A DIMER MODEL FOR THE JONES POLYNOMIAL OF PRETZEL KNOTS

MOSHE COHEN

Abstract. This work gives the construction of an "activity matrix" whose determinant gives Tutte's activity words associated with spanning trees of the signed Tait graph coming from a pretzel knot diagram. Evaluations of these activity words give the signed Tutte polynomial, the Kauffman bracket polynomial, and the Jones polynomial, as well as some applications to the reduced Khovanov homology chain complex. This construction holds for some more general knot diagrams, as well. Furthermore, this discussion conforms to the language of dimers or perfect matchings on a bipartite graph.

1. Introduction

To a chemist, a *dimer* is a polymer with exactly two atoms and one bond; to a graph theorist, it is an edge with two distinct vertices; to a combinatorialist, it is a domino. A *dimer covering* is thus a perfect matching of a graph or a domino tiling of a gameboard on a grid. Kasteleyn [Kas63] and Temperley-Fisher [TF61] developed this subject in the 1960's as a tool for studying statistical physics. A *dimer model* for some invariant is a weighting of the edges of a graph Γ such that

$$\sum_{\mu}\prod_{\varepsilon\in\mu}\epsilon(\varepsilon)$$

gives this invariant after summing over all perfect matchings μ and all edges ε in the perfect matching. For a more formal treatment, see Kenyon's lecture notes [Ken09].

The main work in this paper provides the weightings $\epsilon(\varepsilon) = w(\varepsilon)\alpha(\varepsilon)|_V$ of edges ε of a graph Γ obtained from a knot to compute the Jones polynomial for pretzel knots as well as some more general knots.

As described in Section 2, the Jones polynomial of a knot is essentially a specialization of the signed Tutte polynomial of a graph G obtained from the knot diagram and as such has properties that mirror the classical Tutte polynomial. The deletion-contraction formula gives a recursive definition whose base case is a tree with loops; however, this may become computationally expensive. Alternatively the Tutte polynomial of a graph can be written as a state sum over the smaller set of all spanning trees; however, this requires Tutte's rather cumbersome notion of activity.

The activity of each edge e of each spanning tree S must be determined to perform this computation. An edge is "active" if it is the lowest ordered edge satisfying some property after an arbitrary ordering of the edges of the graph. The unfamiliar reader might first read Chapter X in [Bol98] or should at least pay attention to Definition 2.1 (activity), Example 2.2, and Table 1 (polynomial activity evaluations) found below.

The relatively simple construction of the activity matrix in this work implements activity without having to directly determine it, although some very specific conventions must be followed. One reason to accept such specificity here would be the problem of computational complexity of the Jones polynomial, which by Jaeger-Vertigan-Welsh [JVW90] is not polynomial time for an arbitrary

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knot. Futhermore, the unweighted underlying graph used here is the same as the one in the dimer model for the Alexander polynomial constructed in [CDR10].

This unweighted underlying graph is the balanced overlaid Tait graph Γ defined in Section 3. Its dimers can be computed by the permanent, or unsigned determinant, of a matrix: the bipartite adjacency submatrix, which is a submatrix of the adjacency matrix of the graph. The notion of a Kasteleyn weighting $\kappa(\varepsilon)$ of the graph is discussed here, although this signing of the edges will ultimately only be used for entries in the matrix and not on the graph itself, allowing one to use the classical determinant.

Together with the activity weighting $\alpha(\varepsilon)$ and a writhe term $w(\varepsilon)$, this gives the activity matrix A defined in Section 4. The determinant of this matrix provides the list of activity words of the spanning trees of the Tait graph G of a pretzel knot in the Main Theorem 4.9, and the Main Corollary 4.7 evaluates these to obtain the Jones polynomial of any pretzel knot.

Three examples appear in Section 5: the simplest nontrivial knot, the trefoil, given as the $(1,1,1)$-pretzel knot; the first non-alternating knot 8_{19} appearing on the Rolfsen Knot Table [BNMea] given as the $(-2,3,3)$-pretzel knot; and the $(-2,3,7)$-pretzel knot that appears in 3-manifold constructions, e.g. [Jun04].

The Jones polynomial is the graded euler characteristic of a complex of bigraded abelian groups called Khovanov homology [Kho03]. Reduced Khovanov homology was described by a state sum of spanning trees by Champanerkar-Koffman [CK09] and independently by Wehrli [Weh08]. In particular the first authors utilize Tutte's activity; a discussion of how the results in this paper might be useful in this context can be found in Section 6.

Finally, the central technical lemma is proved in Section 7. It should be noted that although the proof of Lemma 7.1 is similar to the proof of the main theorem in [DHH09], this work was done independently of theirs.

Results of a different nature on the same subject of the Jones polynomial of pretzel knots can be found in [Lan98] and [JZ03].

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2. The Jones polynomial as a specialization of the signed Tutte polynomial

The edges of a *signed graph* come weighted with a $+1$ or -1.

Definition 2.1. (Tutte's Activity [Bol98]) To a spanning tree S of a signed graph whose n edges are ordered, Tutte assigns an *activity word* of length n in the alphabet $L, D, \ell, d, \overline{L}, \overline{D}, \overline{\ell}$, and $\overline{d}$.

A positive edge $e \in S$ is *internally active* (or live, L) if it is the lowest numbered edge that reconnects $S \backslash e$; otherwise it is *internally inactive* (or dead, D).

A positive edge $e \notin S$ is *externally active* (or live, ℓ) if it is the lowest numbered edge in the unique cycle contained in $S \cup e$; otherwise it is *externally inactive* (or dead, d).

Negative edges $\overline{L}, \overline{D}, \overline{\ell}$, and $\overline{d}$ are defined similarly.

Let $a(e,S)$ be the activity letter associated to the edge e for a specified spanning tree S, and let $a(S)$ be the activity word associated to the tree S.

Example 2.2. Let G be the (unsigned, or all positive) graph with two vertices and three parallel edges. There are three spanning trees.

By definition the first edge is always live and the last edge is always dead.

In the first spanning tree, the second edge is dead because it is not the lowest numbered edge in the cycle $\{1,2\}$ it completes. So the activity word of the first spanning tree is (Ldd).

In the second spanning tree, the second edge is dead because it is not the lowest numbered edge amongst those $\{1,2,3\}$ that reconnect the two vertices. So the activity word of the second spanning tree is $(\ell D d)$.

In the third spanning tree, the second edge is live because it *is* the lowest numbered edge in the cycle $\{2,3\}$ it completes. So the activity word of the third spanning tree is $(\ell L d)$.

Let $G\backslash e$ be the deletion of e and G/e the contraction.

Definition 2.3. For a(n unsigned) graph G and an edge e, the (unsigned) *Tutte polynomial* $T(G;x,y)$ is the unique graph polynomial satisfying:

$$T(G;x,y) = \begin{cases} T(G\backslash e;x,y) + T(G/e;x,y) & \text{if } e \text{ is neither a bridge nor a loop,} \\ x^{\#\text{ bridges}}\, y^{\#\text{ loops}} & \text{if all edges are bridges and loops.} \end{cases}$$

Theorem 2.4. *(Tutte* [Bol98]*) For a(n unsigned) graph G, the (unsigned) Tutte polynomial can be written using the activity evaluations $a(e,S)|_T$ given in Table 1:*

$$T(G;x,y) = \sum_S x^{\#L} y^{\#\ell} = \sum_S \prod_{e\in G} a(e,S)|_T$$

summing over all spanning trees S of G and taking the product of the activity of each edge e in G.

Definition 2.5. (Kauffman [Kau89]) For a signed graph G and a signed edge e, the *signed Tutte polynomial* $Q(G;A,B,\delta) =$

$$\begin{cases} AQ(G\backslash e;A,B,\delta) + BQ(G/e;A,B,\delta) & \text{if neg } e \text{ is neither a bridge nor a loop,} \\ BQ(G\backslash e;A,B,\delta) + AQ(G/e;A,B,\delta) & \text{if pos } e \text{ is neither a bridge nor a loop,} \\ x^{\#\text{ pos bridges } + \#\text{ neg loops}}\, y^{\#\text{ neg bridges } + \#\text{ pos loops}} & \text{if all edges are bridges and loops,} \end{cases}$$

setting $x = A + B\delta$ and $y = A\delta + B$.

Theorem 2.6. *(Kauffman* [Kau89]*) The spanning tree expansion of the signed Tutte polynomial $Q(G;A,B,\delta)$ for a graph G can be written using the activity evaluations $a(e,S)|_Q$ given in Table 1:*

$$Q(G;A,B,\delta) = \sum_S \prod_{e\in G} a(e,S)|_Q,$$

summing over all spanning trees S of G and taking the product of the activity of each edge e in G.

Given an unoriented crossing depicted locally in a link diagram L, let L_0 and L_∞ be the two smoothings, also called the A- and the B-smoothings, as in Figure 1.

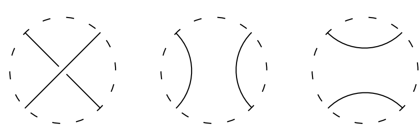


FIGURE 1. An unoriented crossing and the two smoothings.

Definition 2.7. (Kauffman [Kau87]) The *Kauffman bracket polynomial* $\langle L\rangle$ of a link L can be defined by

(1) Smoothing relation: $\langle L\rangle = A\langle L_0\rangle + B\langle L_\infty\rangle$
(2) Stabilization: $\langle U \sqcup L\rangle = \delta\langle L\rangle$
(3) Normalization: $\langle U\rangle = 1$

where U is the unknot and $\sqcup$ is the disjoint union.

Table 1. Polynomial activity evaluations.

$a(e,S)$	activity letter of e w.r.t. S	L	D	ℓ	d	$\overline{L}$	$\overline{D}$	$\overline{\ell}$	$\overline{d}$
$a(e,S)\|_T$	Tutte polynomial $T(G;x,y)$	x	1	y	1	$--$	$--$	$--$	$--$
$a(e,S)\|_Q$	signed Tutte polynomial $Q(G;A,B,\delta)$ with $x=A+B\delta$ and $y=A\delta+B$	x	A	y	B	y	B	x	A
$a(e,S)\|_V$	Jones polynomial $V_K(t)$ with $A=t^{-1/4}$	$-A^{-3}$	A	$-A^3$	A^{-1}	$-A^3$	A^{-1}	$-A^{-3}$	A

Theorem 2.8. *(Thistlethwaite* [Thi87]*) The spanning tree expansion of the Kauffman bracket polynomial $\langle K\rangle$ of a knot K with signed Tait graph G can be written using the activity evaluations $a(e,S)|_V$ given in Table 1:*

$$\langle K\rangle = \sum_S \prod_{e\in G} a(e,S)|_V,$$

summing over all spanning trees S of G and taking the product of the activity of each edge e in G.

Invariance of the bracket polynomial under Reidemeister moves II and III (see for example [Lic97]) comes from setting the signed Tutte polynomial variables as $B=A^{-1}$ and $\delta=(A^2-A^{-2})$. Invariance of the Jones polynomial under Reidemeister move I comes from an additional concept.

Definition 2.9. The *writhe* $w(D)$ of an oriented diagram is the sum over all crossings of the evaluation $+1$ for positive crossings and -1 for negative crossings as in Figure 2.

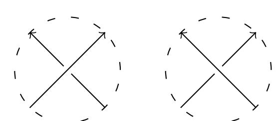
positive  negative

Figure 2. Positive and negative crossings contribute $+1$ and -1 to the writhe.

Definition 2.10. The *Jones polynomial* $V_L(t)$ of a link L given an oriented diagram D can be defined via the Kauffman bracket polynomial by

$$V_L(t) = (-A^-3)^{w(D)}\langle L\rangle,$$

where $w(D)$ is the writhe of the diagram, along with the substitutions $\delta=(A^2-A^{-2})$, $B=A^{-1}$, and $A=t^{-1/4}$.

Theorem 2.11. *(Thistlethwaite* [Thi87]*) The spanning tree expansion of the Jones polynomial $V_K(t)$ of a knot K with diagram D and signed Tait graph G can be written using the activity* *evaluations $a(e,S)|_V$ given in Table 1:*

$$V_K(t) = (-A^{-3})^{w(D)} \sum_S \prod_{e \in G} a(e,S)|_V,$$

summing over all spanning trees S of G and taking the product of the activity of each edge e in G.

3. Constructions of graphs and matrices from knots

Checkerboard color the regions of a knot diagram black and white.

Definition 3.1. The *signed Tait graph G*, also called the *signed medial graph*, associated with a knot diagram has as its vertices the black regions in the checkerboard coloring and as its signed edges the crossings with signs as in Figure 3.

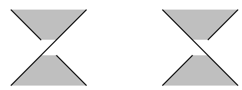


Figure 3. Crossings determine the sign of the edges in the signed Tait graph.

Note that there are actually two graphs here: G for the black regions and its dual G^* for the white regions.

Definition 3.2. The *overlaid Tait graph* $\widehat{\Gamma}$ is the signed bipartite graph whose first vertex set is the set of intersection points of the edges of both Tait graphs and whose second vertex set is the union of the vertex sets of both Tait graphs. That is, $V(\widehat{\Gamma}) = [E(G) \cap E(G^*)] \sqcup [V(G) \sqcup V(G^*)]$. The edges of this graph are the half-edges of both Tait graphs. A similar notion can be found in [HV08].

Figure 4 shows (A) the signed Tait graph G, (B) it's dual G^*, and (C) the overlaid Tait graph $\widehat{\Gamma}$ for the knot 8_{19} considered in Example 5.2 below. The thickened edges in (A) and (B) are negatively signed.

Note that the signs of the edges of the overlaid Tait graph do not arise from the signs of the original Tait graphs but may be assigned somewhat arbitrarily according to a Kasteleyn weighting, which is Definition 3.11 below.

Definition 3.3. The projection of a knot diagram without its crossing information is called the *projection graph*. This four-valent graph has as its vertices the crossings of the diagram.

Property 3.4. Each edge in the projection graph corresponds to a square face in the overlaid Tait graph $\widehat{\Gamma}$.

Proof. Each edge in the projection graph is incident with exactly two vertices and exactly two faces, which correspond to four vertices in the overlaid Tait graph, as in Figure 5. □

Definition 3.5. The *balanced overlaid Tait graph* Γ is obtained from the overlaid Tait graph $\widehat{\Gamma}$ by deleting two vertices from the larger vertex set of $\widehat{\Gamma}$ that lie on the same square face.

This is the graph on which perfect matchings will be considered for the dimer model. A bipartite graph whose vertex sets are not of the same size, that is, which is *unbalanced*, can have no perfect matchings.

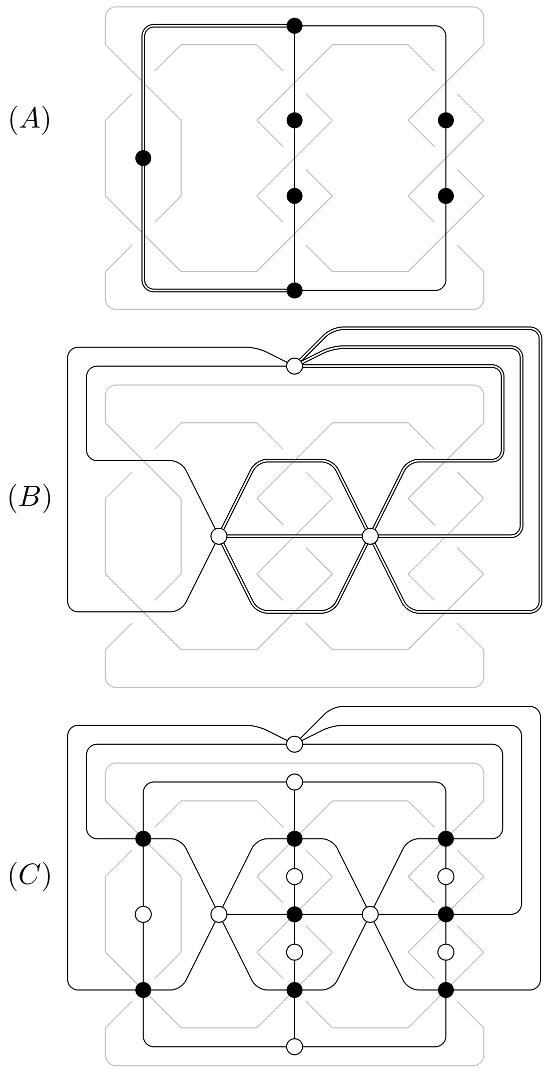


FIGURE 4. (A) The Tait graph G for 8_{19}, (B) its dual G^*, and (C) the overlaid Tait graph $\widehat{\Gamma}$.

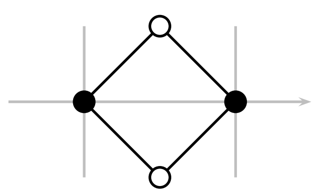


FIGURE 5. A square face of the balanced overlaid Tait graph.

Definition 3.6. The adjacency matrix of the balanced overlaid Tait graph Γ can be written in block form by

$$\left(\begin{array}{c|c} 0 & M \\ \hline M^T & 0 \end{array}\right)$$

with some submatrix M, which shall be called the *bipartite adjacency submatrix* associated with the balanced overlaid Tait graph Γ.

Definition 3.7. Alternatively, the incidence matrix of the original Tait graph G can be amalgamated with that of its dual G^* so that rows correspond to the edges of the graphs, initial columns correspond to the vertices of G, and remaining columns correspond to faces of G. Delete two columns corresponding to a vertex and a face that are incident to obtain what shall be called the *squared incidence matrix*.

Proposition 3.8. *Choose to omit two adjacent faces in the projection graph associated with a knot diagram. Then the squared incidence matrix of the (unweighted) Tait graph G associated with the knot diagram is in fact the bipartite adjacency submatrix of the (unweighted) balanced overlaid Tait graph Γ associated with the knot diagram.*

Proof. By construction, the rows of both matrices represent the original crossings of the knot diagram, and the columns of both matrices represent the faces of the projection graph associated to the knot diagram. Before weighting the edges, the zero and non-zero entries of the matrix are the same. □

The goal of the remainder of this section is to interpret the determinant of this matrix. First consider the unsigned determinant.

Definition 3.9. The *permanent* or *unsigned determinant* of a matrix $M = (m_{ij})$ is

$$perm(M) = \sum_{\sigma} \prod_{i} m_{i\sigma(i)}$$

summing over all permuations σ in the symmetric group.

Proposition 3.10. *The terms in the permanent expansion of a bipartite adjacency submatrix associated with a balanced bipartite graph give the complete list of perfect matchings of the graph.*

Proof. Each term in the permanent expansion is a permutation σ matching each vertex i in the first vertex set to a vertex $\sigma(i)$ in the second vertex set. □

Thus the important object here is the permanent and not the determinant. However, for the purpose of easier calculation, the following notion can be used to switch back and forth between the signed and unsigned version.

Definition 3.11. A *Kasteleyn weighting* of a plane bipartite graph is a choice of sign for each edge such that the number of negatives around a particular face is

- odd if the face has length 0 mod 4 or
- even if the face has length 2 mod 4.

Note that this definition only makes sense for a *plane graph*, that is, an abstract graph with its plane embedding.

Lemma 3.12. *If an edge is deleted from a graph with a Kasteleyn weighting, the resulting graph still has a Kasteleyn weighting.*

Proof. The deletion of an edge incident with two faces of length f_1 and f_2 results in a new face of length $f_1 + f_2 - 2$. The number of negatives in this new face changes by 0 or 2 (an even number) compared with the sum of the number of negatives in f_1 and f_2.

It is left to the reader to check that the four cases preserve the Kasteleyn weighting. □

Definition 3.13. Kauffman's signing convention (appearing [Kau06] in the future republication of [Kau83]) described in Figure 6 provides a canonical way to distribute signs to the edges of the balanced overlaid Tait graph Γ coming from a knot diagram. This shall be called *Kauffman's trick* and shall be denoted $\kappa(\varepsilon)$ for an edge $\epsilon \in \Gamma$.

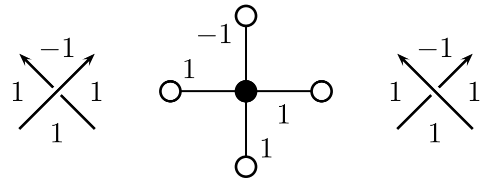


FIGURE 6. Kauffman's trick $\kappa(\varepsilon)$ to produce a Kasteleyn weighting.

Proposition 3.14. *Kauffman's trick $\kappa(\varepsilon)$ provides a Kasteleyn weighting for the balanced overlaid Tait graph Γ coming from an oriented knot diagram.*

Proof. By Property 3.4, each of the faces in the overlaid Tait graph $\widehat{\Gamma}$ is a square as in Figure 5. The assigning of a negative edge according to Figure 6 affects exactly one of the northwest and southwest sides of this square. Thus exactly one edge of every square face is negatively signed.

By Lemma 3.12, the edge deletions that result in the balanced overlaid Tait graph Γ do not affect this weighting. □

In the discussion below, the signs coming from Kauffman's trick $\kappa(\varepsilon)$ giving a Kasteleyn weighting will not appear on the graph; instead these signs will occur only in the associated terms of the matrix.

Proposition 3.15. *Suppose a balanced bipartite graph is a plane graph whose bipartite adjacency submatrix has entries given signs by a Kasteleyn weighting. Then the terms in the determinant expansion of the signed bipartite adjacency submatrix associated with the graph give the complete list of perfect matchings of the unsigned graph, up to an overall sign.*

Proof. By Proposition 3.10, only the sign of each term in the determinant expansion needs to be checked against the signs of the edges in the perfect matchings. It is enough to demonstrate the sign difference between any two terms and the sign difference between the two corresponding perfect matchings are indeed the same.

Suppose two permutations that do not give zero terms differ by exactly one transposition. This holds if and only if there are four non-zero terms arranged as corners of a rectangle in the matrix. This holds if an only if there are two vertices from each of the two vertex sets incident with both of the vertices in the other two set, that is, if and only if there is a square face in the graph.

Since the face has an odd number of signs by the Kasteleyn weighting, the two perfect matchings, which differ only on the opposite sides of this square, must have opposite signs.

Any two permutations differ in some number of transpositions, so this can be extended to all terms. □

Proposition 3.16. *Given a knot diagram and the choice of two omitted faces in the projection graph associated with the knot digram, there is a bijection between perfect matchings of the balanced overlaid Tait graph Γ associated with the knot diagram and rooted spanning trees of the Tait graph G associated with the knot diagram.*

Proof. By Proposition 3.15, there is a bijection between perfect matchings of the balanced overlaid Tait graph Γ associated with the knot diagram and the terms in the permanent expansion of the bipartite adjacency submatrix. By Proposition 3.8, the bipartite adjacency submatrix is the squared incidence matrix of the Tait graph G associated with the knot diagram. Then it is enough to show that the non-zero terms of the permanent expansion of the squared incidence matrix give the complete list of rooted spanning trees of the Tait graph G associated with the knot diagram.

A non-zero term in the permanent expansion of the squared incidence matrix is a bijection between edges of the Tait graph G and the set composed of all but one of the vertices and all but

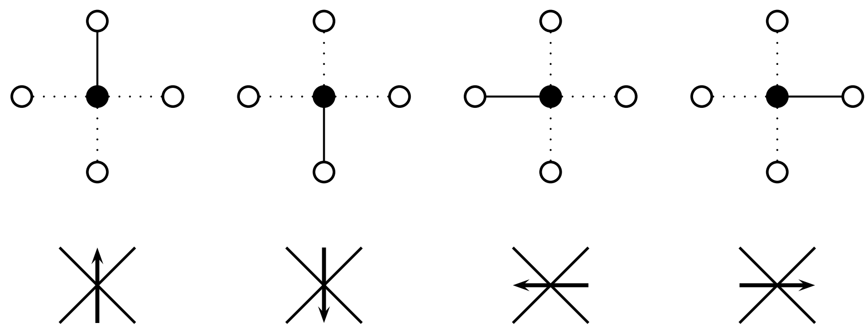


Figure 7. The correspondence between Tait graph and overlaid Tait graph edges.

one of the faces. This partitions the edge set into a collection of edges S in the Tait graph G and the complement $S^c = (E(G) - S)^*$ in the dual graph G^*.

S is a spanning tree if and only if S^c is a spanning tree in the dual graph. To check that S is indeed a spanning tree in G and that S^c is indeed a spanning tree in the dual G^*, first note that by construction both of these span all except the omitted vertices.

If there is a cycle $C \subset S$, then since there are an equal number of vertices and edges, the omitted vertex cannot be part of the cycle. Thus it must be on one side, say the outside, of C. Since C partitions the faces into two non-empty sets, the omitted face must be on one side of C. By construction the omitted vertex must be on the omitted face, so these must be on the same side: the outside. Then so that S^c can span, and since there must be an equal number of dual vertices and edges on the inside of C, there must be a cycle in the dual on the inside. Then so that S can span, there must be a cycle in the original graph G on the inside of that cycle in the dual G^*. Repeating this process yields an infinite graph, which is a contradiction.

Since S does not have any cycles, it must be incident with all of the vertices, specifically the omitted one. The dual tree S^c is similarly incident with the omitted dual vertex.

On the other hand, a rooted spanning tree S and its dual rooted spanning tree S^c yield a matching by associating each edge with the vertex it is directed toward in the graph and the dual graph. $\square$

To a crossing in the knot diagram, there are exactly four configurations of edges that can be incident with the associated vertex in the overlaid Tait graph, and there are exactly four configurations of directed edges that can be associated with it in the Tait graph. It is easy to see the relationship between these, as depicted in Figure 7.

4. Main Results

Definition 4.1. Let $P = P(n_1, n_2, \ldots, n_k)$ be the pretzel knot with k columns, each of $|n_i| \in \mathbb{N}$ crossings, where the sign of n_i determines the crossings in the column.

For the sake of fixing notation, let G be the signed Tait graph and Γ the balanced overlaid Tait graph for a diagram D of P whose whose $n = n_1 + \ldots + n_k$ crossings are labelled from left to right in columns and downward on the first column and then upward on the remaining columns, with the two omitted faces of the projection graph corresponding to the universal face and the upper deck supported by the columns as in Figure 8.

Recall that in the balanced overlaid Tait graph Γ the second vertex set is the union of the vertex sets of both the original Tait graph G and its dual G^*. Thus this graph Γ is more technically a tripartite graph whose three vertex sets correspond to the (ordered) edges, (all but one of the) vertices, and (all but one of the) faces of the specified Tait graph G.

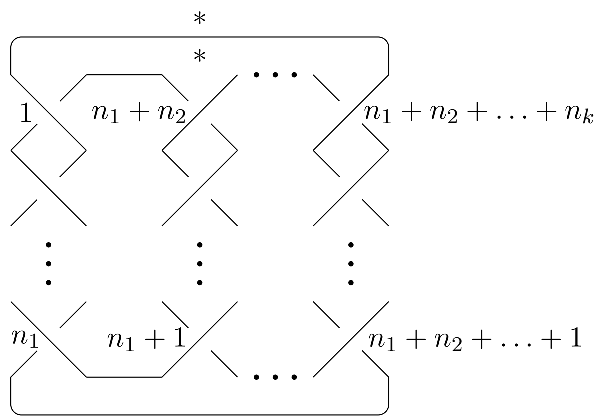


FIGURE 8. The ordering of the crossings and the two omitted regions in pretzel knot P.

Definition 4.2. The *activity weighting* $\alpha(\varepsilon)$ on an edge of ε of a balanced overlaid Tait graph Γ associated with a knot diagram whose n crossings are ordered is determined by three distinctions: positive or negative, internal or external, and live or dead.

The activity weighting for an edge incident with a vertex from the first set is positive or negative if the edge corresponding to the crossing in the chosen Tait graph is signed positive or negative, respectively, according to Figure 3.

The activity weighting for an edge is internal or external depending on whether it is incident with a vertex in the second or third set, respectively, according to the sets above. Note that every edge is incident with a vertex in the first set, so this is a partition.

The activity weighting for an edge is live or dead depending on whether or not it connects the lowest-ordered vertex from the first set to the other vertex with which it is incident. That is, given a vertex in the second or third set, label all of the edges around it as dead except for the one incident with the lowest-ordered vertex in the first set, and this last edge is labelled live.

These three choices determine the activity letter assigned as the weighting of the edge.

The evaluations $\alpha(\varepsilon)|_Q$ and $\alpha(\varepsilon)|_V$ of $\alpha(\varepsilon)$ follow those given in Table 1. Note the distinction between the activity evaluations $\alpha(\varepsilon)$ of edges ε in the balanced overlaid Tait graph Γ and the activity evaluations $a(e, S)$ of edges e in the Tait graph G.

Definition 4.3. The *activity matrix* A associated with a knot diagram is the squared incidence matrix whose non-zero entries are given by the activity weighting $\alpha(\varepsilon)$ above along with the writhe weighting $w(\varepsilon)$ and signs coming from a Kasteleyn weighting given by Kauffman's trick $\kappa(\varepsilon)$.

The activity weighting amounts to the following weighting rules. Ordered rows associated with the original ordered crossings (vertices of the first set) contain only positive or only negative letters following the sign of the original crossing in the specific Tait graph considered. Columns associated with the vertices of the second set are internal, and columns associated with the vertices of the third set are external. The first non-zero entry in a column is live; the rest are dead.

For the diagram of the $(n_1, n_2, \ldots, n_k)$-pretzel knot in Figure 8, see as an example the following (unsigned) bipartite adjacency submatrix with activity weighting $\alpha(\varepsilon)$:

$$
\left(\begin{array}{cccc|cccc|c|cccc|c|cccc}
L &&&&&&&&&&&&&& \ell &&& \\
D & L &&&&&&&&&&&&& d &&& \\
& D & \ddots &&&&&&&&&&&& d &&& \\
&& \ddots & L &&&&&&&&&&& \vdots &&& \\
&&& D &&&&&&&&&& L & d &&& \\
\hline
&&&& L &&&&&&&&& D & d & \ell && \\
&&&& D & L &&&&&&&&& d & d && \\
&&&&& D & \ddots &&&&&&&& d & d && \\
&&&&&& \ddots & L &&&&&&& \vdots & \vdots && \\
&&&&&&& D &&&&&&& d & d && \\
\hline
&&&&&&&& \ddots &&&&& \vdots && \vdots & \ddots & \vdots \\
\hline
&&&&&&&&& L &&&& D &&&& d \\
&&&&&&&&& D & L &&&&&&& d \\
&&&&&&&&&& D & \ddots &&&&&& d \\
&&&&&&&&&&& \ddots & L &&&&& \vdots \\
&&&&&&&&&&&& D &&&&& d
\end{array}\right)
$$

Observe that there are k blocks, where the i-th block is $n_i \times (n_i - 1)$, followed by some $1 + (k-1) = k$ columns. The first column has non-zero entries in the first position of each block, except for the first block which has a non-zero entry in the last position. The remaining columns have non-zero entries in two consecutive blocks.

Some care must be taken to order these activity weights correctly when assembling them into activity words. The ordering of the edges in a perfect matching comes from the ordering of the original crossings, that is, by the ordering of the first vertex set or the rows of the matrix.

Corollary 4.4. *Summing over all perfect matchings μ in Γ and taking the product over all edges ε in the perfect matching,*

$$\sum_{\mu} \prod_{\varepsilon \in \mu} \alpha(\varepsilon)|_Q = Q(G; A, B, \delta)$$

gives the signed Tutte polynomial $Q(G; A, B, \delta)$ of G up to sign.

Proof. See Main Theorem 4.9 for the list of activity words and Table 1 for the evaluations of the activity letters. □

The Kauffman bracket is of course an evaluation of the signed Tutte polynomial, providing the following.

Corollary 4.5. *Summing over all perfect matchings μ in Γ and taking the product over all edges ε in the perfect matching,*

$$\sum_{\mu} \prod_{\varepsilon \in \mu} \alpha(\varepsilon)|_V = \langle P \rangle$$

gives the Kauffman bracket polynomial $\langle P \rangle$ of P up to sign.

The Jones polynomial is the Kauffman bracket together with a writhe term, which can in fact be brought into the dimer above.

Definition 4.6. The *writhe weighting* $w(\varepsilon)$ on an edge ε of a balanced overlaid Tait graph Γ associated with a knot diagram is either $(-A)^{-3}$ or $(-A)^3$, depending on the positive or negative contribution, respectively, to the writhe by the crossing whose associated vertex the edge ε is incident with.

Corollary 4.7. *(Main Corollary) Summing over all perfect matchings μ in Γ and taking the product over all edges ε in the perfect matching,*

$$\sum_{\mu} \prod_{\varepsilon \in \mu} w(\varepsilon)\alpha(\varepsilon)|_V = V_P(t)$$

gives the Jones polynomial $V_P(t)$ of P up to sign.

Corollary 4.8. *Let ε_{ij} be the edge $\varepsilon \in \Gamma$ between the i-th vertex coming from the crossings and the j-th vertex coming from the regions. Let $A = (\kappa(\varepsilon_{ij})w(\varepsilon_{ij})\alpha(\varepsilon_{ij})|_V)$ be the activity weighting on the bipartite adjacency submatrix associated with P. Then*

$$\det(A) = V_P(t)$$

gives the Jones polynomial $V_P(t)$ of P up to sign.

For further applications using these activity words, see Section 6.

Theorem 4.9. *(Main Theorem) Summing over all perfect matchings μ in Γ and taking the product over all edges ε in the perfect matching,*

$$\sum_{\mu} \prod_{\varepsilon \in \mu} \alpha(\epsilon) = \sum_{S} a(S)$$

gives the complete list of activity words $a(S)$ associated with spanning trees S of the Tait graph G associated with the diagram of P.

Proof. By Proposition 3.16, there is a bijection between the perfect matchings of the balanced overlaid Tait graph and the rooted spanning trees of the Tait graph associated with a knot diagram. It is enough to show that the activity weighting of the perfect matching gives the activity word of the associated spanning tree.

Since there are $n = n_1 + n_2 + \ldots + n_k$ edges and $1 + (n_1 - 1) + (n_2 - 1) + \ldots + (n_k - 1) + 1 = n - k + 2$ vertices in the Tait graph, each spanning tree S omits exactly $k - 1$ edges. These $k - 1$ edges must come from distinct columns in order for S to be acyclic and connected.

The activity word associated with S can be decomposed into the activity words of the k columns, which are called paths in this paper, and considered according to Lemma 7.1.

Suppose that an edge from each column except for the first is omitted. Then by the lemma, the activity word of the first column will be $(L \ldots L)$, and the activity words of the remaining columns will range from $(dD \ldots D)$ to $(L \ldots LdD \ldots D)$ to $(L \ldots Ld)$.

Suppose that an edge from each column except for the i-th is omitted. Then by the lemma, the activity words of the first $i - 1$ columns will range from $(\ell D \ldots D)$ to $(L \ldots LdD \ldots D)$ to $(L \ldots Ld)$, the activity word of the i-th column will be $(D \ldots D)$, and the activity words of the remaining columns will range from $(dD \ldots D)$ to $(L \ldots LdD \ldots D)$ to $(L \ldots Ld)$.

As in the argument above, consider the location of the pivots in the last $k - 1$ columns. Note that if they both belong to the same i-th block, there must be a zero pivot in the $n_i \times (n_i - 1)$ block to the left of it, and so this choice does not contribute to the permanent expansion.

When a pivot in each of the last $k - 1$ columns is chosen, it forces the pivots of the corresponding blocks, with L's chosen above the pivot row and D's below in each block. This then forces the pivot in the first of the last k columns, which in turn forces the pivots in the remaining block to be uniform.

This gives the activity words described above. □

This is not the only class for which the activity weighting works. Next is a technique to begin with a knot diagram that works and extend it outside of the class of pretzel knots.

Proposition 4.10. *(Subdivision/Doubling) Suppose the highest ordered edge e_n in the Tait graph G is incident with the omitted vertex and the omitted face. Then if the activity weighting on a balanced overlaid Tait graph Γ provides a dimer model for the knot diagram associated with the Tait graph G, this can be extended to the balanced overlaid Tait graph associated with a Tait graph $G \cup \{e_{n+1}\}$ that subdivides or doubles the edge e_n.*

Note that for the $(n_1, n_2, \ldots, n_k)$-pretzel knot, if $n_k > 1$ and the n-th edge is doubled, the end result is not a pretzel knot or link. See an example constructed from successive doubling and subdivisions on an edge in Figure 9.

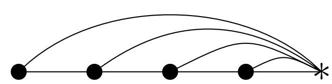


FIGURE 9. Doubling and subdivisions leaves the class of pretzel knots.

Proof. Consider the squared incidence matrix of the original knot diagram. The row corresponding to the non-loop, non-bridge edge e_n, which is incident with both the omitted vertex and the omitted face, has only two non-zero entries.

If the edge is neither a bridge nor a loop, these entries are D and d. After subdividing this edge, the matrix gets a new row corresponding to the edge e_{n+1} and a new column corresponding to the new vertex between e_n and e_{n+1}. The entries in this column are zero except for an L and a D in the n-th and $(n+1)$-st rows, respectively, and the last new entry is another d in the $(n+1)$-st row below the first d in the n-th row mentioned above.

Configurations in the determinant expansion for the final two terms in the new matrix have only three options: DD and dD, which preserve all of the first n choices of pivots, and Ld, where the first $n-1$ choices are preserved and the d of the n-th row gets replaced by the d in the $(n+1)$-st row. These are exactly the three possibilities for the activity words associated with the spanning trees by Lemma 7.1.

The dual case of doubling works similarly. □

The first Reidemeister move adds either bridges or loops to the Tait graph. These are easily handled by the following:

Proposition 4.11. *(Reidemeister I) If the activity weighting on a balanced overlaid Tait graph Γ associated with a knot diagram with signed Tait graph G provides a dimer model, this can be extended to one whose Tait graph $G \cup \{e_{n+1}\}$ is the same as before together with an additional bridge or loop.*

Proof. Since the edge e_{n+1} is a bridge or a loop, this amounts only to adding the terms L or ℓ, respectively, to the end of the activity words, and this appears in the expansion because of a column with only a single non-zero entry. □

Each column of a pretzel knot contains copies of the same signed crossing, but the signed Tutte polynomial of more general signed graphs can also be considered.

Property 4.12. (Reidemeister II) If the activity weighting on a balanced overlaid Tait graph Γ associated with a knot diagram with signed Tait graph G provides a dimer model, this can be extended to one whose Tait graph $G \cup \{e'_i, e'_{i+1}\}$ is the same as before together with two additional oppositely signed edges either in parallel or in series of an edge e_i already in G.

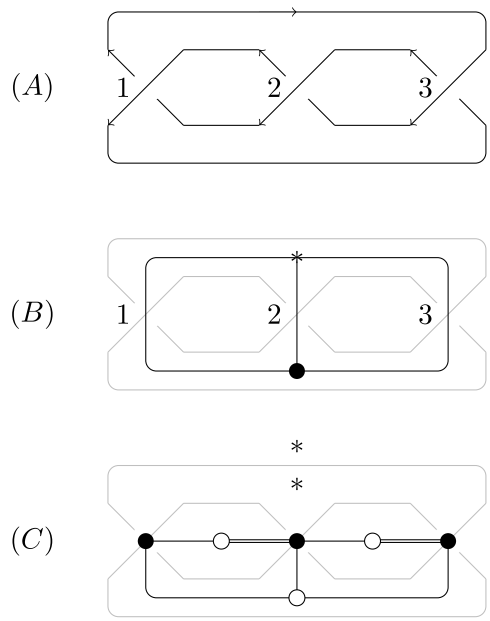


FIGURE 10. (A) The (oriented) trefoil as the $(1,1,1)$-pretzel knot, (B) its corresponding Tait graph, and (C) its corresponding balanced overlaid Tait graph.

Proof. This is a specialization of Lemma 7.1 and its dual case. □

Proposition 4.11 and Property 4.12 lead to the obvious question.

Question 4.13. Under what conditions can Reidemeister move III be used to enlarge the class of signed graphs G whose balanced overlaid version Γ gives a dimer model for the signed Tutte polynomial $Q(G; A, B, \delta)$ of G?

5. Examples

Example 5.1. Figure 10 shows (A) the oriented trefoil depicted as the $(1,1,1)$-pretzel knot, (B) its corresponding Tait graph, and (C) its corresponding balanced overlaid Tait graph. The thickened edges of the balanced overlaid Tait graph correspond to entries in the activity matrix that are negatively signed by Kauffman's trick, giving the Kasteleyn weighting. Note that the writhe of the diagram is $w(D) = -3$.

The three spanning trees of the Tait graph are (obviously) each of the three edges. From Example 2.2, the activity words are (Ldd), (ℓDd), and $(\ell\ell D)$, respectively.

The relatively trivial activity matrix (without the writhe terms $w(\varepsilon)$) has only the last columns and does not have any of the initial blocks:

$$\left(\begin{array}{c|ccc} L & \ell & & \\ \hline D & -d & \ell & \\ \hline D & & -d & \end{array}\right)$$

One can see that the terms in the determinant expansion give the three activity words. After evaluation, the determinant of this matrix is $-A^{-5} - A^3 + A^7$. Together with the term $(-A^{-3})^{-3}$ accounting for the writhe of the diagram, this gives $A^4 + A^{12} - A^{16} = t^{-1} + t^{-3} - t^{-4}$, which is indeed the Jones polynomial of the trefoil.

Example 5.2. Figure 11 shows (A) the oriented $(-2,3,3)$-pretzel knot, also known as 8_{19}, (B) its corresponding Tait graph, and (C) its corresponding balanced overlaid Tait graph. The thickened

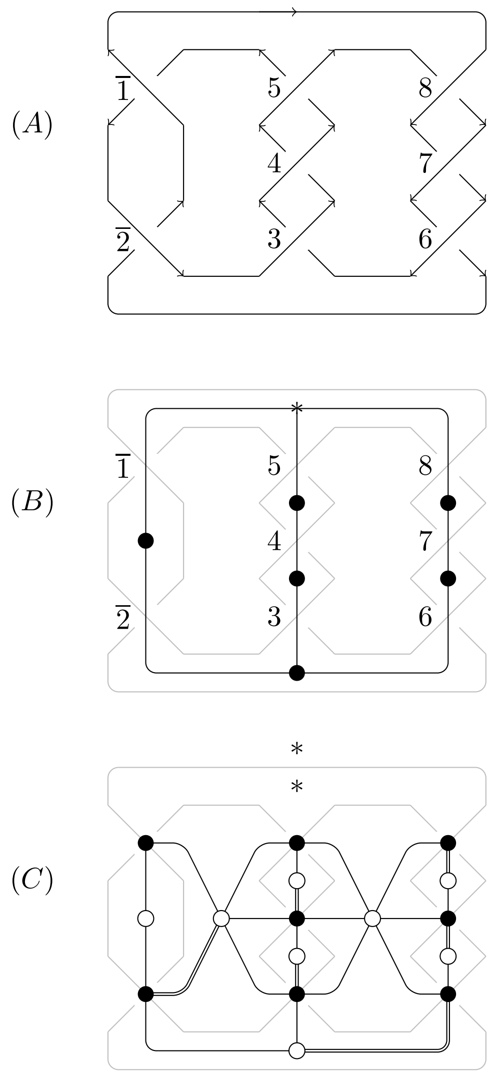


FIGURE 11. (A) The (oriented) $(-2, 3, 3)$-pretzel knot 8_{19}, (B) its corresponding Tait graph, and (C) its corresponding balanced overlaid Tait graph.

edges of the balanced overlaid Tait graph correspond to entries in the activity matrix that are negatively signed by Kauffman's trick, giving the Kasteleyn weighting. Note that the writhe of the diagram is $w(D) = 8$.

The activity matrix (without the writhe terms $w(\varepsilon)$) is the following:

$$\left(\begin{array}{c|cc|cc|c|c|c}
\overline{L} & & & & & & \overline{\ell} & \\
\overline{D} & & & & & \overline{L} & -\overline{d} & \\
\hline
 & -L & & & & D & d & \ell \\
 & D & -L & & & & d & d \\
 & & D & & & & d & d \\
\hline
 & & & L & & -D & & d \\
 & & & -D & L & & & d \\
 & & & & -D & & & d
\end{array}\right)$$

Table 2. Reduced Khovanov homology activity evaluations.

$a(e,S)$	Activity letter	L	D	ℓ	d	$\overline{L}$	$\overline{D}$	$\overline{\ell}$	$\overline{d}$
$a(e,S)\vert_K$	chain complex of reduced Khovanonv homology	uv	v	u^{-1}	1	u^{-1}	1	u	1

Taking the determinant after evaluation and together with the term $(-A^{-3})^8$ accounting for the writhe of the diagram, this gives $-A^{-32}+A^{-20}+A^{-12}=-t^8+t^5+t^3$, which is indeed the Jones polynomial of 8_{19}.

Example 5.3. Given the $(-2,3,7)$-pretzel knot, which is useful in the construction of three manifolds, the activity matrix (without the writhe terms $w(\varepsilon)$) is the following:

$$\left(\begin{array}{c|cc|cccccc|c|c|c}
\overline{L} & & & & & & & & & & \overline{\ell} & \\
\overline{D} & & & & & & & & & \overline{L} & -\overline{d} & \\
\hline
 & -L & & & & & & & & D & d & \ell \\
 & D & -L & & & & & & & & d & d \\
 & & D & & & & & & & & d & d \\
\hline
 & & & L & & & & & & -D & & d \\
 & & & -D & L & & & & & & & d \\
 & & & & -D & L & & & & & & d \\
 & & & & & -D & L & & & & & d \\
 & & & & & & -D & L & & & & d \\
 & & & & & & & -D & L & & & d \\
 & & & & & & & & -D & & & d
\end{array}\right)$$

Taking the determinant after evaluation and together with the term $(-A^{-3})^{12}$ accounting for the writhe of the diagram, this gives $-A^{-40}+A^{-36}-A^{-32}+A^{-16}+A^{-8}=-t^{10}+t^9-t^8+t^2$.

It should be noted that although the negative signs in the matrices above can be easily obtained from the oriented knot diagram using Kauffman's trick, one can avoid this by computing the permanent of the unsigned matrix.

6. Potential application to reduced Khovanov homology

Champanerkar-Koffman [CK09] define two gradings u and v for reduced Khovanov homology based on the activity words of spanning trees S of the Tait graph G of a knot diagram. Let $a(e,S)|_K$ denote this evaluation described in Table 2, and let $\alpha(\varepsilon)|_K$ be the corresponding evaluation in the balanced overlaid Tait graph Γ.

Corollary 6.1. *Summing over all perfect matchings μ in Γ and taking the product over all edges ε in the perfect matching,*

$$\sum_{\mu}\prod_{\varepsilon\in\mu}\alpha(\varepsilon)|_K$$

gives the two-variable polynomial for the reduced Khovanov chain complex of P up to sign.

In order to get reduced Khovanov homology from the chain complex, the differential $\partial : \{u, v\} \to \{u-1, v-1\}$ is needed. There are differently-ordered differential maps given in [CK09]. The first can be easily seen in the biparite adjacency submatrix.

Corollary 6.2. *The first order differential for the spanning tree model of reduced Khovanov homology corresponds to very particular 2×2 blocks of this matrix: two rows and two columns who meet at four nonzero terms in each of the following configurations.*

$$\left(\begin{array}{c|c} L & d \\ \hline \overline{D} & \overline{d} \end{array}\right) \quad \left(\begin{array}{c|c} \overline{d} & \overline{L} \\ \hline d & D \end{array}\right) \quad \left(\begin{array}{c|c} \overline{\ell} & \overline{D} \\ \hline d & D \end{array}\right) \quad \left(\begin{array}{c|c} D & \ell \\ \hline \overline{D} & \overline{d} \end{array}\right)$$

Note that the order of the columns is not important; the second and third submatrices are written in this way to help illustrate the direction of the differential.

Proof. The determinants of these submatrices give the first order differential ∂ given in [CK09]:

$$\begin{aligned} L\overline{d} &\to d\overline{D} \\ \overline{d}D &\to \overline{L}d \\ \overline{\ell}D &\to \overline{D}d \\ D\overline{d} &\to \ell\overline{D}. \end{aligned}$$

□

Perhaps collections of edges that do not give spanning trees can be used to produce the higher-order differentials through the context of the squared incidence matrix. This would lead to a positive answer to the following question.

Question 6.3. Can the reduced Khovanov homology of pretzel knots can be computed via the squared incidence matrix alone?

7. Technical Lemma

The set of edges between the same two vertices will be referred to as a *parallel edge class*. The dual of this parallel edge class will be non-standardly referred to as a *path*, even though this term usually represents something far more general in graph theory.

The following lemma and its other versions will be used to determine the activity words of spanning trees on certain subgraphs of signed graphs whose edges are ordered. The first one is for positive path subgraphs, that is, on subgraphs whose vertices are all two-valent in both the graph and the subgraph except for two single-valent vertices in the subgraph which can be multi-valent in the graph. Another way to view this is by the repeated subdivision of an edge.

Lemma 7.1. *(Classification of activity on paths) Suppose that a path P_{k+1} of $k+1$ positive edges indexed sequentially by $i, \ldots, i+k$ for some $k > 0$ belongs to a graph G such that all interior vertices of the path have degree exactly two in G. Then when determining the activity word for G given a spanning tree S, the portion of the activity word associated with the path P_{k+1} must be one of the following:*

1. *$(L \ldots L)$ or $(D \ldots D)$ when all edges of P_{k+1} are included in S, or*
2. *$(L \ldots LdD \ldots D)$ omitting only the $(i+j)$-th edge for $0 \leq j \leq k$, or*
3. *$(\ell D \ldots D)$ when $i = 1$ and a few other exceptional cases.*

The following proof is similar to the one given for the main theorem in [DHH09], although this work was done independently of theirs.

Proof. First suppose that the path P_{k+1} is contained in the spanning tree S; then each of its edges are labelled by L or D. Supposing the i-th edge is labelled L, this is the least-indexed edge to reconnect the severed tree $T - \{i\}$, and so no edge indexed less than $i+1$ can reconnect the severed tree $T - \{i+1\}$. Iterate j times to get the path labelled by $L \dots L$. Supposing the i-th edge is labelled D, there is an edge indexed less than i that reconnects the severed tree $T - \{i\}$, and so this edge also reconnects the severed tree $T - \{i+1\}$. Iterate j times to get the path labelled by $(D \dots D)$. Take note that only these two possibilities arisse here, as this face is used below.

Now suppose that the edge indexed by $i+j$ of the path P_{k+1} is not contained in the spanning tree S; then it must be labelled by either ℓ or d. The cycle contained in $T \cup \{i+j\}$ must also contain the edge indexed by $i+j-1$, so the $(i+j)$-th edge must be labelled by d unless $j = 0$. Note that no other edge may be omitted without disconnecting the tree, so the rest of the edges of P_{k+1} must be labelled by L or D. Then by the argument above, each path $i, \dots, i+j-1$ and $i+j+1, \dots, i+k$ must be one of $(L \dots L)$ or $(D \dots D)$.

The string before the omitted edge $i+j$ must contain an L (and therefore be $(L \dots L)$ by the argument above) because only the edges $i+j-1$ and $i+j$ reconnect the severed tree $T - \{i+j-1\}$. The string following the omitted edge $i+j$ must contain a D (and therefore be $(D \dots D)$ by the argument above) because only the edges $i+j$ and $i+j+1$ reconnect the severed tree $T - \{i+j+1\}$. ☐

The negative case is similar, with each activity letter replaced by its negative counterpart. The dual version and its negative case work, as well, and the proofs are similar.

Lemma 7.2. *(Classification of activity on parallel edge classes) Suppose a* $k+1$*-parallel edge class* D_{k+1} *of* $k+1$ *positive edges indexed sequentially by* $i, \dots, i+k$ *for some* $k > 0$ *belongs to a graph* G. *Then when determining the activity word for* G *given a spanning tree* S*, the portion of the activity word associated with the parallel edge class* D_{k+1} *must be one of the following:*

(1) *$(\ell \dots \ell)$ or $(d \dots d)$ when no edges of D_{k+1} are included in S, or*
(2) *$(\ell \dots \ell D d \dots d)$ when only the $(i+j)-th$ edge for $0 \leq j \leq k$ in included in S, or*
(3) *$(Ld \dots d)$ when $i = 1$ and a few other exceptional cases.*

Note that the Reidemeister II move allows one to consider only paths and parallel edges classes with the same sign. See also Property 4.12.

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Department of Mathematics and Computer Science, Bar-Ilan University, Ramat Gan 52900, Israel
E-mail address: cohenm10@macs.biu.ac.il