b}}(X)$. If $\mathcal{F}'$ is an object in $\mathbf{D}^{\mathrm{b}}(X)$, then there is an isomorphism $\mathcal{F} \dashrightarrow \mathcal{F}'$ in $\mathbf{D}^{\mathrm{b}}(X)$, for some object $\mathcal{F}$ in $\mathbf{C}^{\mathrm{b}}(X)$. Hence, there exists an isomorphism $F \dashrightarrow \mathcal{F}'$ with $F \in \mathbf{V}^{\mathrm{b}}(X)$. In general, the analogous statements are no longer true if we work with complex manifolds, as shown by the counterexample [46, Appendix, Cor. A.5].

For the sake of completeness, we recall how morphisms in $\mathbf{D}^{\mathrm{b}}(X)$ between bounded complexes of vector bundles can be represented.

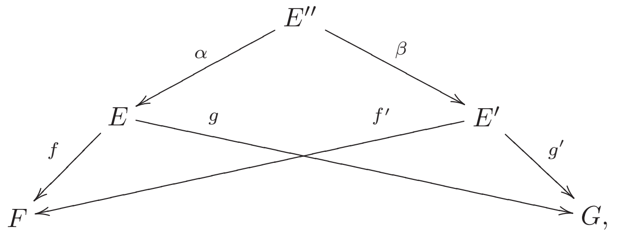
Lemma 2.36. *(i) Let F, G be bounded complexes of vector bundles on X. Every morphism $F \dashrightarrow G$ in $\mathbf{D}^{\mathrm{b}}(X)$ may be represented by a diagram in $\mathbf{C}^{\mathrm{b}}(X)$*

$$\begin{array}{ccc} & E & \\ {}^{f}\swarrow & & \searrow^{g} \\ F & & G, \end{array}$$

where $E \in \operatorname{Ob}\mathbf{V}^{\mathrm{b}}(X)$ and f is a quasi-isomorphism.

(ii) Let E, E', F, G be bounded complex of vector bundles on X and f $(f')\colon E$ $(E') \to F$ quasi-isomorphisms and g $(g')\colon E$ $(E') \to G$ morphisms in $\mathbf{C}^{\mathrm{b}}(X)$.

These data define the same morphism $F \dashrightarrow G$ in $\mathbf{D}^{\mathrm{b}}(X)$ if, and only if, there exists a bounded complex of vector bundles E'' and a diagram



whose squares are commutative up to homotopy and where α and β are quasi-isomorphisms.

Proof. This follows from the equivalence of $\mathbf{D}^{\mathrm{b}}(X)$ with the localization of the homotopy category of $\mathbf{C}^{\mathrm{b}}(X)$ with respect to the class of quasi-isomorphisms and Remark 2.35. ☐

Proposition 2.37. *Let $f\colon \overline{E} \to \overline{E}$ be an endomorphism in $\overline{\mathbf{V}}^{\mathrm{b}}(X)$ that represents id_E in $\mathbf{D}^{\mathrm{b}}(X)$. Then $\overline{\mathrm{cone}}(f)$ is meager.*

Proof. By Lemma 2.36 (ii), the fact that f represents the identity in $\mathbf{D}^{\mathrm{b}}(X)$ means that there are diagrams

$$\begin{array}{ccc} \overline{E}' & \xrightarrow[\sim]{\alpha} & \overline{E} \\ {\scriptstyle\beta}\downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle \mathrm{id}_E} \\ \overline{E} & \xrightarrow[\mathrm{id}_E]{} & \overline{E}, \end{array} \qquad\qquad \begin{array}{ccc} \overline{E}' & \xrightarrow[\sim]{\alpha} & \overline{E} \\ {\scriptstyle\beta}\downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle f} \\ \overline{E} & \xrightarrow[\mathrm{id}_E]{} & \overline{E}, \end{array}$$

that commute up to homotopy. By Theorem 2.27 (iv) and (vi) the equalities

$$[\overline{\mathrm{cone}}(\alpha)] - [\overline{\mathrm{cone}}(\mathrm{id}_E)] = [\overline{\mathrm{cone}}(\beta)] - [\overline{\mathrm{cone}}(\mathrm{id}_E)]$$
$$[\overline{\mathrm{cone}}(\alpha)] - [\overline{\mathrm{cone}}(\mathrm{id}_E)] = [\overline{\mathrm{cone}}(\beta)] - [\overline{\mathrm{cone}}(f)]$$

hold in the group $\mathbf{KA}(X)$ (observe that these relations do not depend on the choice of homotopies).Therefore

$$[\overline{\mathrm{cone}}(f)] = [\overline{\mathrm{cone}}(\mathrm{id}_E)] = 0.$$

Hence $\overline{\mathrm{cone}}(f)$ is meager. ☐

Definition 2.38. Let $\mathcal{F}$ be an object of $\mathbf{D}^{\mathrm{b}}(X)$. A *hermitian metric on $\mathcal{F}$* consists of the following data:

- an isomorphism $E \overset{\sim}{\dashrightarrow} \mathcal{F}$ in $\mathbf{D}^{\mathrm{b}}(X)$, where $E \in \mathrm{Ob}\,\mathbf{V}^{\mathrm{b}}(X)$;
- an object $\overline{E} \in \mathrm{Ob}\,\overline{\mathbf{V}}^{\mathrm{b}}(X)$, whose underlying complex is E.

We write $\overline{E} \dashrightarrow \mathcal{F}$ to refer to the data above and we call it a *metrized object of* $\mathbf{D}^{\mathrm{b}}(X)$.

Our next task is to define the category $\overline{\mathbf{D}}^{\mathrm{b}}(X)$, whose objects are objects of $\mathbf{D}^{\mathrm{b}}(X)$ provided with equivalence classes of metrics. We will show that in this category there is a hermitian cone well defined up to isometries.

Lemma 2.39. *Let* $\overline{E}, \overline{E}' \in \mathrm{Ob}(\overline{\mathbf{V}}^{\mathrm{b}}(X))$ *and consider an arrow* $E \dashrightarrow E'$ *in* $\mathbf{D}^{\mathrm{b}}(X)$*. Then the following statements are equivalent:*

(i) for any diagram

$$\begin{array}{ccc} & E'' & \\ {\scriptstyle\sim}\swarrow & & \searrow \\ E & & E', \end{array}$$

that represents $E \dashrightarrow E'$*, and any choice of hermitian metric on* E''*, we have that*

$$\overline{\mathrm{cone}}(\overline{E}'', \overline{E})[1] \oplus \overline{\mathrm{cone}}(\overline{E}'', \overline{E}') \tag{2.40}$$

is meager;

(ii) there is a diagram

$$\begin{array}{ccc} & E'' & \\ {\scriptstyle\sim}\swarrow & & \searrow \\ E & & E', \end{array}$$

that represents $E \dashrightarrow E'$*, and a choice of hermitian metric on* E''*, such that*

$$\overline{\mathrm{cone}}(\overline{E}'', \overline{E})[1] \oplus \overline{\mathrm{cone}}(\overline{E}'', \overline{E}') \tag{2.41}$$

is meager;

(iii) there is a diagram

$$\begin{array}{ccc} & E'' & \\ \swarrow & & \searrow \\ E & & E', \end{array}$$

that represents $E \dashrightarrow E'$*, and a choice of hermitian metric on* E''*, such that the arrows* $\overline{E}'' \to \overline{E}$ *and* $\overline{E}' \to \overline{E}'$ *are tight morphisms.*

Proof. Clearly (i) implies (ii). To prove the converse we assume the existence of a $\overline{E}''$ that satisfies equation (2.40) and let $\overline{E}'''$ be any complex that satisfies the hypothesis of (i). Then there is a diagram

$$\begin{array}{ccccc} & & E'''' & & \\ & {\scriptstyle\alpha}\swarrow & & \searrow{\scriptstyle\beta} & \\ & E'' & & E''' & \\ {\scriptstyle f}\swarrow & & {\scriptstyle g}\searrow \!\!\!\! \swarrow{\scriptstyle f'} & & \searrow{\scriptstyle g'} \\ E & & & & E' \end{array}$$

whose squares commute up to homotopy. Using acyclic calculus we have

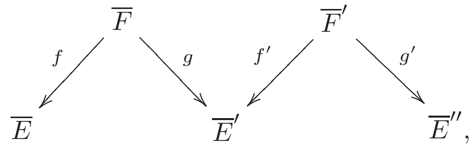
$$\begin{aligned} &[\overline{\mathrm{cone}}(g')] - [\overline{\mathrm{cone}}(f')] = \\ &\quad [\overline{\mathrm{cone}}(\beta)] + [\overline{\mathrm{cone}}(g)] - [\overline{\mathrm{cone}}(\alpha)] - [\overline{\mathrm{cone}}(\beta)] - [\overline{\mathrm{cone}}(f)] + [\overline{\mathrm{cone}}(\alpha)] = \\ &\qquad\qquad [\overline{\mathrm{cone}}(g)] - [\overline{\mathrm{cone}}(f)] = 0. \end{aligned}$$

Now repeat the argument of Lemma 2.20 to prove that (ii) and (iii) are equivalent. The only point is to observe that the diagram constructed in Lemma 2.20 represents the same morphism in the derived category as the original diagram. ☐

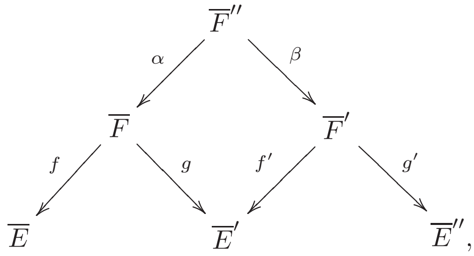
Definition 2.42. Let $\mathcal{F} \in \operatorname{Ob} \mathbf{D}^{\mathrm{b}}(X)$ and let $\overline{E} \dashrightarrow \mathcal{F}$ and $\overline{E}' \dashrightarrow \mathcal{F}$ be two hermitian metrics on $\mathcal{F}$. We say that they *fit tight* if the induced arrow $\overline{E} \dashrightarrow \overline{E}'$ satisfies any of the equivalent conditions of Lemma 2.39

Theorem 2.43. *The relation "to fit tight" is an equivalence relation.*

Proof. The reflexivity and the symmetry are obvious. To prove the transitivity, consider a diagram



where all the arrows are tight morphisms. By Lemma 2.21, this diagram can be completed into a diagram



where all the arrows are tight morphisms and the square commutes up to homotopy. Now observe that $f \circ \alpha$ and $g' \circ \beta$ represent the morphism $E \dashrightarrow E''$ in $\mathbf{D}^{\mathrm{b}}(X)$ and are both tight morphisms by Proposition 2.30. This finishes the proof.

☐

Definition 2.44. We denote by $\overline{\mathbf{D}}^{\mathrm{b}}(X)$ the category whose objects are pairs $\overline{\mathcal{F}} = (\mathcal{F}, h)$ where $\mathcal{F}$ is an object of $\mathbf{D}^{\mathrm{b}}(X)$ and h is an equivalence class of metrics that fit tight, and with morphisms

$$\operatorname{Hom}_{\overline{\mathbf{D}}^{\mathrm{b}}(X)}(\overline{\mathcal{F}}, \overline{\mathcal{G}}) = \operatorname{Hom}_{\mathbf{D}^{\mathrm{b}}(X)}(\mathcal{F}, \mathcal{G}).$$

A class h of metrics will be called *a hermitian structure*, and may be referenced by any representative $\overline{E} \dashrightarrow \mathcal{F}$ or, if the arrow is clear, by the complex $\overline{E}$. We will denote by $\overline{0} \in \operatorname{Ob} \overline{\mathbf{D}}^{\mathrm{b}}(X)$ a zero object of $\mathbf{D}^{\mathrm{b}}(X)$ provided with a trivial hermitian structure given by any meager complex.

If the underlying complex to an object $\overline{\mathcal{F}}$ is acyclic, then its hermitian structure has a well defined class in $\mathbf{KA}(X)$. We will use the notation $[\overline{\mathcal{F}}]$ for this class.

Definition 2.45. A morphism in $\overline{\mathbf{D}}^{\mathrm{b}}(X)$, $f\colon (\overline{E} \dashrightarrow \mathcal{F}) \dashrightarrow (\overline{F} \dashrightarrow \mathcal{G})$, is called *a tight isomorphism* whenever the underlying morphism $f\colon \mathcal{F} \dashrightarrow \mathcal{G}$ is an isomorphism and the metric on $\mathcal{G}$ induced by f and $\overline{E}$ fits tight with $\overline{F}$. An object of $\overline{\mathbf{D}}^{\mathrm{b}}(X)$ will be called *meager* if it is tightly isomorphic to the zero object with the trivial metric.

Remark 2.46. A word of warning should be said about the use of acyclic calculus to show that a particular map is a tight isomorphism. There is an assignment $\operatorname{Ob}\overline{\mathbf{D}}^{\mathrm{b}}(X) \to \overline{\mathbf{V}}^{\mathrm{b}}(X)/\mathscr{M}$ that sends $\overline{E} \dashrightarrow \mathcal{F}$ to $[\overline{E}]$. This assignment is not injective. For instance, let $r > 0$ be a real number and consider the trivial bundle $\mathcal{O}_X$ with the trivial metric $\|1\| = 1$ and with the metric $\|1\|' = 1/r$. Then the product by r induces an isometry between both bundles. Hence, if $\overline{E}$ and $\overline{E}'$ are the complexes that have $\mathcal{O}_X$ in degree 0 with the above hermitian metrics, then $[\overline{E}] = [\overline{E}']$, but they define different hermitian structures on $\mathcal{O}_X$ because the product by r does not represent $\mathrm{id}_{\mathcal{O}_X}$.

Thus the right procedure to show that a morphism $f\colon (\overline{E} \dashrightarrow \mathcal{F}) \dashrightarrow (\overline{F} \dashrightarrow \mathcal{G})$ is a tight isomorphism, is to construct a diagram

$$\begin{array}{ccc} & \overline{G} & \\ {\scriptstyle \alpha}\swarrow {\scriptstyle \sim} & & \searrow{\scriptstyle \beta} \\ \overline{E} & & \overline{F} \end{array}$$

that represents f and then use the acyclic calculus to show that $[\overline{\operatorname{cone}}(\beta)] - [\overline{\operatorname{cone}}(\alpha)] = 0$.

By definition, the forgetful functor $\mathfrak{F}\colon \overline{\mathbf{D}}^{\mathrm{b}}(X) \to \mathbf{D}^{\mathrm{b}}(X)$ is fully faithful. The structure of this functor will be given in the next result that we suggestively summarize by saying that $\overline{\mathbf{D}}^{\mathrm{b}}(X)$ is a principal fibered category over $\mathbf{D}^{\mathrm{b}}(X)$ with structural group $\mathbf{KA}(X)$ provided with a flat connection.

Theorem 2.47. *The functor $\mathfrak{F}\colon \overline{\mathbf{D}}^{\mathrm{b}}(X) \to \mathbf{D}^{\mathrm{b}}(X)$ defines a structure of category fibered in grupoids. Moreover*

(i) The fiber $\mathfrak{F}^{-1}(0)$ is the grupoid associated to the abelian group $\mathbf{KA}(X)$. The object $\overline{0}$ is the neutral element of $\mathbf{KA}(X)$.

(ii) For any object $\mathcal{F}$ of $\mathbf{D}^{\mathrm{b}}(X)$, the fiber $\mathfrak{F}^{-1}(\mathcal{F})$ is the grupoid associated to a torsor over $\mathbf{KA}(X)$. The action of $\mathbf{KA}(X)$ over $\mathfrak{F}^{-1}(\mathcal{F})$ is given by orthogonal direct sum. We will denote this action by $+$.

(iii) Every isomorphism $f\colon \mathcal{F} \dashrightarrow \mathcal{G}$ in $\mathbf{D}^{\mathrm{b}}(X)$ determines an isomorphism of $\mathbf{KA}(X)$-torsors

$$\mathfrak{t}_f\colon \mathfrak{F}^{-1}(\mathcal{F}) \longrightarrow \mathfrak{F}^{-1}(\mathcal{G}),$$

that sends the hermitian structure $\overline{E} \overset{\epsilon}{\dashrightarrow} \mathcal{F}$ to the hermitian structure $\overline{E} \overset{f\circ\epsilon}{\dashrightarrow} \mathcal{G}$. This isomorphism will be called the parallel transport along f.

(iv) Given two isomorphisms $f\colon \mathcal{F} \dashrightarrow \mathcal{G}$ and $g\colon \mathcal{G} \dashrightarrow \mathcal{H}$, the equality

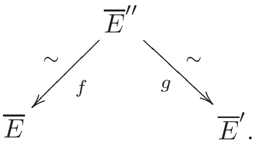
$$\mathfrak{t}_{g\circ f} = \mathfrak{t}_g \circ \mathfrak{t}_f$$

holds.

Proof. Recall that $\mathfrak{F}^{-1}(\mathcal{F})$ is the subcategory of $\overline{\mathbf{D}}^{\mathrm{b}}(X)$ whose objects satisfy $\mathfrak{F}(A) = \mathcal{F}$ and whose morphisms satisfy $\mathfrak{F}(f) = \mathrm{id}_{\mathcal{F}}$. The first assertion is trivial. To prove that $\mathfrak{F}^{-1}(\mathcal{F})$ is a torsor under $\mathbf{KA}(X)$, we need to show that $\mathbf{KA}(X)$ acts freely and transitively on this fiber. For the freeness, it is enough to observe that if for $\overline{E} \in \overline{\mathbf{V}}^{\mathrm{b}}(X)$ and $\overline{M} \in \overline{\mathbf{V}}^{0}(X)$, the complexes $\overline{E}$ and $\overline{E} \oplus \overline{M}$ represent the same hermitian structure, then the inclusion $\overline{E} \hookrightarrow \overline{E} \oplus \overline{M}$ is tight. Hence $\overline{\mathrm{cone}}(\overline{E}, \overline{E} \oplus \overline{M})$ is meager. Since

$$\overline{\mathrm{cone}}(\overline{E}, \overline{E} \oplus \overline{M}) = \overline{\mathrm{cone}}(\overline{E}, \overline{E}) \oplus \overline{M}$$

and $\overline{\mathrm{cone}}(\overline{E}, \overline{E})$ is meager, we deduce that $\overline{M}$ is meager. For the transitivity, any two hermitian structures on $\mathcal{F}$ are related by a diagram



After possibly replacing $\overline{E}''$ by $\overline{E}'' \oplus \overline{\mathrm{cone}}(f)$, we may assume that f is tight. We consider the natural arrow $\overline{E}'' \to \overline{E}' \oplus \overline{\mathrm{cone}}(g)[1]$ induced by g. Observe that $\overline{\mathrm{cone}}(g)[1]$ is acyclic. Finally, we find

$$\overline{\mathrm{cone}}(\overline{E}'', \overline{E}' \oplus \overline{\mathrm{cone}}(g)[1]) = \overline{\mathrm{cone}}(g) \oplus \overline{\mathrm{cone}}(g)[1],$$

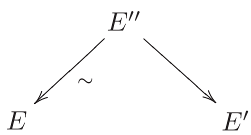
that is meager. Thus the hermitian structure represented by $\overline{E}''$ agrees with the hermitian structure represented by $\overline{E}' \oplus \overline{\mathrm{cone}}(g)[1]$.

The remaining properties are straightforward. □

Our next objective is to define the cone of a morphism in $\overline{\mathbf{D}}^{\mathrm{b}}(X)$. This will be an object of $\overline{\mathbf{D}}^{\mathrm{b}}(X)$ uniquely defined up to tight isomorphism. Let $f\colon (\overline{E} \dashrightarrow \mathcal{F}) \dashrightarrow (\overline{E}' \dashrightarrow \mathcal{G})$ be a morphism in $\overline{\mathbf{D}}^{\mathrm{b}}(X)$, where $\overline{E}$ and $\overline{E}'$ are representatives of the hermitian structures.

Definition 2.48. A *hermitian cone* of f, to be denoted $\overline{\mathrm{cone}}(f)$, is an object $(\mathrm{cone}(f), h_f)$ of $\overline{\mathbf{D}}^{\mathrm{b}}(X)$ where:

- $\mathrm{cone}(f) \in \mathrm{Ob}\,\mathbf{D}^{\mathrm{b}}(X)$ is a choice of cone of f. Namely an object of $\mathbf{D}^{\mathrm{b}}(X)$ completing f into a distinguished triangle;
- h_f is a hermitian structure on $\mathrm{cone}(f)$ constructed as follows. The morphism f induces an arrow $E \dashrightarrow E'$. Choose any bounded complex E'' of vector bundles with a diagram



that represents $E \dashrightarrow E'$, and an arbitrary hermitian metric on E''. Put

$$\overline{C}(f) = \overline{\mathrm{cone}}(\overline{E}'', \overline{E})[1] \oplus \overline{\mathrm{cone}}(\overline{E}'', \overline{E}'). \tag{2.49}$$

There are morphisms defined as compositions

$$\overline{E}' \longrightarrow \overline{\mathrm{cone}}(\overline{E}'', \overline{E}') \longrightarrow \overline{C}(f),$$

where the second arrow is the natural inclusion, and

$$\overline{C}(f) \longrightarrow \overline{\mathrm{cone}}(\overline{E}'', \overline{E}') \longrightarrow \overline{E}''[1] \longrightarrow \overline{E}[1],$$

where the first arrow is the natural projection. These morphisms fit into a natural distinguished triangle completing $\overline{E} \dashrightarrow \overline{E}'$. By the axioms of triangulated category, there is a quasi-isomorphism $\overline{C}(f) \dashrightarrow \mathrm{cone}(f)$ such that the diagram of distinguished triangles

$$\begin{array}{ccccccc}
\overline{E} & \dashrightarrow & \overline{E}' & \dashrightarrow & \overline{C}(f) & \dashrightarrow & \overline{E}[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{F} & \dashrightarrow & \mathcal{G} & \dashrightarrow & \mathrm{cone}(f) & \dashrightarrow & \mathcal{F}[1]
\end{array}$$

commutes. We take the hermitian structure that $\overline{C}(f) \dashrightarrow \mathrm{cone}(f)$ defines on $\mathrm{cone}(f)$. By Theorem 2.51 below, this hermitian structure does not depend on the particular choice of arrow $\overline{C}(f) \dashrightarrow \mathrm{cone}(f)$. Moreover, by Theorem 2.55, the hermitian structure will not depend on the choices of representatives of hermitian structures nor on the choice of $\overline{E}''$.

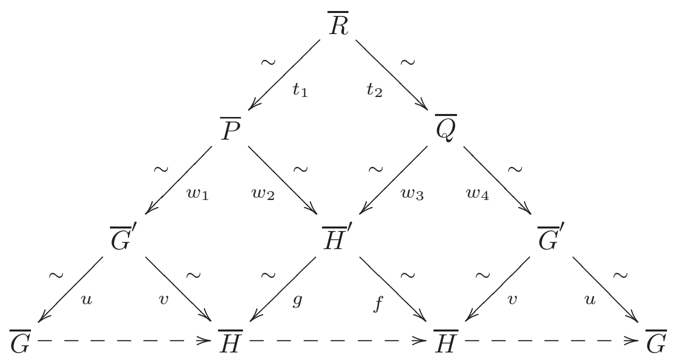
Remark 2.50. The factor $\overline{\mathrm{cone}}(\overline{E}'', \overline{E})[1]$ has to be seen as a correction term to take into account the difference of metrics from $\overline{E}$ and $\overline{E}''$. We would have obtained an equivalent definition using the factor $\overline{\mathrm{cone}}(\overline{E}'', \overline{E})[-1]$.

Theorem 2.51. *Let*

$$\begin{array}{ccccccccc}
\mathcal{F} & \dashrightarrow & \mathcal{G} & \dashrightarrow & \mathcal{H} & \dashrightarrow & \mathcal{F}[1] & \dashrightarrow & \cdots \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
\mathcal{F} & \dashrightarrow & \mathcal{G} & \dashrightarrow & \mathcal{H} & \dashrightarrow & \mathcal{F}[1] & \dashrightarrow & \cdots
\end{array}$$

be a commutative diagram in $\mathbf{D}^{\mathrm{b}}(X)$, *where the rows are the same distinguished triangle. Let* $\overline{H} \dashrightarrow \mathcal{H}$ *be any hermitian structure. Then* $\alpha\colon (\overline{H} \dashrightarrow \mathcal{H}) \dashrightarrow (\overline{H} \dashrightarrow \mathcal{H})$ *is a tight isomorphism.*

Proof. First of all, we claim that if $\gamma : \overline{\mathcal{B}} \dashrightarrow \overline{\mathcal{H}}$ is any isomorphism, then $\gamma^{-1} \circ \alpha \circ \gamma$ is tight if, and only if, α is tight. Indeed, denote by $\overline{G} \dashrightarrow \mathcal{B}$ a representative of the hermitian structure on $\overline{\mathcal{B}}$. Then there is a diagram



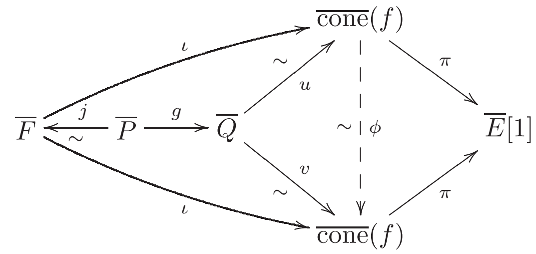
for the liftings of γ^{-1}, α, γ to representatives, as well as for their composites, all whose squares are commutative up to homotopy. By acyclic calculus, we have the following chain of equalities

$$\begin{aligned}
&[\overline{\operatorname{cone}}(u \circ w_1 \circ t_1)[1]] + [\overline{\operatorname{cone}}(u \circ w_4 \circ t_2)] = \\
&\qquad [\overline{\operatorname{cone}}(u)[1]] + [\overline{\operatorname{cone}}(v)] + [\overline{\operatorname{cone}}(g)[1]] + \\
&\qquad [\overline{\operatorname{cone}}(f)] + [\overline{\operatorname{cone}}(v)[1]] + [\overline{\operatorname{cone}}(u)] = \\
&\qquad\qquad [\overline{\operatorname{cone}}(g)[1]] + [\overline{\operatorname{cone}}(f)].
\end{aligned}$$

Thus, the right hand side vanishes if and only if the left hand side vanishes, proving the claim. This observation allows to reduce the proof of the lemma to the following situation: consider a diagram of complexes of hermitian vector bundles

$$\begin{array}{ccccccccc}
\overline{E} & \xrightarrow{f} & \overline{F} & \xrightarrow{\iota} & \overline{\operatorname{cone}}(f) & \xrightarrow{\pi} & \overline{E}[1] & \longrightarrow & \cdots \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \sim \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
\overline{E} & \xrightarrow{f} & \overline{F} & \xrightarrow{\iota} & \overline{\operatorname{cone}}(f) & \xrightarrow{\pi} & \overline{E}[1] & \longrightarrow & \cdots,
\end{array}$$

which commutes in $\mathbf{D}^{\mathrm{b}}(X)$. We need to show that ϕ is a tight isomorphism. The commutativity of the diagram translates into the existence of bounded complexes of hermitian vector bundles $\overline{P}$ and $\overline{Q}$ and a diagram



fulfilling the following properties: (a) j, u, v are quasi-isomorphisms; (b) the squares formed by ι, j, g, u and ι, j, g, v are commutative up to homotopy; (c) the morphisms u, v induce ϕ in the derived category. We deduce a commutative up to homotopy square

$$\begin{array}{ccc}
\overline{\operatorname{cone}}(g) & \xrightarrow[\sim]{\tilde{u}} & \overline{\operatorname{cone}}(\iota) \\
{\scriptstyle \tilde{v}}\downarrow \sim & & \sim \downarrow{\scriptstyle \tilde{\pi}} \\
\overline{\operatorname{cone}}(\iota) & \xrightarrow[\sim]{\tilde{\pi}} & \overline{E}[1].
\end{array}$$

The arrows $\tilde{u}$, $\tilde{v}$ are induced by j, u and j, v respectively. Observe they are quasi-isomorphisms. Also the natural projection $\tilde{\pi}$ is a quasi-isomorphism. By acyclic calculus, we have

$$[\overline{\operatorname{cone}}(\tilde{\pi})] + [\overline{\operatorname{cone}}(\tilde{u})] = [\overline{\operatorname{cone}}(\tilde{\pi})] + [\overline{\operatorname{cone}}(\tilde{v})].$$

Therefore we find

$$[\overline{\operatorname{cone}}(\tilde{u})] = [\overline{\operatorname{cone}}(\tilde{v})]. \tag{2.52}$$

Finally, notice there is an exact sequence

$$0 \longrightarrow \overline{\operatorname{cone}}(u) \longrightarrow \overline{\operatorname{cone}}(\tilde{u}) \longrightarrow \overline{\operatorname{cone}}(j[1]) \longrightarrow 0,$$

whose rows are orthogonally split. Therefore,

$$[\overline{\operatorname{cone}}(\tilde{u})] = [\overline{\operatorname{cone}}(u)] + [\overline{\operatorname{cone}}(j[1])]. \tag{2.53}$$

Similarly we prove

$$[\overline{\operatorname{cone}}(\tilde{v})] = [\overline{\operatorname{cone}}(v)] + [\overline{\operatorname{cone}}(j[1])]. \tag{2.54}$$

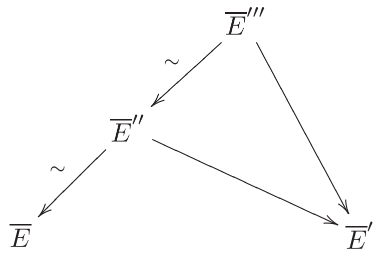
From equations (2.52)–(2.54) we infer

$$[\overline{\operatorname{cone}}(u)[1]] + [\overline{\operatorname{cone}}(v)] = 0,$$

as was to be shown. □

Theorem 2.55. *The object $\overline{C}(f)$ of equation* (2.49) *is well defined up to tight isomorphism.*

Proof. We first show the independence on the choice of $\overline{E}''$, up to tight isomorphism. To this end, it is enough to assume that there is a diagram

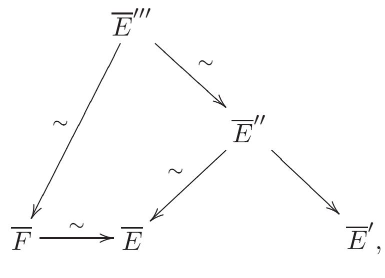


such that the triangle commutes up to homotopy. Fix such a homotopy. Then

$$[\overline{\operatorname{cone}}(\overline{\operatorname{cone}}(\overline{E}''', \overline{E}'), \overline{\operatorname{cone}}(\overline{E}'', \overline{E}'))] = -[\overline{\operatorname{cone}}(E''', E'')],$$
$$[\overline{\operatorname{cone}}(\overline{\operatorname{cone}}(\overline{E}''', \overline{E}), \overline{\operatorname{cone}}(\overline{E}'', \overline{E}))] = -[\overline{\operatorname{cone}}(E''', E'')].$$

By Lemma 2.3, the left hand sides of these relations agree and hence this implies that the hermitian structure does not depend on the choice of $\overline{E}''$.

We now prove the independence on the choice of the representative $\overline{E}$. Let $\overline{F} \to \overline{E}$ be a tight morphism. Then we can construct a diagram



where the square commutes up to homotopy. Choose one homotopy. Taking into account Lemma 2.3, we find

$$\begin{aligned}
[\overline{\operatorname{cone}}(\overline{\operatorname{cone}}(\overline{E}''', \overline{E}'), \overline{\operatorname{cone}}(\overline{E}'', \overline{E}'))] &= -[\overline{\operatorname{cone}}(E''', E'')], \\
[\overline{\operatorname{cone}}(\overline{\operatorname{cone}}(\overline{E}''', \overline{F}), \overline{\operatorname{cone}}(\overline{E}'', \overline{E}))] &= -[\overline{\operatorname{cone}}(E''', E'')] + [\overline{\operatorname{cone}}(\overline{F}, \overline{E})] \\
&= -[\overline{\operatorname{cone}}(E''', E'')].
\end{aligned}$$

Hence the definitions of $\overline{C}(f)$ using $\overline{E}$ or $\overline{F}$ agree up to tight isomorphism. The remaining possible choices of representatives are treated analogously. ☐

Remark 2.56. The construction of $\overline{\operatorname{cone}}(f)$ involves the choice of $\operatorname{cone}(f)$, which is unique up to isomorphism. Since the construction of $\overline{C}(f)$ (2.49) does not depend on the choice of $\operatorname{cone}(f)$, by Theorem 2.51, we see that different choices of $\operatorname{cone}(f)$ give rise to tightly isomorphic hermitian cones. Therefore $\overline{\operatorname{cone}}(f)$ is well defined up to tight isomorphism and we will usually call it *the* hermitian cone of f. When the morphism is clear, we will also write $\overline{\operatorname{cone}}(\overline{\mathcal{F}}, \overline{\mathcal{G}})$ to refer to it.

The hermitian cone satisfies the same relations than the usual cone.

Proposition 2.57. *Let $f\colon \overline{\mathcal{F}} \dashrightarrow \overline{\mathcal{G}}$ be a morphism in $\overline{\mathbf{D}}^{\mathrm{b}}(X)$. Then, the natural morphisms*

$$\begin{aligned}
\overline{\operatorname{cone}}(\overline{\mathcal{G}}, \overline{\operatorname{cone}}(f)) &\dashrightarrow \overline{\mathcal{F}}[1], \\
\overline{\mathcal{G}} &\dashrightarrow \overline{\operatorname{cone}}(\overline{\operatorname{cone}}(f)[-1], \overline{\mathcal{F}})
\end{aligned}$$

are tight isomorphisms.

Proof. After choosing representatives, there are isometries

$$\begin{aligned}
&\overline{\operatorname{cone}}(\overline{\operatorname{cone}}(\overline{\mathcal{G}}, \overline{\operatorname{cone}}(f)), \overline{\mathcal{F}}[1]) \cong \\
&\qquad \overline{\operatorname{cone}}(\overline{\operatorname{cone}}(\mathrm{id}_{\mathcal{F}}), \overline{\operatorname{cone}}(\mathrm{id}_{\mathcal{G}})) \cong \\
&\qquad\qquad \overline{\operatorname{cone}}(\overline{\mathcal{G}}, \overline{\operatorname{cone}}(\overline{\operatorname{cone}}(f)[-1], \overline{\mathcal{F}})).
\end{aligned}$$

Since the middle term is meager, the same is true for the other two. ☐

We next extend some basic constructions in $\mathbf{D}^{\mathrm{b}}(X)$ to $\overline{\mathbf{D}}^{\mathrm{b}}(X)$.

Derived tensor product. Let $\overline{\mathcal{F}}_i = (\overline{E}_i \dashrightarrow \mathcal{F}_i)$, $i = 1, 2$, be objects of $\overline{\mathbf{D}}^{\mathrm{b}}(X)$. The derived tensor product $\mathcal{F}_1 \otimes \mathcal{F}_2$ is endowed with a natural hermitian structure

$$\overline{E}_1 \otimes \overline{E}_2 \dashrightarrow \mathcal{F}_1 \otimes \mathcal{F}_2, \tag{2.58}$$

that is well defined by Theorem 2.17 (iii). We write $\overline{\mathcal{F}}_1 \otimes \overline{\mathcal{F}}_2$ for the resulting object in $\overline{\mathbf{D}}^{\mathrm{b}}(X)$.

Derived internal Hom **and dual objects.** Let $\overline{\mathcal{F}}_i = (\overline{E}_i \dashrightarrow \mathcal{F}_i)$, $i = 1, 2$, be objects of $\overline{\mathbf{D}}^{\mathrm{b}}(X)$. The derived internal Hom, $\underline{\operatorname{Hom}}(\mathcal{F}_1, \mathcal{F}_2)$ is endowed with a natural hermitian structure

$$\underline{\operatorname{Hom}}(\overline{E}_1, \overline{E}_2) \dashrightarrow \underline{\operatorname{Hom}}(\mathcal{F}_1, \mathcal{F}_2), \tag{2.59}$$

that is well defined by Theorem 2.17 (iii). We write $\underline{\mathrm{Hom}}(\overline{\mathcal{F}}_1, \overline{\mathcal{F}}_2)$ for the resulting object in $\overline{\mathbf{D}}^{\mathrm{b}}(X)$.

In particular, denote by $\overline{\mathcal{O}}_X$ the structural sheaf with the standard metric $\|1\| = 1$. Then, for every object $\overline{\mathcal{F}} \in \overline{\mathbf{D}}^{\mathrm{b}}(X)$, the *dual object* is defined to be

$$\overline{\mathcal{F}}^{\vee} = \underline{\mathrm{Hom}}(\overline{\mathcal{F}}, \overline{\mathcal{O}}_X). \tag{2.60}$$

Left derived inverse image. Let $g\colon X' \to X$ be a morphism of smooth noetherian schemes over $\mathbb{C}$ and $\overline{\mathcal{F}} = (\overline{E} \dashrightarrow \mathcal{F}) \in \mathrm{Ob}\,\overline{\mathbf{D}}^{\mathrm{b}}(X)$. Then the left derived inverse image $g^*(\mathcal{F})$ is equipped with the hermitian structure $g^*(\overline{E}) \dashrightarrow g^*(\mathcal{F})$, that is well defined up to tight isomorphism by Theorem 2.17 (iv). As it is customary, we will pretend that g^* is a functor. The notation for the corresponding object in $\overline{\mathbf{D}}^{\mathrm{b}}(X')$ is $g^*(\overline{\mathcal{F}})$. If $f\colon \overline{\mathcal{F}}_1 \dashrightarrow \overline{\mathcal{F}}_2$ is a morphism in $\overline{\mathbf{D}}^{\mathrm{b}}(X)$, we denote by $g^*(f)\colon g^*(\overline{\mathcal{F}}_1) \dashrightarrow g^*(\overline{\mathcal{F}}_2)$ its left derived inverse image by g.

The functor g^* preserves the structure of principal fibered category with flat connection and the formation of hermitian cones. Namely we have the following result that is easily proved.

Theorem 2.61. *Let $g\colon X' \to X$ be a morphism of smooth noetherian schemes over $\mathbb{C}$ and let $f\colon \overline{\mathcal{F}}_1 \dashrightarrow \overline{\mathcal{F}}_2$ be a morphism in $\overline{\mathbf{D}}^{\mathrm{b}}(X)$.*

(i) The functor g^* preserves the forgetful functor:

$$\mathfrak{F} \circ g^* = g^* \circ \mathfrak{F}$$

(ii) The restriction $g^*\colon \mathbf{KA}(X) \to \mathbf{KA}(X')$ is a group homomorphism.

(iii) The functor g^* is equivariant with respect to the actions of $\mathbf{KA}(X)$ and $\mathbf{KA}(X')$.

(iv) The functor g^* preserves parallel transport: if f is an isomorphism, then

$$g^* \circ \mathfrak{t}_f = \mathfrak{t}_{g^*(f)} \circ g^*.$$

(v) The functor g^* preserves hermitian cones:

$$g^*(\overline{\mathrm{cone}}(f)) = \overline{\mathrm{cone}}(g^*(f)).$$

□

Classes of isomorphisms and distinguished triangles. Let $f\colon \overline{\mathcal{F}} \overset{\sim}{\dashrightarrow} \overline{\mathcal{G}}$ be an isomorphism in $\overline{\mathbf{D}}^{\mathrm{b}}(X)$. To it, we attach a class $[f] \in \mathbf{KA}(X)$ that measures the default of being a tight isomorphism. This class is defined using the hermitian cone.

$$[f] = [\overline{\mathrm{cone}}(f)]. \tag{2.62}$$

Observe the abuse of notation: we wrote $[\overline{\mathrm{cone}}(f)]$ for the class in $\mathbf{KA}(X)$ of the hermitian structure of a hermitian cone of f. This is well defined, since the hermitian cone is unique up to tight isomorphism. Alternatively, we can construct $[f]$ using parallel transport as follows. There is a unique element $\overline{A} \in \mathbf{KA}(X)$ such that

$$\overline{\mathcal{G}} = \mathfrak{t}_f \overline{\mathcal{F}} + \overline{A}.$$

We denote this element by $\overline{\mathcal{G}} - \mathfrak{t}_f \overline{\mathcal{F}}$. Then

$$[f] = \overline{\mathcal{G}} - \mathfrak{t}_f \overline{\mathcal{F}}.$$

By the very definition of parallel transport, both definitions clearly agree.

Definition 2.63. A *distinguished triangle in* $\overline{\mathbf{D}}^{\mathrm{b}}(X)$, consists in a diagram

$$\overline{\tau} = (u, v, w) : \overline{\mathcal{F}} \overset{u}{\dashrightarrow} \overline{\mathcal{G}} \overset{v}{\dashrightarrow} \overline{\mathcal{H}} \overset{w}{\dashrightarrow} \overline{\mathcal{F}}[1] \overset{u}{\dashrightarrow} \dots \tag{2.64}$$

in $\overline{\mathbf{D}}^{\mathrm{b}}(X)$, whose underlying morphisms in $\mathbf{D}^{\mathrm{b}}(X)$ form a distinguished triangle. We will say that it is *tightly distinguished* if there is a commutative diagram

$$\begin{array}{ccccccccc} \overline{\mathcal{F}} & \dashrightarrow & \overline{\mathcal{G}} & \dashrightarrow & \overline{\mathrm{cone}}(\overline{\mathcal{F}}, \overline{\mathcal{G}}) & \dashrightarrow & \overline{\mathcal{F}}[1] & \dashrightarrow & \cdots \\ \downarrow \mathrm{id} & & \downarrow \mathrm{id} & & \downarrow \alpha & & \downarrow \mathrm{id} & & \\ \overline{\mathcal{F}} & \dashrightarrow & \overline{\mathcal{G}} & \dashrightarrow & \overline{\mathcal{H}} & \dashrightarrow & \overline{\mathcal{F}}[1] & \dashrightarrow & \cdots, \end{array} \tag{2.65}$$

with α a tight isomorphism.

To every distinguished triangle in $\overline{\mathbf{D}}^{\mathrm{b}}(X)$ we can associate a class in $\mathbf{KA}(X)$ that measures the default of being tightly distinguished. Let $\overline{\tau}$ be a distinguished triangle as in (2.64). Then there is a diagram as (2.65), but with α an isomorphism non-necessarily tight. Then we define

$$[\overline{\tau}] = [\alpha]. \tag{2.66}$$

By Theorem 2.51, the class $[\alpha]$ does not depend on the particular choice of morphism α in $\overline{\mathbf{D}}^{\mathrm{b}}(X)$ for which (2.65) commutes. Hence (2.66) only depends on $\overline{\tau}$.

Theorem 2.67. *(i) Let f be an isomorphism in $\overline{\mathbf{D}}^{\mathrm{b}}(X)$ (respectively $\overline{\tau}$ a distinguished triangle). Then $[f] = 0$ (respectively $[\overline{\tau}] = 0$) if and only if f is a tight isomorphism (respectively $\overline{\tau}$ is tightly distinguished).*

(ii) Let $g \colon X' \to X$ be a morphism of smooth complex varieties, let f be an isomorphism in $\overline{\mathbf{D}}^{\mathrm{b}}(X)$ and $\overline{\tau}$ a distinguished triangle in $\overline{\mathbf{D}}^{\mathrm{b}}(X)$. Then

$$g^*[f] = [g^* f], \qquad g^*[\overline{\tau}] = [g^*\overline{\tau}].$$

In particular, tight isomorphisms and tightly distinguished triangles are preserved under left derived inverse images.

(iii) Let $f \colon \overline{\mathcal{F}} \dashrightarrow \overline{\mathcal{G}}$ and $h \colon \overline{\mathcal{G}} \dashrightarrow \overline{\mathcal{H}}$ be two isomorphisms in $\overline{\mathbf{D}}^{\mathrm{b}}(X)$. Then:

$$[h \circ f] = [h] + [f].$$

In particular, $[f^{-1}] = -[f]$.

(iv) For any distinguished triangle $\overline{\tau}$ in $\overline{\mathbf{D}}^{\mathrm{b}}(X)$ as in Definition 2.63, the rotated triangle

$$\overline{\tau}' \colon \ \overline{\mathcal{G}} \overset{v}{\dashrightarrow} \overline{\mathcal{H}} \overset{w}{\dashrightarrow} \overline{\mathcal{F}}[1] \overset{-u[1]}{\dashrightarrow} \overline{\mathcal{G}}[1] \overset{v[1]}{\dashrightarrow} \dots$$

satisfies

$$[\overline{\tau}'] = -[\overline{\tau}].$$

In particular, rotating preserves tightly distinguished triangles.

(v) For any acyclic complex $\overline{\mathcal{F}}$, we have

$$[\overline{\mathcal{F}} \to 0 \to 0 \to \dots] = [\overline{\mathcal{F}}].$$

(vi) If $f\colon \overline{\mathcal{F}} \dashrightarrow \overline{\mathcal{G}}$ is an isomorphism in $\overline{\mathbf{D}}^{\mathrm{b}}(X)$, then

$$[0 \to \overline{\mathcal{F}} \dashrightarrow \overline{\mathcal{G}} \to \dots] = [f].$$

(vii) For a commutative diagram of distinguished triangles

$$\begin{array}{ccccccccc}
\overline{\tau} & & \overline{\mathcal{F}} & \dashrightarrow & \overline{\mathcal{G}} & \dashrightarrow & \overline{\mathcal{H}} & \dashrightarrow & \overline{\mathcal{F}}[1] \dashrightarrow \cdots \\
\Big\downarrow & & \sim\Big\downarrow f & & \sim\Big\downarrow g & & \sim\Big\downarrow h & & \sim\Big\downarrow f[1] \\
\overline{\tau}' & & \overline{\mathcal{F}}' & \dashrightarrow & \overline{\mathcal{G}}' & \dashrightarrow & \overline{\mathcal{H}}' & \dashrightarrow & \overline{\mathcal{F}}'[1] \dashrightarrow \cdots,
\end{array}$$

the following relation holds:

$$[\overline{\tau}'] - [\overline{\tau}] = [f] - [g] + [h].$$

(viii) For a commutative diagram of distinguished triangles

$$\begin{array}{ccccccccc}
\overline{\tau} & & \overline{\mathcal{F}} & \dashrightarrow & \overline{\mathcal{G}} & \dashrightarrow & \overline{\mathcal{H}} & \dashrightarrow & \overline{\mathcal{F}}[1] \dashrightarrow \cdots \\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
\overline{\tau}' & & \overline{\mathcal{F}}' & \dashrightarrow & \overline{\mathcal{G}}' & \dashrightarrow & \overline{\mathcal{H}}' & \dashrightarrow & \overline{\mathcal{F}}'[1] \dashrightarrow \cdots \\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
\overline{\tau}'' & & \overline{\mathcal{F}}'' & \dashrightarrow & \overline{\mathcal{G}}'' & \dashrightarrow & \overline{\mathcal{H}}'' & \dashrightarrow & \overline{\mathcal{F}}''[1] \dashrightarrow \cdots \\
 & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
 & & \overline{\mathcal{F}}[1] & \dashrightarrow & \overline{\mathcal{G}}[1] & \dashrightarrow & \overline{\mathcal{H}}[1] & \dashrightarrow & \overline{\mathcal{F}}[2] \dashrightarrow \cdots \\
 & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
 & & \vdots & & \vdots & & \vdots & & \vdots \\
 & & & & & & & & \\
 & & \overline{\eta} & \dashrightarrow & \overline{\eta}' & \dashrightarrow & \overline{\eta}'' & &
\end{array} \tag{2.68}$$

the following relation holds:

$$[\overline{\tau}] - [\overline{\tau}'] + [\overline{\tau}''] = [\overline{\eta}] - [\overline{\eta}'] + [\overline{\eta}''].$$

Proof. The first two statements are clear. For the third, we may assume that f and g are realized by quasi-isomorphisms

$$f\colon \overline{F} \longrightarrow \overline{G}, \quad g\colon \overline{G} \longrightarrow \overline{H}.$$

Then the result follows from Theorem 2.27 (vii). The fourth assertion is a consequence of Proposition 2.57. Then (v), (vi) and (vii) follow from equation (2.66) and the fourth statement. The last property is derived from (vii) by comparing the diagram to a diagram of tightly distinguished triangles. □

As an application of the class in $\mathbf{KA}(X)$ attached to a distinguished triangle, we exhibit a natural morphism $K_1(X) \to \mathbf{KA}(X)$. This is included for the sake of completeness, but won't be needed in the sequel.

Proposition 2.69. *There is a natural morphism of groups $K_1(X) \to \mathbf{KA}(X)$.*

Proof. We follow the definitions and notations of [19]. From *loc. cit.* we know it is enough to construct a morphism of groups

$$H_1(\widetilde{\mathbb{Z}}C(X)) \to \mathbf{KA}(X). \tag{2.70}$$

By definition, the piece of degree n of the homological complex $\widetilde{\mathbb{Z}}C(X)$ is

$$\widetilde{\mathbb{Z}}C_n(X) = \mathbb{Z}C_n(X)/D_n.$$

Here $\mathbb{Z}C_n(X)$ stands for the free abelian group on metrized exact n-cubes and D_n is the subgroup of degenerate elements. A metrized exact 1-cube is a short exact sequence of hermitian vector bundles. Hence, for such a 1-cube $\overline{\varepsilon}$, there is a well defined class in $\mathbf{KA}(X)$. Observe that this class coincides with the class of $\overline{\varepsilon}$ thought as a distinguished triangle in $\overline{\mathbf{D}}^{\mathrm{b}}(X)$. Because $\mathbf{KA}(X)$ is an abelian group, it follows the existence of a morphism of groups

$$\mathbb{Z}C_1(X) \longrightarrow \mathbf{KA}(X).$$

From the definition of degenerate cube [19, Def. 3.3] and the construction of $\mathbf{KA}(X)$, this morphism clearly factors through $\widetilde{\mathbb{Z}}C_1(X)$. The definition of the differential d of the complex $\widetilde{\mathbb{Z}}C(X)$ [19, (3.2)] and Theorem 2.67 (viii) ensure that $d\mathbb{Z}C_2(X)$ is in the kernel of the morphism. Hence we derive the existence of a morphism (2.70). □

Classes of complexes and of direct images of complexes. In [18, Section 2] the notion of homological exact sequences of metrized coherent sheaves is treated. In the present article, this situation will arise in later considerations. Therefore we provide the link between the point of view of *loc. cit.* and the formalism adopted here. The reader will find no difficulty to translate it to cohomological complexes.

Consider a homological complex

$$\overline{\varepsilon}: \quad 0 \to \overline{\mathcal{F}}_m \to \cdots \to \overline{\mathcal{F}}_l \to 0$$

of metrized coherent sheaves, namely coherent sheaves provided with hermitian structures $\overline{\mathcal{F}}_i = (\mathcal{F}_i, \overline{F}_i \dashrightarrow \mathcal{F}_i)$. We may equivalently see $\overline{\varepsilon}$ as a cohomological complex, by the usual relabeling $\overline{\mathcal{F}}^{-i} = \overline{\mathcal{F}}_i$. This will be freely used in the sequel, especially in cone constructions.

Definition 2.71. The complex $\overline{\varepsilon}$ defines an object $[\overline{\varepsilon}] \in \operatorname{Ob}\overline{\mathbf{D}}^{\mathrm{b}}(X)$ that is determined inductively by the condition

$$[\overline{\varepsilon}] = \overline{\operatorname{cone}}(\overline{\mathcal{F}}_m[m], [\sigma_{<m}\overline{\varepsilon}]).$$

Here $\sigma_{<m}$ is the homological bête filtration and $\overline{\mathcal{F}}_m$ denotes a cohomological complex concentrated in degree zero. Hence, $\overline{\mathcal{F}}_m[m]$ is a cohomological complex concentrated in degree $-m$.

If $\overline{E}$ is a hermitian vector bundle on X, then there is an equality

$$[\overline{\varepsilon} \otimes \overline{E}] = [\overline{\varepsilon}] \otimes \overline{E}.$$

According to Definition 2.44, if ε is an acyclic complex, then we also have the corresponding class $[[\overline{\varepsilon}]]$ in $\mathbf{KA}(X)$. We will employ the lighter notation $[\overline{\varepsilon}]$ for this class.

Given a morphism $\varphi\colon \overline{\varepsilon} \to \overline{\mu}$ of bounded complexes of metrized coherent sheaves, the pieces of the complex $\operatorname{cone}(\varepsilon, \mu)$ are natural endowed with hermitian metrics. We thus get a complex of metrized coherent sheaves $\overline{\operatorname{cone}(\varepsilon, \mu)}$. Hence Definition 2.71 provides an object $[\overline{\operatorname{cone}(\varepsilon, \mu)}]$ in $\overline{\mathbf{D}}^{\mathrm{b}}(X)$. On the other hand, Definition 2.48 attaches to φ the hermitian cone $\overline{\operatorname{cone}}([\overline{\varepsilon}], [\overline{\mu}])$, which is well defined up to tight isomorphism. Both constructions actually agree.

Lemma 2.72. *Let $\overline{\varepsilon} \to \overline{\mu}$ be a morphism of bounded complexes of metrized coherent sheaves on X. Then there is a tight isomorphism*

$$\overline{\operatorname{cone}}([\overline{\varepsilon}], [\overline{\mu}]) \cong [\overline{\operatorname{cone}(\varepsilon, \mu)}],$$

Proof. The case when ε and μ are both concentrated in a single degree d is clear. The general case follows by induction taking into account Definition 2.71. ☐

Assume now that $f\colon X \to Y$ is a morphism of smooth complex varieties and, for each complex $f_*\mathcal{F}_i$, we have chosen a hermitian structure $\overline{f_*\mathcal{F}_i} = (\overline{E}_i \dashrightarrow f_*\mathcal{F}_i)$. Denote by $\overline{f_*\varepsilon}$ this choice of metrics.

Definition 2.73. The family of hermitian structures $\overline{f_*\varepsilon}$ defines an object $[\overline{f_*\varepsilon}] \in \operatorname{Ob} \overline{\mathbf{D}}^{\mathrm{b}}(Y)$ that is determined inductively by the condition

$$[\overline{f_*\varepsilon}] = \overline{\operatorname{cone}}(\overline{f_*\mathcal{F}_m}[m], [\overline{f_*\sigma_{<m}\varepsilon}]).$$

We remark that the notation $\overline{f_*\varepsilon}$ means that the hemitian structure is choosen after taking the direct image and it is not determined by the hermitian structure on $\overline{\varepsilon}$.

If $\overline{F}$ is a hermitian vector bundle on Y, then the obviously defined object $[\overline{f_*(\varepsilon \otimes f^*F)}]$ satisfies

$$[\overline{f_*(\varepsilon \otimes f^*F)}] = [\overline{f_*\varepsilon}] \otimes \overline{F}.$$

Also, notice that if ε is an acyclic complex on X, then we have the class $[\overline{f_*\varepsilon}] \in \mathbf{KA}(Y)$.

Let $\varepsilon \to \mu$ be a morphism of bounded complexes of coherent sheaves on X and $f\colon X \to Y$ a morphism of smooth complex varieties. Fix choices of metrics $\overline{f_*\varepsilon}$ and $\overline{f_*\mu}$. Then there is an obvious choice of metrics on $f_* \operatorname{cone}(\varepsilon, \mu)$, that we denote $\overline{f_* \operatorname{cone}(\varepsilon, \mu)}$, and hence an object $[\overline{f_* \operatorname{cone}(\varepsilon, \mu)}]$ in $\overline{\mathbf{D}}^{\mathrm{b}}(Y)$. On the other hand, we also have the hermitian cone $\overline{\operatorname{cone}}([\overline{f_*\varepsilon}], [\overline{f_*\mu}])$. Again both definitions agree.

Lemma 2.74. *Let $\varepsilon \to \mu$ be a morphism of bounded complexes of coherent sheaves on X and $f\colon X \to Y$ a morphism of smooth complex varieties. Assume chosen families of metrics $\overline{f_*\varepsilon}$ and $\overline{f_*\mu}$. Then there is a tight isomorphism*

$$\overline{\operatorname{cone}}([\overline{f_*\varepsilon}], [\overline{f_*\mu}]) \cong [\overline{f_* \operatorname{cone}(\varepsilon, \mu)}].$$

Proof. If ε and μ are concentrated in a single degree d, then the statement is obvious. The proof follows by induction and Definition 2.73. $\square$

The objects we have defined are compatible with short exact sequences, in the sense of the following statement.

Proposition 2.75. *Consider a commutative diagram of exact sequences of coherent sheaves on X*

$$\begin{array}{ccccccccc}
 & & & 0 & & & & 0 & \\
 & & & \downarrow & & & & \downarrow & \\
\mu' & & 0 \longrightarrow & \mathcal{F}'_m & \longrightarrow & \cdots & \longrightarrow & \mathcal{F}'_l & \longrightarrow 0 \\
 & & & \downarrow & & & & \downarrow & \\
\mu & & 0 \longrightarrow & \mathcal{F}_m & \longrightarrow & \cdots & \longrightarrow & \mathcal{F}_l & \longrightarrow 0 \\
 & & & \downarrow & & & & \downarrow & \\
\mu'' & & 0 \longrightarrow & \mathcal{F}''_m & \longrightarrow & \cdots & \longrightarrow & \mathcal{F}''_l & \longrightarrow 0 \\
 & & & \downarrow & & & & \downarrow & \\
 & & & 0 & & & & 0 & \\
 & & & & & & & & \\
 & & & \xi_m & & \cdots & & \xi_l. &
\end{array}$$

Let $f\colon X \to Y$ be a morphism of smooth complex varieties and choose hermitian structures on the sheaves $\mathcal{F}'_j$, $\mathcal{F}_j$, $\mathcal{F}''_j$ and on the objects $f_*\mathcal{F}'_j$, $f_*\mathcal{F}_j$ and $f_*\mathcal{F}''_j$, $j = l, \ldots, m$. Then the following equalities hold in $\mathbf{KA}(X)$ and $\mathbf{KA}(Y)$, respectively:

$$\sum_j (-1)^j [\overline{\xi}_j] = [\overline{\mu}'] - [\overline{\mu}] + [\overline{\mu}''],$$
$$\sum_j (-1)^j [\overline{f_*\xi_j}] = [\overline{f_*\mu}'] - [\overline{f_*\mu}] + [\overline{f_*\mu}''].$$

Proof. The lemma follows inductively taking into account definitions 2.71 and 2.73 and Theorem 2.67 (viii). $\square$

Corollary 2.76. *Let $\overline{\varepsilon} \to \overline{\mu}$ be a morphism of exact sequences of metrized coherent sheaves. Let $f\colon X \to Y$ be a morphism of smooth complex varieties and fix families of metrics $\overline{f_*\varepsilon}$ and $\overline{f_*\mu}$. Then there are equalities in $\mathbf{KA}(X)$ and $\mathbf{KA}(Y)$, respectively*

$$[\overline{\operatorname{cone}(\varepsilon, \mu)}] = [\overline{\mu}] - [\overline{\varepsilon}], \tag{2.77}$$
$$[\overline{f_* \operatorname{cone}(\varepsilon, \mu)}] = [\overline{f_*\mu}] - [\overline{f_*\varepsilon}]. \tag{2.78}$$

Proof. The result readily follows from lemmas 2.72, 2.74 and Proposition 2.75. $\square$

Hermitian structures on cohomology. Let $\mathcal{F}$ be an object of $\mathbf{D}^{\mathrm{b}}(X)$ and denote by $\mathcal{H}$ its cohomology complex. Observe that $\mathcal{H}$ is a bounded complex with 0 differentials. By the preceding discussion and because $\mathbf{KA}(X)$ acts transitively on hermitian structures, giving a hermitian structure on $\mathcal{H}$ amounts to give hermitian structures on the individual pieces $\mathcal{H}^i$. We show that to these data there is attached a natural hermitian structure on the complex $\mathcal{F}$. This situation will arise when considering cohomology sheaves endowed with L^2 metric structures. The construction is recursive. If the cohomology complex is trivial, then we endow $\mathcal{F}$ with the trivial hermitian structure. Otherwise, let $\mathcal{H}^m$ be the highest non-zero cohomology sheaf. The canonical filtration $\tau^{\leq m}$ is given by

$$\tau^{\leq m}\mathcal{F}\colon \quad \cdots \to \mathcal{F}^{m-2} \to \mathcal{F}^{m-1} \to \ker(\mathrm{d}^m) \to 0.$$

By the condition on the highest non vanishing cohomology sheaf, the natural inclusion is a quasi-isomorphism:

$$\tau^{\leq m}\mathcal{F} \xrightarrow{\sim} \mathcal{F}. \tag{2.79}$$

We also introduce the subcomplex

$$\widetilde{\mathcal{F}}\colon \quad \cdots \to \mathcal{F}^{m-2} \to \mathcal{F}^{m-1} \to \mathrm{Im}\,(\mathrm{d}^{m-1}) \to 0.$$

Observe that the cohomology complex of $\widetilde{\mathcal{F}}$ is the bête truncation $\mathcal{H}/\sigma^{\geq m}\mathcal{H}$. By induction, $\widetilde{\mathcal{F}}$ carries an induced hermitian structure. Moreover we have an exact sequence

$$0 \to \widetilde{\mathcal{F}} \to \tau^{\leq m}\mathcal{F} \to \mathcal{H}^m[-m] \to 0. \tag{2.80}$$

Taking into account the quasi-isomorphism (2.79) and the exact sequence (2.80), we construct a natural commutative diagram of distinguished triangles in $\mathbf{D}^{\mathrm{b}}(X)$

$$\begin{array}{ccccccc}
\mathcal{H}^m[-m-1] & \overset{0}{\dashrightarrow} & \widetilde{\mathcal{F}} & \dashrightarrow & \mathcal{F} & \dashrightarrow & \mathcal{H}^m[m] \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \mathrm{id}} \\
\mathcal{H}^m[-m-1] & \overset{0}{\dashrightarrow} & \widetilde{\mathcal{F}} & \dashrightarrow & \mathrm{cone}(\mathcal{H}^m[-m-1], \widetilde{\mathcal{F}}) & \dashrightarrow & \mathcal{H}^m[m].
\end{array}$$

By the hermitian cone construction and Theorem 2.51, we see that hermitian structures on $\widetilde{\mathcal{F}}$ and $\mathcal{H}^m$ induce a well defined hermitian structure on $\mathcal{F}$.

Definition 2.81. Let $\mathcal{F}$ be an object of $\mathbf{D}^{\mathrm{b}}(X)$ with cohomology complex $\mathcal{H}$. Assume the pieces $\mathcal{H}^i$ are endowed with hermitian structures. The hermitian structure on $\mathcal{F}$ constructed above will be called the *hermitian structure induced by the hermitian structure on the cohomology complex* and will be denoted $(\mathcal{F}, \overline{\mathcal{H}})$.

The following proposition is easily checked from the very construction, and the proof is left to the reader.

Proposition 2.82. *Let $\varphi\colon \mathcal{F}_1 \dashrightarrow \mathcal{F}_2$ be an isomorphism in $\mathbf{D}^{\mathrm{b}}(X)$. Assume the pieces of the cohomology complexes $\mathcal{H}_1$, $\mathcal{H}_2$ of $\mathcal{F}_1$, $\mathcal{F}_2$ are endowed with hermitian structures. If the induced isomorphism in cohomology $\varphi_*\colon \mathcal{H}_1 \to \mathcal{H}_2$ is tight, then φ is tight for the induced hermitian structures on $\mathcal{F}_1$ and $\mathcal{F}_2$.*

□

2.3 The Deligne complexes of differential forms and currents

The natural context where one can define the Bott-Chern classes and the analytic torsion classes is that of Deligne complexes. For the convenience of the reader we will summarize in this section the basic facts about the Deligne complexes we will use in the sequel. For more details the reader is referred to [16] and [17].

Definition 2.83. A *Dolbeault complex* $A = (A^*_{\mathbb{R}}, \mathrm{d}_A)$ is a graded complex of real vector spaces, which is bounded from below and equipped with a bigrading on $A_{\mathbb{C}} = A_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$, i.e.,

$$A^n_{\mathbb{C}} = \bigoplus_{p+q=n} A^{p,q},$$

satisfying the following properties:

(i) The differential d_A can be decomposed as the sum $\mathrm{d}_A = \partial + \bar{\partial}$ of operators ∂ of type $(1,0)$, respectively $\bar{\partial}$ of type $(0,1)$.

(ii) It satisfies the symmetry property $\overline{A^{p,q}} = A^{q,p}$, where $\overline{}$ denotes complex conjugation.

The basic example of Dolbeault complex is the complex of differential forms on a smooth variety X over $\mathbb{C}$, denoted $E^*(X)_{\mathbb{R}}$.

Following [17, §5.2], to a Dolbeault complex one assigns a Deligne complex denoted $\mathcal{D}^*(A,*)$. In this paper we will only use the following pieces of the Deligne complex:

$$\begin{aligned}
\mathcal{D}^{2p+1}(A,p) &= (A^{p,p+1} \oplus A^{p+1,p}) \cap (2\pi i)^p A^{2p+1}_{\mathbb{R}},\\
\mathcal{D}^{2p}(A,p) &= A^{p,p} \cap (2\pi i)^p A^{2p}_{\mathbb{R}},\\
\mathcal{D}^{2p-1}(A,p) &= A^{p-1,p-1} \cap (2\pi i)^{p-1} A^{2p-2}_{\mathbb{R}},\\
\mathcal{D}^{2p-2}(A,p) &= (A^{p-2,p-1} \oplus A^{p-1,p-2}) \cap (2\pi i)^{p-1} A^{2p-3}_{\mathbb{R}}.
\end{aligned}$$

The differential of the Deligne complex, denoted by $\mathrm{d}_{\mathcal{D}} \colon \mathcal{D}^n(A,p) \to \mathcal{D}^{n+1}(A,p)$ is given, in the above degrees by

$$\begin{aligned}
\text{if } \eta \in \mathcal{D}^{2p}(A,p), &\quad \mathrm{d}_{\mathcal{D}}\,\eta = \mathrm{d}\,\eta,\\
\text{if } \eta \in \mathcal{D}^{2p-1}(A,p), &\quad \mathrm{d}_{\mathcal{D}}\,\eta = -2\partial\bar{\partial}\eta,\\
\text{if } \eta = (u,v) \in \mathcal{D}^{2p-2}(A,p), &\quad \mathrm{d}_{\mathcal{D}}\,\eta = -\partial u - \bar{\partial} v.
\end{aligned}$$

When A is a Dolbeault algebra, that is, A is a graded commutative real differential algebra and the product is compatible with the bigrading, then $\mathcal{D}^*(A,*)$ has a product

$$\bullet \colon \mathcal{D}^n(A,p) \otimes \mathcal{D}^m(A,q) \longrightarrow \mathcal{D}^{n+m}(A,p+q)$$

that is graded commutative with respect to the first degree, it is associative up to homotopy and satisfies the Leibniz rule. The only case where we will need the explicit formula for the product is for $\omega \in \mathcal{D}^{2p}(A,p)$ and $\eta \in \mathcal{D}^m(A,q)$. Then the product is given by

$$\omega \bullet \eta = \omega \wedge \eta.$$

The *Deligne algebra of differential forms* on X is defined to be

$$\mathcal{D}^*(X,*) := \mathcal{D}^*(E^*(X)_{\mathbb{R}},*).$$

Recall that, if X is equi-dimensional of dimension d, there is a natural trace map $\int : H_c^{2d}(X,\mathbb{R}(d)) \to \mathbb{R}$ given by

$$\omega \longmapsto \frac{1}{(2\pi i)^d}\int_X \omega.$$

To take this trace map into account the Dolbeault complex of currents is constructed as follows. Denote by $E_c^*(X)_{\mathbb{R}}$ the space of differential forms with compact support. Then $D_{p,q}(X)$ is the topological dual of $E_c^{p,q}(X)$ and $D_n(X)_{\mathbb{R}}$ is the topological dual of $E_c^n(X)_{\mathbb{R}}$. In this complex the differential is given by

$$\mathrm{d}\, T(\eta) = (-1)^n T(\mathrm{d}\,\eta)$$

for $T \in D_n(X)_{\mathbb{R}}$. For X equi-dimensional of dimension d we write

$$D^{p,q}(X) = D_{d-p,d-q}(X), \qquad D^n(X)_{\mathbb{R}} = (2\pi i)^{-d} D_{2d-n}(X).$$

With these definitions, $D^*(X)_{\mathbb{R}}$ is a Dolbeault complex and it is a Dolbeault module over $E^*(X)_{\mathbb{R}}$. We will denote

$$\mathcal{D}_D^*(X,*) := \mathcal{D}^*(D^*(X)_{\mathbb{R}},*).$$

for the Deligne complex of currents on X.

Observe that the trace map above defines an element

$$\delta_X \in \mathcal{D}_D^0(X,0).$$

More generally, if $Y \subset X$ is a subvariety of pure codimension p, then the current integration along Y, denoted $\delta_Y \in \mathcal{D}_D^{2p}(X,p)$ is given by

$$\delta_Y(\omega) = \frac{1}{(2\pi i)^{d-p}}\int_Y \omega.$$

Moreover, if $S \subset T^*X_0$ is a closed conical subset of the cotangent bundle of X with the zero section removed, we will denote by $(\mathcal{D}_D^*(X,S,*), \mathrm{d}_{\mathcal{D}})$ the Deligne complex of currents on X whose wave front set is contained in S.

For instance, if we denote by N_Y^* the conormal bundle to Y, then

$$\delta_Y \in \mathcal{D}_D^{2p}(X, N_Y^*, p).$$

If ω is a locally integrable differential form, we associate to it a current

$$[\omega](\eta) = \frac{1}{(2\pi i)^{\dim X}}\int_X \eta \wedge \omega.$$

This map gives us an isomorphism $\mathcal{D}^*(X,*) \to \mathcal{D}_D^*(X,\emptyset,*)$ that we can use to identify them. For instance, when in a formula there appear sums of currents and differential forms, we will tacitly assume that the differential forms are converted into currents by this map.

Note also that, if $f\colon X \to Y$ is a proper morphism of smooth complex varieties of relative dimension e, then there are direct image morphisms

$$f_*\colon \mathcal{D}^n_D(X,p) \longrightarrow \mathcal{D}^{n-2e}_D(X,p-e).$$

If f is smooth, the direct image of differential forms is defined by, first converting them into currents and then applying the above direct image of currents. If f is a smooth morphism of relative dimension e we can convert them back into differential forms. This procedure gives us $1/(2\pi i)^e$ times the usual integration along the fiber.

We shall use the notations and definitions of [18]. In particular, we write

$$\widetilde{\mathcal{D}}^n(X,p) = \mathcal{D}^n(X,p)/\operatorname{d}_{\mathcal{D}} \mathcal{D}^{n-1}(X,p),$$
$$\widetilde{\mathcal{D}}^n_D(X,p) = \mathcal{D}^n_D(X,p)/\operatorname{d}_{\mathcal{D}} \mathcal{D}^{n-1}_D(X,p).$$

2.4 Bott-Chern classes for isomorphisms and distinguished triangles in $\overline{\mathbf{D}}^{\mathrm{b}}(X)$

In this section we will define Bott-Chern classes for isomorphisms and distinguished triangles in $\overline{\mathbf{D}}^{\mathrm{b}}(X)$.

When characterizing axiomatically Bott-Chern classes, the basic tool to exploit the functoriality is to use a deformation parametrized by $\mathbb{P}^1$. Since we will need several variants of this argument we next state a version that is general enough for our purposes.

Let $f\colon X \to Y$ be a morphism of smooth complex varieties. The set of normal directions of f is defined as follows (see also [18, Sec. 4]).

Definition 2.84. Let T^*Y_0 be the cotangent bundle to Y with the zero section removed. The *set of normal directions of* f is the conic subset of T^*Y_0 given by

$$N_f = \{(y,v) \in T^*Y_0 |\operatorname{d} f^t v = 0\}.$$

Let $Y'' \xrightarrow{h} Y' \xrightarrow{g} Y$ be morphisms of smooth complex varieties such that g and $g \circ h$ are smooth. We form the cartesian diagram

$$\begin{array}{ccccc} X'' & \longrightarrow & X' & \longrightarrow & X \\ \downarrow{\scriptstyle f''} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y'' & \xrightarrow{h} & Y' & \xrightarrow{g} & Y. \end{array}$$

The smoothness of g implies that $N_{f'} = g^* N_f$. Then the smoothness of $g \circ h$ implies that $N_h \cap N_{f'} = \emptyset$. Therefore, any current $\eta \in \mathcal{D}^*_D(Y', N_{f'}, *)$ can be pulled back to a current $h^*\eta \in \mathcal{D}^*_D(Y'', N_{f''}, *)$.

Theorem 2.85. *Let $f\colon X \to Y$ be a morphism of smooth complex varieties. Let $\widetilde{\varphi}$ be an assignment that, to each smooth morphism of complex varieties $g\colon Y' \to Y$ and each acyclic complex $\overline{A}$ of hermitian vector bundles on $X' := X \underset{Y}{\times} Y'$ assigns a class*

$$\widetilde{\varphi}(\overline{A}) \in \bigoplus_{n,p} \widetilde{\mathcal{D}}^n_D(Y', g^*N_f, p)$$

fulfilling the following properties:

(i) (Differential equation) the equality

$$\mathrm{d}_{\mathcal{D}}\, \widetilde{\varphi}(\overline{A}) = 0$$

holds;

(ii) (Functoriality) for each morphism of smooth complex varieties $h\colon Y'' \to Y'$ *with* $g \circ h$ *smooth, the relation*

$$h^{*}\widetilde{\varphi}(\overline{A}) = \widetilde{\varphi}(h^{*}\overline{A});$$

holds;

(iii) (Normalization) if $\overline{A}$ *is orthogonally split, then* $\widetilde{\varphi}(\overline{A}) = 0$.

Then $\widetilde{\varphi} = 0$.

Proof. The argument of the proof of [18, Thm. 2.3] applies *mutatis mutandis* to the present situation. One only needs to observe that all the operations with differential forms of that argument can be extended to the currents that appear in the present situation thanks to the hypothesis about their wave front sets. □

Remark 2.86. In this section we will use Theorem 2.85 in the case when $f = \mathrm{id}_X$, hence $N_f = \emptyset$. The general case will be needed when studying analytic torsion in the next section.

Definition 2.87. An *additive genus in Deligne cohomology* is a characteristic class φ for vector bundles of any rank in the sense of [18, Def. 1.5] that satisfies the equation

$$\varphi(E_1 \oplus E_2) = \varphi(E_1) + \varphi(E_2). \tag{2.88}$$

Let $\mathbb{D}$ denote the base ring for Deligne cohomology (see [18] before Definition 1.5). A consequence of [18, Thm. 1.8] is that there is a bijection between the set of additive genus in Deligne cohomology and the set of power series in one variable $\mathbb{D}[[x]]$. To each power series $\varphi \in \mathbb{D}[[x]]$ it corresponds the unique additive genus such that

$$\varphi(L) = \varphi(c_1(L))$$

for every line bundle L.

Definition 2.89. A *real additive genus* is an additive genus such that the corresponding power series belong to $\mathbb{R}[[x]]$.

Remark 2.90. It is clear that, if φ is a real additive genus, then for each vector bundle E we have

$$\varphi(E) \in \bigoplus_{p} H^{2p}_{\mathcal{D}}(X, \mathbb{R}(p))$$

We now focus on additive genera, for instance the Chern character is a real additive genus. Let φ be such a genus. Using Chern-Weil theory, to each hermitian vector bundle $\overline{E}$ on X we can attach a closed characteristic form

$$\varphi(\overline{E}) \in \bigoplus_{n,p} \mathcal{D}^{n}(X, p).$$

If $\overline{E}$ is an object of $\overline{\mathbf{V}}^{\mathrm{b}}(X)$, then we define

$$\varphi(\overline{E}) = \sum_i (-1)^i \varphi(\overline{E}^i).$$

If $\overline{E}$ is acyclic, following [18, Sec. 2], we associate to it a Bott-Chern characteristic class

$$\widetilde{\varphi}(\overline{E}) \in \bigoplus_{n,p} \widetilde{\mathcal{D}}^{n-1}(X,p)$$

that satisfies the differential equation

$$\mathrm{d}_{\mathcal{D}}\, \widetilde{\varphi}(\overline{E}) = \varphi(\overline{E}).$$

In fact, [18, Thm. 2.3] for additive genera can be restated as follows.

Proposition 2.91. *Let φ be an additive genus. Then there is a unique group homomorphism*

$$\widetilde{\varphi}\colon \mathbf{KA}(X) \to \bigoplus_{n,p} \widetilde{\mathcal{D}}^{n-1}(X,p)$$

satisfying the properties:

(i) (Differential equation)

$$\mathrm{d}_{\mathcal{D}}\, \widetilde{\varphi}(\overline{E}) = \varphi(\overline{E}).$$

(ii) (Functoriality) If $f\colon X \to Y$ is a morphism of smooth complex varieties, then

$$\widetilde{\varphi}(f^*(\overline{E})) = f^*(\widetilde{\varphi}(\overline{E})).$$

Proof. For the uniqueness, we observe that, if $\widetilde{\varphi}$ is a group homomorphism then $\widetilde{\varphi}(\overline{0}) = 0$. Hence, if $\overline{E}$ is a orthogonally split complex, then it is meager and therefore $\widetilde{\varphi}(\overline{E}) = 0$. Thus, the assignment that, to each acyclic complex bounded $\overline{E}$, associates the class $\widetilde{\varphi}([\overline{E}])$ satisfies the conditions of [18, Thm. 2.3], hence is unique. For the existence, we note that Bott-Chern classes for additive genera satisfy the hypothesis of Theorem 2.32. Hence the result follows. □

Remark 2.92. If

$$\overline{\varepsilon}: \quad 0 \to \overline{\mathcal{F}}_m \to \cdots \to \overline{\mathcal{F}}_l \to 0$$

is an acyclic complex of coherent sheaves on X provided with hermitian structures $\overline{\mathcal{F}}_i = (\mathcal{F}_i, \overline{F}_i \dashrightarrow \mathcal{F}_i)$, by Definition 2.71 we have an object $[\overline{\varepsilon}] \in \mathbf{KA}(X)$, hence a class $\widetilde{\varphi}([\overline{\varepsilon}])$. In the case of the Chern character, in [18, Thm. 2.24] there is defined a class $\widetilde{\mathrm{ch}}(\overline{\varepsilon})$. It follows from [18, Thm. 2.24] that both classes agree. That is, $\widetilde{\mathrm{ch}}([\overline{\varepsilon}]) = \widetilde{\mathrm{ch}}(\overline{\varepsilon})$. For this reason we will denote $\widetilde{\varphi}([\overline{\varepsilon}])$ by $\widetilde{\varphi}(\overline{\varepsilon})$.

Definition 2.93. Let $\overline{\mathcal{F}} = (\overline{E} \overset{\sim}{\dashrightarrow} \mathcal{F})$ be an object of $\overline{\mathbf{D}}^{\mathrm{b}}(X)$. Let φ denote an additive genus. We denote the form

$$\varphi(\overline{\mathcal{F}}) = \varphi(\overline{E}) \in \bigoplus_{n,p} \mathcal{D}^n(X,p)$$

and the class

$$\varphi(\mathcal{F}) = [\varphi(\overline{E})] \in \bigoplus_{n,p} H^n_{\mathcal{D}}(X, \mathbb{R}(p)).$$

Note that the form $\varphi(\overline{\mathcal{F}})$ only depends on the hermitian structure and not on a particular representative thanks to Proposition 2.37 and Proposition 2.91. The class $\varphi(\mathcal{F})$ only depends on the object $\mathcal{F}$ and not on the hermitian structure.

Remark 2.94. The reason to restrict to additive genera when working with the derived category is now clear: there is no canonical way to attach a rank to $\oplus_{i\,\mathrm{even}}\mathcal{F}^i$ (respectively $\oplus_{i\,\mathrm{odd}}\mathcal{F}^i$). The naive choice $\mathrm{rk}(\oplus_{i\,\mathrm{even}}E^i)$ (respectively $\mathrm{rk}(\oplus_{i\,\mathrm{odd}}E^i)$) does depend on $E \dashrightarrow \mathcal{F}$. Thus we can not define Bott-Chern classes by the general rule from [18]. The case of a multiplicative genus such as the Todd genus will be considered later.

Next we will construct Bott-Chern classes for isomorphisms in $\overline{\mathbf{D}}^{\mathrm{b}}(X)$.

Definition 2.95. Let $f\colon \overline{\mathcal{F}} \dashrightarrow \overline{\mathcal{G}}$ be a morphism in $\overline{\mathbf{D}}^{\mathrm{b}}(X)$ and φ an additive genus. We define the differential form

$$\varphi(f) = \varphi(\overline{\mathcal{G}}) - \varphi(\overline{\mathcal{F}}).$$

Theorem 2.96. *Let φ be an additive genus. There is a unique way to attach to every isomorphism in $\overline{\mathbf{D}}^{\mathrm{b}}(X)$*

$$f\colon (\overline{F} \dashrightarrow \mathcal{F}) \overset{\sim}{\dashrightarrow} (\overline{G} \dashrightarrow \mathcal{G})$$

a Bott-Chern class

$$\widetilde{\varphi}(f) \in \bigoplus_{n,p} \widetilde{\mathcal{D}}^{n-1}(X,p)$$

such that the following axioms are satisfied:

(i) (Differential equation)

$$\mathrm{d}_{\mathcal{D}}\, \widetilde{\varphi}(f) = \varphi(f).$$

(ii) (Functoriality) If $g\colon X' \to X$ is a morphism of smooth noetherian schemes over $\mathbb{C}$, then

$$\widetilde{\varphi}(g^*(f)) = g^*(\widetilde{\varphi}(f)).$$

(iii) (Normalization) If f is a tight isomorphism, then

$$\widetilde{\varphi}(f) = 0.$$

Proof. For the existence we define

$$\widetilde{\varphi}(f) = \widetilde{\varphi}([f]), \tag{2.97}$$

where $[f] \in \mathbf{KA}(X)$ is the class of f given by equation (2.62). That $\widetilde{\varphi}$ satisfies the axioms follows from Proposition 2.91 and Theorem 2.61.

We now focus on the uniqueness. Assume such a theory $f \mapsto \widetilde{\varphi}_0(f)$ exists. Fix f as in the statement. Since $\widetilde{\varphi}_0$ is well defined, by replacing $\overline{F}$ by one that is tightly related, we may assume that f is realized by a morphism of complexes

$$f\colon \overline{F} \longrightarrow \overline{G}.$$

We factorize f as

$$\overline{F} \xrightarrow{\alpha} \overline{G} \oplus \overline{\mathrm{cone}}(\overline{F},\overline{G})[-1] \xrightarrow{\beta} \overline{G},$$

where both arrows are zero on the second factor of the middle complex. Since α is a tight morphism and $\overline{\mathrm{cone}}(\overline{F}, \overline{G})[-1]$ is acyclic, we are reduced to the case when $\overline{F} = \overline{G} \oplus \overline{A}$, with $\overline{A}$ an acyclic complex and f is the projection onto the first factor.

For each smooth morphism $g\colon X' \to X$ and each acyclic complex of vector bundles $\overline{E}$ on X', we denote

$$\widetilde{\varphi}_1(\overline{E}) = \widetilde{\varphi}_0(g^*\overline{G} \oplus \overline{E} \to g^*\overline{G}) + \widetilde{\varphi}(\overline{E}),$$

where $\widetilde{\varphi}$ is the usual Bott-Chern form for acyclic complexes of hermitian vector bundles associated to φ. Then $\widetilde{\varphi}_1$ satisfies the hypothesis of Theorem 2.85, so $\widetilde{\varphi}_1 = 0$. Therefore

$$\widetilde{\varphi}(f) = -\widetilde{\varphi}(\overline{A}).$$

□

Proposition 2.98. *Let $f\colon \overline{\mathcal{F}} \dashrightarrow \overline{\mathcal{G}}$ and $g\colon \overline{\mathcal{G}} \dashrightarrow \overline{\mathcal{H}}$ be two isomorphisms in $\overline{\mathbf{D}}^{\mathrm{b}}(X)$. Then:*

$$\widetilde{\varphi}(g \circ f) = \widetilde{\varphi}(g) + \widetilde{\varphi}(f).$$

In particular, $\widetilde{\varphi}(f^{-1}) = -\widetilde{\varphi}(f)$.

Proof. Follows from Theorem 2.67 (iii). □

The Bott-Chern classes behave well under shift.

Proposition 2.99. *Let $f\colon \overline{\mathcal{F}} \dashrightarrow \overline{\mathcal{G}}$ be an isomorphism in $\overline{\mathbf{D}}^{\mathrm{b}}(X)$. Let $f[i]\colon \overline{\mathcal{F}}[i] \dashrightarrow \overline{\mathcal{G}}[i]$ be the shifted isomorphism. Then*

$$(-1)^i\widetilde{\varphi}(f[i]) = \widetilde{\varphi}(f).$$

Proof. The assignment $f \mapsto (-1)^i\widetilde{\varphi}(f[i])$ satisfies the characterizing properties of Theorem 2.96. Hence it agrees with $\widetilde{\varphi}$. □

The following notation will be sometimes used.

Notation 2.100. Let $\mathcal{F}$ be an object of $\mathbf{D}^{\mathrm{b}}(X)$ and consider two choices of hermitian structures $\overline{\mathcal{F}}$ and $\overline{\mathcal{F}}'$. Then we write

$$\widetilde{\varphi}(\overline{\mathcal{F}}, \overline{\mathcal{F}}') = \widetilde{\varphi}(\overline{\mathcal{F}} \overset{\mathrm{id}}{\dashrightarrow} \overline{\mathcal{F}}').$$

Thus $\mathrm{d}_{\mathcal{D}}\, \widetilde{\varphi}(\overline{\mathcal{F}}, \overline{\mathcal{F}}') = \varphi(\overline{\mathcal{F}}') - \varphi(\overline{\mathcal{F}})$.

Example 2.101. Let $\overline{\mathcal{F}} = (\mathcal{F}, \mathcal{F} \dashrightarrow \overline{E})$ be an object of $\overline{\mathbf{D}}^{\mathrm{b}}(X)$. Let $\mathcal{H}^i$ denote the cohomology sheaves of $\mathcal{F}$ and assume that we have chosen hermitian structures $\overline{\mathcal{H}}^i$ of each $\mathcal{H}^i$. In the case when the sheaves $\mathcal{H}^i$ are vector bundles and the hermitian structures are hermitian metrics, X. Ma, in the paper [35], has associated to these data a Bott-Chern class, that we denote $M(\overline{\mathcal{F}}, \overline{\mathcal{H}})$. By the characterization given by Ma of $M(\overline{\mathcal{F}}, \overline{\mathcal{H}})$, it is immediate that

$$M(\overline{\mathcal{F}}, \overline{\mathcal{H}}) = \widetilde{\mathrm{ch}}(\overline{\mathcal{F}}, (\mathcal{F}, \overline{\mathcal{H}})),$$

where $(\mathcal{F}, \overline{\mathcal{H}})$ is as in Definition 2.81.

Our next aim is to construct Bott-Chern classes for distinguished triangles.

Definition 2.102. Let $\overline{\tau}$ be a distinguished triangle in $\overline{\mathbf{D}}^{\mathrm{b}}(X)$,

$$\overline{\tau}\colon\ \overline{\mathcal{F}} \overset{u}{\dashrightarrow} \overline{\mathcal{G}} \overset{v}{\dashrightarrow} \overline{\mathcal{H}} \overset{w}{\dashrightarrow} \overline{\mathcal{F}}[1] \overset{u}{\dashrightarrow} \dots$$

For an additive genus φ, we define the attached differential form

$$\varphi(\overline{\tau}) = \varphi(\overline{\mathcal{F}}) - \varphi(\overline{\mathcal{G}}) + \varphi(\overline{\mathcal{H}}).$$

Notice that if $\overline{\tau}$ is tightly distinguished, then $\varphi(\overline{\tau}) = 0$. Moreover, for any distinguished triangle $\overline{\tau}$ as above, the rotated triangle

$$\overline{\tau}'\colon\ \overline{\mathcal{G}} \overset{v}{\dashrightarrow} \overline{\mathcal{H}} \overset{w}{\dashrightarrow} \overline{\mathcal{F}}[1] \overset{-u[1]}{\dashrightarrow} \overline{\mathcal{G}}[1] \overset{v[1]}{\dashrightarrow} \dots$$

satisfies

$$\varphi(\overline{\tau}') = -\varphi(\overline{\tau}).$$

Theorem 2.103. *Let φ be an additive genus. There is a unique way to attach to every distinguished triangle in $\overline{\mathbf{D}}^{\mathrm{b}}(X)$*

$$\overline{\tau}:\quad \overline{\mathcal{F}} \overset{u}{\dashrightarrow} \overline{\mathcal{G}} \overset{v}{\dashrightarrow} \overline{\mathcal{H}} \overset{w}{\dashrightarrow} \overline{\mathcal{F}}[1] \overset{u[1]}{\dashrightarrow} \dots$$

a Bott-Chern class

$$\widetilde{\varphi}(\overline{\tau}) \in \bigoplus_{n,p} \widetilde{\mathcal{D}}^{n-1}(X,p)$$

such that the following axioms are satisfied:

(i) (Differential equation)

$$\mathrm{d}_{\mathcal{D}}\, \widetilde{\varphi}(\overline{\tau}) = \varphi(\overline{\tau}).$$

(ii) (Functoriality) If $g\colon X' \to X$ is a morphism of smooth noetherian schemes over $\mathbb{C}$, then

$$\widetilde{\varphi}(g^*(\overline{\tau})) = g^*\widetilde{\varphi}(\overline{\tau}).$$

(iii) (Normalization) If $\overline{\tau}$ is tightly distinguished, then

$$\widetilde{\varphi}(\overline{\tau}) = 0.$$

Proof. To show the existence we write

$$\widetilde{\varphi}(\overline{\tau}) = \widetilde{\varphi}([\overline{\tau}]). \tag{2.104}$$

Theorem 2.67 implies that it satisfies the axioms.

To prove the uniqueness, observe that, by replacing representatives of the hermitian structures by tightly related ones, we may assume that the distinguished triangle is represented by

$$\overline{F} \longrightarrow \overline{G} \longrightarrow \overline{\mathrm{cone}}(\overline{F},\overline{G}) \oplus \overline{K} \longrightarrow \overline{F}[1],$$

with $\overline{K}$ acyclic. Then Theorem 2.85 shows that the axioms imply

$$\widetilde{\varphi}(\overline{\tau}) = \widetilde{\varphi}(\overline{K}).$$

□

Remark 2.105. The normalization axiom can be replaced by the apparently weaker condition that $\widetilde{\varphi}(\overline{\tau}) = 0$ for all distinguished triangles of the form

$$\overline{\mathcal{F}} \dashrightarrow \overline{\mathcal{F}} \overset{\perp}{\oplus} \overline{\mathcal{G}} \dashrightarrow \overline{\mathcal{G}} \dashrightarrow$$

where the maps are the natural inclusion and projection.

We leave to the reader the task of translating Theorem 2.67 (iv)-(viii) to Bott-Chern classes.

2.5 Multiplicative genera, the Todd genus and the category $\overline{\mathbf{Sm}}_{*/\mathbb{C}}$

Let ψ be a multiplicative genus, such that the piece of degree zero is $\psi^0 = 1$. We write

$$\varphi = \log(\psi).$$

It is a well defined additive genus, because, by the condition above, the power series $\log(\psi)$ contains only a finite number of terms in each degree.

If $\overline{\theta}$ is either a hermitian vector bundle, a complex of hermitian vector bundles, a morphism in $\overline{\mathbf{D}}^{\mathrm{b}}(X)$ or a distinguished triangle in $\overline{\mathbf{D}}^{\mathrm{b}}(X)$ we can write

$$\psi(\overline{\theta}) = \exp(\varphi(\overline{\theta})).$$

All the results of the previous sections can be translated to the multiplicative genus ψ. In particular, for $\overline{\theta}$ an acyclic complex of hermitian vector bundles, an isomorphism in $\overline{\mathbf{D}}^{\mathrm{b}}(X)$ or a distinguished triangle in $\overline{\mathbf{D}}^{\mathrm{b}}(X)$, we define a Bott-Chern class

$$\widetilde{\psi}_m(\overline{\theta}) = \frac{\exp(\varphi(\overline{\theta})) - 1}{\varphi(\overline{\theta})}\widetilde{\varphi}(\overline{\theta}).$$

Theorem 2.106. *The characteristic class $\widetilde{\psi}_m(\overline{\theta})$ satisfies:*

(i) (Differential equation)

$$\mathrm{d}_{\mathcal{D}}\, \widetilde{\psi}_m(\overline{\theta}) = \psi(\overline{\theta}) - 1.$$

(ii) (Functoriality) If $g\colon X' \to X$ is a morphism of smooth noetherian schemes over $\mathbb{C}$, then

$$\widetilde{\psi}_m(g^*(\overline{\theta})) = g^*\widetilde{\psi}_m(\overline{\theta}).$$

(iii) (Normalization) If $\overline{\theta}$ is either a meager complex, a tight isomorphism or a tightly distinguished triangle, then

$$\widetilde{\psi}_m(\overline{\theta}) = 0.$$

Moreover $\widetilde{\psi}_m$ is uniquely characterized by these properties.

Remark 2.107. For an acyclic complex of vector bundles $\overline{E}$, using the general procedure for arbitrary symmetric power series, we can associate a Bott-Chern class $\widetilde{\psi}(\overline{E})$ (see for instance [18, Thm. 2.3]) that satisfies the differential equation

$$\mathrm{d}_{\mathcal{D}}\, \widetilde{\psi}(\overline{E}) = \prod_{k \text{ even}} \psi(\overline{E}^k) - \prod_{k \text{ odd}} \psi(\overline{E}^k),$$

whereas $\widetilde{\psi}_m$ satisfies the differential equation

$$\mathrm{d}_{\mathcal{D}}\, \widetilde{\psi}_m(\overline{E}) = \prod_k \psi(\overline{E}^k)^{(-1)^k} - 1. \tag{2.108}$$

In fact both Bott-Chern classes are related by

$$\widetilde{\psi}_m(\overline{E}) = \widetilde{\psi}(\overline{E}) \prod_{k \text{ odd}} \psi(\overline{E}^k)^{-1}. \tag{2.109}$$

The main example of a multiplicative genus with the above properties is the Todd genus Td. From now on we will treat only this case. Following the above procedure, to the Todd genus we can associate two Bott-Chern classes for acyclic complexes of vector bundles. The one given by the general theory, denoted by $\widetilde{\mathrm{Td}}$ and the one given by the theory of multiplicative genera, denoted $\widetilde{\mathrm{Td}}_m$. Both are related by the equation (2.109). Note however that, for isomorphisms and distinguished triangles in $\overline{\mathbf{D}}^{\mathrm{b}}(X)$, we only have the multiplicative version.

We now consider morphisms between smooth complex varieties and relative hermitian structures.

Definition 2.110. Let $f\colon X \to Y$ be a morphism of smooth complex varieties. The *tangent complex* of f is the complex

$$T_f\colon \quad 0 \longrightarrow T_X \xrightarrow{df} f^*T_Y \longrightarrow 0$$

where T_X is placed in degree 0 and f^*T_Y is placed in degree 1. It defines an object, also denoted $T_f \in \operatorname{Ob} \mathbf{D}^{\mathrm{b}}(X)$. A *relative hermitian structure on* f is the choice of an object $\overline{T}_f \in \overline{\mathbf{D}}^{\mathrm{b}}(X)$ over T_f.

The following particular situations are of special interest:

– suppose $f\colon X \hookrightarrow Y$ is a closed immersion. Let $N_{X/Y}[-1]$ be the normal bundle to X in Y, considered as a complex concentrated in degree 1. By definition, there is a natural quasi-isomorphism $p\colon T_f \xrightarrow{\sim} N_{X/Y}[-1]$ in $\mathbf{C}^{\mathrm{b}}(X)$, and hence an isomorphism $p^{-1}\colon N_{X/Y}[-1] \overset{\sim}{\dashrightarrow} T_f$ in $\mathbf{D}^{\mathrm{b}}(X)$. Therefore, a hermitian metric h on the vector bundle $N_{X/Y}$ naturally induces a hermitian structure $p^{-1}\colon (N_{X/Y}[-1], h) \dashrightarrow T_f$ on T_f. Let $\overline{T}_f$ be the corresponding object in $\overline{\mathbf{D}}^{\mathrm{b}}(X)$. Then we have

$$\mathrm{Td}(\overline{T}_f) = \mathrm{Td}(N_{X/Y}[-1], h) = \mathrm{Td}(N_{X/Y}, h)^{-1};$$

– suppose $f\colon X \to Y$ is a smooth morphism. Let $T_{X/Y}$ be the relative tangent bundle on X, considered as a complex concentrated in degree 0. By definition, there is a natural quasi-isomorphism $\iota\colon T_{X/Y} \xrightarrow{\sim} T_f$ in $\mathbf{C}^{\mathrm{b}}(X)$. Any choice of hermitian metric h on $T_{X/Y}$ naturally induces a hermitian structure $\iota\colon (T_{X/Y}, h) \dashrightarrow T_f$. If $\overline{T}_f$ denotes the corresponding object in $\overline{\mathbf{D}}^{\mathrm{b}}(X)$, then we find

$$\mathrm{Td}(\overline{T}_f) = \mathrm{Td}(T_{X/Y}, h).$$

Let now $g\colon Y \to Z$ be another morphism of smooth varieties over $\mathbb{C}$. The tangent complexes T_f, T_g and $T_{g\circ f}$ fit into a distinguished triangle in $\mathbf{D}^{\mathrm{b}}(X)$

$$\mathcal{T}\colon T_f \dashrightarrow T_{g\circ f} \dashrightarrow f^*T_g \dashrightarrow T_f[1].$$

Definition 2.111. Let $\overline{\mathbf{Sm}}_{*/\mathbb{C}}$ be the category whose objects are smooth complex varieties and whose morphisms are pairs $\overline{f} = (f, \overline{T}_f)$, where f is a projective morphism and $\overline{T}_f$ is a hermitian structure on T_f. When $\overline{f}$ is given we will denote the hermitian structure by $T_{\overline{f}}$. A hermitian structure on T_f will also be called a hermitian structure on f.

For morphisms $\overline{f}\colon X \to Y$ and $\overline{g}\colon Y \to Z$, the composition is defined as

$$\overline{g} \circ \overline{f} = (g \circ f, \overline{\mathrm{cone}}(f^*T_{\overline{g}}[-1], T_{\overline{f}})).$$

In Proposition 2.118 below we prove the associativity of the composition. Hence $\overline{\mathbf{Sm}}_{*/\mathbb{C}}$ is indeed a category. Moreover, if $\mathbf{Sm}_{*/\mathbb{C}}$ denotes the category of smooth complex varieties and projective morphisms and $\mathfrak{F}\colon \overline{\mathbf{Sm}}_{*/\mathbb{C}} \to \mathbf{Sm}_{*/\mathbb{C}}$ is the forgetful functor, for any object X we have that

$$\begin{aligned} \mathrm{Ob}\,\mathfrak{F}^{-1}(X) &= \{X\}, \\ \mathrm{Hom}_{\mathfrak{F}^{-1}(X)}(X, X) &= \mathbf{KA}(X). \end{aligned}$$

Example 2.112. Let $f\colon X \to Y$ and $g\colon Y \to Z$, be projective morphisms of smooth complex varieties. Assume that we have chosen hermitian metrics on the tangent vector bundles T_X, T_Y and T_Z. Denote by $\overline{f}$, $\overline{g}$ and $\overline{g \circ f}$ the morphism of $\overline{\mathbf{Sm}}_{*/\mathbb{C}}$ determined by these metrics. Then

$$\overline{g} \circ \overline{f} = \overline{g \circ f}.$$

This is seen as follows. By the choice of metrics, there is a tight isomorphism $\overline{\mathrm{cone}}(T_{\overline{f}}, T_{\overline{g\circ f}}) \to f^*T_{\overline{g}}$. Then the natural maps

$$T_{\overline{g}\circ\overline{f}} \to \overline{\mathrm{cone}}(f^*T_{\overline{g}}[-1], T_{\overline{f}}) \to \overline{\mathrm{cone}}(\overline{\mathrm{cone}}(T_{\overline{f}}, T_{\overline{g\circ f}})[-1], T_{\overline{f}}) \to T_{\overline{g\circ f}}$$

are tight isomorphisms.

Example 2.113. Let $f\colon X \to Y$ and $g\colon Y \to Z$, be smooth projective morphisms of smooth complex varieties. Chose hermitian metrics on the relative tangent vector bundles T_f, T_g and $T_{g\circ f}$. Denote by $\overline{f}$, $\overline{g}$ and $\overline{g \circ f}$ the morphism of $\overline{\mathbf{Sm}}_{*/\mathbb{C}}$ determined by these metrics. There is a short exact sequence of hermitian vector bundles

$$\overline{\varepsilon}\colon 0 \longrightarrow \overline{T}_f \longrightarrow \overline{T}_{g\circ f} \longrightarrow f^*\overline{T}_g \longrightarrow 0,$$

that we consider as an acyclic complex declaring $f^*\overline{T}_g$ of degree 0. The morphism $f^*T_{\overline{g}}[-1] \dashrightarrow T_{\overline{f}}$ is represented by the diagram

$$f^*T_{\overline{g}}[-1] \xleftarrow{\ \sim\ } \overline{\mathrm{cone}}(T_{\overline{f}}, T_{\overline{g\circ f}})[-1] \longrightarrow T_{\overline{f}}.$$

Thus, by the definition of a composition we have

$$T_{\overline{g}\circ\overline{f}} = \overline{\mathrm{cone}}(\overline{\mathrm{cone}}(T_{\overline{f}}, T_{\overline{g\circ f}})[-1], f^*T_{\overline{g}}[-1])[1] \oplus \overline{\mathrm{cone}}(\overline{\mathrm{cone}}(T_{\overline{f}}, T_{\overline{g\circ f}})[-1], T_{\overline{f}}).$$

In general this hermitian structure is different to $T_{\overline{g\circ f}}$.

Claim. The equality of hermitian structures

$$T_{\overline{g}\circ\overline{f}} = T_{\overline{g\circ f}} + [\overline{\varepsilon}] \tag{2.114}$$

holds.

Proof of the claim. We have a commutative diagram of distinguished triangles

$$\begin{array}{cccccccc}
\overline{\varepsilon} & T_{\overline{f}} & \longrightarrow & T_{\overline{g\circ f}} & \longrightarrow & f^*T_{\overline{g}} & \dashrightarrow & T_{\overline{f}}[1] \\
& {\scriptstyle \mathrm{id}}\big\downarrow & & \big\downarrow & & {\scriptstyle \mathrm{id}}\big\downarrow & & {\scriptstyle \mathrm{id}}\big\downarrow \\
\overline{\tau} & T_{\overline{f}} & \longrightarrow & T_{\overline{g}\circ\overline{f}} & \longrightarrow & f^*T_{\overline{g}} & \dashrightarrow & T_{\overline{f}}[1].
\end{array}$$

By construction the triangle $\overline{\tau}$ is tightly distinguished, hence $[\overline{\tau}] = 0$. Therefore, according to Theorem 2.67 (vii), we have

$$[T_{\overline{g\circ f}} \to T_{\overline{g}\circ\overline{f}}] = [\overline{\varepsilon}].$$

The claim follows. □

Let $f\colon X \to Y$ and $g\colon Y \to Z$ be projective morphisms of smooth complex varieties. By the definition of composition, hermitian structures on f and g determine a hermitian structure on $g \circ f$. Conversely we have the following result.

Lemma 2.115. *Let $\overline{g}$ and $\overline{g \circ f}$ be hermitian structures on g and $g \circ f$. Then there is a unique hermitian structure $\overline{f}$ on f such that*

$$\overline{g \circ f} = \overline{g} \circ \overline{f}. \tag{2.116}$$

Proof. From the distinguished triangle

$$T_f \dashrightarrow T_{g\circ f} \dashrightarrow f^*T_g \dashrightarrow T_f[1]$$

we see that the unique hermitian structure that satisfies equation (2.116) is $\overline{\mathrm{cone}}(T_{\overline{g\circ f}}, f^*T_{\overline{g}})[-1]$. □

Remark 2.117. By constrast with the preceding result, it is not true in general that hermitian structures $\overline{f}$ and $\overline{g \circ f}$ determine a hemitian structure $\overline{g}$ that satisfies equation (2.116). For instance, if $X = \emptyset$, then any hermitian structure on g will satisfy this equation.

Proposition 2.118. *Let $\overline{f} : X \to Y$, $\overline{g} : Y \to Z$ and $\overline{h} : Z \to W$ be projective morphisms together with hermitian structures. Then $\overline{h} \circ (\overline{g} \circ \overline{f}) = (\overline{h} \circ \overline{g}) \circ \overline{f}$.*

Proof. First of all we observe that if the hermitian structures on $\overline{f}$, $\overline{g}$ and $\overline{h}$ come from fixed hermitian metrics on T_X, T_Y, T_Z and T_W, Example 2.112 ensures that the proposition holds. For the general case, it is enough to see that if the proposition holds for a fixed choice of hermitian structures $\overline{f}$, $\overline{g}$, $\overline{h}$, and we change the metric on f, g or h, then the proposition holds for the new choice of metrics. We treat, for instance, the case when we change the hermitian structure on g, and leave the rest as an analogous exercise. Denote by $\overline{g}'$ the new hermitian structure on g. Then there exists a unique class $\varepsilon \in \mathbf{KA}(Y)$ such that $T_{\overline{g}'} = T_{\overline{g}} + \varepsilon$. According to the definitions, we have

$$T_{\overline{h}\circ(\overline{g}'\circ\overline{f})} = \overline{\text{cone}}((g\circ f)^* T_{\overline{h}}[-1], \overline{\text{cone}}(f^*(T_{\overline{g}}+\varepsilon)[-1], T_{\overline{f}})) = T_{\overline{h}\circ(\overline{g}\circ\overline{f})} + f^*\varepsilon.$$

Similarly, we find

$$T_{(\overline{h}\circ\overline{g}')\circ\overline{f}} = \overline{\text{cone}}(f^*\,\overline{\text{cone}}(g^* T_{\overline{h}}[-1], T_{\overline{g}})[-1] + f^*(-\varepsilon), T_{\overline{f}}) = T_{(\overline{h}\circ\overline{g})\circ\overline{f}} + f^*\varepsilon.$$

By assumption, $T_{\overline{h}\circ(\overline{g}\circ\overline{f})} = T_{(\overline{h}\circ\overline{g})\circ\overline{f}}$. Hence the relations above show

$$T_{\overline{h}\circ(\overline{g}'\circ\overline{f})} = T_{(\overline{h}\circ\overline{g}')\circ\overline{f}}.$$

This concludes the proof. □

To any arrow $\overline{f}\colon X \to Y$ in $\overline{\mathbf{Sm}}_{*/\mathbb{C}}$ we associate a Todd form

$$\text{Td}(\overline{f}) := \text{Td}(T_{\overline{f}}) \in \bigoplus_p \mathcal{D}^{2p}(X, p).$$

If $\overline{g}\colon Y \to Z$ is another morphism, it follows from the definition of $\overline{g}\circ\overline{f}$ that

$$\text{Td}(\overline{g}\circ\overline{f}) = f^*\,\text{Td}(\overline{g}) \bullet \text{Td}(\overline{f}).$$

If we choose two hermitian structures on $f\colon X \to Y$, say $\overline{f}$ and $\overline{f}'$, then one obtains an isomorphism $\overline{\theta}\colon T_{\overline{f}} \to T_{\overline{f}'}$ whose Bott-Chern class satisfies

$$\text{d}_{\mathcal{D}}\,\widetilde{\text{Td}}_m(\overline{\theta}) = \text{Td}(T_{\overline{f}'})\,\text{Td}(T_{\overline{f}})^{-1} - 1.$$

We will usually write

$$\widetilde{\text{Td}}_m(\overline{f}, \overline{f}') := \widetilde{\text{Td}}_m(\overline{\theta}).$$

We finish this section by introducing a notation for the direct image of currents twisted by the Todd genus. This notation will simplify many formulas related with analytic torsion classes.

Let $f\colon X \to Y$ be a morphism of smooth complex varieties. Let S be a closed conic subset of T^*X_0. Then we denote

$$f_*(S) = \{(f(x), \eta) \in T^*Y_0 \mid (x, (\text{d}\,f)^t\eta) \in S\} \cup N_f. \tag{2.119}$$

If $g\colon Y \to Z$ is another morphism of smooth complex varieties, it is easy to see that we have $(g\circ f)_*(S) \subseteq g_* f_*(S)$.

Definition 2.120. Let $\overline{f}\colon X \to Y$ be a morphism in $\overline{\mathbf{Sm}}_{*/\mathbb{C}}$ of relative dimension e. For each closed conical subset $S \subset T^*X_0$ and each pair of integers n, d, we denote by

$$\overline{f}_\flat\colon \mathcal{D}^n_D(X, S, p) \to \mathcal{D}^{n-2e}_D(Y, f_*S, p-e)$$

the map given by

$$\overline{f}_\flat(\omega) = f_*(\omega \bullet \text{Td}(\overline{f})).$$

Proposition 2.121. *Let* $\overline{f}\colon X \to Y$ *and* $\overline{g}\colon Y \to Z$ *be morphisms in* $\overline{\mathbf{Sm}}_{*/\mathbb{C}}$ *of relative dimensions* e_1 *and* e_2 *respectively. Let* $S \subset T^*X_0$, $T \subset T^*Y_0$ *be closed conical subsets and let* $\overline{h} = \overline{f} \circ \overline{g}$, *of relative dimension* $e = e_1 + e_2$.

(i) *The following diagram is commutative*

$$\begin{array}{ccc} \mathcal{D}^n_D(X,S,p) & \xrightarrow{\overline{f}_\flat} & \mathcal{D}^{n-2e_1}_D(Y, f_*S, p-e_1) \\ \Big\downarrow{\scriptstyle \overline{h}_\flat} & & \Big\downarrow{\scriptstyle \overline{g}_\flat} \\ \mathcal{D}^{n-2e}_D(Z, h_*S, p-e) & \hookrightarrow & \mathcal{D}^{n-2e}_D(Z, g_*f_*S, p-e). \end{array}$$

(ii) *let* $\theta \in \mathcal{D}^m_D(X,S,q)$ *and* $\omega \in \mathcal{D}^n_D(Y,T,p)$. *Assume* $T \cap N_f = \emptyset$ *and that* $T + f_*S$ *is disjoint with the zero section in* T^*Y_0. *Then* $f^*T + S$ *is disjoint with the zero section and there is an equality of currents*

$$\overline{f}_\flat(f^*(\omega) \bullet \theta) = \omega \bullet \overline{f}_\flat(\theta)$$

in $\mathcal{D}^{n+m}_D(Y, W, p+q)$, *with*

$$W = f_*(S + f^*T) \cup f_*S \cup f_*f^*T.$$

Proof. For the first assertion, it is enough to notice the equality of currents

$$\overline{g}_\flat(\overline{f}_\flat(\omega)) = (g \circ f)_*(\omega \bullet f^* \operatorname{Td}(\overline{g}) \bullet \operatorname{Td}(\overline{f}))).$$

For the second, it is easy to see that $f^*T + S$ does not cross the zero section, and hence both sides of the equality are defined. It then suffices to establish the equality of currents

$$f_*(f^*\omega \bullet \theta) = f_*(\omega) \bullet \theta.$$

If θ and ω are smooth, then the equality holds true as an elementary application of the definitions shows. The general case follows by approximation of θ and ω by smooth currents and the continuity of the operators f^* and f_*. □

3 Analytic torsion

3.1 Transverse morphisms and relative metrized complexes

In this section we recall the definition of transverse morphisms and we review some basic properties. Then we introduce the notion of relative metrized complex, and explain some basic constructions.

Definition 3.1. Let $f\colon X \to Y$ and $g\colon Z \to Y$ be morphisms of smooth complex varieties. We say that f and g are *transverse* if

$$N_f \cap N_g = \emptyset,$$

where N_f and N_g are the sets of normal directions to f and g respectively as in Definition 2.84.

It is easily seen that, if f is a closed immersion, this definition of transverse morphisms agrees with the definition given in [31, IV-17.13].

If f and g are transverse, then the cartesian product $X \underset{Y}{\times} Z$ is smooth.

For lack of a good reference we prove the following result.

Proposition 3.2. *Let $f\colon X \to Y$ and $g\colon Z \to Y$ be transverse morphisms of smooth complex varieties. Then they are tor-independent.*

Proof. Since the conditions of being transverse and being tor-independent are both local on Y, X and Z we may assume that the map f factorizes as $X \xrightarrow{i} Y \times \mathbb{A}^n \xrightarrow{p} Y$, where i is a closed immersion and p is the projection. Let $g'\colon Z \times \mathbb{A}^n \to Y \times \mathbb{A}^n$ be the morphism $g \times \mathrm{id}$. If f and g are transverse then i and g' are also transverse. While, if i and g' are tor-independent then f and g are tor-independent. Therefore we are reduced to the case when f is a closed immersion.

Since every closed immersion between smooth schemes is regular, we may assume that $Y = \operatorname{Spec} A$, $X = \operatorname{Spec} A/I$, where I is an ideal generated by a regular sequence $(s_1, \dots, s_k)$ and $Z = \operatorname{Spec} B$. The transversality condition implies that $(s_1, \dots, s_k)$ is a regular sequence generating IB. Let K be the Koszul resolution of A/I attached to the above sequence. Then $K \otimes_A B$ is the Koszul resolution of B/IB, hence exact. Therefore, $\operatorname{Tor}^i_A(A/I, B) = 0$ for all $i \geq 1$. Thus f and g are tor-independent. □

Definition 3.3. Let $f\colon X \to Y$ be a projective morphism of smooth complex varieties and $\overline{f} \in \operatorname{Hom}_{\overline{\mathbf{Sm}}_{*/\mathbb{C}}}(X, Y)$ an arrow over f. Let $\overline{\mathcal{F}} \in \operatorname{Ob} \overline{\mathbf{D}}^{\mathrm{b}}(X)$ and let $\overline{f_*\mathcal{F}} \in \operatorname{Ob} \overline{\mathbf{D}}^{\mathrm{b}}(Y)$ be an object over $f_*\mathcal{F}$. The triple $\overline{\xi} = (\overline{f}, \overline{\mathcal{F}}, \overline{f_*\mathcal{F}})$ will be called *a relative metrized complex.* When f is a closed immersion we will also call it an *embedded metrized complex.* When $\overline{\mathcal{F}}$ and $\overline{f_*\mathcal{F}}$ are clear from the context we will denote the relative metrized complex $\overline{\xi}$ by the arrow $\overline{f}$.

Let $\overline{\xi} = (\overline{f}, \overline{\mathcal{F}}, \overline{f_*\mathcal{F}})$ be a relative metrized complex and let $h\colon Y' \to Y$ be a morphism of smooth complex varieties that is transverse to f. Consider the cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{h'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[h]{} & Y. \end{array} \tag{3.4}$$

Then f' is still projective. Moreover, the transversality condition also implies that the canonical arrow $h'^*T_{\overline{f}} \dashrightarrow T_{f'}$ is a hermitian structure on $T_{f'}$. We define

$$h^*\overline{f} = (f', h'^*T_{\overline{f}}) \in \operatorname{Hom}_{\overline{\mathbf{Sm}}_{*/\mathbb{C}}}(X', Y'). \tag{3.5}$$

By tor-independence, there is a canonical isomorphism

$$h^* f_* \mathcal{F} \dashrightarrow f'_* h'^* \mathcal{F}.$$

Therefore $h^*\overline{f_*\mathcal{F}}$ induces a hermitian structure on $f'_* h'^* \mathcal{F}$.

Definition 3.6. The *pull-back* of $\overline{\xi}$ by h is the relative metrized complex

$$h^*\overline{\xi} = (h^*\overline{f}, h'^*\overline{\mathcal{F}}, h^*\overline{f_*\mathcal{F}}).$$

We introduce two elementary operations with relative metrized complexes.

Definition 3.7. Let $\overline{\xi} = (\overline{f}\colon X \to Y, \overline{\mathcal{F}}, \overline{f_*\mathcal{F}})$ be a relative metrized complex. Let $\overline{\mathcal{G}}$ be an object of $\overline{\mathbf{D}}^{\mathrm{b}}(Y)$. There is a canonical isomorphism

$$f_*(\mathcal{F} \otimes f^*\mathcal{G}) \dashrightarrow f_*\mathcal{F} \otimes \mathcal{G}.$$

Therefore the hermitian structures on $\overline{f_*\mathcal{F}}$ and $\overline{\mathcal{G}}$ induce a natural hermitian structure on $f_*(\mathcal{F} \otimes f^*\mathcal{G})$. The *tensor product of* $\overline{\xi}$ *by* $\overline{\mathcal{G}}$ is then defined to be the relative metrized complex

$$\overline{\xi} \otimes \overline{\mathcal{G}} = (\overline{f}, \overline{\mathcal{F}} \otimes f^*\overline{\mathcal{G}}, \overline{f_*\mathcal{F}} \otimes \overline{\mathcal{G}}).$$

Definition 3.8. Let $\overline{\xi}_1$ and $\overline{\xi}_2$ be relative metrized coherent complexes on X, with

$$\begin{aligned} \overline{\xi}_1 &= (\overline{f}, \overline{\mathcal{F}_1}, \overline{f_*\mathcal{F}_1}) \\ \overline{\xi}_2 &= (\overline{f}, \overline{\mathcal{F}_2}, \overline{f_*\mathcal{F}_2}). \end{aligned}$$

Then *the direct sum relative metrized complex* is defined to be

$$\overline{\xi}_1 \oplus \overline{\xi}_2 := (\overline{f}, \overline{\mathcal{F}_1} \oplus \overline{\mathcal{F}_2}, \overline{f_*\mathcal{F}_1} \oplus \overline{f_*\mathcal{F}_2}).$$

We finish this section with a base change type formula for currents, involving transverse morphisms and direct images of the form $\overline{f}_\flat$.

Proposition 3.9. *Let* $\overline{f}$ *be a morphism in* $\overline{\mathbf{Sm}}_{*/\mathbb{C}}$ *of relative dimension* e *and* S *a closed conical subset of* T^*X_0. *Let* $g\colon Y' \to Y$ *be a morphism of smooth complex varieties transverse to* f. *Consider the cartesian diagram*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y. \end{array} \tag{3.10}$$

Equip f' *with the hermitian structure induced by the natural isomorphism* $g^*T_f \dashrightarrow T_{f'}$ *(cf. equation* (3.5) *above). Finally, suppose that* $N_{g'}$ *is disjoint with* S. *Then:*

(i) N_g *and* f_*S *are disjoint and* $g^*f_*S \subset f'_*g'^*S$*;*

(ii) the following diagram commutes:

$$\begin{array}{ccc} \mathcal{D}^n_D(X, S, p) & \xrightarrow{\overline{f}_\flat} & \mathcal{D}^{n-2e}_D(Y, f_*S, p-e) \\ {\scriptstyle g'^*}\downarrow & & \downarrow{\scriptstyle g^*} \\ \mathcal{D}^n_D(X', g'^*S, p) & \xrightarrow{\overline{f}'_\flat} & \mathcal{D}^{n-2e}_D(Y', f'_*g'^*S, p-e) \end{array}$$

Proof. The first claim is straightforward from the definitions. In particular the diagram makes sense. For the commutativity of the diagram, we observe that, since

$$g'^*\operatorname{Td}(\overline{f}) = \operatorname{Td}(\overline{f}'),$$

it suffices to check the equality of currents

$$g^* f_*(\theta) = {f'}_* {g'}^*(\theta)$$

for $\theta \in \mathcal{D}^n_D(X, S, p)$.

By the continuity of the operators g^* (respectively ${g'}^*$) and f_* (respectively ${f'}^*$), it is enough to prove the relation whenever θ is smooth. Moreover, using a partition of unity argument we are reduced to the following local analytic statement.

Lemma 3.11. *Let $f\colon X \to Y$ and $g\colon Y' \to Y$ be transverse morphisms of complex manifolds. Let θ be a smooth differential form on X with compact support. Consider the diagram* (3.10)*. Then*

$$g^* f_*(\theta) = {f'}_* {g'}^*(\theta). \tag{3.12}$$

Proof. The map f can be factored as $X \xrightarrow{\varphi} X \times Y \xrightarrow{p_2} Y$, where $\varphi(x) = (x, f(x))$ is a closed immersion and p_2, the second projection, is smooth. Using again the continuity of the operators g^* (respectively ${g'}^*$) and f_* (respectively ${f'}^*$), we are reduced to prove the equation (3.12) in the case when f is smooth and in the case when f is a closed immersion. The case when f is smooth is clear. Assume now that f is a closed immersion. By transversality, f' is also a closed immersion of complex manifolds. We may assume that $\theta = f^*\widetilde{\theta}$ for some smooth form $\widetilde{\theta}$ on Y. Then equation (3.12) follows from the chain of equalities

$$g^* f_*\theta = g^* f_* f^*\widetilde{\theta} = g^*(\widetilde{\theta} \wedge \delta_X) = g^*(\widetilde{\theta}) \wedge \delta_{X'} = f'_* f'^* g^* \widetilde{\theta} = f'_* g'^* f^* \widetilde{\theta} = f'_* g'^* \theta.$$

This concludes the proof of the lemma. □

The proposition follows from the lemma. □

3.2 Analytic torsion for closed immersions

In the paper [18] the authors study the singular Bott-Chern classes associated to closed immersions of smooth complex varieties. The singular Bott-Chern classes are the analogue for closed immersions of the analytic torsion for smooth morphisms. In fact, they are a particular case of the analytic torsion classes for arbitrary projective morphisms. For this reason, we will call them also analytic torsion classes.

In *loc. cit.* only the singular Bott-Chern classes associated to a single coherent sheaf were studied. The aim of this section is to recall the main results of [18] and to translate them into the language of derived categories.

Definition 3.13. A *theory of analytic torsion classes for closed immersions* is an assignment that, to each embedded metrized complex $\overline{\xi} = (\overline{f}, \overline{\mathcal{F}}, \overline{f_*\mathcal{F}})$, where $f\colon X \to Y$ is a closed immersion of smooth complex varieties, associates a class

$$T(\overline{\xi}) \in \bigoplus_p \widetilde{\mathcal{D}}^{2p-1}_D(Y, N_f, p)$$

satisfying the following conditions.

(i) (Differential equation) The equality

$$\mathrm{d}_{\mathcal{D}}\, T(\overline{\xi}) = \mathrm{ch}(\overline{f_*\mathcal{F}}) - \overline{f}_\flat[\mathrm{ch}(\overline{\mathcal{F}})]$$

holds.

(ii) (Functoriality) For every morphism of smooth complex varieties $h\colon Y' \to Y$ that is transverse to f we have the equality

$$h^* T(\overline{\xi}) = T(h^*\overline{\xi}).$$

(iii) (Normalization) When $X = \emptyset$ (hence $\overline{\mathcal{F}} = \overline{0}$), $Y = \operatorname{Spec}\mathbb{C}$, and $\overline{f_*\mathcal{F}} = \overline{0}$, then

$$T(\overline{f}, \overline{0}, \overline{0}) = 0.$$

Definition 3.14. Let T be a theory of analytic torsion classes for closed immersions.

(i) We say that T is *compatible with the projection formula* if, for every embedded metrized complex $\overline{\xi} = (\overline{f}, \overline{\mathcal{F}}, \overline{f_*\mathcal{F}})$, and every object $\overline{\mathcal{G}} \in \overline{\mathbf{D}}^{\mathrm{b}}(Y)$, the equation

$$T(\overline{\xi} \otimes \overline{\mathcal{G}}) = T(\overline{\xi}) \bullet \mathrm{ch}(\overline{\mathcal{G}}) \tag{3.15}$$

holds.

(ii) We say that T is *additive* if, given two embedded metrized complexes of the form

$$\begin{aligned} \overline{\xi}_1 &= (\overline{f}, \overline{\mathcal{F}_1}, \overline{f_*\mathcal{F}_1}) \\ \overline{\xi}_2 &= (\overline{f}, \overline{\mathcal{F}_2}, \overline{f_*\mathcal{F}_2}), \end{aligned}$$

the equation

$$T(\overline{\xi}_1 \oplus \overline{\xi}_2) = T(\overline{\xi}_1) + T(\overline{\xi}_2) \tag{3.16}$$

holds.

(iii) We say that T is *transitive* if, for every embedded metrized complex $\overline{\xi} = (\overline{f}, \overline{\mathcal{F}}, \overline{f_*\mathcal{F}})$, every closed immersion of smooth complex varieties $g\colon Y \to Z$, each choice of an arrow $\overline{g} \in \operatorname{Hom}_{\overline{\mathbf{Sm}}_{*/\mathbb{C}}}(Y, Z)$ over g, and each choice of an object $\overline{(g \circ f)_*\mathcal{F}} \in \operatorname{Ob}\overline{\mathbf{D}}^{\mathrm{b}}(Z)$ over $(g \circ f)_*\mathcal{F}$, the equation

$$T(\overline{g} \circ \overline{f}) = T(\overline{g}) + \overline{g}_\flat(T(\overline{f})) \tag{3.17}$$

holds. Note that in this equation we are using the convention at the end of Definition 3.3.

Remark 3.18. (i) The normalization condition here and the normalization condition in [18, Def. 6.9] are equivalent once one assumes that T is well defined for objects of $\overline{\mathbf{D}}^{\mathrm{b}}$. Clearly, the compatibility with the projection formula implies the normalization condition. Moreover the compatibility with the projection formula also implies the additivity (see [18, Prop. 10.9])

(ii) It is easy to see that, to check if a theory is compatible with the projection formula, it is enough to consider complexes $\overline{\mathcal{G}}$ consisting of a single hermitian vector bundle $\overline{G}$ placed in degree 0.

Let X be a smooth complex variety and let $\overline{N}$ be a hermitian vector bundle of rank r. We denote by $P = \mathbb{P}(N \oplus \mathbf{1})$ the projective bundle obtained by completing N. Let $\pi_P \colon P \to X$ be the projection and let $s \colon X \to P$ be the zero section. Since N can be identified with the normal bundle to X on P, the hermitian metric of $\overline{N}$ induces a hermitian structure on s. We will denote it by $\overline{s}$. On P we have a tautological quotient vector bundle with an induced metric $\overline{Q}$. For each hermitian vector bundle $\overline{F}$ on X we have the Koszul resolution $K(F, N)$ of s_*F. We denote by $K(\overline{F}, \overline{N})$ the Koszul resolution with the induced metrics. See [18, Def. 5.3] for details.

Definition 3.19. Let T be a theory of analytic torsion classes for closed immersions. We say that T is *homogeneous* if, for every pair of hermitian vector bundles $\overline{N}$ and $\overline{F}$ with $\operatorname{rk} N = r$, there exists a homogeneous class of bidegree $(2r-1, r)$ in the Deligne complex

$$\widetilde{e}(\overline{F}, \overline{N}) \in \widetilde{\mathcal{D}}_D^{2r-1}(P, N_s, r)$$

such that

$$T(\overline{s}, \overline{F}, K(\overline{F}, \overline{N})) \bullet \operatorname{Td}(\overline{Q}) = \widetilde{e}(\overline{F}, \overline{N}) \bullet \operatorname{ch}(\pi_P^* \overline{F}). \tag{3.20}$$

Remark 3.21. Observe that Definition 3.19 is equivalent to [18, Def. 9.2]. The advantage of the definition in this paper is that it treats on equal footing the case when $\operatorname{rk} F = 0$.

The main result of [18] can be translated into the language of derived categories as follows. Denote by $\mathbf{1}_1 \in \mathbb{D}$ the element represented by the constant function 1 of $\mathcal{D}^1(\operatorname{Spec} \mathbb{C}, 1) = \mathbb{R}$.

Theorem 3.22. *(i) There is a unique homogeneous theory of analytic torsion classes for closed immersions, that we denote T^h. This theory is compatible with the projection formula, additive and transitive.*

(ii) Let T be any transitive theory of analytic torsion classes for closed immersions, that is compatible with the projection formula. Then there is a unique real additive genus S_T (Definition 2.89) such that, for any embedded metrized complex $\overline{\xi} := (\overline{f}, \overline{\mathcal{F}}, \overline{f_*\mathcal{F}})$, we have

$$T(\overline{\xi}) - T^h(\overline{\xi}) = -f_*[\operatorname{ch}(\mathcal{F}) \bullet \operatorname{Td}(T_f) \bullet S_T(T_f) \bullet \mathbf{1}_1]. \tag{3.23}$$

(iii) Conversely, any real additive genus S defines, by means of equation (3.23)*, a unique transitive theory of analytic torsion classes T_S for closed immersions, that is compatible with the projection formula and additive.*

Proof. Existence and uniqueness for both T^h and T_S is the content of [18] when restricting to triples $\overline{\xi}$ with $T_{\overline{f}} = \overline{N}_{X/Y}[-1]$, $\overline{\mathcal{F}}$ a hermitian vector bundle placed in degree 0 and $\overline{f_*\mathcal{F}}$ given by a finite locally free resolution. For the general case, we thus need to prove that the theories of analytic torsion classes for closed immersions in the sense of *loc. cit.* uniquely extend to arbitrary $\overline{\xi}$, fulfilling the desired properties.

Assume given a theory T in the sense of [18], compatible with the projection formula and transitive. We will call T the initial theory. First, let us consider a triple $\overline{\xi}$ with $T_{\overline{f}} = \overline{N}_{X/Y}[-1]$ and $\overline{\mathcal{F}} \in \operatorname{Ob} \overline{\mathbf{D}}^{\mathrm{b}}(X)$. Choose a representative $\overline{F} \dashrightarrow \mathcal{F}$ of the hermitian structure on $\overline{\mathcal{F}}$. We then define $T(\overline{\xi})$ by induction on the length of the complex F. First suppose that $F = F^d[-d]$ consists of a single vector bundle placed in degree d. Choose a finite locally free resolution

$$\cdots \to E^1 \to E^0 \to f_* F^d \to 0.$$

Endow the vector bundles E^i with smooth hermitian metrics. Observe that there is an induced isomorphism in $\overline{\mathbf{D}}^{\mathrm{b}}(Y)$

$$\overline{E}[-d] \overset{\sim}{\dashrightarrow} \overline{f_*\mathcal{F}},$$

whose Bott-Chern classes have already been defined. We then put

$$T(\overline{\xi}) = (-1)^d T(\overline{N}_{X/Y}, \overline{F}^d, \overline{E}) + \widetilde{\operatorname{ch}}(\overline{E}[-d] \overset{\sim}{\dashrightarrow} \overline{f_*\mathcal{F}}). \tag{3.24}$$

This definition does not depend on the particular choice of representative of the hermitian structure on $\overline{\mathcal{F}}$, nor on the choice of $\overline{E}$, due to Theorem 2.96 (iii), Proposition 2.98 and [18, Cor. 6.14]. The differential equation is satisfied as a consequence of the differential equations for $T(\overline{N}_{X/Y}, \overline{F}^d, \overline{E})$ and $\widetilde{\operatorname{ch}}(\overline{E}[-d] \overset{\sim}{\dashrightarrow} \overline{f_*\mathcal{F}})$. The compatibility with pull-back by morphisms $h\colon Y' \to Y$ transverse to f is immediate as well. Finally, notice that by construction, if $\overline{\xi}' = (\overline{N}_{X/Y}, \overline{\mathcal{F}}, \overline{f_*\mathcal{F}}')$, then

$$T(\overline{\xi'}) = T(\overline{\xi}) + \widetilde{\operatorname{ch}}(\overline{f_*\mathcal{F}}', \overline{f_*\mathcal{F}}). \tag{3.25}$$

Now suppose that $T(\overline{\xi})$ has been defined for F of length l, satisfying in addition (3.25). If F has length $l+1$, let F^d be the first non-zero vector bundle of F. Consider the exact sequence of complexes

$$(\overline{\varepsilon}) \qquad 0 \to \sigma^{>d}\overline{F} \to \overline{F} \to \overline{F}^d[-d] \to 0,$$

where $\sigma^{>d}$ is the bête filtration. Observe that as a distinguished triangle, $(\overline{\varepsilon})$ is tightly distinguished, hence $\widetilde{\operatorname{ch}}(\overline{\varepsilon}) = 0$. Choose hermitian metrics on $f_*\sigma^{>d}F$ and $f_*F^d[-d]$. We thus have a distinguished triangle in $\overline{\mathbf{D}}^{\mathrm{b}}(Y)$

$$(\overline{\tau}) \qquad \overline{f_*\sigma^{>d}F} \to \overline{f_*F} \to \overline{f_*F^d}[-d] \to \overline{f_*\sigma^{>d}F}[1] \to \dots.$$

We define

$$T(\overline{\xi}) = T(\overline{N}_{X/Y}, \sigma^{>d}\overline{F}, \overline{f_*\sigma^{>d}F}) + (-1)^d T(\overline{N}_{X/Y}, \overline{F}^d, \overline{f_*F^d}) - \widetilde{\operatorname{ch}}(\overline{\tau}). \tag{3.26}$$

This does not depend on the choice of hermitian structures on $f_*\sigma^{>d}F$ and f_*F^d, by the analogue to Theorem 2.67 (vii) for $\widetilde{\operatorname{ch}}$ and because (3.25) holds by assumption for $T(\overline{N}_{X/Y}, \sigma^{>d}\overline{F}, \overline{f_*\sigma^{>d}F})$ and $T(\overline{N}_{X/Y}, \overline{F}^d, \overline{f_*F^d})$. Similarly, (3.25) holds for the defined $T(\overline{\xi})$. The differential equation and compatibility with pull-back are proven as in the first case. This concludes the proof of the existence in case that $T_{\overline{f}} = \overline{N}_{X/Y}[-1]$.

To conclude with the existence, we may now consider a general $\overline{\xi}$. Choose a hermitian metric on the normal bundle $N_{X/Y}$. Put $\overline{\xi}' = (\overline{N}_{X/Y}[-1], \overline{\mathcal{F}}, \overline{f_*\mathcal{F}})$. We define

$$T(\overline{\xi}) = T(\overline{\xi}') + \overline{f}_\flat[\mathrm{ch}(\overline{\mathcal{F}}) \bullet \widetilde{\mathrm{Td}}_m(T_{\overline{f}} \dashrightarrow \overline{N}_{X/Y}[-1])]. \tag{3.27}$$

It is straightforward from the definition that $T(\overline{\xi})$ satisfies the differential equation and is compatible with pull-back by morphisms transverse to f. We call T the extended theory.

We now proceed to prove that the extended theory T is transitive and compatible with the projection formula. For the projection formula, it suffices by Remark 3.18 (ii) to prove

$$T(\overline{\xi} \otimes \overline{G}) = T(\overline{\xi}) \bullet \mathrm{ch}(\overline{G})$$

for $\overline{G}$ a hermitian vector bundle placed in degree 0. This readily follows from the inductive construction of the extended theory T and the assumptions on the initial theory T. One similarly establishes the transitivity and the additivity

We conclude by observing that, since Theorem 2.85 implies that the equations (3.24), (3.25), (3.26) and (3.27) hold, the theory $T(\overline{\xi})$ thus constructed for arbitrary $\overline{\xi}$ is completely determined by the values $T(\overline{\xi}')$, with $\overline{\xi}'$ of the form $(\overline{N}_{X/Y}, \overline{F}, \overline{E})$ where $\overline{F}$ is a hermitian vector bundle and $E \to f_*F$ is a finite locally free resolution.

Once we have seen that any theory of singular Bott-Chern classes as in [18] can be uniquely extended, then statements (ii) and (iii) follow combining equation (7.3) and Corollary 9.43 in [18]. Note that the minus sign in equation (3.23) comes from the fact that $S(T_f) = -S(N_{X/Y})$. □

In [18, §6] there are proved several anomaly formulas satisfied by analytic torsion classes for closed immersions. We now indicate the translation of these formulas to the current setting.

Proposition 3.28. *Let T be a theory of analytic torsion classes for closed immersions. Let*

$$\overline{\xi} = (\overline{f}\colon X \to Y, \overline{\mathcal{F}}, \overline{f_*\mathcal{F}})$$

be an embedded metrized complex.

(i) If $\overline{\mathcal{F}}'$ is another choice of hermitian structure on $\mathcal{F}$ and $\overline{\xi}_1$ is the corresponding relative metrized complex, then

$$T(\overline{\xi}_1) = T(\overline{\xi}) + \overline{f}_\flat[\widetilde{\mathrm{ch}}(\overline{\mathcal{F}}', \overline{\mathcal{F}})].$$

(ii) If $\overline{f}'$ is another choice of hermitian structure on T_f and $\overline{\xi}_2$ is the corresponding relative metrized complex, then

$$T(\overline{\xi}_2) = T(\overline{\xi}) + \overline{f}'_\flat[\mathrm{ch}(\overline{\mathcal{F}}) \bullet \widetilde{\mathrm{Td}}_m(\overline{f}', \overline{f})]. \tag{3.29}$$

(iii) If $\overline{f_*\mathcal{F}}'$ is a different choice of hermitian structure on $f_*\mathcal{F}$, and $\overline{\xi}_3$ is the corresponding relative metrized complex, then

$$T(\overline{\xi}_3) = T(\overline{\xi}) - \widetilde{\mathrm{ch}}(\overline{f_*\mathcal{F}}', \overline{f_*\mathcal{F}}).$$

Proof. We first prove the second assertion. We will reduce to an application of Theorem 2.85. Let $\overline{E} \dashrightarrow T_f$ be a representative of the hermitian structure on $T_{\overline{f}}$. By Theorem 2.47 (ii), we may assume the hermitian structure on $T_{\overline{f}'}$ is represented by the composition

$$\overline{E} \oplus \overline{A} \longrightarrow \overline{E} \dashrightarrow T_f$$

where $\overline{A}$ is a bounded acyclic complex of hermitian vector bundles and $\overline{E} \oplus \overline{A} \to \overline{E}$ is the projection. For every smooth morphism $g \colon Y' \to Y$ of complex varieties, consider the cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y. \end{array} \tag{3.30}$$

We introduce the assignment that, to every such g and each bounded acyclic complex of hermitian vector bundles $\overline{B}$ on X', assigns the class

$$\begin{aligned} \widetilde{\varphi}(\overline{B}) =& T(g'^*\overline{\xi}) - T\left((f_1, g'^*T_{\overline{f}} + [\overline{B}]), g'^*\overline{\mathcal{F}}, g^*\overline{f_*\mathcal{F}}\right) \\ &+ f_{1*}\left[\mathrm{ch}(g'^*\overline{\mathcal{F}})\widetilde{\mathrm{Td}}_m\left((g'^*T_{\overline{f}} + [\overline{B}]), g'^*T_{\overline{f}}\right)\mathrm{Td}(g'^*T_{\overline{f}} + [\overline{B}])\right]. \end{aligned}$$

Here we recall that $[\overline{B}]$ stands for the class of $\overline{B}$ in $\mathbf{KA}(X')$ and $+$ denotes the action of $\mathbf{KA}(X')$ on $\overline{\mathbf{D}}^{\mathrm{b}}(X')$. It is readily seen that $\widetilde{\varphi}$ satisfies the hypothesis of Theorem 2.85. Hence $\widetilde{\varphi} = 0$. This concludes the proof of (ii).

To prove (i), we consider again the cartesian diagram (3.30). We put on f_1 the hermitian structure induced by that of $\overline{f}$. Let $\widetilde{\varphi}_1$ be the assignment that, to each bounded acyclic complex of hermitian vector bundles $\overline{B}$ on X', assigns the class

$$\widetilde{\varphi}_1(\overline{B}) = T(g'^*\overline{\xi}) - T\left(\overline{f}_1, g'^*\overline{\mathcal{F}} + [B], g^*\overline{f_*\mathcal{F}}\right) - \overline{f}_{1\flat}[\widetilde{\mathrm{ch}}(B)].$$

The assignment $\widetilde{\varphi}_1$ satisfies the hypothesis of Theorem 2.85. Hence $\widetilde{\varphi}_1 = 0$. This concludes the proof of (i).

Finally, to prove (iii), to each morphism $g \colon Y' \to Y$, transverse to f, we associate the cartesian diagram (3.30) and we consider the assignment $\widetilde{\varphi}_2$ that, to each bounded acyclic complex of hermitian vector bundles $\overline{B}$ on Y', assigns the class

$$\widetilde{\varphi}_2(\overline{B}) = T(g'^*\overline{\xi}) - T\left(\overline{f}_1, g'^*\overline{\mathcal{F}}, g^*\overline{f_*\mathcal{F}} + [B]\right) + \widetilde{\mathrm{ch}}(B).$$

Again, the assignment $\widetilde{\varphi}_2$ satisfies the hypothesis of Theorem 2.85. Hence $\widetilde{\varphi}_2 = 0$. This concludes the proof of (iii). □

The following result provides a compatibility relation for analytic torsion classes for closed immersions with respect to distinguished triangles. The statement is valid for additive theories, in particular the ones we are concerned with.

Proposition 3.31. *Let T be an additive theory of analytic torsion classes for closed immersions. Let $f\colon X \to Y$ be a closed immersion of smooth complex varieties. Consider distinguished triangles in $\overline{\mathbf{D}}^{\mathrm{b}}(X)$ and $\overline{\mathbf{D}}^{\mathrm{b}}(Y)$ respectively,*

$$\begin{aligned}(\overline{\tau}): &\quad \overline{\mathcal{F}}_2 \to \overline{\mathcal{F}}_1 \to \overline{\mathcal{F}}_0 \to \overline{\mathcal{F}}_2[1],\\ (\overline{f_*\tau}): &\quad \overline{f_*\mathcal{F}}_2 \to \overline{f_*\mathcal{F}}_1 \to \overline{f_*\mathcal{F}}_0 \to \overline{f_*\mathcal{F}}_2[1],\end{aligned}$$

and define relative hermitian complexes

$$\begin{aligned}\overline{\xi}_0 &= (\overline{f}, \overline{\mathcal{F}}_0, \overline{f_*\mathcal{F}}_0),\\ \overline{\xi}_1 &= (\overline{f}, \overline{\mathcal{F}}_1, \overline{f_*\mathcal{F}}_1),\\ \overline{\xi}_2 &= (\overline{f}, \overline{\mathcal{F}}_2, \overline{f_*\mathcal{F}}_2).\end{aligned}$$

Then, the following relation holds:

$$\sum_j (-1)^j T(\overline{\xi}_j) = \widetilde{\mathrm{ch}}(\overline{f_*\tau}) - \overline{f}_\flat(\widetilde{\mathrm{ch}}(\overline{\tau})).$$

Proof. We can assume that the distinguished triangles $\overline{\tau}$ and $\overline{f_*\tau}$ can be represented by short exact sequences of complexes of hermitian vector bundles

$$\begin{aligned}\overline{\varepsilon}: &\quad 0 \longrightarrow \overline{E}_2 \longrightarrow \overline{E}_1 \longrightarrow \overline{E}_0 \longrightarrow 0,\\ \overline{\nu}: &\quad 0 \longrightarrow \overline{V}_2 \longrightarrow \overline{V}_1 \longrightarrow \overline{V}_0 \longrightarrow 0.\end{aligned}$$

Applying the explicit construction at the beginning of the proof of [18, Theorem 2.3] to each row of the above exact sequences, we obtain exact sequences

$$\begin{aligned}\widetilde{\varepsilon}^i: &\quad 0 \longrightarrow \widetilde{E}^i_2 \longrightarrow \widetilde{E}^i_1 \longrightarrow \widetilde{E}^i_0 \longrightarrow 0,\\ \widetilde{\nu}^i: &\quad 0 \longrightarrow \widetilde{V}^i_2 \longrightarrow \widetilde{V}^i_1 \longrightarrow \widetilde{V}^i_0 \longrightarrow 0\end{aligned}$$

over $X \times \mathbb{P}^1$ and $Y \times \mathbb{P}^1$ respectively. The restriction of $\widetilde{\varepsilon}^i$ (respectively $\widetilde{\nu}^i$) to $X \times \{0\}$ (respectively $Y \times \{0\}$) agrees with $\overline{\varepsilon}$ (respectively $\overline{\nu}$). Whereas the restriction to $X \times \{\infty\}$ (respectively $Y \times \{\infty\}$) is orthogonally split. The sequences $\widetilde{\varepsilon}^i$ and $\widetilde{\nu}^i$ form exact sequences of complexes that we denote $\widetilde{\varepsilon}$ and $\widetilde{\nu}$. It is easy to verify that the restriction to $X \times \{\infty\}$ (respectively $Y \times \{\infty\}$) are orthogonally split as sequences of complexes. Moreover, there are isomorphisms $\widetilde{V}_j \dashrightarrow f_*\widetilde{E}_j$, $j = 0, 1, 2$. We denote

$$\widetilde{\xi}_j = (\overline{f} \times \mathrm{id}_{\mathbb{P}^1}, \widetilde{E}_j, \widetilde{V}_j).$$

Then, in the group $\bigoplus_p \widetilde{\mathcal{D}}^{2p-1}_D(Y, N_f, p)$, we have the equality

$$\begin{aligned}0 &= \mathrm{d}_{\mathcal{D}} \frac{1}{2\pi i}\int_{\mathbb{P}^1} \frac{-1}{2}\log t\bar{t} \bullet \sum_j (-1)^j T(\widetilde{\xi}_j)\\ &= T(\overline{\xi}_1) - T(\overline{\xi}_0 \oplus \overline{\xi}_2) - \frac{1}{2\pi i}\int_{\mathbb{P}^1} \frac{-1}{2}\log t\bar{t} \bullet \sum_j (-1)^j \mathrm{ch}(\widetilde{V}_j)\\ &+ \frac{1}{2\pi i}\int_{\mathbb{P}^1} \frac{-1}{2}\log t\bar{t} \bullet \sum_j (-1)^j (f \times \mathrm{id}_{\mathbb{P}^1})_*(\mathrm{ch}(\widetilde{E}_j) \bullet \mathrm{Td}(\overline{f} \times \mathrm{id}_{\mathbb{P}^1}))\\ &= T(\overline{\xi}_1) - T(\overline{\xi}_0 \oplus \overline{\xi}_2) + \widetilde{\mathrm{ch}}(\overline{f_*\tau}) - f_*(\widetilde{\mathrm{ch}}(\overline{\tau})\,\mathrm{Td}(\overline{f})).\end{aligned}$$

Thus the result follows from the additivity. ☐

We end this chapter with the relation between the singular Bott-Chern classes of Bismut-Gillet-Soulé [12] and the theory of homogeneous analytic torsion classes. We draw attention to the difference of normalizations. Let us momentarily denote by τ the theory of singular Bott-Chern classes of Bismut-Gillet-Soulé. By the anomaly formulas, it may be extended to arbitrary embedded metrized complexes. Let $\overline{\xi} = (\overline{f} : X \to Y, \overline{\mathcal{F}}, \overline{f_*\mathcal{F}})$ be a relative metrized complex, with Y of dimension d. If $\tau^{(p-1,p-1)}$ denotes the component of degree $(p-1, p-1)$ of the current τ, we define

$$T^{BGS}(\overline{\xi})^{(2p-1,p)} = -\frac{1}{2(2\pi i)^{d-(p-1)}} \tau^{(p-1,p-1)} \in \widetilde{\mathcal{D}}_D^{2p-1}(Y, N_f, p). \qquad (3.32)$$

In the above equation, the factor $(2\pi i)^{(p-1)}$ comes from the difference in the normalization of characteristic classes. In [12] the authors use real valued classes while we use twisted coefficients. The factor $(2\pi i)^d$ comes from our convention about the Deligne complex of currents. The factor 2 comes from the fact that the second order differential operator that appears in the Deligne complex is $-2\partial\bar{\partial} = 2(2\pi i)dd^c$, while the second order differential operator that appears in the differential equation considered by Bismut, Gillet and Soulé is dd^c. The main reason behind this change is that we want the Bott-Chern classes to be related to Beilinson's regulator and not to twice Beilinson's regulator (see [27] Theorem 3.5.4). Finally, the minus sign comes from the discrepancy of the differential equations of the singular Bott-Chern forms of Bismut-Gillet-Soulé and the analytic torsion forms of Bismut-Köhler. Note that we are forced to change this sign because we want to merge singular Bott-Chern forms and analytic torsion forms on a single theory.

We put

$$T^{BGS}(\overline{\xi}) = \sum_{p \geq 1} T^{BGS}(\overline{\xi})^{(2p-1,p)}.$$

We have the following comparison theorem [18, Thm. 9.25].

Theorem 3.33. *For every embedded metrized complex ξ we have the equality*

$$T^{BGS}(\overline{\xi}) = T^h(\overline{\xi})$$

in $\bigoplus_p \widetilde{\mathcal{D}}_D^{2p-1}(Y, N_f, p)$.

3.3 Regular coherent sheaves

In this section we recall some properties of regular sheaves. Let X be a scheme and let $\mathbb{P}^n_X = \mathbb{P}_X(V)$ be the projective space of lines of the trivial bundle V of rank $n+1$ on X. Let $\pi \colon \mathbb{P}^n_X \to X$ be the natural projection. By abuse of notation, if $\mathcal{G}$ is a sheaf on X, we will denote also by $\mathcal{G}$ the inverse image $\pi^*\mathcal{G}$.

Definition 3.34 ([38], Lecture 14)**.** A quasi-coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n_X$ is called *regular* if $R^q\pi_*\mathcal{F}(-q) = 0$ for all $q > 0$.

The following properties of regular sheaves will be used in the sequel of this paper (see [39]).

(i) If $\mathcal{G}$ is a quasi-coherent sheaf on X, then $\mathcal{G} \otimes_X \mathcal{O}_{\mathbb{P}^n_X}(k)$ is regular for all $k \geq 0$.

(ii) If the scheme X is noetherian and $\mathcal{F}$ is a coherent sheaf on $\mathbb{P}^n_X$, then, Serre's vanishing theorem implies that for d large enough $\mathcal{F}(d)$ is a regular sheaf.

(iii) Let $0 \to \mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F}_0 \to 0$ be an exact sequence of quasi-coherent sheaves on $\mathbb{P}^n_X$ and d be an integer.

   (a) If $\mathcal{F}_2(d)$ and $\mathcal{F}_0(d)$ are regular, then $\mathcal{F}_1(d)$ is regular.

   (b) If $\mathcal{F}_2(d+1)$ and $\mathcal{F}_1(d)$ are regular, then $\mathcal{F}_0(d)$ is regular.

   (c) If $\mathcal{F}_0(d)$ and $\mathcal{F}_1(d+1)$ are regular and the map $R^0\pi_*(\mathcal{F}_1(d)) \to R^0\pi_*(\mathcal{F}_0(d))$ is surjective, then $\mathcal{F}_2(d+1)$ is regular.

(iv) If $\mathcal{F}$ is a regular quasi-coherent sheaf on $\mathbb{P}^n_X$, then $\mathcal{F}(k)$ is regular for all $k > 0$.

(v) If $\mathcal{F}$ is a regular quasi-coherent sheaf on $\mathbb{P}^n_X$ the canonical map $R^0\pi_*\mathcal{F} \otimes_X \mathcal{O}_{\mathbb{P}^n_X} \to \mathcal{F}$ is surjective.

The main property of regular sheaves is the next theorem [39, §8.1].

Theorem 3.35. *Let $\mathcal{F}$ be a regular quasi-coherent sheaf on $\mathbb{P}^n_X$. Then there exists a canonical resolution*

$$\gamma_{\mathrm{can}}(\mathcal{F}) \;:\; 0 \to \mathcal{G}_n(-n) \to \mathcal{G}_{n-1}(-n+1) \to \cdots \to \mathcal{G}_0 \to \mathcal{F} \to 0$$

where $\mathcal{G}_i$ $(i = 0, \dots, n)$ are quasi-coherent sheaves on X. Moreover, for every $k \geq 0$, the sequence

$$0 \to \mathcal{G}_k \to \mathcal{G}_{k-1} \otimes \operatorname{Sym}^1 V^\vee \to \cdots \to \mathcal{G}_0 \otimes \operatorname{Sym}^k V^\vee \to R^0\pi_*(\mathcal{F}(k)) \to 0$$

is exact. In particular, the sheaves $\mathcal{G}_k$ are determined by $\mathcal{F}$ up to unique isomorphism.

Corollary 3.36. *Let X be a noetherian scheme and $\mathcal{F}$ a coherent sheaf on $\mathbb{P}^n_X$. Then, for d large enough, we have a resolution*

$$\gamma_d(\mathcal{F}) \;:\; 0 \to \mathcal{G}_n(-n-d) \to \mathcal{G}_{n-1}(-n-d+1) \to \cdots \to \mathcal{G}_0(-d) \to \mathcal{F} \to 0$$

where $\mathcal{G}_i$, $i = 0, \dots, n$ are coherent sheaves on X.

Example 3.37. The sheaf $\mathcal{O}(1)$ is regular. The canonical resolution of this sheaf is

$$0 \to \Lambda^{n+1}V^\vee(-n) \to \Lambda^n V^\vee(-n+1) \to \cdots \to \Lambda^2 V^\vee(-1) \to V^\vee \to \mathcal{O}(1) \to 0.$$

Twisting this exact sequence by $\mathcal{O}(-1)$ we obtain the Koszul exact sequence

$$0 \to \Lambda^{n+1}V^\vee(-n-1) \to \Lambda^n V^\vee(-n) \to \cdots \to \Lambda^2 V^\vee(-2) \to V^\vee(-1) \to \mathcal{O} \to 0,$$

that we denote K. We will denote by $K(k)$ the twist of the Koszul exact sequence by $\mathcal{O}(k)$.

The next theorem can be found in [51]. We provide a proof for the sake of completeness.

Theorem 3.38. *(i) Let $\mathcal{F}$ be a regular coherent sheaf on $\mathbb{P}^n_X$, and let $\gamma_{\mathrm{can}}(\mathcal{F})$ be the canonical resolution of $\mathcal{F}$ as in Theorem 3.35. Let*

$$\varepsilon_1 \ : \ 0 \to \mathcal{F}_{n+k}(-n-k) \to \cdots \to \mathcal{F}_1(-1) \to \mathcal{F}_0 \to \mathcal{F} \to 0$$

be an exact sequence of coherent sheaves, where the $\mathcal{F}_i$ are sheaves on X. Then there exist natural surjective morphisms of sheaves $\mathcal{F}_i \to \mathcal{G}_i$ on X, $0 \leq i \leq n$ such that the diagram

$$\begin{array}{ccccccccccc}
\mathcal{F}_{n+1}(-n-1) & \longrightarrow & \mathcal{F}_n(-n) & \longrightarrow & \cdots & \longrightarrow & \mathcal{F}_0 & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{G}_n(-n) & \longrightarrow & \cdots & \longrightarrow & \mathcal{G}_0 & \longrightarrow & \mathcal{F} & \longrightarrow & 0
\end{array}$$

is commutative.

(ii) Let $\mathcal{F}$ be a regular coherent sheaf on X, and $\gamma_{\mathrm{can}}(\mathcal{F})$ the canonical resolution. There exists a resolution of $\mathcal{F}(1)$ of the form

$$\varepsilon_2 \ : \ 0 \to \mathcal{S}_{n+k}(-n-k) \to \cdots \to \mathcal{S}_1(-1) \to \mathcal{S}_0 \to \mathcal{F}(1) \to 0$$

such that $\mathcal{S}_0 \ldots, \mathcal{S}_{n+k}$ are coherent sheaves on X and the following diagram of exact sequences is commutative:

$$\begin{array}{ccccccccccc}
\mathcal{S}_{n+1}(-n-1) & \longrightarrow & \mathcal{S}_n(-n) & \longrightarrow & \cdots & \longrightarrow & \mathcal{S}_0 & \longrightarrow & \mathcal{F}(1) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{G}_n(-n+1) & \longrightarrow & \cdots & \longrightarrow & \mathcal{G}_0(1) & \longrightarrow & \mathcal{F}(1) & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \downarrow & & \\
 & & 0 & & & & 0 & & 0. & &
\end{array}$$

Proof. For the first assertion, let us introduce the sheaves $\mathcal{N}_j$ and $\mathcal{K}_j$ defined as the kernels at each term of the sequences γ_{can} and ε_1, respectively. Hence, there are exact sequences

$$\begin{aligned}
0 \to \mathcal{N}_{j+1}(j+1) \to \mathcal{G}_{j+1} \to \mathcal{N}_j(j+1) \to 0, \\
0 \to \mathcal{K}_{j+1}(j+1) \to \mathcal{F}_{j+1} \to \mathcal{K}_j(j+1) \to 0.
\end{aligned}$$

With these notations, observe that $\mathcal{N}_{-1} = \mathcal{K}_{-1} = \mathcal{F}$. By induction, starting from the left hand side of the long exact sequences, it is easily checked that $\mathcal{N}_j(j+1)$ and $\mathcal{K}_j(j+1)$ are regular sheaves, for $j \geq -1$. Also, by Theorem 3.35, we find that $\mathcal{G}_{j+1} = \pi_*(\mathcal{N}_j(j+1))$ for $j \geq -1$. We proceed by induction. Assume for some fixed $k \geq -1$ there is a commutative diagram of exact sequences

$$
\begin{CD}
@. @. 0 @. 0 @. 0 \\
@. @. @VVV @VVV @VVV \\
0 @>>> \mathcal{P}_{k+1} @>>> \mathcal{H}_{k+1} @>>> \mathcal{P}_k(1) \\
@. @VVV @VVV @VVV \\
0 @>>> \mathcal{K}_{k+1}(k+1) @>>> \mathcal{F}_{k+1} @>>> \mathcal{K}_k(k+1) @>>> 0 \\
@. @VVV @VVV @VVV \\
0 @>>> \mathcal{N}_{k+1}(k+1) @>>> \mathcal{G}_{k+1} @>>> \mathcal{N}_k(k+1) @>>> 0 \\
@. @. @VVV @VVV \\
@. @. 0 @. 0,
\end{CD}
\tag{3.39}
$$

where $\mathcal{H}_{k+1}$, $\mathcal{P}_k$ and $\mathcal{P}_{k+1}$ are defined as the kernels of the corresponding morphisms. Suppose in addition that $\mathcal{P}_k(1)$ is regular. In order to proceed with the induction, we need to prove

(i) the map $\mathcal{K}_{k+1}(k+2) \to \mathcal{N}_{k+1}(k+2)$ is surjective,

(ii) the sheaf $\mathcal{P}_{k+1}(1)$ is regular and

(iii) $\mathcal{F}_{k+2}$ surjects onto $\mathcal{G}_{k+2}$.

We first claim that $\mathcal{H}_{k+1}$ surjects onto $\mathcal{P}_k(1)$. Indeed, we apply π_* to the last two columns of diagram (3.39). Observing that $\mathcal{F}_{k+1}$, $\mathcal{G}_{k+1}$ and $\mathcal{H}_{j+1}$ are actually sheaves on X and recalling that $\mathcal{K}_{k+1}(k+2)$ is regular (so that $R^1\pi_*\mathcal{K}_{k+1}(k+1) = 0$), we find a commutative diagram of exact sequences

$$
\begin{CD}
0 @. 0 \\
@VVV @VVV \\
\mathcal{H}_{k+1} @>>> \pi_*(\mathcal{P}_k(1)) \\
@VVV @VVV \\
\mathcal{F}_{k+1} @>>> \pi_*(\mathcal{K}_k(k+1)) @>>> 0 \\
@VVV @VVV \\
\mathcal{G}_{k+1} @= \pi_*(\mathcal{N}_k(k+1)) \\
@VVV @VVV \\
0 @. 0.
\end{CD}
$$

It follows a that the map $\mathcal{H}_{k+1} \twoheadrightarrow \pi_*(\mathcal{P}_k(1))$ is a surjection. Since $\mathcal{P}_k(1)$ is regular, we have that $\pi_*(\mathcal{P}_k(1)) \otimes \mathcal{O}_{\mathbb{P}^n_X} \twoheadrightarrow \mathcal{P}_k(1)$ is also a surjection, thus proving the claim. This property implies that the sequence

$$0 \to \mathcal{P}_{k+1} \to \mathcal{H}_{k+1} \to \mathcal{P}_k(1) \to 0$$

is exact. Hence the sequence

$$0 \to \mathcal{P}_{k+1} \to \mathcal{K}_{k+1}(k+1) \to \mathcal{N}_{k+1}(k+1) \to 0$$

is also exact. We deduce that $\mathcal{K}_{k+1}(k+2) \to \mathcal{N}_{k+1}(k+2)$ is surjective. Now the regularity of $\mathcal{H}_{k+1}$ and $\mathcal{P}_k(1)$, and the surjectivity of $\mathcal{H}_{k+1} \twoheadrightarrow \pi_*(\mathcal{P}_k(1))$ ensure the regularity of $\mathcal{P}_{k+1}(1)$. In its turn, this shows that the sequence

$$0 \to \pi_*(\mathcal{P}_{k+1}(1)) \to \pi_*(\mathcal{K}_{k+1}(k+2)) \to \pi_*(\mathcal{N}_{k+1}(k+2)) \to 0 \tag{3.40}$$

is exact. Finally, we observe that there is a surjective map

$$\mathcal{F}_{k+2} \longrightarrow\!\!\!\!\!\rightarrow \pi_*(\mathcal{K}_{k+1}(k+2)), \tag{3.41}$$

by the regularity of $\mathcal{K}_{k+2}(k+3)$. From the sequences (3.40)–(3.41), we finally obtain a surjection

$$\mathcal{F}_{k+2} \longrightarrow\!\!\!\!\!\rightarrow \pi_*(\mathcal{N}_{k+1}(k+2)) = \mathcal{G}_{k+2}.$$

This completes the proof of the inductive step. Note that the first step of the induction ($k = -1$) is part of the data. Hence we deduce (i).

For the second item, assume that we have constructed the sequence ε_2 up to $\mathcal{S}_k(-k)$. Let $\mathcal{K}_k$ be the kernel of the map $\mathcal{S}_k(-k) \to \mathcal{S}_{k-1}(-k+1)$. We denote by $\mathcal{N}_k$ the successive kernels of the canonical resolution of $\mathcal{F}$ as in the proof of the first statement. Let us assume furthermore that $\mathcal{K}_k(k+1)$ is regular and that we have an exact sequence

$$0 \to \mathcal{P}_k(1) \to \mathcal{K}_k(k+1) \to \mathcal{N}_k(k+2) \to 0$$

with $\mathcal{P}_k(1)$ regular. Recall that we already know that $\mathcal{N}_k(k+1)$ is regular. We consider as well the surjection

$$\mathcal{G}_{k+1}(1) \longrightarrow\!\!\!\!\!\rightarrow \mathcal{N}_k(k+2).$$

We form the fiber product

$$\mathcal{T}_{k+1} := \operatorname{Ker}(\mathcal{K}_k(k+1) \oplus \mathcal{G}_{k+1}(1) \to \mathcal{N}_k(k+2)).$$

Observe that $\mathcal{T}_{k+1}$ is regular, because both $\mathcal{N}_k(k+1)$, $\mathcal{K}_k(k+1) \oplus \mathcal{G}_{k+1}(1)$ are regular and the morphism

$$\pi_*(\mathcal{K}_k(k) \oplus \mathcal{G}_{k+1}) \longrightarrow\!\!\!\!\!\rightarrow \mathcal{G}_{k+1} = \pi_*(\mathcal{N}_k(k+1))$$

is surjective. Moreover the arrows $\mathcal{T}_{k+1} \to \mathcal{G}_{k+1}(1)$ and $\mathcal{T}_{k+1} \to \mathcal{K}_k(k+1)$ are surjective. Therefore, if we define $\mathcal{S}_{k+1} = \pi_*(\mathcal{T}_{k+1})$, we have a commutative diagram of exact sequences

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{P}_{k+1} & \longrightarrow & \mathcal{H}_{k+1} & \longrightarrow & \mathcal{P}_k(1) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{K}_{k+1}(k+1) & \longrightarrow & \mathcal{S}_{k+1} & \longrightarrow & \mathcal{K}_k(k+1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{N}_{k+1}(k+2) & \longrightarrow & \mathcal{G}_{k+1}(1) & \longrightarrow & \mathcal{N}_k(k+2) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0, & &
\end{array}
$$

where $\mathcal{H}_{k+1}$ and $\mathcal{P}_{k+1}$ are defined as the kernels of the corresponding morphisms. To proceed the induction we need to show that

(i) the morphism $\mathcal{K}_{k+1}(k+2) \to \mathcal{N}_{k+1}(k+3)$ is surjective,

(ii) the sheaf $\mathcal{P}_{k+1}(1)$ is regular and

(iii) the sheaf $\mathcal{K}_{k+1}(k+2)$ is regular.

First, we observe that, by the definition of $\mathcal{S}_{k+1}$ and the left exactness of direct images, the map $\pi_*(\mathcal{S}_{k+1}) \to \pi_*(\mathcal{G}_{k+1}(1))$ is surjective. Therefore $\mathcal{H}_{k+1}(1)$ is regular. Moreover, one can check that $\mathcal{S}_{k+1}$ is the fiber product

$$\mathcal{S}_{k+1} = \operatorname{Ker}\left(\pi_*(\mathcal{G}_{k+1}(1)) \oplus \pi_*(\mathcal{K}_k(k+1)) \to \pi_*(\mathcal{N}_k(k+2))\right).$$

This implies easily that $\pi_*(\mathcal{H}_{k+1}) = \pi_*(\mathcal{P}_k(1))$. We also observe that, by definition of fiber product, $\mathcal{P}_k(1) = \operatorname{Ker}(\mathcal{T}_{k+1} \to \mathcal{G}_{k+1}(1))$. Since $\mathcal{S}_{k+1}$ surjects onto $\mathcal{T}_{k+1}$, we deduce that the morphism $\mathcal{H}_{k+1} \to \mathcal{P}_k(1)$ is surjective. From this we conclude that the morphism $\mathcal{K}_{k+1}(k+2) \to \mathcal{N}_{k+1}(k+3)$ is surjective and that the sheaf $\mathcal{P}_{k+1}(1)$ is regular. Since $\mathcal{N}_{k+1}(k+3)$ is regular, we deduce that $\mathcal{K}_{k+1}(k+2)$ is regular.

This concludes the proof of the inductive step and, as with the previous statement, also the proof of the theorem. □

We end this section recalling the notion of generating class of a triangulated category.

Definition 3.42. Let $\mathbf{D}$ be a triangulated category. A *generating class* is a subclass $\mathbf{C}$ of $\mathbf{D}$ such that the smallest triangulated subcategory of $\mathbf{D}$ that contains $\mathbf{C}$ is equivalent to $\mathbf{D}$ via the inclusion.

A direct consequence of Theorem 3.35 is the following result.

Corollary 3.43. *The class of objects of the form $\mathcal{G}(k)$, with $\mathcal{G}$ a coherent sheaf in X and $-n \leq k \leq 0$, is a generating class of $\mathbf{D}^{\mathrm{b}}(\mathbb{P}^n_X)$.*

3.4 Analytic torsion for projective spaces

Let n be a non-negative integer, V the $n+1$ dimensional vector space $\mathbb{C}^{n+1}$ and $\mathbb{P}^n := \mathbb{P}^n(V)$ the projective space of lines in V. We write $\overline{V}$ for the vector space V together with the trivial metric. We will denote by V the trivial vector bundle of fiber V over any base.

We may construct natural relative hermitian complexes that arise by considering the invertible sheaves $\mathcal{O}(k)$, their cohomology and the usual Fubini-Study metric.

If we endow the trivial sheaf with the trivial metric and $\mathcal{O}(1)$ with the Fubini-Study metric, then the tangent bundle T_π carries a quotient hermitian structure via the short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}} \to \mathcal{O}(1)^{n+1} \to T_\pi \to 0. \tag{3.44}$$

We will denote the resulting hermitian vector bundle by $\overline{T_\pi}^{\mathrm{FS}}$ and call it the Fubini-Study metric of T_π. We remark that the Fubini-Study metric is the quotient metric and not the hermitian structure obtained by considering the exact sequence (3.44) as a resolution of T_π. The arrow $(\pi, \overline{T_\pi}^{\mathrm{FS}})$ in $\overline{\mathbf{Sm}}_{*/\mathbb{C}}$ will be written $\overline{\pi}^{\mathrm{FS}}$.

We endow the invertible sheaves $\mathcal{O}(k)$ with the k-th tensor power of the Fubini-Study metric on $\mathcal{O}(1)$. We refer to them by $\overline{\mathcal{O}(k)}$.

We now describe natural hermitian structures on the complexes $\pi_*\mathcal{O}(k)$. First assume $k \geq 0$. Then the sheaf $\mathcal{O}(k)$ is π-acyclic, hence

$$\pi_*\mathcal{O}(k) = \mathrm{H}^0(\mathbb{P}^n_{\mathbb{C}}, \mathcal{O}(k))$$

as a complex concentrated in degree 0. This space is naturally equipped with the L^2 metric with respect to the Fubini-Study metric on $\mathcal{O}(k)$ and the volume form $\mu = c_1(\overline{\mathcal{O}(1)})/n!$ on $\mathbb{P}^n_{\mathbb{C}}$. Namely, given global sections s, t of $\mathcal{O}(k)$, we put

$$\langle s, t\rangle_{L^2} = \int_{\mathbb{P}^n_{\mathbb{C}}} \langle s(x), t(x)\rangle_x \mu(x).$$

Secondly, suppose $-n \leq k < 0$. Then $\pi_*\mathcal{O}(k) = 0$ and we put the trivial metric on it.

Finally, let $k \leq -n-1$. Then the cohomology of $\pi_*\mathcal{O}(k)$ is concentrated in degree n and there is an isomorphism,

$$\pi_*\mathcal{O}(k) \cong \mathrm{H}^0(\mathbb{P}^n_{\mathbb{C}}, \mathcal{O}(-k-n-1))^\vee[-n].$$

Notice that this isomorphism is canonical due to Grothendieck duality and to the natural identification $\omega_{\mathbb{P}^n_{\mathbb{C}}} = \mathcal{O}(-n-1)$. Hence we may endow $\pi_*\mathcal{O}(k)$ with the dual of the L^2 metric on $\mathrm{H}^0(\mathbb{P}^n_{\mathbb{C}}, \mathcal{O}(-k-n-1))$.

The following notation will be useful.

Notation 3.45. For every integer k, we introduce the relative metrized complex

$$\overline{\xi_n}(k) = (\overline{\pi}^{\mathrm{FS}}, \overline{\mathcal{O}(k)}, \overline{\pi_*\mathcal{O}(k)}), \tag{3.46}$$

where $\mathcal{O}(k)$ is endowed with the Fubini-Study metric and $\pi_*\mathcal{O}(k)$ with the hermitian structure above.

If X is a smooth complex variety, then $\overline{\xi}_n(k)$ can be pulled back to $\mathbb{P}^n_X$. We will employ the same notation to refer to this pull-back relative hermitian complex.

Let $\overline{\mathcal{F}}$ be a metrized coherent sheaf on X. Then we define $\overline{\mathcal{F}}(k)$ and $\overline{\pi_*\mathcal{F}(k)}$ by the equality

$$\overline{\xi}_n(k) \otimes \overline{\mathcal{F}} = (\overline{\pi}^{\mathrm{FS}}, \overline{\mathcal{F}}(k), \overline{\pi_*\mathcal{F}(k)}).$$

Definition 3.47. Let X be a complex smooth variety and $\pi\colon \mathbb{P}^n_X \to X$ the projection. An *analytic torsion class* for the relative hermitian complex $\overline{\xi} = (\overline{\pi}, \overline{\mathcal{F}}, \overline{\pi_*\mathcal{F}})$ is a class $\widetilde{\eta} \in \bigoplus_p \widetilde{\mathcal{D}}^{2p-1}(X,p)$ such that

$$\mathrm{d}_{\mathcal{D}}\, \widetilde{\eta} = \mathrm{ch}(\overline{\pi_*\mathcal{F}}) - \overline{\pi}_\flat[\mathrm{ch}(\overline{\mathcal{F}})]. \tag{3.48}$$

The existence of this class is guaranteed by the Grothendieck-Riemann-Roch theorem, which implies that the two currents at the right hand side of equation (3.48) are cohomologous. Since the map π is smooth, the analytic torsion class is the class of a smooth form.

Definition 3.49. Let n be a non-negative integer. A *theory of analytic torsion classes for projective spaces of dimension* n is an assignment that, to each relative metrized complex

$$\overline{\xi} = (\overline{\pi}\colon \mathbb{P}^n_X \to X, \overline{\mathcal{F}}, \overline{\pi_*\mathcal{F}})$$

of relative dimension n, assigns a class of differential forms

$$T(\overline{\xi}) \in \bigoplus_p \widetilde{\mathcal{D}}^{2p-1}(X,p),$$

satisfying the following properties.

(i) (Differential equation) The relation

$$\mathrm{d}_{\mathcal{D}}\, T(\overline{\xi}) = \mathrm{ch}(\overline{\pi_*\mathcal{F}}) - \overline{\pi}_\flat[\mathrm{ch}(\overline{\mathcal{F}})] \tag{3.50}$$

holds.

(ii) (Functoriality) Given a morphism $f\colon Y \longrightarrow X$, we form the cartesian diagram

$$\begin{array}{ccc} \mathbb{P}^n_Y & \xrightarrow{f'} & \mathbb{P}^n_X \\ {\scriptstyle \pi_Y}\downarrow & & \downarrow{\scriptstyle \pi_X} \\ Y & \xrightarrow[f]{} & X. \end{array}$$

Then the equality

$$T(f^*\overline{\xi}) = f^*T(\overline{\xi})$$

holds.

(iii) (Additivity and normalization) If $\overline{\xi}_1$ and $\overline{\xi}_2$ are relative metrized complexes on X, then

$$T(\overline{\xi}_1 \oplus \overline{\xi}_2) = T(\overline{\xi}_1) + T(\overline{\xi}_2).$$

(iv) (Projection formula) For any hermitian vector bundle $\overline{G}$ on X, and an integer $-n \leq k \leq 0$, the equality

$$T(\overline{\xi}_n(k) \otimes \overline{G}) = T(\overline{\xi}_n(k)) \bullet \operatorname{ch}(\overline{G}).$$

holds.

A *theory of analytic torsion classes for projective spaces* is an assignment as before, for all non-negative integers n.

Definition 3.51. Let T be a theory of analytic torsion classes for projective spaces of dimension n. Fix as base space the point $\operatorname{Spec}\mathbb{C}$. The *characteristic numbers* of T are

$$t_{n,k}(T) := T(\overline{\xi}_n(k)) \in \widetilde{\mathcal{D}}^1(\operatorname{Spec}\mathbb{C}, 1) = \mathbb{R},\ k \in \mathbb{Z}. \tag{3.52}$$

The characteristic numbers $t_{n,k}(T)$, $-n \leq k \leq 0$ will be called the *main characteristic numbers* of T.

The central result of this section is the following classification theorem.

Theorem 3.53. *Let n be a non-negative integer and let $\mathfrak{t} = (t_{n,k})_{k=-n,\dots,0}$ be a family of arbitrary real numbers. Then there exists a unique theory $T_{\mathfrak{t}}$ of analytic torsion classes for projective spaces of dimension n, such that $t_{n,k}(T_{\mathfrak{t}}) = t_{n,k}$.*

Before proving Theorem 3.53, we show some consequences of the definition of the analytic torsion classes.

First we state some anomaly formulas that determine the dependence of the analytic torsion classes with respect to different choices of metrics.

Proposition 3.54. *Let T be a theory of analytic torsion classes for projective spaces of dimension n. Let*

$$\overline{\xi} = (\overline{\pi}\colon \mathbb{P}^n_X \to X, \overline{\mathcal{F}}, \overline{\pi_*\mathcal{F}})$$

be a relative metrized complex.

(i) If $\overline{\mathcal{F}}'$ is another choice of metric on $\mathcal{F}$ and $\overline{\xi}_1$ is the corresponding relative metrized complex, then

$$T(\overline{\xi}_1) = T(\overline{\xi}) + \overline{\pi}_\flat[\widetilde{\operatorname{ch}}(\overline{\mathcal{F}}', \overline{\mathcal{F}})].$$

(ii) If $\overline{\pi}'$ is another choice of hermitian structure on π and $\overline{\xi}_2$ is the corresponding relative metrized complex, then

$$T(\overline{\xi}_2) = T(\overline{\xi}) + \overline{\pi}'_\flat[\operatorname{ch}(\overline{\mathcal{F}}) \bullet \widetilde{\operatorname{Td}}_m(\overline{\pi}', \overline{\pi})]. \tag{3.55}$$

(iii) If $\overline{\pi_*\mathcal{F}}'$ is a different choice of metric on $\pi_*\mathcal{F}$, and $\overline{\xi}_3$ is the corresponding relative metrized complex, then

$$T(\overline{\xi}_3) = T(\overline{\xi}) - \widetilde{\operatorname{ch}}(\overline{\pi_*\mathcal{F}}', \overline{\pi_*\mathcal{F}}).$$

Proof. The proof is the same than the proof of Proposition 3.28. ◻

Next we state the behavior of analytic torsion classes for projective spaces with respect to distinguished triangles.

Proposition 3.56. *Let T be a theory of analytic torsion classes for projective spaces of dimension n. Let X be a smooth complex variety and $\pi\colon \mathbb{P}^n_X \to X$ the projection. Consider distinguished triangles in $\overline{\mathbf{D}}^{\mathrm{b}}(\mathbb{P}^n_X)$ and $\overline{\mathbf{D}}^{\mathrm{b}}(X)$ respectively:*

$$\begin{aligned}(\overline{\tau}):\quad & \overline{\mathcal{F}}_2 \to \overline{\mathcal{F}}_1 \to \overline{\mathcal{F}}_0 \to \overline{\mathcal{F}}_2[1],\\ (\overline{\pi_*\tau}):\quad & \overline{\pi_*\mathcal{F}_2} \to \overline{\pi_*\mathcal{F}_1} \to \overline{\pi_*\mathcal{F}_0} \to \overline{\pi_*\mathcal{F}_2}[1],\end{aligned}$$

and define relative metrized complexes

$$\begin{aligned}\overline{\xi}_0 &= (\overline{\pi}, \overline{\mathcal{F}}_0, \overline{\pi_*\mathcal{F}_0}),\\ \overline{\xi}_1 &= (\overline{\pi}, \overline{\mathcal{F}}_1, \overline{\pi_*\mathcal{F}_1}),\\ \overline{\xi}_2 &= (\overline{\pi}, \overline{\mathcal{F}}_2, \overline{\pi_*\mathcal{F}_2}).\end{aligned}$$

Then, the following relation holds:

$$\sum_j (-1)^j T(\overline{\xi}_j) = \widetilde{\mathrm{ch}}(\overline{\pi_*\tau}) - \overline{\pi}_\flat(\widetilde{\mathrm{ch}}(\overline{\tau})).$$

Proof. The proof is similar to the argument for Proposition 3.31, and is thus left as an exercise. □

In view of this proposition, we see that the additivity axiom is equivalent to the apparently stronger statement of the next corollary.

Corollary 3.57. *With the assumptions of Proposition 3.56, if $\overline{\tau}$ and $\overline{\pi_*\tau}$ are tightly distinguished, then*

$$T(\overline{\xi}_1) = T(\overline{\xi}_0) + T(\overline{\xi}_2).$$

Corollary 3.58. *Let $\overline{\xi} = (\overline{\pi}, \overline{\mathcal{F}}, \overline{\pi_*\mathcal{F}})$ be a relative metrized complex and let $\overline{\xi}[i] = (\overline{\pi}, \overline{\mathcal{F}}[i], \overline{\pi_*\mathcal{F}}[i])$ be the shifted relative metrized complex. Then*

$$T(\overline{\xi}) = (-1)^i T(\overline{\xi}[i]).$$

Proof. It is enough to treat the case $i = 1$. We consider the tightly distinguished triangle

$$\overline{\mathcal{F}} \dashrightarrow \overline{\mathrm{cone}}(\mathrm{id}_{\overline{\mathcal{F}}}) \dashrightarrow \overline{\mathcal{F}}[1] \dashrightarrow$$

and the analogous triangle for direct images. Since $\overline{\mathrm{cone}}(\mathrm{id}_{\overline{\mathcal{F}}})$ and $\overline{\mathrm{cone}}(\mathrm{id}_{\overline{\pi_*\mathcal{F}}})$ are meager, we have, by the anomaly formulas and the additivity axiom,

$$T(\overline{\pi}, \overline{\mathrm{cone}}(\mathrm{id}_{\overline{\mathcal{F}}}), \overline{\mathrm{cone}}(\mathrm{id}_{\overline{\pi_*\mathcal{F}}})) = T(\overline{\pi}, \overline{0}, \overline{0}) = 0.$$

Hence, the result follows from Corollary 3.57. □

Next we rewrite Proposition 3.56 in the language of complexes of metrized coherent sheaves. Let

$$\overline{\varepsilon}:\quad 0 \to \overline{\mathcal{F}}_m \to \cdots \to \overline{\mathcal{F}}_l \to 0$$

be a bounded complex of coherent sheaves on $\mathbb{P}^n_X$ provided with hermitian structures as in Definition 2.71 and assume that there are chosen hermitian structures on the complexes $\pi_*\mathcal{F}_j$, $j = l, \dots, m$. Let $[\overline{\varepsilon}]$ and $[\overline{\pi_*\varepsilon}]$ be as in definitions 2.71 and 2.73.

Corollary 3.59. *With the above hypothesis,*

$$T(\overline{\pi}, [\overline{\varepsilon}], [\overline{\pi_*\varepsilon}]) = \sum_{j=l}^{m} (-1)^j T(\overline{\pi}, \overline{\mathcal{F}}_j, \overline{\pi_*\mathcal{F}_j}).$$

Moreover, if ε is acyclic, then

$$T(\overline{\pi}, [\overline{\varepsilon}], [\overline{\pi_*\varepsilon}]) = \widetilde{\mathrm{ch}}(\overline{\pi_*\varepsilon}) - \overline{\pi}_\flat[\widetilde{\mathrm{ch}}(\overline{\varepsilon})].$$

Finally, we show that the projection formula holds in greater generality:

Proposition 3.60. *Let T be a theory of analytic torsion classes for projective spaces of dimension n. Let X be a smooth complex variety, let $\overline{\xi} = (\overline{\pi}, \overline{\mathcal{F}}, \overline{\pi_*\mathcal{F}})$ be a relative metrized complex and let $\overline{\mathcal{G}}$ be an object in $\overline{\mathbf{D}}^{\mathrm{b}}(X)$. Then*

$$T(\overline{\xi} \otimes \overline{\mathcal{G}}) = T(\overline{\xi}) \bullet \mathrm{ch}(\overline{\mathcal{G}}). \tag{3.61}$$

Proof. By the anomaly formulas, if equation (3.61) holds for a particular choice of hermitian structures on π, $\mathcal{F}$ and $\pi_*\mathcal{F}$ then it holds for any other choice. Moreover, if we are in the situation of Proposition 3.56 and equation (3.61) holds for two of $\overline{\xi}_0$, $\overline{\xi}_1$, $\overline{\xi}_2$, then it holds for the third. Using that the objects of the form $\mathcal{H}(k)$, where $\mathcal{H}$ is a coherent sheaf on X and $k = -n, \dots, 0$, constitute a generating class of $\mathbf{D}^{\mathrm{b}}(\mathbb{P}^n_X)$, we are reduced to prove the equation

$$T(\overline{\xi}_n(k) \otimes \overline{\mathcal{G}}) = T(\overline{\xi}_n(k)) \bullet \mathrm{ch}(\overline{\mathcal{G}}).$$

for $k = -n, \dots, 0$.

Now, if

$$\overline{\mathcal{G}}_2 \dashrightarrow \overline{\mathcal{G}}_1 \dashrightarrow \overline{\mathcal{G}}_0 \dashrightarrow$$

is a distinguished triangle in $\overline{\mathbf{D}}^{\mathrm{b}}(X)$ and equation (3.61) is satisfied for two of $\overline{\mathcal{G}}_2$, $\overline{\mathcal{G}}_1$, $\overline{\mathcal{G}}_0$, then it is satisfied also by the third. Therefore, since the complexes of vector bundles concentrated in a single degree constitute a generating class of $\mathbf{D}^{\mathrm{b}}(X)$, the projection formula axiom implies the proposition. □

Proof of Theorem 3.53 . To begin with, we prove the uniqueness assertion. Assume a theory of analytic torsion classes T, with main characteristic numbers $t_{n,k}$, $-n \leq k \leq 0$, exists. Then, the anomaly formulas (Proposition 3.54) imply that, if the value of $T(\overline{\pi}, \overline{\mathcal{F}}, \overline{\pi_*\mathcal{F}})$ is fixed for a particular choice of hermitian structures on π, $\mathcal{F}$ and $\pi_*\mathcal{F}$ then the value of $T(\overline{\pi}', \overline{\mathcal{F}}', \overline{\pi_*\mathcal{F}}')$ for any other choice of hermitian structures is also fixed. By Proposition 3.56, if we know the value of $T(\overline{\pi}, \overline{\mathcal{F}}, \overline{\pi_*\mathcal{F}})$, for $\mathcal{F}$ in a generating class, then T is determined. By the projection formula (Proposition 3.60), the characteristic numbers determine the values of $T(\overline{\xi}(k) \otimes \mathcal{G})$, $k = -n, \dots, 0$. Finally, since by Corollary 3.43, the objects of the form $\mathcal{G}(k)$, $k = -n, \dots, 0$ form a generating class, we deduce that the characteristic numbers determine the theory T. Thus, if it exists, the theory $T_{\mathfrak{t}}$ is unique.

In particular, from the above discussion we see the main characteristic numbers determine all the characteristic numbers. We now derive an explicit inductive formula for them.

Consider the metrized Koszul resolution

$$\overline{K} : 0 \to \Lambda^{n+1}\overline{V}^{\vee}(-n-1) \to \cdots \to \Lambda^1\overline{V}^{\vee}(-1) \to \overline{\mathcal{O}}_{\mathbb{P}^n_{\mathbb{C}}} \to 0, \tag{3.62}$$

where $\mathcal{O}(k)$, for $k \neq 0$, has the Fubini-Study metric and $\overline{\mathcal{O}}_{\mathbb{P}^n_{\mathbb{C}}}$ has the trivial metric. We will denote by $\overline{K}(k)$ the above exact sequence twisted by $\overline{\mathcal{O}(k)}$, $k \in \mathbb{Z}$, again with the Fubini-Study metric. Recall the definition of the relative metrized complexes $\overline{\xi}_n(k)$ (3.46). In particular, for every k, we have fixed natural hermitian structures on the objects $\pi_*\mathcal{O}(k-j)$. According to definitions 2.71 and 2.73, we may consider the classes $[\overline{K}(k)]$ and $[\overline{\pi_* K(k)}]$ in $\overline{\mathbf{D}}^{\mathrm{b}}(\mathbb{P}^n_{\mathbb{C}})$ and $\overline{\mathbf{D}}^{\mathrm{b}}(\operatorname{Spec}\mathbb{C})$, respectively. By Corollary 3.59, for each $k \in \mathbb{Z}$ we find

$$\sum_{j=0}^{n+1}(-1)^j T(\overline{\xi}_n(k-j)\otimes \Lambda^j\overline{V}^\vee) = \widetilde{\mathrm{ch}}(\overline{\pi_* K(k)}) - \overline{\pi}_\flat^{\mathrm{FS}}[\widetilde{\mathrm{ch}}(\overline{K}(k))].$$

Because $\Lambda^j\overline{V}^\vee$ is isometric to $\mathbb{C}^{\binom{n+1}{j}}$ with the trivial metric, the additivity axiom for the theory T and the definition of the characteristic numbers $t_{n,k-j}$ provide

$$T(\overline{\xi}_n(k-j)\otimes \Lambda^j\overline{V}^\vee) = t_{n,k-j}\binom{n+1}{j}.$$

Therefore we derive

$$\sum_{j=0}^{n+1}(-1)^j \binom{n+1}{j} t_{n,k-j} = \widetilde{\mathrm{ch}}(\overline{\pi_* K(k)}) - \overline{\pi}_\flat^{\mathrm{FS}}[\widetilde{\mathrm{ch}}(\overline{K}(k))]. \tag{3.63}$$

This equation gives us an inductive formula for all the characteristic numbers $t_{n,k}$ once we have fixed $n+1$ consecutive characteristic numbers and, in particular, once we have fixed the main characteristic numbers.

To prove the existence, we follow the proof of the uniqueness to obtain a formula for $T(\overline{\xi})$. We start with the main characteristic numbers $\mathfrak{t} = (t_{n,k})_{-n\leq k\leq 0}$. We define the characteristic numbers $t_{n,k}$ for $k \in \mathbb{Z}$ inductively using equation (3.63).

We will need the following results.

Lemma 3.64. *Let*

$$\overline{\eta} : 0 \to \overline{\mathcal{F}}_2 \to \overline{\mathcal{F}}_1 \to \overline{\mathcal{F}}_0 \to 0$$

be a short exact sequence of metrized coherent sheaves on X. Let k be an integer let $\overline{\mathcal{F}}(k)$ and $\overline{\pi_*\mathcal{F}(k)}$ be as in Notation 3.45. Thus we have an exact sequence $\overline{\eta}(k)$ of metrized coherent sheaves on $\mathbb{P}^n_X$ and a distinguished triangle $\overline{\pi_*\eta(k)}$. Then, in the group $\bigoplus_p \widetilde{\mathcal{D}}^{2p-1}(X,p)$, it holds

$$\widetilde{\mathrm{ch}}(\overline{\pi_*\eta(k)}) = \overline{\pi}_\flat^{\mathrm{FS}}(\widetilde{\mathrm{ch}}(\overline{\eta}(k))). \tag{3.65}$$

Proof. By the Riemann-Roch theorem for the map $\mathbb{P}^n_{\mathbb{C}} \to \operatorname{Spec}\mathbb{C}$ we have

$$\mathrm{ch}(\overline{\pi_*\mathcal{O}(k)}) = \pi_*(\mathrm{ch}(\overline{\mathcal{O}(k)})\,\mathrm{Td}(\overline{\pi}^{\mathrm{FS}})). \tag{3.66}$$

Hence, by the properties of Bott-Chern classes and the choice of metrics

$$\begin{aligned}\widetilde{\mathrm{ch}}(\overline{\pi_*\eta(k)}) &= \widetilde{\mathrm{ch}}(\overline{\eta}) \bullet \mathrm{ch}(\overline{\pi_*\mathcal{O}(k)})\\ &= \widetilde{\mathrm{ch}}(\overline{\eta}) \bullet \pi_*(\mathrm{ch}(\overline{\mathcal{O}(k)})\,\mathrm{Td}(\overline{\pi}^{\mathrm{FS}}))\\ &= \pi_*\left(\widetilde{\mathrm{ch}}(\overline{\eta}(k)) \bullet \mathrm{Td}(\overline{\pi}^{\mathrm{FS}})\right)\\ &= \overline{\pi}_\flat^{\mathrm{FS}}(\widetilde{\mathrm{ch}}(\overline{\eta}(k))).\end{aligned}$$

□

Lemma 3.67. *Let*

$$\overline{\mu} : 0 \to \overline{\mathcal{M}}_m(-m-d) \to \cdots \to \overline{\mathcal{M}}_l(-l-d) \to 0 \tag{3.68}$$

be an exact sequence of metrized coherent sheaves on $\mathbb{P}^n_X$, where, for each $i = l, \dots, m$, $\overline{\mathcal{M}}_i$ is a metrized coherent sheaf on X, and $\overline{\mathcal{M}}_i(k)$ is as in Notation 3.45. On $\pi_* \mathcal{M}_i(k)$ we consider the hermitian structures given also by Notation 3.45. Then, in the group $\bigoplus_p \widetilde{\mathcal{D}}^{2p-1}(X,p)$ it holds

$$\sum_{i=l}^{m} (-1)^i t_{n,-d-i} \operatorname{ch}(\overline{\mathcal{M}}_i) = \widetilde{\operatorname{ch}}(\overline{\pi_* \mu}) - \pi_\flat^{\mathrm{FS}}(\widetilde{\operatorname{ch}}(\overline{\mu})). \tag{3.69}$$

Proof. We first observe that, if there is a commutative diagram of exact sequences

$$\begin{array}{cccccccccc}
 & & & 0 & & & & 0 & & \\
 & & & \downarrow & & & & \downarrow & & \\
\overline{\mu}' & & 0 \longrightarrow & \overline{\mathcal{M}}'_m(-m-d) & \longrightarrow & \cdots & \longrightarrow & \overline{\mathcal{M}}'_l(-l-d) & \longrightarrow 0 & \\
 & & & \downarrow & & & & \downarrow & & \\
\overline{\mu} & & 0 \longrightarrow & \overline{\mathcal{M}}_m(-m-d) & \longrightarrow & \cdots & \longrightarrow & \overline{\mathcal{M}}_l(-l-d) & \longrightarrow 0 & \\
 & & & \downarrow & & & & \downarrow & & \\
\overline{\mu}'' & & 0 \longrightarrow & \overline{\mathcal{M}}''_m(-m-d) & \longrightarrow & \cdots & \longrightarrow & \overline{\mathcal{M}}''_l(-l-d) & \longrightarrow 0 & \\
 & & & \downarrow & & & & \downarrow & & \\
 & & & 0 & & & & 0 & & \\
 & & & & & & & & & \\
 & & & \overline{\xi}_m & & \cdots & & \overline{\xi}_l, & &
\end{array}$$

and equation (3.69) is true for two of $\overline{\mu}$, $\overline{\mu}'$ and $\overline{\mu}''$, then it is true for the third. Indeed, on the one hand we have

$$\sum_{i=l}^{m} (-1)^i t_{n,-d-i} \left(\operatorname{ch}(\overline{\mathcal{M}}'_i) - \operatorname{ch}(\overline{\mathcal{M}}_i) + \operatorname{ch}(\overline{\mathcal{M}}''_i) \right) = \sum_{i=l}^{m} (-1)^i t_{n,-d-i} \, \mathrm{d}_{\mathcal{D}} \widetilde{\operatorname{ch}}(\overline{\xi}_i).$$

But, if $t \in \mathcal{D}^1(\operatorname{Spec} \mathbb{C}, 1) = \mathbb{R}$ is a real number, in the group $\bigoplus_p \widetilde{\mathcal{D}}^{2p-1}(X,p)$ we have

$$t \, \mathrm{d}_{\mathcal{D}} \widetilde{\operatorname{ch}}(\overline{\xi}_i) = -\mathrm{d}_{\mathcal{D}}(t \bullet \widetilde{\operatorname{ch}}(\overline{\xi}_i)) = 0.$$

On the other hand

$$\widetilde{\operatorname{ch}}(\overline{\pi_* \mu'}) - \widetilde{\operatorname{ch}}(\overline{\pi_* \mu}) + \widetilde{\operatorname{ch}}(\overline{\pi_* \mu''}) = \\ \pi_\flat^{\mathrm{FS}}(\widetilde{\operatorname{ch}}(\overline{\mu}')) - \pi_\flat^{\mathrm{FS}}(\widetilde{\operatorname{ch}}(\overline{\mu})) + \pi_\flat^{\mathrm{FS}}(\widetilde{\operatorname{ch}}(\overline{\mu}''))$$

by Lemma 3.64.

Now the proof of the lemma is done by induction on the length of the complex $r = m - l$. If $r \leq n$ then $\mu(d+l)$ has the same shape as the canonical resolution of the zero coherent sheaf. By the uniqueness of the canonical resolution, it follows that $\mathcal{M}_i = 0$, for $i = l, \dots, m$. Using the above observation to take into account the possibility to consider non trivial metrics on the zero sheaf, the lemma is proved in this case.

Assume now that $r > n$. Let K be the Koszul exact sequence (3.62). Then $K(1) \otimes \mathcal{M}_l$ is the canonical resolution of the regular coherent sheaf $\mathcal{M}_l(1)$. By Theorem 3.38 (i) there is a surjection of exact sequences $\mu \to K(-l-d) \otimes \mathcal{M}_l$ whose kernel is an exact sequence of the form

$$\mu' : 0 \to \mathcal{M}'_m(-m-d) \to \cdots \to \mathcal{M}'_{l+1}(-d-l-1) \to 0.$$

We consider on K the metrics of (3.62), for $i = l+1, \dots, m$, we choose arbitrary metrics on $\mathcal{M}'_i$ and denote by $\overline{\mu}'$ the corresponding exact sequence of metrized coherent sheaves.

By induction hypothesis, $\overline{\mu}'$ satisfies equation (3.69). Moreover, since the characteristic numbers $t_{n,k}$ for $k \notin [0, n]$ are defined using equation (3.63), the exact sequence $\overline{K}(-l-d) \otimes \overline{\mathcal{M}}_l$ also satisfies equation (3.69). Hence the lemma follows from the previous observation. □

We now treat the case of complexes concentrated in a single degree. Let $\overline{\mathcal{F}}$ be a coherent sheaf on $\mathbb{P}^n_X$ with a hermitian structure and let $\overline{\pi_* \mathcal{F}}$ be a choice of a hermitian structure on the direct image complex. Write

$$\overline{\xi} = (\overline{\pi}^{\mathrm{FS}}, \overline{\mathcal{F}}, \overline{\pi_* \mathcal{F}})$$

for the corresponding relative metrized complex.

Choose an integer d such that $\mathcal{F}(d)$ is regular. Then we have the resolution $\gamma_d(\mathcal{F})$ of Corollary 3.36. More generally, let μ be an exact sequence of the form

$$0 \to \mathcal{S}_m(-d-m) \to \cdots \to \mathcal{S}_1(-d-1) \to \mathcal{S}_0(-d) \to \mathcal{F} \to 0,$$

where the $\mathcal{S}_i$, $i = 0, \dots, m$ are coherent sheaves on X. Assume that we have chosen hermitian structures on the sheaves $\mathcal{S}_i$. Using Notation 3.45 and definitions 2.71 and 2.73 we have objects $[\overline{\mu}]$ in $\mathbf{KA}(\mathbb{P}^n_X)$ and $[\overline{\pi_* \mu}]$ in $\mathbf{KA}(X)$. Then we write

$$T_{\mathfrak{t},\overline{\mu}}(\overline{\xi}) = \sum_{j=0}^{m} (-1)^j t_{n,j-d} \operatorname{ch}(\overline{\mathcal{S}}_j) - \widetilde{\operatorname{ch}}(\overline{\pi_* \mu}) + \overline{\pi}^{\mathrm{FS}}_{\flat}(\widetilde{\operatorname{ch}}(\overline{\mu})) \tag{3.70}$$

Lemma 3.71. *Given any choice of metrics on the sheaves $\mathcal{G}_i$, (respectively $\mathcal{G}'_i$) $i = 0, \dots, n$, that appear in the resolution $\gamma_d(\mathcal{F})$ (respectively $\gamma_{d+1}(\mathcal{F})$), denote by $\overline{\gamma}_d$ and $\overline{\gamma}_{d+1}$ the corresponding exact sequences of metrized coherent sheaves. Then*

$$T_{\mathfrak{t},\overline{\gamma}_{d+1}}(\overline{\xi}) = T_{\mathfrak{t},\overline{\gamma}_d}(\overline{\xi}).$$

In particular, $T_{\mathfrak{t},\overline{\gamma}_d}(\overline{\xi})$ does not depend on the choice of metric on the sheaves $\mathcal{G}_i$.

Proof. By Theorem 3.38 (ii), there is an exact sequence

$$\overline{\mu} : 0 \to \overline{\mathcal{S}}_{n+k}(-n-k-d-1) \to \cdots \to \overline{\mathcal{S}}_0(-d-1) \to \overline{\mathcal{F}} \to 0, \tag{3.72}$$

and a surjection of exact sequences $f\colon \overline{\mu} \to \overline{\gamma}_d$ extending the identity on $\overline{\mathcal{F}}$. Here $\overline{\mathcal{S}}_i$, $i=0,\dots,n+k$ are coherent sheaves on X with hermitian structures.

By Theorem 3.38 (i) there is a surjection of exact sequences $\overline{\mu} \longrightarrow \overline{\gamma}_{d+1}$ extending the identity on $\overline{\mathcal{F}}$, whose kernel is an exact sequence

$$\overline{\varepsilon} : 0 \to \overline{\mathcal{M}}_{n+k}(-n-k-d-1) \to \cdots \to \overline{\mathcal{M}}_0(-d-1) \to 0, \qquad (3.73)$$

where $\overline{\mathcal{M}}_i$, $i=0,\dots,n+k$ are coherent sheaves on X, and we have chosen arbitrarily an hermitian structure on them. Denote by $\overline{\eta}_i$ the rows of the exact sequence

$$0 \to \overline{\varepsilon} \to \overline{\mu} \to \overline{\gamma}_{d+1} \to 0.$$

Observe that $\overline{\eta}_i = \overline{\eta}'_i(-i-d-1)$ for some short exact sequence $\overline{\eta}'_i$ on X. When $j \geq n$ we denote $\overline{\mathcal{G}}'_j = \overline{0}$. Then, we have

$$\sum_{j=0}^{n+k}(-1)^j t_{n,j-d-1}\left(\mathrm{ch}(\overline{\mathcal{G}}'_j) - \mathrm{ch}(\overline{\mathcal{S}}_j) + \mathrm{ch}(\overline{\mathcal{M}}_j)\right) = \sum_{j=0}^{n+k}(-1)^j t_{n,j-d-1}\, \mathrm{d}_{\mathcal{D}}\, \widetilde{\mathrm{ch}}(\overline{\eta}'_i) = 0. \qquad (3.74)$$

By Proposition 2.75, we have

$$\widetilde{\mathrm{ch}}(\overline{\pi_*\gamma_{d+1}}) - \widetilde{\mathrm{ch}}(\overline{\pi_*\mu}) + \widetilde{\mathrm{ch}}(\overline{\pi_*\varepsilon}) = \sum_{j=0}^{n+k}(-1)^j\, \widetilde{\mathrm{ch}}(\overline{\pi_*\eta_j}) \qquad (3.75)$$

and

$$\widetilde{\mathrm{ch}}(\overline{\gamma}_{d+1}) - \widetilde{\mathrm{ch}}(\overline{\mu}) + \widetilde{\mathrm{ch}}(\overline{\varepsilon}) = \sum_{j=0}^{n+k}(-1)^j\, \widetilde{\mathrm{ch}}(\overline{\eta}_j). \qquad (3.76)$$

Combining equations (3.74), (3.75) and (3.76) and lemmas 3.64 and 3.67 we obtain

$$T_{\mathfrak{t},\overline{\mu}}(\overline{\xi}) = T_{\mathfrak{t},\overline{\gamma}_{d+1}}(\overline{\xi}). \qquad (3.77)$$

We consider now $\mathrm{cone}(\mu,\gamma_d)$. On it we put the obvious hermitian structure induced by $\overline{\mu}$ and $\overline{\gamma_d}$, $\overline{\mathrm{cone}(\mu,\gamma_d)}$. On $\pi_*\,\mathrm{cone}(\mu,\gamma_d)$, we put the obvious family of hermitian metrics induced by $\overline{\pi_*\mu}$ and $\overline{\pi_*\gamma_d}$, and denote it as $\overline{\pi_*\,\mathrm{cone}(\mu,\gamma_d)}$. By Corollary 2.76 we have

$$\widetilde{\mathrm{ch}}(\overline{\mathrm{cone}(\mu,\gamma_d)}) = \widetilde{\mathrm{ch}}(\overline{\gamma}_d) - \widetilde{\mathrm{ch}}(\overline{\mu}), \qquad (3.78)$$

and

$$\widetilde{\mathrm{ch}}(\overline{\pi_*\,\mathrm{cone}(\mu,\gamma_d)}) = \widetilde{\mathrm{ch}}(\overline{\pi_*\gamma_d}) - \widetilde{\mathrm{ch}}(\overline{\pi_*\mu}). \qquad (3.79)$$

Observe that $\overline{\mathrm{cone}(\mu,\gamma_d)}^i = \overline{\mathcal{S}}_{-i-1}(i-d) \oplus \overline{\mathcal{G}}_{-i}(i-d)$. Combining Lemma 3.67 for $\overline{\mathrm{cone}(\mu,\gamma_d)}$ with equations (3.78) and (3.79), we obtain

$$T_{\mathfrak{t},\overline{\mu}}(\overline{\xi}) = T_{\mathfrak{t},\overline{\gamma}_d}(\overline{\xi}), \qquad (3.80)$$

Together with equation (3.77) this proves the lemma. □

After this lemma, we are in position to give a definition of $T_{\mathfrak{t}}$.

Definition 3.81. Let $n \geq 0$ be an integer and let $t_{n,j}$, $0 \leq j \leq n$ be real numbers. Let $\overline{\xi} = (\overline{\pi}^{\mathrm{FS}}, \overline{\mathcal{F}}, \overline{\pi_* \mathcal{F}})$ be a relative metrized complex. We next define $T_{\mathfrak{t}}(\overline{\xi})$. Let $t_{n,j}$, $j < 0$ and $j > n$ be determined by equation (3.63).

The definition is given by induction on the length of the cohomology of $\overline{\mathcal{F}}$ If the cohomology of $\mathcal{F}$ has at most a single non zero coherent sheaf $\mathcal{H}$ sitting at degree k, then $\overline{\mathcal{F}}$ and $\overline{\pi_* \mathcal{F}}$ determine hermitian structures on $\mathcal{H}[-k]$ and $\pi_* \mathcal{H}[-k]$ respectively. We choose an integer d such that $\mathcal{H}(d)$ is regular and we write

$$T_{\mathfrak{t}}(\overline{\xi}) = (-1)^k T_{\mathfrak{t}, \overline{\gamma}_d(\mathcal{H})}(\overline{\pi}^{\mathrm{FS}}, \overline{\mathcal{H}}, \overline{\pi_* \mathcal{H}}).$$

By Lemma 3.71, this does not depend on the choice of d nor on the choice of metrics on $\overline{\gamma}_d(\mathcal{H})$.

Assume that we have already defined the analytic torsion classes for all complexes whose cohomology has length less than l and that the cohomology of $\mathcal{F}$ has length l. Let $\mathcal{H}$ be the highest cohomology sheaf of $\mathcal{F}$, say of degree k. Choose auxiliary hermitian structures on $\mathcal{H}[-k]$ and $\pi_* \mathcal{H}[-k]$. There is a unique natural map $\mathcal{H}[-k] \dashrightarrow \mathcal{F}$. Then we define

$$\begin{aligned} T_{\mathfrak{t}}(\overline{\xi}) = T_{\mathfrak{t}}(\overline{\pi}^{\mathrm{FS}}, \overline{\mathcal{H}[-k]}, \overline{\pi_* \mathcal{H}[-k]}) \\ + T_{\mathfrak{t}}(\overline{\pi}^{\mathrm{FS}}, \overline{\mathrm{cone}}(\overline{\mathcal{H}[-k]}, \overline{\mathcal{F}}), \overline{\mathrm{cone}}(\overline{\pi_* \mathcal{H}[-k]}, \overline{\pi_* \mathcal{F}})). \end{aligned}$$

It follows from Theorem 2.27 (iv) that this definition does not depend on the choice of the auxiliary hermitian structures.

Finally, we consider the case when $T_{\overline{\pi}}$ has a metric different from the Fubini-Study metric. Thus, let $\overline{\xi} = (\overline{\pi}, \overline{\mathcal{F}}, \overline{\pi_* \mathcal{F}})$ and write $\overline{\xi}' = (\overline{\pi}^{\mathrm{FS}}, \overline{\mathcal{F}}, \overline{\pi_* \mathcal{F}})$. Then we put

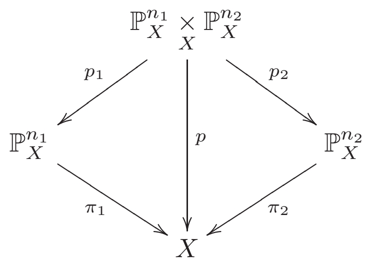
$$T_{\mathfrak{t}}(\overline{\xi}) = T_{\mathfrak{t}}(\overline{\xi}') + \overline{\pi}_\flat[\mathrm{ch}(\overline{F}) \bullet \widetilde{\mathrm{Td}}_m(\overline{\pi}, \overline{\pi}^{\mathrm{FS}})]. \tag{3.82}$$

This ends the definition of $T_{\mathfrak{t}}$.

It remains to prove that $T_{\mathfrak{t}}$ satisfies axioms (i) to (iv). Axiom (i) follows from the differential equations satisfied by the Bott-Chern classes. Axiom (ii) follows from the functoriality of the canonical resolution, the Chern forms and the Bott-Chern classes. Axiom (iii) follows from the additivity of the canonical resolution and the additivity of the Chern character. Finally Axiom (iv) follows from the multiplicativity of the Chern character. □

We finish this section showing the compatibility of analytic torsion classes with the composition of projective bundles.

Let X be a smooth complex variety. Consider the commutative diagram with cartesian square



On π_1 and π_2 we introduce arbitrary hermitian structures and on p_1 and p_2 the hermitian structures induced by the cartesian diagram.

Proposition 3.83. *Let $\overline{\mathcal{F}}$ be an object of $\overline{\mathbf{D}}^{\mathrm{b}}(\mathbb{P}^{n_1}_X \underset{X}{\times} \mathbb{P}^{n_2}_X)$. Put arbitrary hermitian structures on $(p_1)_*\mathcal{F}$, $(p_2)_*\mathcal{F}$, and $p_*\mathcal{F}$. Then*

$$T(\overline{\pi}_1) + (\overline{\pi}_1)_\flat(T(\overline{p}_1)) = T(\overline{\pi}_2) + (\overline{\pi}_2)_\flat(T(\overline{p}_2)), \tag{3.84}$$

where we are using the convention at the end of Definition 3.3.

Proof. By the anomaly formulas (Proposition 3.54), if equation (3.84) holds for a particular choice of hermitian structures on $\mathcal{F}$, $(p_1)_*\mathcal{F}$, $(p_2)_*\mathcal{F}$, and $p_*\mathcal{F}$, then it holds for any other choice.

Let

$$\overline{\mathcal{F}}_2 \dashrightarrow \overline{\mathcal{F}}_1 \dashrightarrow \overline{\mathcal{F}}_0 \dashrightarrow$$

be a distinguished triangle and put hermitian structures on the direct images as before. Then Proposition 3.56 implies that, if equation (3.84) holds for two of them, then it also holds for the third. Since the objects of the form $\mathcal{G}(k,l) := p^*\mathcal{G} \otimes p_1^*\mathcal{O}(k) \otimes p_2^*\mathcal{O}(l)$ are a generating class of $\mathbf{D}^{\mathrm{b}}(\mathbb{P}^{n_1}_X \underset{X}{\times} \mathbb{P}^{n_2}_X)$, the previous discussion shows that it is enough to prove the case $\mathcal{F} = \mathcal{G}(k,l)$, with the hermitian structure of $\mathcal{F}$ induced by a hermitian structure of $\mathcal{G}$ and the Fubini-Study metric on $\mathcal{O}(k)$ and $\mathcal{O}(l)$, and the hermitian structures on the direct images defined as in (3.46). In this case the result follows easily from the functoriality and the projection formula. □

3.5 Compatible analytic torsion classes

In this section we study the compatibility between analytic torsion classes for closed immersions and analytic torsion classes for projective spaces. It turns out that, once the compatibility between the diagonal embedding of $\mathbb{P}^n$ into $\mathbb{P}^n \times \mathbb{P}^n$ and the second projection of $\mathbb{P}^n \times \mathbb{P}^n$ onto $\mathbb{P}^n$ is established, then all the other possible compatibilities follow. Essentially this observation can be traced back to [15].

Let n, V, $\overline{V}$ and $\mathbb{P}^n(V)$ be as in the previous section. We consider the following diagram

$$\begin{array}{ccccc}
\mathbb{P}^n & \xrightarrow{\Delta} & \mathbb{P}^n \times \mathbb{P}^n & \xrightarrow{p_1} & \mathbb{P}^n \\
 & \searrow^{\mathrm{id}} & \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle \pi} \\
 & & \mathbb{P}^n & \xrightarrow[\pi_1]{} & \operatorname{Spec}\mathbb{C}\ .
\end{array}$$

On $\mathbb{P}^n$ we have the tautological short exact sequence

$$0 \to \mathcal{O}(-1) \to V \to Q \to 0\ .$$

This induces on $\mathbb{P}^n \times \mathbb{P}^n$ the exact sequence

$$0 \to p_2^*\mathcal{O}(-1) \to V \to p_2^*Q \to 0\ .$$

There is also an injection

$$p_1^*\mathcal{O}(-1) \hookrightarrow V.$$

By composition, we obtain a morphism

$$p_1^*\mathcal{O}(-1) \to p_2^*Q,$$

hence a section of $p_2^*Q \otimes p_1^*\mathcal{O}(1)$. The zero locus of this section is the image of the diagonal. Moreover, the associated Koszul complex is quasi-isomorphic to $\Delta_*\mathcal{O}_{\mathbb{P}^n}$. That is, the sequence

$$0 \to \Lambda^n(p_2^*Q^\vee) \otimes p_1^*\mathcal{O}_{\mathbb{P}^n}(-n) \to \dots \\ \dots \to \Lambda^1(p_2^*Q^\vee) \otimes p_1^*\mathcal{O}_{\mathbb{P}^n}(-1) \to \mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n} \to \Delta_*\mathcal{O}_{\mathbb{P}^n} \to 0 \quad (3.85)$$

is exact.

On $T_{\mathbb{P}^n}$ and $T_{\mathbb{P}^n\times\mathbb{P}^n}$ we consider the Fubini-Study metrics. We denote by $\overline{\Delta}$ and $\overline{p}_2$ the morphisms of $\overline{\mathbf{Sm}}_{*/\mathbb{C}}$ determined by these metrics. As in Example 2.112, we have that $\overline{p_2} \circ \overline{\Delta} = \overline{\mathrm{id}}_{\mathbb{P}^n}$, where $T_{\overline{\mathrm{id}_{\mathbb{P}^n}}} = \overline{0}$.

The Fubini-Study metric on $\mathcal{O}(-1)$ and the metric induced by the tautological exact sequence on Q, induce a metric on the Koszul complex that we denote $\overline{K}(\Delta)$. This is a hermitian structure on $\Delta_*\mathcal{O}_{\mathbb{P}^n}$.

Finally on $\mathcal{O}_{\mathbb{P}^n}$ we consider the trivial metric. This is a hermitian structure on $(p_2)_*K(\Delta)$.

Fix a real additive genus S and denote by T_S the theory of analytic torsion classes for closed immersions that is compatible with the projection formula and transitive, associated to S ([18, Cor. 9.40], see Theorem 3.22). Moreover, fix a family of real numbers $\mathfrak{t} = \{t_{nk} \mid n \geq 0,\ -n \leq k \leq 0\}$ and denote $T_{\mathfrak{t}}$ the theory of generalized analytic torsion classes for projective spaces associated to this family.

Definition 3.86. The theories of analytic torsion classes T_S and $T_{\mathfrak{t}}$ are called *compatible* if the following formula holds:

$$T_{\mathfrak{t}}(\overline{p_2}, \overline{K}(\Delta), \overline{\mathcal{O}}_{\mathbb{P}^n}) + (\overline{p}_2)_\flat(T_S(\overline{\Delta}, \overline{\mathcal{O}}_{\mathbb{P}^n}, \overline{K}(\Delta))) = 0. \quad (3.87)$$

The reason behind this definition is that compatible analytic torsion classes for closed immersions and projective spaces should combine to provide analytic torsion classes for arbitrary projective morphisms, and these classes should be transitive. In particular the transitivity condition for the composition $\mathrm{id}_{\mathbb{P}^n} = p_2 \circ \Delta$ should give us

$$0 = T(\overline{\mathrm{id}}_{\mathbb{P}^n}, \overline{\mathcal{O}}_{\mathbb{P}^n}, \overline{\mathcal{O}}_{\mathbb{P}^n}) = T_{\mathfrak{t}}(\overline{p_2}, \overline{K}(\Delta), \overline{\mathcal{O}}_{\mathbb{P}^n}) + (\overline{p}_2)_\flat(T_S(\overline{\Delta}, \overline{\mathcal{O}}_{\mathbb{P}^n}, \overline{K}(\Delta))).$$

Theorem 3.88. *Let S be a real additive genus. Then there exists a unique family of real numbers $\mathfrak{t} = \{t_{n,k} \mid n \geq 0,\ -n \leq k \leq 0\}$ such that the theories of analytic torsion classes T_S and $T_{\mathfrak{t}}$ are compatible. The theory $T_{\mathfrak{t}}$ will also be denoted T_S.*

Proof. The first step is to make explicit equation (3.87) in terms of the main characteristic numbers $\mathfrak{t}$. To this end, first observe that, since the exact sequence

$$0 \to T_{p_2} \to T_{\mathbb{P}^n\times\mathbb{P}^n} \to p_2^*T_{\mathbb{P}^n} \to 0 \quad (3.89)$$

is split and the hermitian metric on $T_{\mathbb{P}^n\times\mathbb{P}^n}$ is the orthogonal direct sum metric, we deduce that $\overline{p_2} = \pi_1^*(\overline{\pi}^{\mathrm{FS}})$. Next, we denote by $\overline{K}(\Delta)_i$ the component of degree i of the Koszul complex, and we define

$$\overline{(p_2)_*K(\Delta)_i} = \begin{cases} \overline{\mathcal{O}}_{\mathbb{P}^n}, & \text{for } i = 0, \\ \overline{0}, & \text{for } i > 0. \end{cases}$$

Finally using Corollary 3.59, functoriality and the compatibility with the projection formula, we derive

$$
\begin{aligned}
T_{\mathfrak{t}}(\overline{p_2}, \overline{K}(\Delta), \overline{\mathcal{O}}_{\mathbb{P}^n}) &= \sum_{i=0}^{n} (-1)^i T_{\mathfrak{t}}(\overline{p_2}, \overline{K}(\Delta)_i, \overline{(p_2)_* K(\Delta)_i}) \\
&= \sum_{i=0}^{n} (-1)^i T_{\mathfrak{t}}(\pi_1^* \overline{\xi}_n(-i) \otimes \Lambda^i \overline{Q}^\vee) \\
&= \sum_{i=0}^{n} (-1)^i t_{n,-i} \operatorname{ch}(\Lambda^i \overline{Q}^\vee).
\end{aligned}
$$

Thus, the second and last step is to solve the equation

$$
\sum_{i=0}^{n} (-1)^i t_{n,-i} \operatorname{ch}(\Lambda^i \overline{Q}^\vee) = -(p_2)_*(T_S(\overline{\Delta}, \overline{\mathcal{O}}_{\mathbb{P}^n}, \overline{K}(\Delta)) \bullet \operatorname{Td}(\overline{p_2})). \tag{3.90}
$$

Since the left hand side of equation (3.90) is closed, in order to be able to solve this equation we have to show that the right hand side is also closed. We compute

$$
\begin{aligned}
&\mathrm{d}_{\mathcal{D}} (p_2)_*(T_S(\overline{\Delta}, \overline{\mathcal{O}}_{\mathbb{P}^n}, \overline{K}(\Delta)) \bullet \operatorname{Td}(\overline{p_2})) \\
&\qquad = (p_2)_* \left(\sum_{i=0}^{n} (-1)^i \operatorname{ch}(\overline{K}(\Delta)_i) \operatorname{Td}(\overline{p}_2) - \Delta_*(\operatorname{ch}(\overline{\mathcal{O}}_{\mathbb{P}^n}) \operatorname{Td}(\overline{\Delta})) \operatorname{Td}(\overline{p}_2) \right) \\
&\qquad = (p_2)_* \left(\sum_{i=0}^{n} (-1)^i p_2^*(\operatorname{ch}(\Lambda^i \overline{Q}^\vee)) p_1^*(\operatorname{ch}(\overline{\mathcal{O}}(-i))) \operatorname{Td}(\overline{p}_2) \right) - 1 \\
&\qquad = \sum_{i=0}^{n} (-1)^i \operatorname{ch}(\Lambda^i \overline{Q}^\vee) (p_2)_* \left(p_1^*(\operatorname{ch}(\overline{\mathcal{O}}(-i))) \operatorname{Td}(\overline{p}_2) \right) - 1 \\
&\qquad = \sum_{i=0}^{n} (-1)^i \operatorname{ch}(\Lambda^i \overline{Q}^\vee) \pi_1^* \pi_* \left(\operatorname{ch}(\overline{\mathcal{O}}(-i)) \operatorname{Td}(\overline{\pi}) \right) - 1 \\
&\qquad = 1 - 1 = 0.
\end{aligned}
$$

In the first equality we have used the differential equation of T_S. In the second equality we have used the definition of the Koszul complex, the equation $\operatorname{ch}(\overline{\mathcal{O}}_{\mathbb{P}^n}) = 1$ and the fact that, by the choice of hermitian structures on $T_{\overline{\Delta}}$ and $T_{\overline{p}_2}$ we have $\operatorname{Td}(\overline{\Delta}) \bullet \Delta^*(\operatorname{Td}(\overline{p}_2)) = 1$. The third equality is the projection formula and the fourth is base change for cohomology. For the last equality we have used equation (3.66).

Both sides of equation (3.90) are closed, and are defined up to boundaries, hence this is an equation in cohomology classes. From the tautological exact sequence we obtain exact sequences

$$
0 \to \Lambda^k Q^\vee \to \Lambda^k V^\vee \to \Lambda^{k-1} Q^\vee \otimes \mathcal{O}(1) \to 0,
$$

that give us equations

$$
\operatorname{ch}(\Lambda^k Q^\vee) = \binom{n+1}{k} - \operatorname{ch}(\Lambda^{k-1} Q^\vee) \operatorname{ch}(\mathcal{O}(1)).
$$

Hence

$$\mathrm{ch}(\Lambda^k Q^\vee) = \sum_{i=0}^{k} (-1)^i \binom{n+1}{k-i} \mathrm{ch}(\mathcal{O}(i)).$$

Since the classes $\mathrm{ch}(\mathcal{O}(i))$ for $i = 0, \dots, n$ form a basis of $\bigoplus_p H^{2p}_{\mathcal{D}}(\mathbb{P}^n, \mathbb{R}(p))$, the same is true for the classes $\mathrm{ch}(\Lambda^i Q^\vee)$, $i = 0, \dots, n$. Therefore, if $\mathbf{1}_1 \in H^1_{\mathcal{D}}(\mathbb{P}^n, \mathbb{R}(1))$ is the class represented by the constant function 1, the classes $\mathbf{1}_1 \bullet \mathrm{ch}(\Lambda^i Q^\vee)$, $i = 0, \dots, n$ form a basis of $\bigoplus_{p=1}^{n+1} H^{2p-1}_{\mathcal{D}}(\mathbb{P}^n, \mathbb{R}(p))$, which implies that equation (3.90) has a unique solution. □

Remark 3.91. Given a theory T of analytic torsion classes for projective spaces, obtained from an arbitrary choice of characteristic numbers, in general, it does not exist an additive genus such that the associated theory of singular Bott-Chern classes is compatible with T. It would be interesting to characterize the collections of characteristic numbers that arise from Theorem 3.88.

By definition, compatible analytic torsion classes for closed immersions and projective spaces satisfy a compatibility condition for the trivial vector bundle and the diagonal embedding. When adding the functoriality and the projection formula, we obtain compatibility relations for arbitrary sections of the trivial projective bundle and arbitrary objects.

Let X be a smooth complex variety, let $\pi\colon \mathbb{P}^n_X \to X$ be the projective space over X and let $s\colon X \to \mathbb{P}^n_X$ be a section. Choose any hermitian structure on T_π. Since we have an isomorphism $T_s \dashrightarrow s^*T_\pi[-1]$, this hermitian structure induces an hermitian structure on s. Denote by $\overline{\pi}$ and $\overline{s}$ the corresponding morphisms in $\overline{\mathbf{Sm}}_{*/\mathbb{C}}$. With this choice of hermitian structures, we have

$$\overline{\pi} \circ \overline{s} = (\pi \circ s, \overline{\mathrm{cone}}(s^*T_{\overline{\pi}}[-1], s^*T_{\overline{\pi}}[-1])) = (\mathrm{id}_X, \overline{0}),$$

because the cone of the identity is meager.

Proposition 3.92. *Let S be a real additive genus. Let T_S denote both, the theory of analytic torsion classes for closed immersions determined by S, and the theory of analytic torsion classes for projective spaces compatible with it. Let $\overline{\mathcal{F}}$ be an object of $\overline{\mathbf{D}}^{\mathrm{b}}(X)$. Put a hermitian structure on $s_*\mathcal{F}$. Then*

$$T_S(\overline{\pi}, \overline{s_*\mathcal{F}}, \overline{\mathcal{F}}) + \overline{\pi}_\flat(T_S(\overline{s}, \overline{\mathcal{F}}, \overline{s_*\mathcal{F}})) = 0. \tag{3.93}$$

Proof. By the anomaly formulas Proposition 3.28 and Proposition 3.54, if equation (3.93) holds for a particular choice of hermitian structure of $s_*\mathcal{F}$ then it holds for any other choice. Therefore we can assume that the hermitian structure of $s_*\mathcal{F}$ is given by $\overline{K}(s) \otimes \pi^*\overline{\mathcal{F}}$, where $\overline{K}(s)$ is the Koszul complex associated to the section s.

By the projection formulas, if equation (3.93) holds for the trivial bundle $\mathcal{O}_X$ then it holds for arbitrary objects of $\overline{\mathbf{D}}^{\mathrm{b}}(X)$.

We now prove that, if equation (3.93) holds for a particular choice of hermitian structure $\overline{\pi}$, then it holds for any other choice. Thus, assume that equation (3.93) is satisfied for $\overline{\pi}$ and $\overline{s}$. Let $\overline{\pi}'$ another choice of hermitian structure on π and let $\overline{s}'$ be the hermitian structure induced on s. On the one hand, we have

$$T_S(\overline{\pi}', \overline{K}(s), \overline{\mathcal{O}}_X) = T_S(\overline{\pi}, \overline{K}(s), \overline{\mathcal{O}}_X) + \pi_* \left(\mathrm{ch}(\overline{K}(s) \bullet \widetilde{\mathrm{Td}}_m(\overline{\pi}', \overline{\pi}) \bullet \mathrm{Td}(\overline{\pi}')) \right). \tag{3.94}$$

On the other hand, we have

$$\begin{aligned}
T_S(\overline{s}', \overline{\mathcal{F}}, \overline{s_*\mathcal{F}}) \bullet \operatorname{Td}(\overline{\pi}') \\
&= \Big(T_S(\overline{s}, \overline{\mathcal{F}}, \overline{s_*\mathcal{F}}) + s_*(\widetilde{\operatorname{Td}}_m(\overline{s}', \overline{s}) \operatorname{Td}(\overline{s}'))\Big) \\
&\qquad \bullet \Big(\operatorname{Td}(\overline{\pi}) - \mathrm{d}_{\mathcal{D}}(\widetilde{\operatorname{Td}}_m(\overline{\pi}', \overline{\pi}) \bullet \operatorname{Td}(\overline{\pi}'))\Big) \\
&= T_S(\overline{s}, \overline{\mathcal{F}}, \overline{s_*\mathcal{F}}) \bullet \operatorname{Td}(\overline{\pi}) \\
&\quad + s_*(\widetilde{\operatorname{Td}}_m(\overline{s}', \overline{s}) \operatorname{Td}(\overline{s}')) \bullet \operatorname{Td}(\overline{\pi}') \\
&\quad - T_S(\overline{s}, \overline{\mathcal{F}}, \overline{s_*\mathcal{F}}) \bullet \mathrm{d}_{\mathcal{D}}(\widetilde{\operatorname{Td}}_m(\overline{\pi}', \overline{\pi}) \bullet \operatorname{Td}(\overline{\pi}')) \qquad (3.95)
\end{aligned}$$

In the group $\bigoplus_p \widetilde{D}_D^{2p-1}(\mathbb{P}^n_X, N_s, p)$ we have

$$\begin{aligned}
T_S(\overline{s}, \overline{\mathcal{F}}, \overline{s_*\mathcal{F}}) \bullet \mathrm{d}_{\mathcal{D}}(\widetilde{\operatorname{Td}}_m(\overline{\pi}', \overline{\pi}) \bullet \operatorname{Td}(\overline{\pi}')) \\
&= \big(\operatorname{ch}(\overline{K}(s)) - s_*(\operatorname{Td}(\overline{s}))\big) \bullet \Big(\widetilde{\operatorname{Td}}_m(\overline{\pi}', \overline{\pi}) \bullet \operatorname{Td}(\overline{\pi}')\Big). \qquad (3.96)
\end{aligned}$$

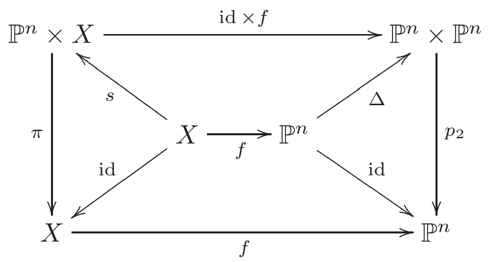
We now observe that, by the definition of the hermitian structure of $\overline{s}$ and $\overline{s}'$ we have the relation

$$\operatorname{Td}(\overline{s}) \bullet s^*\widetilde{\operatorname{Td}}_m(\overline{\pi}', \overline{\pi}) = -\widetilde{\operatorname{Td}}_m(\overline{s}', \overline{s}) \bullet \operatorname{Td}(\overline{s}'). \qquad (3.97)$$

By combining equations (3.93), (3.94), (3.96) and (3.97) we obtain

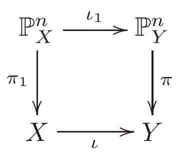
$$T_S(\overline{\pi}', \overline{s_*\mathcal{F}}, \overline{\mathcal{F}}) = -\pi_*\left(T_S(\overline{s}', \overline{\mathcal{F}}, \overline{s_*\mathcal{F}}) \bullet \operatorname{Td}(\overline{\pi}')\right). \qquad (3.98)$$

We now prove equation (3.93) for a particular choice of hermitian structures. Let $f\colon X \to \mathbb{P}^n$ denote the composition of s with the projection $\mathbb{P}^n_X \to \mathbb{P}^n$. Then we have a commutative diagram with cartesian squares



Let $\overline{\Delta}$ and $\overline{p}_2$ be as in Definition 3.86. On $\overline{\pi}$ and $\overline{s}$ we put the hermitian structures induced by $\overline{\Delta}$. Since the Koszul complex $\overline{K}(s) = (\mathrm{id}_{\mathbb{P}^n} \times f)^*\overline{K}(\Delta)$, by Proposition 3.9 and functoriality, equation (3.93) in this case follows from equation (3.87). □

We now study another compatibility between analytic torsion classes for closed immersions and projective spaces. Let $\iota\colon X \to Y$ be a closed immersion of smooth complex varieties. Consider the cartesian square



Choose hermitian structures on π and ι and put on π_1 and ι_i the induced hermitian structures.

Proposition 3.99. *Let S be a real additive genus. Let T_S denote both, the theory of analytic torsion classes for closed immersions determined by S, and the theory of analytic torsion classes for projective spaces compatible with it. Let $\overline{\mathcal{F}}$ be and object of $\overline{\mathbf{D}}^{\mathrm{b}}(\mathbb{P}^n_X)$. Put hermitian structures on $(\pi_1)_*\mathcal{F}$, $(\iota_1)_*\mathcal{F}$ and $(\pi\circ\iota_1)_*\mathcal{F}$. Then*

$$T_S(\overline{\pi})+\overline{\pi}_\flat(T_S(\overline{\iota_1}))=T_S(\overline{\iota})+\overline{\iota}_\flat(T_S(\overline{\pi_1})). \tag{3.100}$$

Proof. By the anomaly formulas, if equation (3.100) holds for a particular choice of metrics on $(\pi_1)_*\mathcal{F}$, $(\iota_1)_*\mathcal{F}$ and $(\pi\circ\iota_1)_*\mathcal{F}$, then it holds for any choice. Because the sheaves $\mathcal{G}(k)$, with $\mathcal{G}$ a coherent sheaf on X, constitute a generating class of $\mathbf{D}^{\mathrm{b}}(\mathbb{P}^n_X)$ and by propositions 3.31 and 3.56, we reduce to the case $\mathcal{F}$ is of the form $\mathcal{G}(k)$. We choose arbitrary hermitian structures on $\mathcal{G}$ and $\iota_*\mathcal{G}$. Furthermore, we assume $\mathcal{O}(k)$, $(\pi_1)_*\mathcal{O}(k)$ and $\pi_*\mathcal{O}(k)$ endowed with the hermitian structures of Notation 3.45. From these choices and the projection formula, the objects $(\pi_1)_*\mathcal{F}$, $(\iota_1)_*\mathcal{F}$ and $(\pi\circ\iota_1)_*\mathcal{F}$ automatically inherit hermitian structures. Indeed, it is enough to observe the natural isomorphisms

$$(\pi_1)_*\mathcal{F}\cong\mathcal{G}\otimes(\pi_1)_*\mathcal{O}(k) \tag{3.101}$$

$$(\iota_1)_*(\pi_1^*\mathcal{G}\otimes\iota_1^*\mathcal{O}(k))\cong\pi^*(\iota_*\mathcal{G})\otimes\mathcal{O}(k) \tag{3.102}$$

$$(\pi\circ\iota_1)_*\mathcal{F}\cong\pi_*(\pi^*\iota_*\mathcal{G}\otimes\mathcal{O}(k))\cong\iota_*\mathcal{G}\otimes\pi_*\mathcal{O}(k). \tag{3.103}$$

We now work out the left hand side of equation (3.100). By the projection formula for the theory T_S for projective spaces, and taking equations (3.101)–(3.103) into account, we easily find

$$T_S(\overline{\pi})=t_{n,k}\bullet\mathrm{ch}(\overline{\iota_*\mathcal{G}}). \tag{3.104}$$

By the functoriality of the theory T_S for closed immersions and the projection formula, we also have

$$T_S(\overline{\iota}_1)=\pi^*T_S(\iota,\overline{\mathcal{G}},\overline{\iota_*\mathcal{G}})\bullet\mathrm{ch}(\overline{\mathcal{O}(k)}).$$

Hence we infer

$$\overline{\pi}_\flat(T_S(\overline{\iota}_1))=T_S(\iota,\overline{\mathcal{G}},\overline{\iota_*\mathcal{G}})\bullet\pi_*(\mathrm{ch}(\overline{\mathcal{O}(k)})\bullet\mathrm{Td}(\overline{\pi})). \tag{3.105}$$

Now for the right hand side of (3.100). By the projection formula for the theory T_S for closed immersions, we have

$$T_S(\overline{\iota})=T_S(\iota,\overline{\mathcal{G}},\overline{\iota\mathcal{G}})\bullet\mathrm{ch}(\overline{\pi_*\mathcal{O}(k)}). \tag{3.106}$$

Similarly, we obtain

$$T_S(\overline{\pi}_1)=t_{n,k}\bullet\mathrm{ch}(\overline{\mathcal{G}}),$$

and hence

$$\overline{\iota}_\flat(T_S(\overline{\pi}_1))=t_{n,k}\bullet\iota_*(\mathrm{ch}(\overline{\mathcal{G}})\bullet\mathrm{Td}(\overline{\iota})). \tag{3.107}$$

Finally, the difference of the left hand side and the right hand side of (3.100) equals (3.104)−(3.107)+(3.105)−(3.106), that can be equivalently written as

$$t_{n,k}\bullet\mathrm{d}_{\mathcal{D}}\,T_S(\iota,\overline{\mathcal{G}},\overline{\iota_*\mathcal{G}})-T_S(\iota,\overline{\mathcal{G}},\overline{\iota_*\mathcal{G}})\bullet\mathrm{d}_{\mathcal{D}}\,t_{n,k}=\\ -\mathrm{d}_{\mathcal{D}}(t_{n,k}\bullet T_S(\iota,\overline{\mathcal{G}},\overline{\iota_*\mathcal{G}}))=0$$

in the group $\oplus_p\widetilde{\mathcal{D}}_D^{2p-1}(Y,N_\iota,p)$. □

3.6 Generalized analytic torsion classes

In this section we will extend the definition of analytic torsion classes to arbitrary morphisms of smooth complex varieties. Our construction is based on the construction of analytic torsion classes by Zha in [51].

Definition 3.108. A *theory of generalized analytic torsion classes* is an assignment that, to each morphism $\overline{f}\colon X \to Y$ in $\overline{\mathbf{Sm}}_{*/\mathbb{C}}$ and each relative metrized complex

$$\overline{\xi} = (\overline{f}, \overline{\mathcal{F}}, \overline{f_*\mathcal{F}}),$$

assigns a class of currents

$$T(\overline{\xi}) \in \bigoplus_{p=1}^{n+1} \widetilde{\mathcal{D}}_D^{2p-1}(Y, N_f, p)$$

satisfying the following properties:

(i) (Differential equation) The following equality holds

$$\mathrm{d}_{\mathcal{D}}\,\eta = \mathrm{ch}(\overline{f_*\mathcal{F}}) - \overline{f}_\flat[\mathrm{ch}(\overline{\mathcal{F}})] \tag{3.109}$$

for any current $\eta \in T(\overline{\xi})$.

(ii) (Functoriality) For every morphism $g\colon Y' \to Y$ that is transverse to f, the equation

$$g^*T(\overline{\xi}) = T(g^*\overline{\xi})$$

holds.

(iii) (Additivity and normalization) If $\overline{\xi}_1$ and $\overline{\xi}_2$ are relative metrized complexes on X, then

$$T(\overline{\xi}_1 \oplus \overline{\xi}_2) = T(\overline{\xi}_1) + T(\overline{\xi}_2).$$

(iv) (Projection formula) If $\overline{\xi}$ is a relative metrized complex, $\overline{\mathcal{G}}$ an object of $\overline{\mathbf{D}}^{\mathrm{b}}(Y)$ then

$$T(\overline{\xi} \otimes \overline{\mathcal{G}}) = T(\overline{\xi}) \bullet \mathrm{ch}(\overline{\mathcal{G}}).$$

(v) (Transitivity) If $\overline{f}\colon X \to Y$ and $\overline{g}\colon Y \to Z$ are morphisms in $\overline{\mathbf{Sm}}_{*/\mathbb{C}}$ and $(\overline{f}, \overline{\mathcal{F}}, \overline{f_*\mathcal{F}})$ and $(\overline{g}, \overline{f_*\mathcal{F}}, \overline{(g\circ f)_*\mathcal{F}})$ are relative metrized complexes, then

$$T(\overline{g} \circ \overline{f}) = T(\overline{g}) + \overline{g}_\flat(T(\overline{f})). \tag{3.110}$$

If T is a theory of generalized analytic torsion classes, then by properties (i)–(iii) it automatically satisfies several anomaly formulas analogue to those in propositions 3.28 and 3.54, as well as compatibility formulas with respect to distinguished triangles as in propositions 3.31 and 3.56. The proofs are the same in the general case.

Proposition 3.111. *Let T be a theory of generalized analytic torsion classes. Let*

$$\overline{\xi} = (\overline{f}, \overline{\mathcal{F}}, \overline{f_*\mathcal{F}})$$

be a relative metrized complex.

(i) If $\overline{\mathcal{F}}'$ *is another choice of metric on* $\mathcal{F}$ *and* $\overline{\xi}_1$ *is the corresponding relative metrized complex, then*

$$T(\overline{\xi}_1) = T(\overline{\xi}) + \overline{f}_\flat[\widetilde{\mathrm{ch}}(\overline{\mathcal{F}}', \overline{\mathcal{F}})].$$

(ii) If $\overline{f}'$ *is another choice of hermitian structure on* f *and* $\overline{\xi}_2$ *is the corresponding relative metrized complex, then*

$$T(\overline{\xi}_2) = T(\overline{\xi}) + \overline{f}'_\flat[\mathrm{ch}(\overline{\mathcal{F}}) \bullet \widetilde{\mathrm{Td}}_m(\overline{f}', \overline{f})]. \tag{3.112}$$

(iii) If $\overline{f_*\mathcal{F}}'$ *is a different choice of metric on* $f_*\mathcal{F}$*, and* $\overline{\xi}_3$ *is the corresponding relative metrized complex, then*

$$T(\overline{\xi}_3) = T(\overline{\xi}) - \widetilde{\mathrm{ch}}(\overline{f_*\mathcal{F}}', \overline{f_*\mathcal{F}}).$$

□

Proposition 3.113. *Let* T *be a theory of generalized analytic torsion classes. Let* $\overline{f}\colon X \to Y$ *be a morphism in* $\overline{\mathbf{Sm}}_{*/\mathbb{C}}$*. Consider distinguished triangles in* $\overline{\mathbf{D}}^{\mathrm{b}}(X)$ *and* $\overline{\mathbf{D}}^{\mathrm{b}}(Y)$ *respectively:*

$$\begin{aligned} (\overline{\tau}) : &\quad \overline{\mathcal{F}}_2 \to \overline{\mathcal{F}}_1 \to \overline{\mathcal{F}}_0 \to \overline{\mathcal{F}}_2[1], \\ (\overline{f_*\tau}) : &\quad \overline{f_*\mathcal{F}_2} \to \overline{f_*\mathcal{F}_1} \to \overline{f_*\mathcal{F}_0} \to \overline{f_*\mathcal{F}_2}[1], \end{aligned}$$

and define relative metrized complexes

$$\begin{aligned} \overline{\xi}_0 &= (\overline{f}, \overline{\mathcal{F}}_0, \overline{f_*\mathcal{F}_0}), \\ \overline{\xi}_1 &= (\overline{f}, \overline{\mathcal{F}}_1, \overline{f_*\mathcal{F}_1}), \\ \overline{\xi}_2 &= (\overline{f}, \overline{\mathcal{F}}_2, \overline{f_*\mathcal{F}_2}). \end{aligned}$$

Then, the following relation holds:

$$\sum_j (-1)^j T(\overline{\xi}_j) = \widetilde{\mathrm{ch}}(\overline{\pi_*\tau}) - \overline{f}_\flat(\widetilde{\mathrm{ch}}(\overline{\tau})).$$

□

The main result of this section is the following classification theorem.

Theorem 3.114. *Let* S *be a real additive genus. Then there exists a unique theory of analytic torsion classes that agrees with* T_S *when restricted to the class of closed immersions. We will denote such theory by* T_S*. In particular, there is a unique theory of generalized analytic torsion classes that agrees with* T^h *when restricted to the class of closed immersions. This theory will be called homogeneous. Moreover, if* T *is a theory of generalized analytic torsion classes, then there exists a real additive genus* S *such that* $T = T_S$*.*

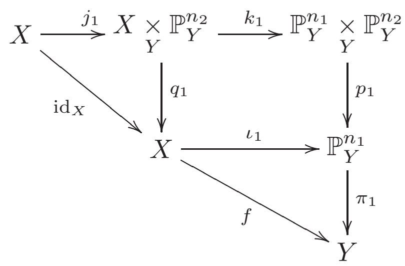
Proof. We first prove the uniqueness. Let T be a theory of analytic torsion classes that agrees with T_S for the class of closed immersions. Since the restriction of T to projective spaces, by the transitivity axiom, is compatible with T_S, by Theorem 3.88, it also agrees with T_S. Finally, the transitivity axiom implies that T is determined by its values for closed immersions and projective spaces.

We now prove the existence. For the moment, let T_S be the theory of analytic torsion classes for closed immersions and projective spaces determined by S. Let $\overline{f}\colon X \to Y$ be a morphism in $\overline{\mathbf{Sm}}_{*/\mathbb{C}}$, and let $\overline{\xi} = (\overline{f}, \overline{\mathcal{F}}, \overline{f_*\mathcal{F}})$ be a relative metrized complex. Since f is assumed to be projective, there is a factorization $f = \pi \circ \iota$, where $\iota\colon X \to \mathbb{P}^n_Y$ is a closed immersion and $\pi\colon \mathbb{P}^n_Y \to Y$ is the projection. Choose auxiliary hermitian structures on ι, π and $\iota_*\mathcal{F}$. Then we define

$$T_S(\overline{\xi}) = T_S(\overline{\pi}) + \overline{\pi}_\flat(T_S(\overline{\iota})) + \overline{f}_\flat\left[\operatorname{ch}(\overline{\mathcal{F}}) \bullet \widetilde{\operatorname{Td}}_m(\overline{f}, \overline{\pi} \circ \overline{\iota})\right] \tag{3.115}$$

To simplify the notations, in the sequel we will also refer to it simply by $T(\overline{\xi})$. The anomaly formulas easily imply that this definition does not depend on the choice of hermitian structures on ι, π and $\iota_*\mathcal{F}$. We next show that this definition is independent of the factorization of f. Let $f = \pi_1 \circ \iota_1 = \pi_2 \circ \iota_2$ be two different factorizations, being $\mathbb{P}^{n_i}$, the target of ι_i, $i = 1, 2$. Since equation (3.115) is independent of the choice of auxiliary hermitian structures, by Lemma 2.115, we may assume that $\overline{f} = \overline{\pi}_1 \circ \overline{\iota}_1 = \overline{\pi}_2 \circ \overline{\iota}_2$.

We consider the commutative diagram with cartesian square



where $j_1(x) = (x, \iota_2(x))$, p_1 is the first projection and q_1 and k_1 are defined by the cartesian square. The hermitian structure of $\overline{\pi}_2$ induces a hermitian structure on p_1 that, in turn, induces a hermitian structure on q_1. The hermitian structure of ι_1 induces a hermitian structure on k_1 and the hermitian structure of ι_2 induces one on j_1. We will denote the corresponding morphisms of $\overline{\mathbf{Sm}}_{*/\mathbb{C}}$ by $\overline{p}_1$, $\overline{q}_1$, $\overline{k}_1$ and $\overline{j}_1$. We consider also the analogous diagram obtained swapping 1 and 2. Finally, we write $\overline{p} = \overline{\pi}_1 \circ \overline{p}_1 = \overline{\pi}_2 \circ \overline{p}_2$ and $\overline{j} = \overline{k}_1 \circ \overline{j}_1 = \overline{k}_2 \circ \overline{j}_2$. Then we have

$$\begin{aligned}
T(\overline{\pi}_1) + (\overline{\pi}_1)_\flat(T(\overline{\iota}_1)) &= T(\overline{\pi}_1) + (\overline{\pi}_1)_\flat(T(\overline{\iota}_1)) + \overline{f}_\flat\left(T(\overline{q}_1) + (\overline{q}_1)_\flat(T(\overline{j}_1))\right) \\
&= T(\overline{\pi}_1) + (\overline{\pi}_1)_\flat\big(T(\overline{\iota}_1) + (\overline{\iota}_1)_\flat(T(\overline{q}_1))\big) + \overline{p}_\flat(\overline{k}_1)_\flat(T(\overline{j}_1)) \\
&= T(\overline{\pi}_1) + (\overline{\pi}_1)_\flat\big(T(\overline{p}_1) + (\overline{p}_1)_\flat(T(\overline{k}_1))\big) + \overline{p}_\flat(\overline{k}_1)_\flat(T(\overline{j}_1)) \\
&= T(\overline{p}) + \overline{p}_\flat(T(\overline{j})).
\end{aligned}$$

Analogously, we obtain

$$T(\overline{\pi}_2) + (\overline{\pi}_2)_\flat(T(\overline{\iota}_2)) = T(\overline{p}) + \overline{p}_\flat(T(\overline{j})).$$

Hence T_S is well defined for all relative metrized complexes. It remains to prove that it satisfies the properties of a theory of analytic torsion classes. The properties (i) to (iv) are clear. We thus focus on property (v).

Let $\overline{f}\colon X \to Y$ and $\overline{g}\colon Y \to Z$ be morphisms in $\overline{\mathbf{Sm}}_{*/\mathbb{C}}$. We choose factorizations of $\overline{g} \circ \overline{f}$ and $\overline{g}$:

$$\begin{array}{ccc} X & \xhookrightarrow{\;\overline{i}\;} & \mathbb{P}^m_Z \\ & \overset{\overline{g}\circ\overline{f}}{\searrow} & \downarrow{\scriptstyle \overline{p}} \\ & & Z \end{array} \qquad\qquad \begin{array}{ccc} Y & \xhookrightarrow{\;\overline{\ell}\;} & \mathbb{P}^n_Z \\ & \overset{\overline{g}}{\searrow} & \downarrow{\scriptstyle \overline{r}} \\ & & Z, \end{array}$$

where the hermitian structures on $\overline{p}$ and $\overline{r}$ come from fixed hermitan structures on the tangent bundles $T_{\mathbb{P}^m_{\mathbb{C}}}$ and $T_{\mathbb{P}^n_{\mathbb{C}}}$, and the hermitian structures $\overline{i}$ and $\overline{\ell}$ are obtained by using Lemma 2.115. We define $\varphi : X \to \mathbb{P}^m_{\mathbb{C}}$ to be the arrow obtained from i by composing with the projection to $\mathbb{P}^m_{\mathbb{C}}$. Then we see that the morphism $j := (\varphi, f)\colon X \to \mathbb{P}^m_Y$ is a closed immersion. Indeed, it is enough to realize that the composition

$$X \xrightarrow{(\varphi,f)} \mathbb{P}^m_Y \xrightarrow{(\mathrm{id},g)} \mathbb{P}^m_Z$$

agrees with the closed immersion i and that $G := (\mathrm{id}, g)$ is separated (since proper). We can thus decompose $\overline{f}$ as

$$\begin{array}{ccc} X & \xhookrightarrow{\;\overline{j}\;} & \mathbb{P}^m_Y \\ & \overset{\overline{f}}{\searrow} & \downarrow{\scriptstyle \overline{q}} \\ & & Y. \end{array}$$

Again, in this factorization the hermitian structure $\overline{q}$ comes from the previously fixed hermitian structure on $T_{\mathbb{P}^m_{\mathbb{C}}}$ and the hermitian structure $\overline{j}$ is obtained by using Lemma 2.115. Because $\overline{g} \circ \overline{f} = \overline{p} \circ \overline{i}$ and by the very construction of T for arbitrary projective morphisms (3.115), we have

$$T(\overline{g} \circ \overline{f}) = T(\overline{p}) + \overline{p}_\flat(T(\overline{i})). \tag{3.116}$$

We proceed to work on $T(\overline{i})$. For this we write the commutative diagram

$$\begin{array}{ccccccc} X & \xhookrightarrow{\;j\;} & \mathbb{P}^m_Y & \xhookrightarrow{\;k\;} & \mathbb{P}^m_{\mathbb{P}^n_Z} & = & \mathbb{P}^n_{\mathbb{P}^m_Z} \\ & \overset{i}{\searrow} & \downarrow{\scriptstyle G} & & & & \downarrow{\scriptstyle \pi} \\ & & \mathbb{P}^m_Z & & \xrightarrow{\quad\mathrm{id}\quad} & & \mathbb{P}^m_Z. \end{array}$$

Here we recall that $G = (\mathrm{id}, g)$ and we have also put $k = (\mathrm{id}, \ell)$. Below, G, k and π will be endowed with the obvious hermitian structures. With these choices, we observe that $\overline{i} = \overline{G} \circ \overline{j}$ and $\overline{G} = \overline{\pi} \circ \overline{k}$. Taking this into account, the construction of T and the fact that $T = T_S$ is transitive for compositions of closed immersions, we find

$$\begin{aligned} T(\overline{i}) =& T(\overline{\pi} \circ \overline{k} \circ \overline{j}) \\ =& T(\overline{\pi}) + \overline{\pi}_\flat(T(\overline{k})) + \overline{G}_\flat(T(\overline{j})) \\ =& T(\overline{G}) + \overline{G}_\flat(T(\overline{j})). \end{aligned} \tag{3.117}$$

Therefore, from equations (3.116), (3.117) and applying the identity $\overline{p}_\flat \overline{G}_\flat = \overline{g}_\flat \overline{q}_\flat$ we derive

$$T(\overline{g} \circ \overline{f}) = T(\overline{p}) + \overline{p}_\flat(T(\overline{G})) + \overline{g}_\flat \overline{q}_\flat(T(\overline{\jmath})). \tag{3.118}$$

We claim that

$$T(\overline{p}) + \overline{p}_\flat(T(\overline{G})) = T(\overline{g}) + \overline{g}_\flat(T(\overline{q})). \tag{3.119}$$

Assuming this for a while, we combine (3.118) and (3.119) into

$$T(\overline{g} \circ \overline{f}) = T(\overline{g}) + \overline{g}_\flat(T(\overline{q}) + \overline{q}_\flat(T(\overline{\jmath}))) = T(\overline{g}) + \overline{g}_\flat(T(\overline{f})). \tag{3.120}$$

Hence we are lead to prove (3.119). For this we construct the commutative diagram with cartesian squares

$$\begin{array}{ccccc}
\mathbb{P}^m_Y & \xhookrightarrow{\tilde{\ell}} & \mathbb{P}^m_Z \times_Z \mathbb{P}^n_Z & \xrightarrow{\tilde{r}} & \mathbb{P}^m_Z \\
\downarrow{\scriptstyle \overline{q}} & & \downarrow{\scriptstyle \tilde{p}} & & \downarrow{\scriptstyle \overline{p}} \\
Y & \xhookrightarrow{\overline{\ell}} & \mathbb{P}^n_Z & \xrightarrow{\overline{r}} & Z.
\end{array}$$

Observe that $\overline{G} = \tilde{r} \circ \tilde{\ell}$. Recall now Proposition 3.83 and Proposition 3.99. We then have the chain of equalities

$$\begin{aligned}
T(\overline{p}) + \overline{p}_\flat(T(\overline{G})) =& T(\overline{p}) + \overline{p}_\flat(T(\tilde{r}) + \tilde{r}_\flat(T(\tilde{\ell})))\\
=& T(\overline{r}) + \overline{r}_\flat(T(\tilde{p}) + \tilde{p}_\flat(T(\tilde{\ell}))\\
=& T(\overline{r}) + \overline{r}_\flat(T(\overline{\ell}) + \overline{\ell}_\flat(T(\overline{q})))\\
=& T(\overline{g}) + \overline{g}_\flat(T(\overline{q})).
\end{aligned}$$

This proves the claim.

The last assertion of the statement of the theorem is obvious from the uniqueness part. ☐

Theorem 3.121. *(i) Let T be a theory of generalized analytic torsion classes. Then there is a unique real additive genus S such that, for any relative metrized complex $\overline{\xi} := (\overline{f}, \overline{\mathcal{F}}, \overline{f_*\mathcal{F}})$, we have*

$$T(\overline{\xi}) - T^h(\overline{\xi}) = -f_*[\mathrm{ch}(\mathcal{F}) \bullet \mathrm{Td}(T_f) \bullet S(T_f) \bullet \mathbf{1}_1]. \tag{3.122}$$

(ii) Conversely, any real additive genus S defines, by means of equation (3.122), *a unique theory of generalized analytic torsion classes T_S.*

Proof. We prove the first item, the second being immediate. Let S be the real additive genus corresponding to T, provided by Theorem 3.114. Then (3.122) holds for embedded metrized complexes. Because T and T^h are both transitive, it suffices to show that (3.122) holds whenever $f\colon \mathbb{P}^n_X \to X$ is a trivial projective bundle. Observe T and T^h satisfy the same anomaly formulas. Then, since the sheaves $\mathcal{G}(k)$, $k = -n, \dots, 0$ form a generating class for $\mathbf{D}^{\mathrm{b}}(\mathbb{P}^n_X)$, and by the projection formula for T and T^h, we easily reduce to the case $\overline{\xi} = \overline{\xi}(k)$. Let $t_{n,k}$, $t^h_{n,k}$ be the characteristic numbers of T, T^h respectively. We have to establish the equality

$$t_{n,-i} - t^h_{n,-i} = -\pi_*(\mathrm{ch}(\overline{\mathcal{O}}(-i))\,\mathrm{Td}(\overline{\pi})S(T_{\overline{\pi}})), \quad i = -n, \dots, 0. \tag{3.123}$$

This is an equation of real numbers. By functoriality, this equation is equivalent to the analogous equation in $\oplus_p H^{2p-1}_{\mathcal{D}}(\mathbb{P}^n_{\mathbb{C}}, \mathbb{R}(p))$, for the second projection $p_2 : \mathbb{P}^n_{\mathbb{C}} \times \mathbb{P}^n_{\mathbb{C}} \to \mathbb{P}^n_{\mathbb{C}}$ instead of π. Because the classes $\mathrm{ch}(\Lambda^i Q^\vee)$ constitute a basis for $\oplus_p H^{2p-1}_{\mathcal{D}}(\mathbb{P}^n_{\mathbb{C}}, \mathbb{R}(p))$, (3.123) is equivalent to the equation in cohomology

$$\sum_i (-1)^i (t_{n,-i} - t^h_{n,-i}) \,\mathrm{ch}(\Lambda^i \overline{Q}^\vee) = \\ - p_{2*}\Big(\sum_i (-1)^i \,\mathrm{ch}(p_1^* \overline{\mathcal{O}}(-i) \otimes \Lambda^i p_2^* \overline{Q}^\vee) \,\mathrm{Td}(\overline{p}_2) S(T_{\overline{p}_2}) \bullet \mathbf{1}_1\Big). \quad (3.124)$$

Recalling the exact sequence (3.85), minus the right hand side of (3.124) becomes

$$p_{2*}(\mathrm{ch}(\overline{\Delta_* \mathcal{O}_{\mathbb{P}^n}}) \,\mathrm{Td}(\overline{p}_2) S(T_{\overline{p}_2}) \bullet \mathbf{1}_1) = \\ p_{2*}(\Delta_*(\mathrm{ch}(\overline{\mathcal{O}}_{\mathbb{P}^n}) \,\mathrm{Td}(\overline{\Delta})) \,\mathrm{Td}(\overline{p}_2) S(T_{\overline{p}_2}) \bullet \mathbf{1}_1) = S(T_{\mathbb{P}^n}) \bullet \mathbf{1}_1.$$

On the other hand, using the compatibility condition (Definition 3.86), the left hand side of (3.124) can be equivalently written as

$$T(\overline{p}_2, \overline{\Delta_* \mathcal{O}_{\mathbb{P}^n}}, \overline{\mathcal{O}_{\mathbb{P}^n}}) - T^h(\overline{p}_2, \overline{\Delta_* \mathcal{O}_{\mathbb{P}^n}}, \overline{\mathcal{O}_{\mathbb{P}^n}}) = \\ - p_{2\flat}(T(\Delta, \overline{\mathcal{O}}_{\mathbb{P}^n}, \overline{\Delta_* \mathcal{O}_{\mathbb{P}^n}}) - T^h(\Delta, \overline{\mathcal{O}}_{\mathbb{P}^n}, \overline{\Delta_* \mathcal{O}_{\mathbb{P}^n}})). \quad (3.125)$$

Since the statement is known for closed immersions, the right hand side of (3.125) can be written

$$p_{2*}(\Delta_*(\mathrm{ch}(\overline{\mathcal{O}_{\mathbb{P}^n}}) \,\mathrm{Td}(T_{\overline{\Delta}}) S(T_{\overline{\Delta}}) \bullet \mathbf{1}_1) \,\mathrm{Td}(\overline{p}_2)) = -S(T_{\mathbb{P}^n}) \bullet \mathbf{1}_1.$$

To derive this equality we used that the genus S is additive, so we have the relation in Deligne cohomology

$$S(T_{\overline{\Delta}}) = S(T_{\mathbb{P}^n}) - \Delta^* S(T_{\mathbb{P}^n \times \mathbb{P}^n}) = \\ S(T_{\mathbb{P}^n}) - \Delta^* p_1^* S(T_{\mathbb{P}^n}) - \Delta^* p_2^* S(T_{\mathbb{P}^n}) = -S(T_{\mathbb{P}^n}).$$

This concludes the proof. □

3.7 Higher analytic torsion forms of Bismut and Köhler

We now explain the relationship between the theory of analytic torsion forms of Bismut-Köhler [13] and the theory of generalized analytic torsion classes developed so far.

Let $\pi \colon X \to Y$ be a smooth projective morphism (a projective submersion) of smooth complex varieties. Let ω be a closed $(1,1)$ form on X that induces a Kähler metric on the fibers of π. Then (π, ω) is called a Kähler fibration. The form ω defines a hermitian structure on T_π, and we will abusively write $\overline{\pi} = (\pi, \omega)$ for the corresponding morphism in $\overline{\mathbf{Sm}}_{*/\mathbb{C}}$.

Let $\overline{F}$ be a hermitian vector bundle on X such that for every $i \geq 0$, $R^i\pi_* F$ is locally free. We consider on $R^i\pi_* F$ the L^2 metric obtained using Hodge theory on the fibers of π. The object of the derived category $\pi_* F$ together with the hermitian structure induced by the L^2 metric (Definition 2.81) will be denoted by $\overline{\pi_* F}_{L^2}$. Then $\overline{\xi} = (\overline{\pi}, \overline{F}, \overline{\pi_* F}_{L^2})$ is an example of relative metrized complex. The relative metrized complexes that arise in this way will be said to be *Kähler*.

In the paper [13], Bismut and Köhler associate to every Kähler relative metrized complex $\overline{\xi}$ a differential form, that we temporarily denote by $\tau(\overline{\xi})$. Since in [13] the authors use real valued characteristic classes, while we use characteristic classes in the Deligne complex, we have to change the normalization of this form. To this end, if $\tau(\overline{\xi})^{(p-1,p-1)}$ is the component of degree $(p-1,p-1)$ of $\tau(\overline{\xi})$, then we put

$$T^{BK}(\overline{\xi})^{(2p-1,p)} = \frac{1}{2}(2\pi i)^{p-1}[\tau(\overline{\xi})^{(p-1,p-1)}] \in \widetilde{\mathcal{D}}_D^{2p-1}(Y, \emptyset, p).$$

We recall that $[\cdot]$ converts differential forms into currents according with the conventions in Section 2.3 (compare with equation (3.32)). We define

$$T^{BK}(\overline{\xi}) = \sum_{p \geq 1} T^{BK}(\overline{\xi})^{(2p-1,p)}.$$

The first main result of [13] is that this class satisfies the differential equation

$$d_{\mathcal{D}} T^{BK}(\overline{\xi}) = \operatorname{ch}(\overline{\pi_* F}_{L^2}) - \overline{\pi}_\flat[\operatorname{ch}(\overline{F})].$$

Thus, $T^{BK}(\overline{\xi})$ is an example of analytic torsion class.

Let now ω' be another closed $(1,1)$ form on X that induces a Kähler metric on the fibers of π. We denote $\overline{\pi}' = (\pi, \omega')$. Let $\overline{F}'$ be the vector bundle F with another choice of metric and define $\overline{\pi_* F}'_{L^2}$ to be the object $\pi_* F$ with the L^2 metric induced by ω' and $\overline{F}'$. We write $\overline{\xi}'$ for the Kähler relative metrized complex $(\overline{\pi}', \overline{F}', \overline{\pi_* F}'_{L^2})$.

The second main result of [13] is the following anomaly formula.

Theorem 3.126 ([13] Theorem 3.10)**.** *The following formula holds in the group* $\bigoplus_p \widetilde{\mathcal{D}}_D^{2p-1}(Y, \emptyset, p)$*:*

$$T^{BK}(\overline{\xi}') - T^{BK}(\overline{\xi}) = \widetilde{\operatorname{ch}}(\overline{\pi_* F}_{L^2}, \overline{\pi_* F}'_{L^2}) + \overline{\pi}'_\flat \left[\operatorname{ch}(\overline{F}) \bullet \widetilde{\operatorname{Td}}_m(\overline{\pi}', \overline{\pi}) - \widetilde{\operatorname{ch}}(\overline{F}, \overline{F}') \right].$$

In the book [3], Bismut studies the compatibility of higher analytic torsion forms with complex immersions. Before stating his result we have to recall the definition of the R-genus of Gillet and Soulé [29]. It is the additive genus attached to the power series

$$R(x) = \sum_{\substack{m \text{ odd} \\ m \geq 1}} \left(2\zeta'(-m) + \left(1 + \frac{1}{2} + \cdots + \frac{1}{m}\right) \zeta(-m) \right) \frac{x^m}{m!}. \qquad (3.127)$$

Let $T_{-R/2}$ be the theory of analytic torsion classes for closed immersions associated to $\frac{-1}{2}R$.

Remark 3.128. The fact that we obtain the additive genus $-R/2$ instead of R is due to two facts. The signs comes from the minus sign in equation (3.23), while the factor 1/2 comes from the difference of the normalization of Green forms used in this paper and the one used in [27]. Note however that the arithmetic intersection numbers computed using both normalizations agree, because the definition of arithmetic degree in [27, §3.4.3] has a factor 1/2 while the definition of arithmetic degree in [17, (6.24)] does not.

Consider a commutative diagram of smooth complex varieties

$$\begin{array}{ccc} X & \xrightarrow{\iota} & Y \\ & \searrow^{f} & \downarrow{\scriptstyle g} \\ & & Z \end{array}$$

where f and g are projective submersions and ι is a closed immersion. Let $\overline{F}$ be a hermitian vector bundle on X such that the sheaves $R^i f_* F$ are locally free and let

$$0 \to \overline{E}_n \to \cdots \to \overline{E}_0 \to \iota_* F \to 0$$

be a resolution of $\iota_* F$ by hermitian vector bundles. We assume that for all i, j, $R^i g_* E_j$ is locally free. We will denote by E the complex $E_n \to \cdots \to E_0$. Let ω^X and ω^Y be closed $(1,1)$ forms that define a structure of Kähler fibration on f and g respectively. As before we write $\overline{f} = (f, \omega^X)$ and $\overline{g} = (g, \omega^Y)$. The exact sequence

$$0 \longrightarrow T_f \longrightarrow f^* T_g \longrightarrow N_{X/Y} \longrightarrow 0$$

induces a hermitian structure on $N_{X/Y}$. We will denote $\overline{\iota}$ the inclusion ι with this hermitian structure. Finally we denote by $\overline{f_* F}_E$ the hermitian structure on $f_* F$ induced by the hermitian structures $\overline{g_* E_j}_{L^2}$, $j = 0, \dots, n$.

Then, adapted to our language, the main result of [3] can be stated as follows.

Theorem 3.129 ([3] Theorem 0.1 and 0.2)**.** *The following equation holds in the group $\bigoplus_p \widetilde{\mathcal{D}}_D^{2p-1}(Z, \emptyset, p)$:*

$$\begin{aligned} T^{BK}(\overline{f}, \overline{F}, \overline{f_* F}_{L_2}) = \sum_{j=0}^{n} (-1)^j T^{BK}(\overline{g}, \overline{E}_j, \overline{f_* E_j}_{L_2}) \\ + \overline{g}_\flat(T_{-R/2}(\overline{\iota}, \overline{F}, \overline{E})) + \widetilde{\mathrm{ch}}(\overline{f_* F}_E, \overline{f_* F}_{L^2}). \end{aligned}$$

We can particularize the previous result to the case when $F = 0$. Then E and $g_* E$ are acyclic objects. The hermitian structures of $\overline{E}_j$ and $\overline{g_* E_j}_{L^2}$ induce hermitian structures on them. We denote these hermitian structures as $\overline{E}$ and $\overline{g_* E}_{L^2}$.

Corollary 3.130. *Let $\overline{E}$ be a bounded acyclic complex of hermitian vector bundles on Y such that the direct images $R^i g_* E_j$ are locally free on Z. Then there is an equality in $\bigoplus_p \widetilde{\mathcal{D}}_D^{2p-1}(Z, \emptyset, p)$*

$$\sum_{j=0}^{n} (-1)^j T^{BK}(\overline{g}, \overline{E_j}, \overline{g_* E_j}_{L^2}) = \widetilde{\mathrm{ch}}(\overline{g_* E}_{L^2}) - \overline{g}_\flat(\widetilde{\mathrm{ch}}(\overline{E})).$$

We will also need a particular case of functoriality and projection formula for the higher analytic torsion forms of Bismut-Köhler proved by Rössler [43].

The relative metrized complexes $\overline{\xi}_n(k)$ of Notation 3.45 are Kähler. Therefore we can apply the construction of Bismut-Köhler to them. We denote

$$t_{n,k}^{BK} = T^{BK}(\overline{\xi}_n(k)). \tag{3.131}$$

By Corollary 3.130, the numbers $t_{n,k}^{BK}$ satisfy the relation (3.63). Hence they are determined by the main characteristic numbers $t_{n,k}^{BK}$ for $-n \leq k \leq 0$.

Theorem 3.132 ([43] Lemma 7.15). *Let $\pi\colon \mathbb{P}^n_X \to X$ be a trivial projective bundle. Let $\overline{G}$ be a hermitian vector bundle on X. Then*

$$T^{BK}(\overline{\xi}_n(k)\otimes \overline{G}) = t^{BK}_{n,k} \bullet \operatorname{ch}(\overline{G}).$$

Proof. In [43] this result is proved for $k \gg 0$. Using Corollary 3.130 and the Koszul resolution (3.62) one can extend the result to all $k \in \mathbb{Z}$. □

We have all the ingredients we need to prove the main result of this section.

Theorem 3.133. *Let $T_{-R/2}$ be the theory of generalized analytic torsion classes associated to the additive genus $\frac{-1}{2}R$. Then, for every Kähler relative metrized complex $\overline{\xi}$, we have*

$$T^{BK}(\overline{\xi}) = T_{-R/2}(\overline{\xi}).$$

In particular $T_{-R/2}$ extends the construction of Bismut-Köhler to arbitrary projective morphisms of smooth complex varieties and arbitrary smooth metrics.

Proof. Let $\mathfrak{t}^{BK} = \{t^{BK}_{n,k} \mid n \geq 0, -n \leq k \leq 0\}$ and let $T_{\mathfrak{t}^{BK}}$ be the theory of analytic torsion classes for projective spaces associated to it. Let $\pi\colon \mathbb{P}^n_X \to X$ be a relative projective space and let $\overline{\xi} = (\overline{\pi}, \overline{E}, \overline{\pi_* E}_{L^2})$ be a Kähler relative metrized complex. By choosing $d \gg 0$ we may assume that all the coherent sheaves of the resolution $\gamma_d(F)$ of Corollary 3.36 are locally free. Using theorems 3.132 and 3.126, Proposition 3.54 and corollaries 3.59 and 3.130 we obtain that

$$T^{BK}(\overline{\xi}) = T_{\mathfrak{t}^{BK}}(\overline{\xi}).$$

By Theorem 3.129 we see that the theories $T_{\mathfrak{t}^{BK}}$ and $T_{-R/2}$ are compatible in the sense of Definition 3.86. Therefore, $T^{BK} = T_{-R/2}$ when restricted to projective spaces.

Finally, by factoring a smooth projective morphism as a closed immersion followed by the projection of a relative projective space, Theorem 3.129 implies that $T^{BK} = T_{-R/2}$ for all smooth projective morphisms. □

Remark 3.134. (i) The construction of Bismut-Köhler applies to a wider class of varieties and morphisms: complex analytic manifolds and proper Kähler submersions. However for the comparison we have to restrict to smooth algebraic varieties and smooth projective morphisms.

(ii) The results of Bismut and his coworkers are more precise. Here the class $T^{BK}(\overline{\xi})$ is well defined up to the image of $\mathrm{d}_{\mathcal{D}}$. In contrast, the higher analytic torsion form of Bismut and Köhler is a well defined differential form, local on the base and whose class modulo $\mathrm{d}_{\mathcal{D}}$ agrees with $T^{BK}(\overline{\xi})$.

As a consequence of Theorem 3.133, we obtain the following results that, although they should follow from the definition of higher analytic torsion classes, we have not been able to find them explicitly in the literature.

Corollary 3.135. *Let $f\colon X \to Y$ be a smooth projective morphism of smooth complex varieties, and let $\overline{\xi} = (\overline{f}, \overline{E}, \overline{f_* E}_{L^2})$ be a Kähler relative metrized complex.*

(i) Let $g\colon Y' \to Y$ be a morphism of smooth complex varieties. Then

$$T^{BK}(g^*\overline{\xi}) = g^* T^{BK}(\overline{\xi}).$$

(ii) Let $\overline{G}$ be a hermitian vector bundle on Y. Then

$$T^{BK}(\overline{\xi}\otimes\overline{G}) = T^{BK}(\overline{\xi})\bullet \operatorname{ch}(\overline{G}).$$

□

The last consequence we want to discuss generalizes results already proved by Berthomieu-Bismut [1, Thm 3.1] and Ma [35, Thm. 0.1], [36, Thm. 0.1]. However we note that while we stay within the algebraic category and work with projective morphisms, these authors deal with proper Kähler holomorphic submersions of complex manifolds. Let $\overline{g}\colon X\to Y$ and $\overline{h}\colon Y\to Z$ be morphisms in the category $\overline{\mathbf{Sm}}_{*/\mathbb{C}}$, such that the composition $\overline{f}=\overline{h}\circ\overline{g}$ is a smooth morphism. We choose a structure of Kähler fibration on f, that we denote $\overline{f}'$. Let $\overline{E}$ be a hermitian vector bundle on X and assume that the higher direct images R^if_*E are locally free. Then we may consider the analytic torsion $T^{BK}(\overline{f}')$ attached to the Kähler relative metrized complex $(\overline{f}',\overline{E},\overline{f_*E}_{L^2})$. Also, we choose an auxiliary hermitian structure on g_*E. We can consider the torsion classes $T_{R/2}(\overline{g})$ and $T_{R/2}(\overline{h})$ of the relative metrized complexes $(\overline{g},\overline{E},\overline{g_*E})$ and $(\overline{h},\overline{g_*E},\overline{f_*E}_{L^2})$.

We make the following additional assumption in some particular situations:

(*) The morphisms g and h are Kähler fibrations, the higher direct images R^ig_*E and $R^jh_*R^ig_*E$ are locally free and the auxiliary hermitian structure on g_*E is the L^2 hermitian structure.

When the hypothesis **(*)** is satisfied we denote by $\overline{h_*g_*E}_{L^2}$ the L^2 hermitian structure attached to the Kähler structure on $\overline{h}$ and the L^2 metric on $\overline{g_*E}_{L^2}$. Observe then that this last structure may differ from the L^2 structure on $\overline{f_*E}_{L^2}$. In this situation we can consider the torsion classes $T^{BK}(\overline{g})$ and $T^{BK}(\overline{h}')$ attached to $(\overline{g},\overline{E},\overline{g_*E}_{L^2})$ and $(\overline{h},\overline{g_*E}_{L^2},\overline{h_*g_*E}_{L^2})$. Observe that by Proposition 3.54, we have the relation

$$T^{BK}(\overline{h}') = T_{R/2}(\overline{h}) - \widetilde{\operatorname{ch}}(\overline{h_*g_*E}_{L^2},\overline{f_*E}_{L^2}).$$

By the properties of the generalized analytic torsion classes, the following statement is immediate.

Corollary 3.136. *Let the assumptions be as above. Then we have the equality*

$$T^{BK}(\overline{f}') = T_{-R/2}(\overline{h}) + \overline{h}_\flat(T_{-R/2}(\overline{g})) + \overline{f}'_\flat(\operatorname{ch}(\overline{E})\bullet\widetilde{\operatorname{Td}}_m(\overline{f}',\overline{f})).$$

If in addition the hypothesis **(*)** *is satisfied, then we have an equality*

$$\begin{aligned}T^{BK}(\overline{f}') = T^{BK}(\overline{h}') + \overline{h}_\flat(T^{BK}(\overline{g}))\\ + \overline{f}'_\flat(\operatorname{ch}(\overline{E})\bullet\widetilde{\operatorname{Td}}_m(\overline{f}',\overline{f})) + \widetilde{\operatorname{ch}}(\overline{h_*g_*E}_{L^2},\overline{f_*E}_{L^2}).\end{aligned}$$

□

Since $T_{-R/2}$ extends the theory of analytic torsion classes T^{BK}, we will denote $T_{-R/2}$ by T^{BK} for arbitrary relative metrized complexes.

3.8 Grothendieck duality and analytic torsion

The aim of this section is to study the compatibility of the analytic torsion with Grothendieck duality.

Definition 3.137. Let $\overline{\mathcal{F}} = (\mathcal{F}, \overline{E} \dashrightarrow \mathcal{F})$ be an object of $\overline{\mathbf{D}}^{\mathrm{b}}(X)$. Then the *rank* of $\overline{\mathcal{F}}$ is

$$\operatorname{rk}(\overline{\mathcal{F}}) = \sum_i (-1)^i \dim(E_i).$$

This is just the Euler characteristic of the complex. *The determinant* of $\overline{\mathcal{F}}$ is the complex

$$\det(\overline{\mathcal{F}}) = \bigotimes_i \left(\Lambda^{\dim E^i} \overline{E}^i\right)^{(-1)^i} [-\operatorname{rk}(\overline{\mathcal{F}})].$$

It consists of a single line bundle concentrated in degree $\operatorname{rk}(\overline{\mathcal{F}})$.

Definition 3.138. Let $\overline{f}\colon X \to Y$ be a morphism in $\overline{\mathbf{Sm}}_{*/\mathbb{C}}$ of relative dimension e. The *metrized dualizing complex*, is the complex given by

$$\boldsymbol{\omega}_{\overline{f}} = (\det T_{\overline{f}})^{\vee}.$$

This complex is concentrated in degree $-e$. The underlying object of $\mathbf{D}^{\mathrm{b}}(X)$ will be denoted by $\boldsymbol{\omega}_f$. If we are interested in the dualizing sheaf as a sheaf and not as an element of $\mathbf{D}^{\mathrm{b}}(X)$ we will denote it by ω_f or $\omega_{X/Y}$. Finally, if $T = \operatorname{Spec}\mathbb{C}$, we will denote $\boldsymbol{\omega}_f$ (respectively ω_f) by $\boldsymbol{\omega}_X$ (respectively ω_X).

Definition 3.139. Let $\mathcal{D}^*(*)$ be the Deligne complex associated to a Dolbeault complex. Then the *sign operator*

$$\sigma\colon \mathcal{D}^*(*) \longrightarrow \mathcal{D}^*(*)$$

is given by $\sigma(\omega) = (-1)^p\omega$ for $\omega \in \mathcal{D}^n(p)$.

The sign operator satisfies the following compatibilities.

Proposition 3.140. *(i) Let $(\mathcal{D}^*(*), \mathrm{d}_{\mathcal{D}})$ be a Deligne algebra. Then the sign operator is a morphism of differential algebras. That is*

$$\mathrm{d}_{\mathcal{D}} \circ \sigma = \sigma \circ \mathrm{d}_{\mathcal{D}},$$
$$\sigma(\omega \bullet \eta) = \sigma(\omega) \bullet \sigma(\eta).$$

(ii) Let $\overline{\mathcal{F}}$ be an object of $\overline{\mathbf{D}}^{\mathrm{b}}(X)$. Then the following equalities are satisfied

$$\sigma \operatorname{ch}(\overline{\mathcal{F}}) = \operatorname{ch}(\overline{\mathcal{F}}^{\vee}), \tag{3.141}$$
$$\sigma \operatorname{ch}(\det(\overline{\mathcal{F}})) = \operatorname{ch}(\det(\overline{\mathcal{F}})^{\vee}) = \operatorname{ch}(\det(\overline{\mathcal{F}}))^{-1}, \tag{3.142}$$
$$\sigma \operatorname{Td}(\overline{\mathcal{F}}) = (-1)^{\operatorname{rk}(\overline{\mathcal{F}})} \operatorname{Td}(\overline{\mathcal{F}}) \bullet \operatorname{ch}(\det(\overline{\mathcal{F}})^{\vee}. \tag{3.143}$$

Proof. The first statement is clear because if $\omega \in \mathcal{D}^n(p)$ and $\eta \in \mathcal{D}^m(q)$ then $\mathrm{d}_{\mathcal{D}}\,\omega \in \mathcal{D}^{n+1}(p)$ and $\omega \bullet \eta \in \mathcal{D}^{n+m}(p+q)$.

For the second statement, let $\overline{E} \dashrightarrow \mathcal{F}$ be the hermitian structure of $\mathcal{F}$. Write

$$\begin{aligned}\overline{E}^{+} &= \bigoplus_{i \text{ even}} \overline{E}^{i},\\ \overline{E}^{-} &= \bigoplus_{i \text{ odd}} \overline{E}^{i}.\end{aligned}$$

Since this statement is local on X, we can chose trivializations of $\overline{E}^{+}$ and $\overline{E}^{-}$ over an open subset U. Let H^{+} and H^{-} be the matrices of the hermitian metrics on $\overline{E}^{+}$ and $\overline{E}^{-}$. The curvature matrices of $\overline{E}^{+}$ and $\overline{E}^{-}$ are given by

$$K^{\pm} = K^{\pm}(\overline{\mathcal{F}}) = -\bar{\partial}(H^{\pm})^{-1}\partial H^{\pm}.$$

The entries of these two matrices are elements of $\mathcal{D}^{2}(U,1)$. The characteristic forms can be computed from the curvature matrix:

$$\begin{aligned}\mathrm{ch}(\overline{\mathcal{F}}) &= \mathrm{tr}(\exp(K^{+})) - \mathrm{tr}(\exp(K^{-})),\\ \mathrm{ch}(\det(\overline{\mathcal{F}})) &= (-1)^{\mathrm{rk}(\overline{\mathcal{F}})}\det(\exp(K^{+})) \bullet \det(\exp(K^{-}))^{-1},\\ \mathrm{Td}(\overline{\mathcal{F}}) &= \det\left(\frac{K^{+}}{1-\exp(-K^{+})}\right) \bullet \det\left(\frac{K^{-}}{1-\exp(-K^{-})}\right)^{-1}.\end{aligned}$$

The sign in the second equation comes from the fact that $\det(\overline{\mathcal{F}})$ is concentrated in degree $\mathrm{rk}(\overline{\mathcal{F}})$. Therefore, since $\sigma(K^{\pm}) = -K^{\pm} = K^{\pm}(\overline{\mathcal{F}}^{\vee})$, we have

$$\begin{aligned}\sigma\,\mathrm{ch}(\overline{\mathcal{F}}) &= \sigma\,\mathrm{tr}(\exp(K^{+})) - \sigma\,\mathrm{tr}(\exp(K^{-}))\\ &= \mathrm{tr}(\exp(K^{+}(\overline{\mathcal{F}}^{\vee}))) - \mathrm{tr}(\exp(K^{-}(\overline{\mathcal{F}}^{\vee}))) = \mathrm{ch}(\overline{\mathcal{F}}^{\vee}),\end{aligned}$$

$$\sigma\,\mathrm{ch}(\det(\overline{\mathcal{F}})) = \det(\exp(-K^{+})) \bullet \det(\exp(-K^{-}))^{-1} = \mathrm{ch}(\det(\overline{\mathcal{F}}))^{-1},$$

and

$$\begin{aligned}\sigma\,\mathrm{Td}(\overline{\mathcal{F}}) &= \det\left(\frac{-K^{+}}{1-\exp(K^{+})}\right) \bullet \det\left(\frac{-K^{-}}{1-\exp(K^{-})}\right)^{-1}\\ &= \det\left(\frac{K^{+}}{1-\exp(-K^{+})}\right) \bullet \det(\exp(-K^{+}))\\ &\quad \bullet \det\left(\frac{K^{-}}{1-\exp(-K^{-})}\right)^{-1} \bullet \det(\exp(-K^{-}))^{-1}\\ &= \mathrm{Td}(\overline{\mathcal{F}}) \bullet \mathrm{ch}(\det(\overline{\mathcal{F}}))^{-1}.\end{aligned}$$

□

Corollary 3.144. *Let $[\overline{E}] \in \mathbf{KA}(X)$. Then we have the equality*

$$\widetilde{\mathrm{ch}}(\overline{E}^{\vee}) = \sigma\widetilde{\mathrm{ch}}(\overline{E}).$$

Proof. Due to Proposition 3.140, the assignment sending $[\overline{E}]$ to $\sigma\widetilde{\mathrm{ch}}(\overline{E})$ satisfies the characterizing properties of $\widetilde{\mathrm{ch}}$. □

In the particular case of a projective morphism between smooth complex varieties or, more generally, smooth varieties over a field, Grothendieck duality takes a very simple form (see for instance [32, §3.4] and the references therein). If $\mathcal{F}$ is an object of $\mathbf{D}^{\mathrm{b}}(X)$ and $f\colon X \to Y$ is a projective morphism of smooth complex varieties, then there is a natural functorial isomorphism

$$f_*(\mathcal{F}^\vee \otimes \boldsymbol{\omega}_f) \cong (f_*\mathcal{F})^\vee. \tag{3.145}$$

The compatibility between analytic torsion and Grothendieck duality is given by the following result.

Theorem Definition 3.146. *Let T be a theory of generalized analytic torsion classes. Then the assignment that, to each relative metrized complex $\overline{\xi} = (\overline{f}, \overline{\mathcal{F}}, \overline{f_*\mathcal{F}})$, associates the class*

$$T^\vee(\overline{\xi}) = \sigma T(\overline{f}, \overline{\mathcal{F}}^\vee \otimes \boldsymbol{\omega}_{\overline{f}}, \overline{f_*\mathcal{F}}^\vee)$$

is a theory of generalized analytic torsion classes that we call the theory dual to T.

Proof. We have to show that, if T satisfies the conditions of Definition 3.108, then the same is true for $T^\vee$. We first check the differential equation. Let e be the relative dimension of f.

$$\begin{aligned}
\mathrm{d}_{\mathcal{D}}\, T^\vee(\overline{\xi}) &= \mathrm{d}_{\mathcal{D}}\, \sigma T(\overline{f}, \overline{\mathcal{F}}^\vee \otimes \boldsymbol{\omega}_{\overline{f}}, \overline{f_*\mathcal{F}}^\vee)\\
&= \sigma\, \mathrm{d}_{\mathcal{D}}\, T(\overline{f}, \overline{\mathcal{F}}^\vee \otimes^{\mathbb{L}} \boldsymbol{\omega}_{\overline{f}}, \overline{f_*\mathcal{F}}^\vee)\\
&= \sigma\, \mathrm{ch}(\overline{f_*\mathcal{F}}^\vee) - \sigma f_* \left[\mathrm{ch}(\overline{\mathcal{F}}^\vee \otimes^{\mathbb{L}} \boldsymbol{\omega}_{\overline{f}}) \bullet \mathrm{Td}(\overline{f})\right]\\
&= \mathrm{ch}(\overline{f_*\mathcal{F}}) - (-1)^e f_* \left[\sigma\, \mathrm{ch}(\overline{\mathcal{F}}^\vee) \bullet \sigma(\mathrm{ch}(\det(T_{\overline{f}})^\vee) \bullet \mathrm{Td}(T_{\overline{f}}))\right]\\
&= \mathrm{ch}(\overline{f_*\mathcal{F}}) - f_* \left[\mathrm{ch}(\overline{\mathcal{F}}) \bullet \mathrm{Td}(\overline{f})\right]
\end{aligned}$$

The functoriality and the additivity are clear. We next check the projection formula. Let $\overline{\mathcal{G}}$ be an object of $\overline{\mathbf{D}}^{\mathrm{b}}(Y)$. Then

$$\begin{aligned}
T^\vee(\overline{\xi} \otimes \overline{\mathcal{G}}) &= \sigma T(\overline{f}, \overline{\mathcal{F}}^\vee \otimes f^*\overline{\mathcal{G}}^\vee \otimes^{\mathbb{L}} \boldsymbol{\omega}_{\overline{f}}, \overline{f_*\mathcal{F}}^\vee \otimes^{\mathbb{L}} \overline{\mathcal{G}}^\vee)\\
&= \sigma \left(T(\overline{f}, \overline{\mathcal{F}}^\vee \otimes^{\mathbb{L}} \boldsymbol{\omega}_{\overline{f}}, \overline{f_*\mathcal{F}}^\vee) \bullet \mathrm{ch}(\overline{\mathcal{G}}^\vee)\right)\\
&= T^\vee(\overline{\xi}) \bullet \mathrm{ch}(\overline{\mathcal{G}}).
\end{aligned}$$

Finally we check the transitivity. Let $\overline{g}\colon Y \to Z$ be another morphism in $\overline{\mathbf{Sm}}_{*/\mathbb{C}}$. By the definition of $\overline{g} \circ \overline{f}$ we have

$$\boldsymbol{\omega}_{\overline{g}\circ\overline{f}} = f^*\boldsymbol{\omega}_{\overline{g}} \otimes^{\mathbb{L}} \boldsymbol{\omega}_{\overline{f}}. \tag{3.147}$$

Therefore,

$$\begin{aligned}
f_*\left(\mathcal{F}^\vee \otimes^{\mathbb{L}} \boldsymbol{\omega}_{g\circ f}\right) &= f_*\left(\mathcal{F}^\vee \otimes^{\mathbb{L}} f^*\boldsymbol{\omega}_g \otimes^{\mathbb{L}} \boldsymbol{\omega}_f)\right)\\
&= f_*\left(\mathcal{F}^\vee \otimes^{\mathbb{L}} \boldsymbol{\omega}_f)\right) \otimes^{\mathbb{L}} \boldsymbol{\omega}_g\\
&= (f_*\mathcal{F})^\vee \otimes^{\mathbb{L}} \boldsymbol{\omega}_g.
\end{aligned}$$

On $f_*(\mathcal{F}^\vee \otimes \boldsymbol{\omega}_{g\circ f})$ we put the hermitian structure of $\overline{f_*\mathcal{F}}^\vee \otimes \boldsymbol{\omega}_{\overline{g}}$. With this choice we have

$$\begin{aligned}
T^\vee(\overline{g}\circ\overline{f}) &= \sigma T(\overline{g}\circ\overline{f}, \overline{\mathcal{F}}^\vee \otimes^{\mathbb{L}} \boldsymbol{\omega}_{\overline{g}\circ\overline{f}}, \overline{(g\circ f)_*\mathcal{F}}^\vee) \\
&= \sigma T(\overline{g}, \overline{f_*\mathcal{F}}^\vee \otimes^{\mathbb{L}} \boldsymbol{\omega}_{\overline{g}}, \overline{(g\circ f)_*\mathcal{F}}^\vee) \\
&\qquad + \sigma\overline{g}_\flat T(\overline{f}, \overline{\mathcal{F}}^\vee \otimes^{\mathbb{L}} \boldsymbol{\omega}_{\overline{f}} \otimes^{\mathbb{L}} f^*\boldsymbol{\omega}_{\overline{g}}, \overline{f_*\mathcal{F}}^\vee \otimes^{\mathbb{L}} \boldsymbol{\omega}_{\overline{g}}) \\
&= T^\vee(\overline{g}, \overline{f_*\mathcal{F}}, \overline{(g\circ f)_*\mathcal{F}}) \\
&\qquad + \sigma g_*(T(\overline{f}, \overline{\mathcal{F}}^\vee \otimes^{\mathbb{L}} \boldsymbol{\omega}_{\overline{f}}, \overline{f_*\mathcal{F}}^\vee) \bullet \mathrm{ch}(\boldsymbol{\omega}_{\overline{g}}) \bullet \mathrm{Td}(\overline{g})) \\
&= T^\vee(\overline{g}) + \overline{g}_\flat T^\vee(\overline{f}).
\end{aligned}$$

Therefore, $T^\vee$ satisfies also the transitivity property. Hence is a generalized theory of analytic torsion classes. □

Definition 3.148. A theory of generalized analytic torsion classes T is called *self-dual* when the equation

$$T^\vee = T \tag{3.149}$$

holds.

We want to characterize the self-dual theories of generalized analytic torsion classes.

Theorem 3.150. *The homogeneous theory of generalized analytic torsion classes is self-dual.*

Proof. By the uniqueness of the homogeneous theory, it is enough to prove that, if T is homogeneous then $T^\vee$ is homogeneous. Let X be a smooth complex variety and let $\overline{N}$ be a hermitian vector bundle of rank r on X. Put $P = \mathbb{P}(N\oplus 1)$ and let $s\colon X \to P$ be the zero section and $\pi\colon P \to X$ the projection. Let $\overline{Q}$ be the tautological quotient bundle with the induced metric and $\overline{K}(s)$ the Koszul resolution associated to the section s. Since the normal bundle $N_{X/P}$ can be identified with N, on the map s we can consider the hermitian structure given by the hermitian metric on N. Then $\det\overline{Q}$ is a complex concentrated in degree r. Moreover

$$s^*\det\overline{Q} = \det\overline{N} = \boldsymbol{\omega}_{\overline{s}}.$$

The Koszul resolution satisfies the duality property

$$\overline{K}(s)^\vee = \overline{K}(s) \otimes \det\overline{Q}.$$

The theory T is homogeneous if and only if the class

$$T(\overline{s}, \overline{\mathcal{O}}_X, \overline{K}(s)) \bullet \mathrm{Td}(\overline{Q})$$

is homogeneous of bidegree $(2r-1, r)$ in the Deligne complex. Then

$$\begin{aligned}
T^\vee(\overline{s}, \overline{\mathcal{O}}_X, \overline{K}(s)) \bullet \mathrm{Td}(\overline{Q}) &= \sigma T(\overline{s}, \boldsymbol{\omega}_{\overline{s}}, \overline{K}(s)^\vee) \bullet \mathrm{Td}(\overline{Q}) \\
&= \sigma T(\overline{s}, s^*\det\overline{Q}, \overline{K}(s)\otimes\det\overline{Q}) \bullet \mathrm{Td}(\overline{Q}) \\
&= \sigma(T(\overline{s}, \overline{\mathcal{O}}_X, \overline{K}(s)) \bullet \mathrm{ch}(\det\overline{Q})) \bullet \mathrm{Td}(\overline{Q}) \\
&= \sigma T(\overline{s}, \overline{\mathcal{O}}_X, \overline{K}(s)) \bullet \mathrm{ch}(\det\overline{Q}^\vee) \bullet \mathrm{Td}(\overline{Q}) \\
&= (-1)^r\sigma(T(\overline{s}, \overline{\mathcal{O}}_X, \overline{K}(s)) \bullet \mathrm{Td}(\overline{Q}))
\end{aligned}$$

is homogeneous of bidegree $(2r-1,r)$ in the Deligne complex, which proves the theorem. □

Proposition 3.151. *Let*

$$S(x)=\sum_{n=0}^{\infty}a_n x^n\in\mathbb{R}[[x]]$$

be a power series in one variable with real coefficients. Denote by S the corresponding real additive genus and by T_S the associated theory of analytic torsion classes. Then the dual theory $T_S^{\vee}$ has corresponding real additive genus $S^{\sigma}(x):=-S(-x)$.

Proof. Let $\overline{\xi}=(\overline{f},\overline{\mathcal{F}},\overline{f_*\mathcal{F}})$ be a relative metrized complex. If e is the relative dimension of f, then we have $\sigma f_*=(-1)^e f_*\sigma$. Then the proposition readily follows from the definition of $T_S^{\vee}$, the self-duality of T^h and Proposition 3.140. □

We can now characterize the self-dual theories of analytic torsion classes.

Corollary 3.152. *The theory of analytic torsion classes T_S attached to the real additive genus $S(x)=\sum_{n\geq 0}a_n x^n$ is self-dual if and only if $a_n=0$ for n even.*

Proof. By the proposition, $T_S^{\vee}=T_{S^{\sigma}}$, hence T is self-dual if, and only if, $S^{\sigma}=S$. The corollary follows. □

In particular we recover the following fact, which is well known if we restrict to Kähler relative metrized complexes.

Corollary 3.153. *The theory of analytic torsion classes of Bismut-Köhler T^{BK} is self-dual.*

Proof. We just remark that the even coefficients of the R-genus vanish (3.127), and apply the preceding characterization of self-duality. □

We now elaborate on an intimate relation between self-duality phenomena and the analytic torsion of de Rham complexes. Let $f\colon X\to Y$ be a smooth projective morphism of smooth algebraic varieties, of relative dimension e. Let $\overline{T}_{X/Y}$ denote the vertical tangent bundle, endowed with a hermitian metric. Write $\overline{f}$ for the corresponding morphism in $\overline{\mathbf{Sm}}_{*/\mathbb{C}}$. On the locally free sheaves $\Omega^p_{X/Y}=\Lambda^p\Omega_{X/Y}$ we put the induced hermitian structures. We introduce the metrized de Rham complex

$$0\to\overline{\mathcal{O}}_X\xrightarrow{0}\overline{\Omega}_{X/Y}\xrightarrow{0}\overline{\Omega}^2_{X/Y}\xrightarrow{0}\dots\xrightarrow{0}\overline{\Omega}^e_{X/Y}\to 0$$

with 0 differentials. In fact, we are really considering the de Rham graded sheaf and converting it into a complex in a trivial way. We refer to the corresponding object of $\overline{\mathbf{D}}^{\mathrm{b}}(X)$ by $\overline{\Omega}^{\bullet}_{X/Y}$ (Definition 2.71). The individual terms $\overline{\Omega}^p_{X/Y}$ will be considered as complexes concentrated in degree p.

Lemma 3.154. *The objects $(\overline{\Omega}^{\bullet}_{X/Y})^{\vee}\otimes\boldsymbol{\omega}_{\overline{f}}$ and $\overline{\Omega}^{\bullet}_{X/Y}[2e]$ are tightly isomorphic.*

Proof. This is obvious. □

For every p, q, the cohomology sheaf $R^q f_*\Omega^p_{X/Y}$ is locally free, because the Hodge numbers $h^{p,q}$ of the fibers of f (which are projective, hence Kähler) are known to be locally constant. Every stalk of this sheaf is endowed with the usual L^2 metric of Hodge theory. This family of L^2 metrics on $R^q f_*\Omega^p_{X/Y}$ glue into a smooth metric. Because the Hodge star operators $*$ act by isometries, it is easily shown that Serre duality becomes an isometry for the L^2 structures: the isomorphism

$$(R^q f_*\Omega^p_{X/Y})^\vee \xrightarrow{\sim} R^{e-q} f_*((\Omega^p_{X/Y})^\vee \otimes \boldsymbol{\omega}_f) = R^{e-q} f_*\Omega^{e-p}_{X/Y}$$

preserves the L^2 hermitian structures. For every p, let $\overline{f_*\Omega^p_{X/Y}}$ denote the object of $\overline{\mathbf{D}}^{\mathrm{b}}(Y)$ with the metric induced by the L^2 metrics on its cohomology pieces (Definition 2.81). Here f_* stands for the derived direct image. By Proposition 2.82, Grothendieck duality

$$(f_*\Omega^p_{X/Y})^\vee \xrightarrow{\sim} f_*\Omega^{e-p}_{X/Y}[2e]$$

is a tight isomorphism. Finally, let $[\overline{f_*\Omega^\bullet_{X/Y}}]$ be the object of $\overline{\mathbf{D}}^{\mathrm{b}}(Y)$ provided by Definition 2.73. The next statement is easily checked from the very construction of Definition 2.73.

Lemma 3.155. *Grothendieck duality defines a tight isomorphism* $[\overline{f_*\Omega^\bullet_{X/Y}}]^\vee \cong [\overline{f_*\Omega^\bullet_{X/Y}}][2e]$ *in* $\overline{\mathbf{D}}^{\mathrm{b}}(Y)$.

Theorem 3.156. *Let T be a theory of analytic torsion classes. The following assertions are equivalent:*

(i) the theory T is self-dual;

(ii) for every f, $\overline{T}_f$, $\overline{\Omega}^\bullet_{X/Y}$ and $[\overline{f_*\Omega^\bullet_{X/Y}}]$ as above and for every odd integer $p \geq 1$, the part of bidegree $(2p-1,p)$ (in the Deligne complex) of $T(\overline{f}, \overline{\Omega}^\bullet_{X/Y}, [\overline{f_*\Omega^\bullet_{X/Y}}])$ vanishes.

Proof. Assume first of all that T is self-dual. We apply the definition of $T^\vee$, the self-duality assumption and lemmas 3.154 and 3.155. We find the equality

$$\begin{aligned} T(\overline{f}, \overline{\Omega}^\bullet_{X/Y}, [\overline{f_*\Omega^\bullet_{X/Y}}]) &= \sigma T(\overline{f}, \overline{\Omega}^\bullet_{X/Y}[2e], [\overline{f_*\Omega^\bullet_{X/Y}}][2e]) \\ &= (-1)^{2e}\sigma T(\overline{f}, \overline{\Omega}^\bullet_{X/Y}, [\overline{f_*\Omega^\bullet_{X/Y}}]) \\ &= \sigma T(\overline{f}, \overline{\Omega}^\bullet_{X/Y}, [\overline{f_*\Omega^\bullet_{X/Y}}]). \end{aligned}$$

The sign operator σ changes the sign of the components of bidegree $(2p-1,p)$ for odd p. Hence $T(\overline{f}, \overline{\Omega}^\bullet_{X/Y}, [\overline{f_*\Omega^\bullet_{X/Y}}])^{(2p-1,p)}$ vanishes for $p \geq 1$ odd.

For the converse implication, let $S(x) = \sum_{n\geq 0} a_n x^n$ be the real additive genus attached to T via Theorem 3.121. After Corollary 3.152, we have to show that the coefficients a_n with n even vanish. Let us look at a smooth morphism $f\colon X \to Y$ of relative dimension 1, with an arbitrary metric on T_f. Then, developing the power series of ch and Td and taking into account that $\Omega^1_{X/Y} = T_f^\vee = \omega_{X/Y}$, we compute

$$f_*[\mathrm{ch}(\Omega^\bullet_{X/Y})\,\mathrm{Td}(T_f)S(T_f) \bullet \mathbf{1}_1] = \sum_{n\geq 0}(-1)^{n+1} a_n f_*[c_1(\omega_{X/Y})^{n+1} \bullet \mathbf{1}_1].$$

Therefore, for $p \geq 1$ odd, we have

$$(-1)^p a_{p-1} f_*[c_1(\omega_{X/Y})^p \bullet \mathbf{1}_1] = \\ (T(\overline{f}, \overline{\Omega^\bullet_{X/Y}}, [\overline{f_*\Omega^\bullet_{X/Y}}]) - T^h(\overline{f}, \overline{\Omega^\bullet_{X/Y}}, [\overline{f_*\Omega^\bullet_{X/Y}}]))^{(2p-1,p,p)} = 0. \quad (3.157)$$

Hence it is enough that for every odd integer $p \geq 1$, we find a relative curve $f\colon X \to Y$ such that $f_*(c_1(\omega_{X/Y})^p) \neq 0$ in the cohomology group $H^{2p}(Y, \mathbb{C})$. Let $d = p-1$ and choose Y to be a smooth projective variety of dimension d. Let L be an ample line bundle on Y and take $X = \mathbb{P}(L \oplus \mathcal{O}_Y)$. Consider the tautological exact sequence

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow f^*(L \oplus \mathcal{O}_Y) \longrightarrow Q \longrightarrow 0.$$

We easily derive the relations

$$\pi^* c_1(L) = c_1(Q) - c_1(\mathcal{O}(1)) \quad (3.158)$$

$$c_1(\mathcal{O}(-1)) c_1(Q) = 0. \quad (3.159)$$

Moreover we have

$$c_1(\omega_{X/Y}) = -c_1(Q) - c_1(\mathcal{O}(1)). \quad (3.160)$$

From (3.158)–(3.160) and because $d = p-1$ is even, we compute

$$c_1(\omega_{X/Y})^d = c_1(Q)^d + c_1(\mathcal{O}(1))^d = \pi^* c_1(L)^d.$$

Therefore we find

$$c_1(\omega_{X/Y})^p = \pi^* c_1(L)^d c_1(\omega_{X/Y}). \quad (3.161)$$

Finally, f is a fibration in curves of genus 0, hence $f_*(c_1(\omega_{X/Y})) = -2$. We infer that (3.161) leads to

$$f_*(c_1(\omega_{X/Y})^p) = -2c_1(L)^d.$$

This class does not vanish, since Y is projective of dimension d and L is ample. □

We end with a characterization of the theory of analytic torsion classes of Bismut-Köhler.

Theorem 3.162. *The theory of analytic torsion classes of Bismut-Köhler T^{BK} is the unique theory of generalized analytic torsion classes such that, for every Kähler fibration $\overline{f}\colon X \to Y$ in $\overline{\mathbf{Sm}}_{*/\mathbb{C}}$, we have the vanishing*

$$T^{BK}(\overline{f}, \overline{\Omega}^\bullet_{X/Y}, [\overline{f_*\Omega^\bullet_{X/Y}}]) = 0.$$

Proof. That the theory T^{BK} vanishes for de Rham complexes of Kähler fibrations is a theorem of Bismut [5]. For the uniqueness, let T be a theory of generalized analytic torsion classes vanishing on de Rham complexes of Kähler fibrations. Denote by $S(x) = \sum_{k\geq 0} a_k x^k$ its corresponding genus. If $\overline{f}$ is a relative curve with a structure of Kähler fibration, then by Theorem 3.121

$$T^h(\overline{f}, \overline{\Omega}^\bullet_{X/Y}, [\overline{f_*\Omega^\bullet_{X/Y}}]) = \sum_{k\geq 0} (-1)^k a_k f_*[c_1(\omega_{X/Y})^{k+1} \bullet \mathbf{1}_1]. \quad (3.163)$$

It is enough to find, for every $k \geq 0$, a relative curve f such that $f_*(c_1(\omega_{X/Y})^{k+1})$ does not vanish. The elementary construction in the proof of Theorem 3.156 works whenever k is even, but one easily sees it fails for k odd. Fortunately, there is an alternative argument. Let $g \geq 2$ and $n \geq 3$ be integers. Consider the fine moduli scheme of smooth curves of genus g with a Jacobi structure of level n [21, Def. 5.4], to be denoted $\mathcal{M}_g^n$. Let $\pi\colon \mathcal{C}_g^n \to \mathcal{M}_g^n$ be the universal curve. An example of Kähler fibration structure on π is provided by Teichmüller theory (see for instance [49, Sec. 5]). By [22, Thm. 1], the tautological class $\kappa_{g-2} := \pi_*(c_1(\omega_\pi)^{g-1}) \in H^{2(g-2)}(\mathcal{M}_g^n, \mathbb{C})$ does not vanish. Taking $g = k+2$ and $f = \pi$, we conclude the proof of the theorem. □

We note that in the previous theorem, the existence is provided by Bismut's theorem. It would be interesting to have a proof of the existence of a theory satisfying the condition of Theorem 3.162 from the axiomatic point of view.

3.9 The category $\widehat{\mathbf{D}}^{\mathrm{b}}(X, S)$

As a first example of application of a theory of generalized analytic torsion classes, we construct direct images of metrized complexes. It turns out that the natural place to define direct images is not the category $\overline{\mathbf{D}}^{\mathrm{b}}(\cdot)$ but a new category $\widehat{\mathbf{D}}^{\mathrm{b}}(\cdot)$ that is the analogue to the arithmetic K-theory of Gillet and Soulé [28].

Let X be a smooth complex variety. In sections 2.1 and 2.2 we introduced the group $\mathbf{KA}(X)$ and the category $\overline{\mathbf{D}}^{\mathrm{b}}(X)$. The fibers of the forgetful morphism $\overline{\mathbf{D}}^{\mathrm{b}}(X) \to \mathbf{D}^{\mathrm{b}}(X)$ have a structure of $\mathbf{KA}(X)$-torsor, for the action of $\mathbf{KA}(X)$ by translation of the hermitian structures (Theorem 2.47). At the same time, the group $\mathbf{KA}(X)$ acts on the group $\oplus_p \widetilde{\mathcal{D}}_D^{2p-1}(X, p)$ by translation, via the Bott-Chern character $\widetilde{\mathrm{ch}}$ (Proposition 2.91). Observe that all Bott-Chern classes live in these groups, as for the analytic torsion classes. It is thus natural to build a product category over $\mathbf{KA}(X)$. This product category is the natural place to define direct images of hermitian vector bundles.

Definition 3.164. Let $S \subset T^*X_0$ be a closed conical subset. We define

$$\widehat{\mathbf{D}}^{\mathrm{b}}(X, S) = \overline{\mathbf{D}}^{\mathrm{b}}(X) \times_{\mathbf{KA}(X)} \bigoplus_p \widetilde{\mathcal{D}}_D^{2p-1}(X, S, p)$$

to be the category whose objects are equivalence classes $[\overline{\mathcal{F}}, \eta]$ of pairs $(\overline{\mathcal{F}}, \eta)$ in $\operatorname{Ob} \overline{\mathbf{D}}^{\mathrm{b}}(X) \times \oplus_p \widetilde{\mathcal{D}}_D^{2p-1}(X, S, p)$, under the equivalence relation

$$(\overline{\mathcal{F}}, \eta) \sim (\overline{\mathcal{F}} + [\overline{E}], \eta - \widetilde{\mathrm{ch}}(\overline{E}))$$

for $[\overline{E}] \in \mathbf{KA}(X)$, and with morphisms

$$\operatorname{Hom}_{\widehat{\mathbf{D}}^{\mathrm{b}}(X)}([\overline{\mathcal{F}}, \eta], [\overline{\mathcal{G}}, \nu]) = \operatorname{Hom}_{\mathbf{D}^{\mathrm{b}}(X)}(\mathcal{F}, \mathcal{G}).$$

Observe that if $S \subset T$ are closed conical subsets of T^*X_0, then $\widehat{\mathbf{D}}^{\mathrm{b}}(X, S)$ is naturally a full subcategory of $\widehat{\mathbf{D}}^{\mathrm{b}}(X, T)$.

In the sequel, we extend the main operations in $\mathbf{D}^{\mathrm{b}}(X)$ to the categories $\widehat{\mathbf{D}}^{\mathrm{b}}(X, S)$. In particular, we use the theory of generalized analytic torsion classes to construct push-forward morphisms attached to morphisms in $\overline{\mathbf{Sm}}_{*/\mathbb{C}}$.

The category $\widehat{\mathbf{D}}^{\mathrm{b}}(X,S)$ has a natural additive structure. More generally, if S,T are closed conical subsets of T^*X_0, then there is an obvious addition functor

$$\widehat{\mathbf{D}}^{\mathrm{b}}(X,S)\times\widehat{\mathbf{D}}^{\mathrm{b}}(X,T)\xrightarrow{\oplus}\widehat{\mathbf{D}}^{\mathrm{b}}(X,S\cup T).$$

Observe the object $[\overline{0},0]$ is a neutral element for this operation. Assume that $S+T$ is disjoint with the zero section in T^*X. Then there is a product defined by the functor

$$\begin{aligned}\operatorname{Ob}\widehat{\mathbf{D}}^{\mathrm{b}}(X,S)\times\operatorname{Ob}\widehat{\mathbf{D}}^{\mathrm{b}}(X,T)&\xrightarrow{\otimes}\operatorname{Ob}\widehat{\mathbf{D}}^{\mathrm{b}}(X,(S+T)\cup S\cup T)\\ ([\overline{\mathcal{F}},\eta],[\overline{\mathcal{G}},\nu])&\longmapsto[\overline{\mathcal{F}}\otimes\overline{\mathcal{G}},\operatorname{ch}(\overline{\mathcal{F}})\bullet\nu+\eta\bullet\operatorname{ch}(\overline{\mathcal{G}})+\mathrm{d}_D\,\eta\bullet\nu]\end{aligned}\tag{3.165}$$

and the obvious assignment for morphisms. This product is commutative up to natural isomorphism. It induces on $\widehat{\mathbf{D}}^{\mathrm{b}}(X,\emptyset)$ a structure of commutative and associative ring category. Also, $[\overline{\mathcal{O}}_X,0]$ is a unity object for the product structure. More generally, the category $\widehat{\mathbf{D}}^{\mathrm{b}}(X,S)$ becomes a left and right $\widehat{\mathbf{D}}^{\mathrm{b}}(X,\emptyset)$ module. Under the same assumptions on S, T we may define an internal Hom. For this, let $[\overline{\mathcal{F}},\eta]\in\operatorname{Ob}\widehat{\mathbf{D}}^{\mathrm{b}}(X,S)$ and $[\overline{\mathcal{G}},\nu]\in\operatorname{Ob}\widehat{\mathbf{D}}^{\mathrm{b}}(X,T)$. Then we put

$$\underline{\operatorname{Hom}}([\overline{\mathcal{F}},\eta],[\overline{\mathcal{G}},\nu])=[\underline{\operatorname{Hom}}(\overline{\mathcal{F}},\overline{\mathcal{G}}),(\sigma\operatorname{ch}(\overline{\mathcal{F}}))\bullet\nu+(\sigma\eta)\bullet\operatorname{ch}(\overline{\mathcal{G}})+(\mathrm{d}_D\,\sigma\eta)\bullet\nu],$$

where we recall that σ is the sign operator (Definition 3.139). Using Corollary 3.144, it is easily seen this is well defined. In particular, we put

$$[\overline{\mathcal{F}},\eta]^{\vee}:=\underline{\operatorname{Hom}}([\overline{\mathcal{F}},\eta],[\overline{\mathcal{O}}_X,0])=[\overline{\mathcal{F}}^{\vee},\sigma\eta].$$

The shift [1] on $\overline{\mathbf{D}}^{\mathrm{b}}(X)$ induces a well defined shift functor on $\widehat{\mathbf{D}}^{\mathrm{b}}(X,S)$, whose action on objects is

$$[\overline{\mathcal{F}},\eta][1]=[\overline{\mathcal{F}}[1],-\eta].$$

There is a Chern character

$$\begin{aligned}\operatorname{ch}:\operatorname{Ob}\widehat{\mathbf{D}}^{\mathrm{b}}(X,S)&\longrightarrow\bigoplus_p\widetilde{\mathcal{D}}^{2p}_D(X,S,p)\\ [\overline{\mathcal{F}},\eta]&\longmapsto\operatorname{ch}(\overline{\mathcal{F}})+\mathrm{d}_D\,\eta,\end{aligned}$$

which is well defined because $\mathrm{d}_D\,\widetilde{\operatorname{ch}}(\overline{E})=\operatorname{ch}(\overline{E})$ for $[\overline{E}]\in\mathbf{KA}(X)$. The Chern character is additive and compatible with the product structure:

$$\operatorname{ch}([\overline{\mathcal{F}},\eta]\otimes[\overline{\mathcal{G}},\nu])=\operatorname{ch}([\overline{\mathcal{F}},\eta])\bullet\operatorname{ch}([\overline{\mathcal{G}},\nu]).$$

Notice the relations

$$\begin{aligned}\operatorname{ch}([\overline{\mathcal{F}},\eta]^{\vee})&=\sigma\operatorname{ch}([\overline{\mathcal{F}},\eta]),\\ \operatorname{ch}([\overline{\mathcal{F}},\eta][1])&=-\operatorname{ch}([\overline{\mathcal{F}},\eta]).\end{aligned}$$

We may also define Bott-Chern classes for isomorphisms and distinguished triangles. Let $\widehat{\varphi}\colon [\overline{\mathcal{F}},\eta] \dashrightarrow [\overline{\mathcal{G}},\nu]$ be an isomorphism in $\widehat{\mathbf{D}}^{\mathrm{b}}(X,S)$, whose underlying morphism in $\mathbf{D}^{\mathrm{b}}(X)$ we denote φ. While the class $\widetilde{\mathrm{ch}}(\varphi\colon \overline{\mathcal{F}} \dashrightarrow \overline{\mathcal{G}})$ depends on the representatives $(\overline{\mathcal{F}},\eta)$, $(\overline{\mathcal{G}},\nu)$, the class

$$\widetilde{\mathrm{ch}}(\widehat{\varphi}) := \widetilde{\mathrm{ch}}(\varphi\colon \overline{\mathcal{F}} \dashrightarrow \overline{\mathcal{G}}) + \nu - \eta$$

is well defined.

Lemma 3.166. *Let $\widehat{\varphi} : [\overline{\mathcal{F}},\eta] \dashrightarrow [\overline{\mathcal{G}},\nu]$ be an isomorphism in $\widehat{\mathbf{D}}^{\mathrm{b}}(X,S)$, with underlying morphism φ in $\mathbf{D}^{\mathrm{b}}(X)$. Then, the following conditions are equivalent:*

(i) there exists $[\overline{E}] \in \mathbf{KA}(X)$ such that φ induces a tight isomorphism between $\overline{\mathcal{F}} + [\overline{E}]$ and $\overline{\mathcal{G}}$, and $\nu = \eta - \widetilde{\mathrm{ch}}(\overline{E})$;

(ii) we have

$$\widetilde{\mathrm{ch}}(\widehat{\varphi}) = 0.$$

Proof. This is actually a tautology. Because $\mathbf{KA}(X)$ acts freely transitively on the possible hermitian structures on $\mathcal{F}$, there exists a unique $[\overline{E}] \in \mathbf{KA}(X)$ such that $\overline{\mathcal{F}} + [\overline{E}]$ is tightly isomorphic to $\overline{\mathcal{G}}$ via the morphism φ. Then we have

$$\widetilde{\mathrm{ch}}(\widehat{\varphi}) = \widetilde{\mathrm{ch}}(\overline{E}) + \nu - \eta.$$

The lemma follows. □

Definition 3.167. Let $\widehat{\varphi}$ be an isomorphism in $\widehat{\mathbf{D}}^{\mathrm{b}}(X,S)$. We say that $\widehat{\varphi}$ is tight if the equivalent conditions of Lemma 3.166 are satisfied.

In particular, if $\varphi\colon \overline{\mathcal{F}} \dashrightarrow \overline{\mathcal{G}}$ is a tight isomorphism in $\overline{\mathbf{D}}^{\mathrm{b}}(X)$, then φ induces a tight isomorphism $[\overline{\mathcal{F}},\eta] \dashrightarrow [\overline{\mathcal{G}},\nu]$ if and only if $\eta = \nu$.

The following lemma provides an example involving the notion of tight isomorphism.

Lemma 3.168. *Let $[\overline{\mathcal{F}},\eta] \in \widehat{\mathbf{D}}^{\mathrm{b}}(X,S)$ and $[\overline{\mathcal{G}},\nu] \in \widehat{\mathbf{D}}^{\mathrm{b}}(X,T)$. Assume that $S+T$ does not cross the zero section. Then there is a functorial tight isomorphism*

$$[\overline{\mathcal{F}},\eta]^{\vee} \otimes [\overline{\mathcal{G}},\nu] \cong \underline{\mathrm{Hom}}([\overline{\mathcal{F}},\eta],[\overline{\mathcal{G}},\nu]).$$

□

Assume now given a distinguished triangle

$$\widehat{\tau}\colon \quad [\overline{\mathcal{F}},\eta] \dashrightarrow [\overline{\mathcal{G}},\nu] \dashrightarrow [\overline{\mathcal{H}},\mu] \dashrightarrow [\overline{\mathcal{F}},\eta][1].$$

Let $\overline{\tau}$ denote the distinguished triangle $\overline{\mathcal{F}} \dashrightarrow \overline{\mathcal{G}} \dashrightarrow \overline{\mathcal{H}} \dashrightarrow$ in $\overline{\mathbf{D}}^{\mathrm{b}}(X)$. Then we put

$$\widetilde{\mathrm{ch}}(\widehat{\tau}) = \widetilde{\mathrm{ch}}(\overline{\tau}) + \eta - \nu + \mu.$$

By Theorem 2.67 (vii), this class does not depend on the representatives and is thus well defined.

We next consider the functoriality of $\widehat{\mathbf{D}}^{\mathrm{b}}(X,S)$ with respect to inverse and direct images.

Let $f\colon X \to Y$ be a morphism of smooth complex varieties. Let $T \subset T^*Y_0$ be a closed conical subset disjoint with N_f. Then we have a left inverse image functor f^* whose action on objects is

$$\begin{aligned} f^*\colon \operatorname{Ob}\widehat{\mathbf{D}}^{\mathrm{b}}(Y,T) &\longrightarrow \operatorname{Ob}\widehat{\mathbf{D}}^{\mathrm{b}}(X,f^*T) \\ [\overline{\mathcal{F}},\eta] &\longmapsto [f^*\overline{\mathcal{F}}, f^*\eta]. \end{aligned}$$

That this assignment is well defined amounts to the functoriality of $\widetilde{\mathrm{ch}}$.

Let $\overline{f}$ be a morphism in the category $\overline{\mathbf{Sm}}_{*/\mathbb{C}}$. The definition of a direct image functor attached to $\overline{f}$ depends upon the choice of a theory of generalized analytic torsion classes. Let T be such a theory. Then we define a functor $\overline{f}_*$ whose action on objects is

$$\begin{aligned} \overline{f}_*\colon \operatorname{Ob}\widehat{\mathbf{D}}^{\mathrm{b}}(X,S) &\longrightarrow \operatorname{Ob}\widehat{\mathbf{D}}^{\mathrm{b}}(Y,f_*S) \\ [\overline{\mathcal{F}},\eta] &\longmapsto [\overline{f_*\mathcal{F}}, \overline{f}_\flat(\eta) - T(\overline{f},\overline{\mathcal{F}},\overline{f_*\mathcal{F}})], \end{aligned} \tag{3.169}$$

where $\overline{f_*\mathcal{F}}$ is an arbitrary choice of hermitian structure on $f_*\mathcal{F}$. By the anomaly formulas, this definition does not depend on the representative $(\overline{\mathcal{F}},\eta)$ nor on the choice of hermitian structure on $\overline{f_*\mathcal{F}}$.

Theorem 3.170. *Let $\overline{f}: X \to Y$ and $\overline{g}: Y \to Z$ be morphisms in $\overline{\mathbf{Sm}}_{*/\mathbb{C}}$. Let $S \subset T^*X_0$ and $T \subset T^*Y_0$ be closed conical subsets.*

(i) Let $[\overline{\mathcal{F}},\eta] \in \operatorname{Ob}\widehat{\mathbf{D}}^{\mathrm{b}}(X,S)$. Then there is a functorial tight isomorphism

$$(\overline{g}\circ\overline{f})_*([\overline{\mathcal{F}},\eta]) \cong \overline{g}_*\overline{f}_*([\overline{\mathcal{F}},\eta]).$$

(ii) (Projection formula) Assume that $T \cap N_f = \emptyset$ and that $T + f_*S$ does not cross the zero section of T^*Y. Let $[\overline{\mathcal{F}},\eta] \in \operatorname{Ob}\widehat{\mathbf{D}}^{\mathrm{b}}(X,S)$ and $[\overline{\mathcal{G}},\nu] \in \operatorname{Ob}\widehat{\mathbf{D}}^{\mathrm{b}}(Y,T)$. Then there is a functorial tight isomorphism

$$\overline{f}_*([\overline{\mathcal{F}},\eta] \otimes f^*[\overline{\mathcal{G}},\nu]) \cong \overline{f}_*[\overline{\mathcal{F}},\eta] \otimes [\overline{\mathcal{G}},\nu]$$

in $\widehat{\mathbf{D}}^{\mathrm{b}}(Y,W)$, where

$$W = f_*(S + f^*T) \cup f_*S \cup f_*f^*T.$$

(iii) (Base change) Consider a cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{h'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{h} & Y. \end{array}$$

Suppose that f and h are transverse and that $N_{h'}$ is disjoint with S. Equip f' with the hermitian structure induced by the natural isomorphism $h^*T_f \dashrightarrow T_{f'}$. Let $[\overline{\mathcal{F}},\eta] \in \operatorname{Ob}\widehat{\mathbf{D}}^{\mathrm{b}}(X,S)$. Then there is a functorial tight isomorphism

$$h^*\overline{f}_*[\overline{\mathcal{F}},\eta] \cong \overline{f}'_*h'^*[\overline{\mathcal{F}},\eta]$$

in $\widehat{\mathbf{D}}^{\mathrm{b}}(Y', f'_*h'^*S)$.

Proof. For the proof of the first and the second assertions, it is enough to take into account Proposition 2.121, the transitivity and the projection formula for T. For the third item, one uses the functoriality of the analytic torsion classes together with Proposition 3.9. $\square$

We close this section with an extension of Grothendieck duality to $\widehat{\mathbf{D}}^{\mathrm{b}}$. We need to introduce a last functor. Let $\overline{f} : X \to Y$ be a morphism is $\overline{\mathbf{Sm}}_{*/\mathbb{C}}$. To enlighten notations, we denote by $\boldsymbol{\omega}_{\overline{f}}$ the object $[\boldsymbol{\omega}_{\overline{f}}, 0]$ in $\widehat{\mathbf{D}}^{\mathrm{b}}(X, \emptyset)$ (Definition 3.138). Suppose given a closed conical subset $T \subset T^*Y_0$ such that $T \cap N_f = \emptyset$. Then we define the functor $\overline{f}^!$ whose action on objects is

$$\begin{aligned}\overline{f}^! : \operatorname{Ob}\widehat{\mathbf{D}}^{\mathrm{b}}(Y, T) &\longrightarrow \operatorname{Ob}\widehat{\mathbf{D}}^{\mathrm{b}}(X, f^*T)\\ [\overline{\mathcal{F}}, \eta] &\longmapsto f^*[\overline{\mathcal{F}}, \eta] \otimes \boldsymbol{\omega}_{\overline{f}}.\end{aligned}$$

Observe the equality

$$[\overline{\mathcal{G}}, \nu] \otimes \boldsymbol{\omega}_{\overline{f}} = [\overline{\mathcal{G}} \otimes \boldsymbol{\omega}_{\overline{f}}, \nu \bullet \operatorname{ch}(\boldsymbol{\omega}_{\overline{f}})]. \tag{3.171}$$

Now fix a theory of generalized analytic torsion classes. To the morphism $\overline{f}$ there is an attached direct image functor $\overline{f}_*$. We may as well consider the dual theory (Theorem Definition 3.146). Then we denote by $\overline{f}_*^{\vee}$ the direct image functor associated to $\overline{f}$ and the dual theory.

Theorem 3.172 (Grothendieck duality for $\widehat{\mathbf{D}}^{\mathrm{b}}$)**.** *Let $\overline{f} : X \to Y$ be a morphism in $\overline{\mathbf{Sm}}_{*/\mathbb{C}}$. Let $S \subset T^*X_0$ and $T \subset T^*Y_0$ be closed conical subsets such that $T \cap N_f = \emptyset$ and $T + f_*S$ is disjoint with the zero section. Let $[\overline{\mathcal{F}}, \eta] \in \operatorname{Ob}\widehat{\mathbf{D}}^{\mathrm{b}}(X, S)$ and $[\overline{\mathcal{G}}, \sigma] \in \operatorname{Ob}\widehat{\mathbf{D}}^{\mathrm{b}}(Y, T)$. Then there is a functorial tight isomorphism*

$$\underline{\operatorname{Hom}}(\overline{f}_*[\overline{\mathcal{F}}, \eta], [\overline{\mathcal{G}}, \nu]) \cong \overline{f}_*^{\vee}\underline{\operatorname{Hom}}([\overline{\mathcal{F}}, \eta], \overline{f}^![\overline{\mathcal{G}}, \nu])$$

in $\widehat{\mathbf{D}}^{\mathrm{b}}(Y, W)$, where

$$W = f_*(S + f^*T) \cup f_*S \cup f_*f^*T.$$

In particular, we have

$$(\overline{f}_*[\overline{\mathcal{F}}, \eta])^{\vee} \cong \overline{f}_*^{\vee}([\overline{\mathcal{F}}, \eta]^{\vee} \otimes \boldsymbol{\omega}_{\overline{f}}). \tag{3.173}$$

Proof. By Lemma 3.168 and Proposition 3.170, we are reduced to establish the functorial tight isomorphism (3.173). The proof follows readily from the definitions, Grothendieck duality and the following two observations. First of all, if T is the theory of analytic torsion classes, then by the very definition of $T^{\vee}$ we find

$$\sigma T(\overline{f}, \overline{\mathcal{F}}, \overline{f_*\mathcal{F}}) = T^{\vee}(\overline{f}, \overline{\mathcal{F}}^{\vee} \otimes \boldsymbol{\omega}_{\overline{f}}, \overline{f_*(\mathcal{F}^{\vee} \otimes \boldsymbol{\omega}_f)}),$$

where the metric on $\overline{f_*(\mathcal{F}^{\vee} \otimes \boldsymbol{\omega}_f)}$ is chosen so that Grothendieck duality provides a tight isomorphism

$$\overline{f_*\mathcal{F}}^{\vee} \cong \overline{f_*(\mathcal{F}^{\vee} \otimes \boldsymbol{\omega}_f)}.$$

Secondly, for direct images of currents, we compute

$$\begin{aligned}\sigma\overline{f}_\flat(\eta) = \sigma f_*(\eta \bullet \operatorname{Td}(T_{\overline{f}})) &=(-1)^e f_*(\sigma\eta \bullet \sigma \operatorname{Td}(T_{\overline{f}}))\\ &= f_*(\sigma\eta \bullet \operatorname{ch}(\boldsymbol{\omega}_{\overline{f}}) \bullet \operatorname{Td}(T_{\overline{f}})).\end{aligned}$$

Here e is the relative dimension of f, and to derive the last equality we appeal to Proposition 3.140. To conclude, we recall equation (3.171). □

Corollary 3.174. *Let T be a self-dual theory of generalized analytic torsion classes.*

(i) Then there is a functorial isomorphism

$$(\overline{f}_*[\overline{\mathcal{F}},\eta])^\vee \cong \overline{f}_*([\overline{\mathcal{F}},\eta]^\vee \otimes \boldsymbol{\omega}_{\overline{f}}).$$

(ii) If the hermitian structure of $\overline{f}$ comes from chosen metrics on T_X, T_Y and $\boldsymbol{\omega}_X$, $\boldsymbol{\omega}_Y$ are equipped with the induced metrics, then there is a commutative diagram

$$\begin{array}{ccc}\widehat{\mathbf{D}}^{\mathrm{b}}(X,S) & \xrightarrow{(\cdot)^\vee\otimes\overline{\boldsymbol{\omega}}_X} & \widehat{\mathbf{D}}^{\mathrm{b}}(X,S)\\ \Big\downarrow{\scriptstyle \overline{f}_*} & & \Big\downarrow{\scriptstyle \overline{f}_*}\\ \widehat{\mathbf{D}}^{\mathrm{b}}(Y,f_*S) & \xrightarrow{(\cdot)^\vee\otimes\overline{\boldsymbol{\omega}}_Y} & \widehat{\mathbf{D}}^{\mathrm{b}}(Y,f_*S).\end{array}$$

Proof. The first claim is immediate from Theorem 3.172. The second item follows from the first one and the projection formula (Proposition 3.170). □

3.10 Analytic torsion for degenerating families of curves

As a second example of application of the theory developed in this article, we describe the singularities of the analytic torsion for degenerating families of curves. The results we prove are particular instances of those obtained by Bismut-Bost [6], Bismut [4] and Yoshikawa [50]. However our approach is more elementary: we combine the geometry of such families and the existence of our analytic torsion classes for arbitrary projective morphisms. The existence of analytic torsion classes for arbitrary projective morphisms allows us to avoid the use of the spectral theory of the laplacian operator. For simplicity we will restrict to fibrations in curves over a curve.

Relative curves with ordinary double points. Let S be a smooth complex curve and $f\colon X \to S$ a projective morphism of smooth complex varieties, whose fibers are reduced curves with at most ordinary double singular points. We assume that f is generically smooth. Following Bismut-Bost [6, Sec. 2(b)], we call such a family an f.s.o. (*famille à singularités ordinaires*). The singular locus of f, to be denoted Σ, is a zero dimensional reduced closed subset of X. Its direct image $\Delta = f_*(\Sigma)$ is the Weil divisor

$$\Delta = \sum_{p\in S} n_p p,$$

where n_p is the number of singular points of the fiber $f^{-1}(p)$. We will abusively identify Δ with its support. With these notations, we put $V = S \setminus \Delta$. Locally for the analytic topology, the morphism f can be written in complex coordinates either as $f(z_0, z_1) = z_0$ or $f(z_0, z_1) = z_0 z_1$ [6, Sec. 3(a)]. In the second case, the point of coordinates $(z_0, z_1) = (0, 0)$ belongs to the singular locus Σ.

For a vector bundle F over X, we denote by $\mathbb{P}(F)$ the projective space of lines in F. The differential $df : T_X \to f^*T_S$ induces a section $\mathcal{O}_X \to \Omega_X \otimes f^*T_S$. Because f is smooth over $X \setminus \Sigma$, this section does not vanish on $X \setminus \Sigma$. Therefore there is an induced map

$$\mu : X \setminus \Sigma \longrightarrow \mathbb{P}(\Omega_X \otimes f^*T_S) \cong \mathbb{P}(\Omega_X),$$

called the *Gauss map*. Notice that this map was already used in the works of Bismut [4] and Yoshikawa [50].

We next study the blow-up of X at Σ and relate it to the Gauss map. Let $\widetilde{X} = \mathrm{Bl}_\Sigma(X)$ be this blow-up and denote by $\pi : \widetilde{X} \to X$ the natural projection. Let E be the exceptional divisor of π. This is a disjoint union

$$E = \bigsqcup_{p \in \Sigma} E_p, \quad E_p \cong \mathbb{P}(T_pX),$$

with the reduced scheme structure.

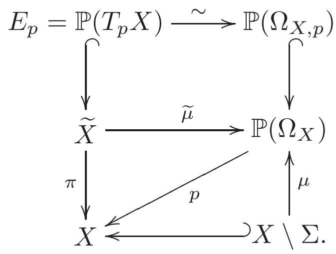
Lemma 3.175. *Denote by $\widetilde{f} : \widetilde{X} \to S$ the composite $f \circ \pi$. Then $\widetilde{f}$ is an f.s.o. If Σ' is the singular locus of $\widetilde{f}$, then $\Sigma' \to \Sigma$ is a 2 to 1 finite covering. In particular, we have*

$$\widetilde{f}_*(\Sigma') = 2\Delta.$$

Proof. The local description of the blow-up easily shows that if $p \in E$ is a singular point of $\widetilde{f}$, then there exist analytic coordinates (z_0, z_1) centered at p such that E is the divisor $z_0 = 0$ and $f(z_0, z_1) = z_0 z_1$. Furthermore, for every $p \in \Sigma$, there are exactly two points $p_1, p_2 \in \Sigma'$ with $\pi(p_1) = \pi(p_2) = p$. They correspond to the directions in T_pX of the two branches of $f^{-1}(f(p))$ through p. □

For every $p \in \Sigma$, there is an identification $T_pX \cong \Omega_{X,p}$ provided by the hessian of f, which is a non-degenerate bilinear form on T_pX.

Lemma 3.176. *There is a commutative diagram*



Denote by $\mathcal{O}(-1)$ the tautological divisor either on $\mathbb{P}(\Omega_X)$ or on E_p. Then there is a natural isomorphism

$$\widetilde{\mu}^*\mathcal{O}(-1)\,|_{E_p} \cong \mathcal{O}(-1).$$

Proof. This is easily checked by the local description of the blow-up at a point. □

Consider now the short exact sequence of vector bundles on $\mathbb{P}(\Omega_X)$

$$0 \to \mathcal{O}(-1) \to p^*\Omega_X \to Q \to 0,$$

where Q is the universal quotient bundle. Observe that Q is of rank 1. The dual exact sequence is

$$0 \to U \to p^*T_X \to \mathcal{O}(1) \to 0,$$

where U is the universal vector subsheaf. We will denote by η the induced exact sequence on $\widetilde{X}$

$$\eta\colon 0 \to \widetilde{\mu}^*U \to \pi^*T_X \to \widetilde{\mu}^*\mathcal{O}(1) \to 0, \tag{3.177}$$

From this exact sequence and the definition $\omega_{X/S} = \omega_X \otimes f^*T_S$, we derive a natural isomorphism

$$\widetilde{\mu}^*U \otimes \pi^*\omega_{X/S} \cong \widetilde{\mu}^*\mathcal{O}(-1) \otimes \widetilde{f}^*T_S. \tag{3.178}$$

Lemma 3.179. *We have*

$$\widetilde{\mu}^*\mathcal{O}(-1) \otimes \widetilde{f}^*T_S = \mathcal{O}(E). \tag{3.180}$$

Proof. First of all we observe that $\widetilde{\mu}^*U \otimes \pi^*\omega_{X/S}$ is trivial on the open $W = \widetilde{X} \setminus E$. Indeed, by construction of the Gauss map we have

$$\widetilde{\mu}^*U \mid_W = \ker(df\colon T_X \to f^*T_S) \mid_W = \omega^\vee_{X/S} \mid W.$$

Hence by equation (3.178) we can write

$$\widetilde{\mu}^*\mathcal{O}(-1) \otimes \widetilde{f}^*T_S = \mathcal{O}(\sum_{p\in\Sigma} m_p E_p).$$

To compute the multiplicities m_p we take into account that $\widetilde{\mu}^*\mathcal{O}(-1) \mid_{E_p} = \mathcal{O}(-1)$, $(E_p \cdot \widetilde{f}^*T_S) = 0$ and $(E_p \cdot E_p) = -1$:

$$-m_p = \deg(\widetilde{\mu}^*\mathcal{O}(-1) \otimes \widetilde{f}^*T_S) \mid_{E_p} = -1 + 0 = -1.$$

The lemma follows. □

Later we will need the commutative diagram of exact sequences

$$\begin{array}{lccccccccc}
\eta \mid_W\colon & 0 & \longrightarrow & \widetilde{\mu}^*U \mid_W & \longrightarrow & T_X \mid_W & \longrightarrow & \widetilde{\mu}^*\mathcal{O}(1) \mid_W & \longrightarrow & 0 \\
& & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\
\varepsilon\colon & 0 & \longrightarrow & \omega^\vee_{X/S} \mid_W & \longrightarrow & T_X \mid_W & \longrightarrow & f^*T_S \mid_W & \longrightarrow & 0.
\end{array} \tag{3.181}$$

After the identification $\widetilde{\mu}^*\mathcal{O}(-1) \otimes \widetilde{f}^*T_S = \mathcal{O}(E)$ provided by the lemma, the morphism γ is the restriction to W of the natural inclusion $\widetilde{\mu}^*\mathcal{O}(1) \to \widetilde{\mu}^*\mathcal{O}(1) \otimes \mathcal{O}(E)$. This fact will be used below.

Hermitian structures and analytic torsion classes. We now proceed to introduce the hermitian vector bundles and the analytic torsion classes we aim to study. We fix a theory of generalized analytic torsion classes T.

Let $f\colon X \to S$, $\widetilde{f}\colon \widetilde{X} \to S$ be f.s.o. as above. Recall that we write $W = X \setminus \Sigma = \widetilde{X} \setminus E$ and $V = S \setminus \Delta$, so that $f^{-1}(V) \subset W$. We endow the tangent spaces T_X and T_S with smooth hermitian metrics. We will denote by $\overline{f}$ the corresponding morphism in the category $\overline{\mathbf{Sm}}_{*/\mathbb{C}}$. On the open subset $W = X\setminus\Sigma$, there is a quasi-isomorphism

$$\omega^{\vee}_{X/S} \mid_W = \boldsymbol{\omega}^{\vee}_{X/S}[1] \mid_W \to T_f$$

induced by the identification $\omega^{\vee}_{X/S} \mid_W = \ker(T_X \mid_W \to f^* T_S)$. On $\omega^{\vee}_{X/S} \mid_W$, and in particular on $\omega^{\vee}_{f^{-1}(V)/V}$, we will put the metric induced by $\overline{T_X} \mid_W$. We will write $\overline{f}'\colon f^{-1}(V) \to V$ for the corresponding morphism in $\overline{\mathbf{Sm}}_{*/\mathbb{C}}$. Observe that the restriction of f to W, and hence to $f^{-1}(V)$, may be identified with the restriction of $\widetilde{f}$. Let $\overline{\mathcal{F}}$ be an object in $\overline{\mathbf{D}}^{\mathrm{b}}(X)$ and fix a hermitian structure on $f_*\mathcal{F}$. Then we consider the relative metrized complexes

$$\overline{\xi} = (\overline{f}, \overline{\mathcal{F}}, \overline{f_*\mathcal{F}}),$$
$$\overline{\xi}' = (\overline{f}', \overline{\mathcal{F}} \mid_{f^{-1}(V)}, \overline{f_*\mathcal{F}} \mid_V),$$

and the corresponding analytic torsion classes

$$T(\overline{\xi}) \in \bigoplus_p \widetilde{\mathcal{D}}^{2p-1}_D(S, N_f, p),$$
$$T(\overline{\xi}') \in \bigoplus_p \widetilde{\mathcal{D}}^{2p-1}_D(V, \emptyset, p).$$

By the functoriality of analytic torsion classes and the anomaly formulas, we have

$$T(\overline{\xi}') = T(\overline{\xi}) \mid_V - \overline{f}_\flat[\mathrm{ch}(\overline{\mathcal{F}} \mid_{f^{-1}(V)}) \widetilde{\mathrm{Td}}_m(\overline{\varepsilon} \mid_{f^{-1}(V)})]. \tag{3.182}$$

Here $\overline{\varepsilon}$ is the exact sequence in (3.181), with the hermitian metrics we have just defined. From now on we will lighten the notations by omitting the reference to $f^{-1}(V)$ and V in the formulas.

We consider the hermitian structures on the sheaves U and $\mathcal{O}(1)$ on $\mathbb{P}(\Omega_X)$ induced by $p^*\overline{T_X}$. We will write $\overline{\eta}$ for the exact sequence in (3.177) and $\overline{\alpha}$, $\overline{\beta}$ and $\overline{\gamma}$ for the vertical isomorphisms in diagram (3.181), all provided with the corresponding metrics. Notice that $\overline{\alpha}$ and $\overline{\beta}$ are isometries. By the properties of the Bott-Chern class $\widetilde{\mathrm{Td}}_m$, we have

$$\widetilde{\mathrm{Td}}_m(\overline{\varepsilon}) = \widetilde{\mathrm{Td}}_m(\overline{\eta}) + \mathrm{Td}(\overline{\eta})\widetilde{\mathrm{Td}}_m(\overline{\gamma}). \tag{3.183}$$

Hence, from (3.182)–(3.183) and identifying f with $\widetilde{f}$ over V, we have

$$\begin{aligned} T(\overline{\xi}') = T(\overline{\xi}) &- \widetilde{f}_*[\pi^* \mathrm{ch}(\overline{\mathcal{F}})\pi^* \mathrm{Td}(\overline{f})\widetilde{\mathrm{Td}}_m(\overline{\eta})] \\ &- \widetilde{f}_*[\pi^* \mathrm{ch}(\overline{\mathcal{F}})\pi^* \mathrm{Td}(\overline{f})\, \mathrm{Td}(\overline{\eta})\widetilde{\mathrm{Td}}_m(\overline{\gamma})]. \end{aligned} \tag{3.184}$$

It will be convenient to have a precise description of $\widetilde{\mathrm{Td}}_m(\overline{\gamma})$ at our disposal.

The class $\widetilde{\mathrm{Td}}_m(\overline{\gamma})$. For shorthand, we write $L := \widetilde{\mu}^*\mathcal{O}(1)$ and $\|\cdot\|_0$ for its hermitian structure constructed before. We denote by $\|\cdot\|_1$ the metric on $\mathcal{O}(E)$ such that the isomorphism $\overline{\mathcal{O}(E)}_1 = \overline{L}_0^{-1} \otimes \widetilde{f}^*\overline{T_S}$ (Lemma 3.179) is an isometry. Recall that γ gets identified with the restriction to W of the natural inclusion $L \to L \otimes \mathcal{O}(E)$. We let $\|\cdot\|_\infty$ be the hermitian metric on $L\mid_W$ such that γ is an isometry. Hence, if $\mathbf{1}$ denotes the canonical section of $\mathcal{O}(E)$ and ℓ is any section of $L\mid_W$, then we have

$$\|\ell\|_\infty = \|\ell\|_0 \|\mathbf{1}\|_1.$$

To simplify the notations, we will skip the reference to W. With this convention, we have on W

$$\widetilde{\mathrm{Td}}_m(\overline{\gamma}) = \widetilde{\mathrm{Td}}_m(\overline{L}_0 \overset{\mathrm{id}}{\to} \overline{L}_\infty).$$

To compute a representative of this class, we fix a smooth function $h\colon \mathbb{P}^1_{\mathbb{C}} \to \mathbb{R}$ such that $h(0) = 0$ and $h(\infty) = 1$. Then we proceed by a deformation argument. Let $q\colon W \times \mathbb{P}^1_{\mathbb{C}} \to W$ be the projection to the first factor. On the line bundle q^*L we put the metric that, on the fiber at the point $(w,t) \in W \times \mathbb{P}^1_{\mathbb{C}}$, is determined by the formula

$$\|\ell\|_{(w,t)} = \|\ell\|_{0,w} \|\mathbf{1}\|_{1,w}^{h(t)}.$$

We will write $\|\cdot\|_t$ for this family of metrics parametrized by $\mathbb{P}^1_{\mathbb{C}}$. Define

$$\overline{\mathrm{Td}}(\overline{L}_0 \to \overline{L}_\infty) = \frac{1}{2\pi i}\int_{\mathbb{P}^1_{\mathbb{C}}} \frac{-1}{2}\log(t\bar{t})(\mathrm{Td}(\overline{q^*L_t}) - \mathrm{Td}(\overline{q^*L_0})).$$

Then

$$\overline{\mathrm{Td}}_m(\overline{\gamma}) = \mathrm{Td}^{-1}(\overline{L}_0)\overline{\mathrm{Td}}(\overline{L}_0 \to \overline{L}_\infty) \tag{3.185}$$

represents the class $\widetilde{\mathrm{Td}}_m(\gamma)$. Let us develop $\overline{\mathrm{Td}}_m(\overline{\gamma})$. If $\overline{\mathcal{O}}_t$ denotes the trivial line bundle on $W \times \mathbb{P}^1_{\mathbb{C}}$ with the norm $\|\mathbf{1}\|_t = \|\mathbf{1}\|_1^{h(t)}$, then we compute

$$\mathrm{Td}(\overline{q^*L_t}) - \mathrm{Td}(\overline{q^*L_0}) = \frac{1}{2}c_1(\overline{\mathcal{O}}_t) + \frac{1}{6}c_1(\overline{\mathcal{O}}_t)q^*c_1(\overline{L}_0) + \frac{1}{12}c_1(\overline{\mathcal{O}}_t)^2.$$

By the very definition of c_1, we find

$$\begin{aligned} c_1(\overline{\mathcal{O}}_t) = \partial\bar{\partial}\log\|\mathbf{1}\|_t^2 &= \partial\bar{\partial}(h(t)\log\|\mathbf{1}\|_1^2) \\ &= h(t)c_1(\overline{\mathcal{O}(E)}_1) + \log\|\mathbf{1}\|_1^2\,\partial\bar{\partial}h(t) \\ &\quad + \partial h(t) \wedge \bar{\partial}\log\|\mathbf{1}\|_1^2 + \partial\log\|\mathbf{1}\|_1^2 \wedge \bar{\partial}h(t). \end{aligned}$$

We easily obtain

$$\frac{1}{2\pi i}\int_{\mathbb{P}^1_{\mathbb{C}}} \frac{-1}{2}\log(t\bar{t})\frac{1}{2}c_1(\overline{\mathcal{O}}_t) = -\frac{1}{2}\log\|\mathbf{1}\|_1, \tag{3.186}$$

$$\frac{1}{2\pi i}\int_{\mathbb{P}^1_{\mathbb{C}}} \frac{-1}{2}\log(t\bar{t})\frac{1}{6}q^*c_1(\overline{L}_0)c_1(\overline{\mathcal{O}}_t) = -\frac{1}{6}\log\|\mathbf{1}\|_1 c_1(\overline{L}_0). \tag{3.187}$$

With some more work, we have

$$\begin{aligned} \frac{1}{2\pi i}\int_{\mathbb{P}^1_{\mathbb{C}}} \frac{-1}{2}\log(t\bar{t})\frac{1}{12}c_1(\overline{\mathcal{O}}_t)^2 = &-\frac{a}{6}\log\|\mathbf{1}\|_1 c_1(\overline{\mathcal{O}(E)}_1) \\ &+\frac{b}{3}\,\partial(\log\|\mathbf{1}\|_1\ \bar{\partial}\log\|\mathbf{1}\|_1), \end{aligned} \tag{3.188}$$

where

$$a = \frac{1}{2\pi i}\int_{\mathbb{P}^1_{\mathbb{C}}} \log(t\bar{t})\frac{1}{2}\,\partial\,\overline{\partial}(h(t)^2),$$

$$b = \frac{1}{2\pi i}\int_{\mathbb{P}^1_{\mathbb{C}}} \log(t\bar{t})\,\partial\, h(t)\wedge\overline{\partial}\, h(t). \tag{3.189}$$

We observe that

$$a = \frac{1}{2\pi i}\int_{\mathbb{P}^1_{\mathbb{C}}} \log(t\bar{t})\frac{1}{2}\,\partial\,\overline{\partial}(h(t)^2) = \frac{1}{2},$$

which is independent of h. All in all, equations (3.185)–(3.188) provide the following expression for the representative $\overline{\mathrm{Td}}_m(\overline{\gamma})$ of $\widetilde{\mathrm{Td}}_m(\overline{\gamma})$:

$$\begin{aligned}\overline{\mathrm{Td}}_m(\overline{\gamma}) = \mathrm{Td}^{-1}(\overline{L}_0)\Big(-\frac{1}{2}\log\|\mathbf{1}\|_1 - \frac{1}{6}\log\|\mathbf{1}\|_1 c_1(\overline{L}_0)\\ - \frac{1}{12}\log\|\mathbf{1}\|_1 c_1(\overline{\mathcal{O}(E)}_1)\\ + \frac{b}{3}\,\partial(\log\|\mathbf{1}\|_1\,\overline{\partial}\log\|\mathbf{1}\|_1)\Big)\end{aligned} \tag{3.190}$$

where b is given by (3.189).

The component of Deligne bidegree $(1,1)$**.** Given a current $\eta \in \mathcal{D}^n_D(X,p)$, we will call (n,p) its *Deligne bidegree*, while we will call the *Dolbeault bidegree* to the bidegree in the Dolbeault complex. When it is clear from the context to which bidegree we are referring, we call it bidegree.

We now study the singularities of the component of Deligne bidegree $(1,1)$ of $T(\overline{\xi}')$ near the divisor Δ. For this we first recall the decomposition of equation (3.184). Observe that $\widetilde{\mathcal{D}}^1_D(V,\emptyset,1)$ gets identified with the space of smooth real functions on V. In the sequel, for an element $\vartheta \in \oplus_p \widetilde{D}^{2p-1}_D(*,p)$, we write $\vartheta^{(2r-1,r)}$ to refer to its component of bidegree $(2r-1,r)$. By construction of the Deligne complex, an element of Deligne bidegree $(2r-1,r)$ is just a current of Dolbeault bidegree $(r-1,r-1)$.

Lemma 3.191. *Let $\Omega \subset \mathbb{C}$ be an open subset and ϑ a current of Dolbeault bidegree $(0,0)$ on Ω. Let Δ be the standard laplacian. If the current $\Delta\vartheta$ is represented by a locally bounded measurable function, then ϑ is represented by a continuous function.*

Proof. The assertion is a well-known fact. See for instance [48, Lemma 2.1, Cor. 2.2]. □

Proposition 3.192. *The current $T(\overline{\xi})^{(1,1)} \in \widetilde{\mathcal{D}}^1_D(S,N_f,1)$ is represented by a continuous function on S.*

Proof. The differential equation satisfied by $T(\overline{\xi})^{(1,1)}$ is

$$\mathrm{d}_{\mathcal{D}}\, T(\overline{\xi})^{(1,1)} = \mathrm{ch}(\overline{f_*\mathcal{F}})^{(2,1)} - f_*[\mathrm{ch}(\overline{\mathcal{F}})\,\mathrm{Td}(\overline{f})]^{(2,1)}. \tag{3.193}$$

In local coordinates, the operator $\mathrm{d}_{\mathcal{D}} = -2\,\partial\,\overline{\partial}$ is a rescaling of the laplacian Δ. By the lemma, it is enough we prove that the current at the right hand side of (3.193) is represented by a locally bounded measurable differential form. Because $\mathrm{ch}(\overline{f_*\mathcal{F}})^{(2,1)}$ and $\mathrm{ch}(\overline{\mathcal{F}})\,\mathrm{Td}(\overline{f})$ are smooth differential forms, we are reduced to study currents of the form $f_*[\theta]^{(2,1)}$, where θ is a smooth differential form. By a partition of unity argument, we reduce to the case where $f\colon \mathbb{C}^2 \to \mathbb{C}$ is the morphism $f(z_0, z_1) = z_0 z_1$ and θ is a differential form of Dolbeault bidegree (2,2) with compact support. Then we need to prove that the fiber integral

$$G(w) = \int_{z_0 z_1 = w} \theta$$

is a bounded form in a neighborhood of $w = 0$. Write

$$\theta = h(z_0, z_1) dz_0 \wedge d\overline{z}_0 \wedge dz_1 \wedge d\overline{z}_1.$$

We reduce to study integrals of the form

$$G(w) = \left(\int_{|w|<|z_0|<1} h(z_0, z_0/w) \frac{|w|^2}{|z_0|^4} dz_0 \wedge \overline{z}_0 \right) dw \wedge d\overline{w}.$$

The property follows from an easy computation in polar coordinates. □

Proposition 3.194. *Let θ be a differential form of Dolbeault bidegree (1,1) on $\widetilde{X}$. Then the current $\widetilde{f}_*[\theta]$ is represented by a continuous function on S.*

Proof. This is [6, Prop. 5.2]. □

Corollary 3.195. *The current $\widetilde{f}_*[\pi^* \mathrm{ch}(\overline{\mathcal{F}})\pi^* \mathrm{Td}(\overline{f})\widetilde{\mathrm{Td}}_m(\overline{\eta})]$ is represented by a continuous function on S.*

Proof. It suffices to observe that the differential form $\pi^* \mathrm{ch}(\overline{\mathcal{F}})\pi^* \mathrm{Td}(\overline{f})\widetilde{\mathrm{Td}}_m(\overline{\eta})$ is actually smooth on the whole $\widetilde{X}$. □

According to (3.184), it remains to study the current

$$\widetilde{f}_*[\pi^* \mathrm{ch}(\overline{\mathcal{F}})\pi^* \mathrm{Td}(\overline{f})\,\mathrm{Td}(\overline{\eta})\widetilde{\mathrm{Td}}_m(\overline{\gamma})]\ |_V\ .$$

The main difference with the situation in Corollary 3.195 is that the class $\widetilde{\mathrm{Td}}_m(\overline{\gamma})$ is not defined on the whole $\widetilde{X}$, but only on $W = \widetilde{X} \setminus E$. In the following discussion we will use the representative $\overline{\mathrm{Td}}_m(\overline{\gamma})$ defined in (3.185) at the place of $\widetilde{\mathrm{Td}}_m(\overline{\gamma})$. In view of equations (3.186)–(3.188), the first result we need is the following statement.

Proposition 3.196. *Let θ be a smooth and $\partial, \overline{\partial}$ closed differential form on $\widetilde{X}$, of Dolbeault bidegree $(1,1)$. Let w be an analytic coordinate in a neighborhood of $p \in \Delta$ with $w(p) = 0$. Write $D_p = E \cap \widetilde{f}^{-1}(p)$. Then, the current*

$$\widetilde{f}_*[\log \|\mathbf{1}\|_1 \theta] - \left(\frac{1}{2\pi i} \int_{D_p} \theta \right) [\log |w|]$$

is represented by a continuous function in a neighborhood of p. In particular, if θ is cohomologous to a form $\pi^*\vartheta$, where ϑ is a smooth and $\partial, \overline{\partial}$ closed differential form on X, then $\widetilde{f}_*[\log \|\mathbf{1}\|_1 \theta]$ is represented by a continuous funcion on S.

Proof. Recall that the Poincaré-Lelong formula provides the equality of currents

$$\mathrm{d}_{\mathcal{D}}[\log \|\mathbf{1}\|_1^{-1}] = [c_1(\overline{\mathcal{O}(E)}_1)] - \delta_E.$$

Moreover, the operator $\mathrm{d}_{\mathcal{D}}$ commutes with proper push-forward. Therefore, taking into account that θ is ∂ and $\overline{\partial}$ closed, in a small neighborhood of p one derives the equation

$$\mathrm{d}_{\mathcal{D}}\, \widetilde{f}_*[\log \|\mathbf{1}\|_1 \theta] = \left(\frac{1}{2\pi i}\int_{D_p} \theta\right) \delta_p - \widetilde{f}_*[c_1(\overline{\mathcal{O}(E)}_1)\theta]. \tag{3.197}$$

On the other hand, again the Poincaré-Lelong equation gives

$$\mathrm{d}_{\mathcal{D}}[\log |w|] = \delta_p. \tag{3.198}$$

From (3.197)–(3.198), we see that

$$\mathrm{d}_{\mathcal{D}}\left(\widetilde{f}_*[\log \|\mathbf{1}\|_1\theta] - \left(\frac{1}{2\pi i}\int_{D_p}\theta\right)[\log |w|]\right) = -\frac{1}{2}\widetilde{f}_*[c_1(\overline{\mathcal{O}(E)}_1)\theta].$$

Finally, by Proposition 3.194, the current $\widetilde{f}_*[c_1(\overline{\mathcal{O}(E)}_1)\theta]$ is represented by a continuous function on S. Hence the first assertion follows from Lemma 3.191. For the second assertion, we just observe that, in this case,

$$\int_{D_p}\theta = \int_{D_p}\pi^*\vartheta = 0.$$

The proof is complete □

Corollary 3.199. *Notations being as above, the following estimates hold in a neighborhood of p*

$$\begin{aligned}
&\widetilde{f}_*[\log \|\mathbf{1}\|_1 c_1(\pi^*\overline{T_X})] = O(1),\\
&\widetilde{f}_*[\log \|\mathbf{1}\|_1 c_1(\overline{\mathcal{O}(E)}_1)] = -n_p[\log |w|] + O(1),\\
&\widetilde{f}_*[\log \|\mathbf{1}\|_1 c_1(\overline{L}_0)] = n_p[\log |w|] + O(1),\\
&\widetilde{f}_*[\log \|\mathbf{1}\|_1 c_1(\widetilde{\mu}^*\overline{U})] = -n_p[\log |w|] + O(1),
\end{aligned}$$

where n_p is the multiplicity of Δ at p and $O(1)$ denotes the current represented by a locally bounded function.

Proof. After the proposition, it is enough to recall equations (3.178)–(3.180) and the intersection numbers $(D_p \cdot D_p) = (D_p \cdot E) = -n_p$. □

Corollary 3.200. *With the notations above, we have the development in a neighborhood of p,*

$$\widetilde{f}_*[\pi^*\operatorname{ch}(\overline{\mathcal{F}})\pi^*\operatorname{Td}(\overline{f})\operatorname{Td}(\overline{\eta})\widetilde{\operatorname{Td}}_m(\overline{\gamma})]^{(3,2)} = \operatorname{rk}(\mathcal{F})\frac{n_p}{6}[\log |w|] + O(1),$$

where $O(1)$ denotes the current represented by a locally bounded function.

Proof. We take into account the expression (3.190) for the representative $\overline{\mathrm{Td}}_m(\overline{\gamma})$, the developments of the smooth differential forms $\mathrm{ch}(\overline{\mathcal{F}})$, $\mathrm{Td}(\overline{f})$, $\mathrm{Td}(\overline{\eta})$ and $\mathrm{Td}^{-1}(\overline{L}_0)$, and then apply Corollary 3.199. We find

$$\widetilde{f}_*[\pi^* \mathrm{ch}(\overline{\mathcal{F}})\pi^* \mathrm{Td}(\overline{f}) \mathrm{Td}(\overline{\eta})\widetilde{\mathrm{Td}}_m(\overline{\gamma})]^{(3,2)} = \\ \mathrm{rk}(\overline{\mathcal{F}})\frac{n_p}{6}[\log |w|] + \mathrm{rk}(\overline{\mathcal{F}})\frac{b}{3}\widetilde{f}_*[\partial(\log \|\mathbf{1}\|_1 \overline{\partial} \log \|\mathbf{1}\|_1)] + O(1).$$

To conclude we observe that, on V, the term $\widetilde{f}_*[\partial(\log \|\mathbf{1}\|_1 \overline{\partial} \log \|\mathbf{1}\|_1)]$ vanishes. Indeed, the morphism $\widetilde{f}_*$ is smooth on V with one dimensional fibers. Hence this current is represented by the function

$$V \ni s \mapsto \frac{1}{2\pi i}\int_{\widetilde{f}^{-1}(s)} \partial(\log \|\mathbf{1}\|_1 \overline{\partial} \log \|\mathbf{1}\|_1) = \\ \frac{1}{2\pi i}\int_{\widetilde{f}^{-1}(s)} d(\log \|\mathbf{1}\|_1 \overline{\partial} \log \|\mathbf{1}\|_1) = 0.$$

This ends the proof. □

The results of this section are summarized in the following statement.

Theorem 3.201. *Let $p \in \Delta$ and let n_p be the number of singular points of $f\colon X \to S$ lying above p. Let w be a local coordinate on S, centered at p. Then, in a neighborhood of p, we have the estimate*

$$T(\overline{\xi}')^{(1,1)} = -\frac{\mathrm{rk}\,\overline{\mathcal{F}}}{6}n_p[\log |w|] + O(1),$$

where $O(1)$ is the current represented by a locally bounded function.

Proof. It is enough to put together equation (3.184), Proposition 3.192, Corollary 3.195 and 3.200. □

Corollary 3.202. *Assume that $\overline{\mathcal{F}} = \overline{E}$ is a vector bundle placed in degree 0, and that $R^1 f_*E = 0$ on S. Endow f_*E with the L^2 metric on V depending on $\overline{E}$ and the metric on $\overline{\omega_{f^{-1}(V)/V}}$. Write $\xi'' = (\overline{f}', \overline{E}, \overline{f_*E}_{L^2})$ for the corresponding relative metrized complex on V. Let p and w be as in the theorem. Then we have*

$$T(\overline{\xi}'')^{(1,1)} = -\frac{\mathrm{rk}(\overline{\mathcal{F}})}{6}n_p[\log |w|] + O(\log\log |w|^{-1})$$

as $w \to 0$.

Proof. Introduce an auxiliary smooth hermitian metric on the vector bundle f_*E on S, and let $\overline{\xi}' = (\overline{f}', \overline{E}, \overline{f_*E})$ be the corresponding relative metrized complex. Then the theorem applies to $\overline{\xi}'$. By the anomaly formulas, on V we have

$$T(\overline{\xi}'')^{(1,1)} = T(\overline{\xi}')^{(1,1)} + \widetilde{\mathrm{ch}}(\overline{f_*E}, \overline{f_*E}_{L^2})^{(1,1)}.$$

By [6, Prop. 7.1], the L^2 metric has logarithmic singularities near $w = 0$ and

$$\widetilde{\mathrm{ch}}(\overline{f_*E}, \overline{f_*E}_{L^2}) = O(\log\log |w|^{-1})$$

as $w \to 0$. This proves the corollary. □

Remark 3.203. The corollary is to be compared with [6, Thm. 9.3]. The difference of sign is due to the fact that Bismut and Bost work with the inverse of the usual determinant line bundle. The approach of *loc. cit.* is more analytic and requires the spectral description of the Ray-Singer analytic torsion.

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