THE REPRESENTATION DIMENSION OF HECKE ALGEBRAS AND SYMMETRIC GROUPS

PETTER ANDREAS BERGH & KARIN ERDMANN

ABSTRACT. We establish a lower bound for the representation dimension of all the classical Hecke algebras of types A, B and D. For all the type A algebras, and "most" of the algebras of types B and D, we also establish upper bounds. Moreover, we establish bounds for the representation dimension of group algebras of some symmetric groups.

1. INTRODUCTION

The representation dimension of a finite dimensional algebra was introduced by Auslander in [Au1]. His aim was an invariant which would somehow measure how far an algebra is from having finite representation type. As a first step, he showed that a non-semisimple algebra is of finite representation type if and only if its representation dimension is exactly two, whereas it is of infinite type if and only if the representation dimension is at least three.

For more than three decades, no example was produced of an algebra whose representation dimension exceeds three. However, in 2006 Rouquier showed in [Ro1] that the representation dimension of the exterior algebra on a d-dimensional vector space is $d+1$, using the notion of the dimension of a triangulated category (cf. [Ro2]). Thus there do exist finite dimensional algebras of arbitrarily large representation dimension. Other examples illustrating this were subsequently given in [Ber], [BO1], [BO2], [KrK], [Op1], [Op2], [Op3], [OpM].

Naturally, these papers focused on finding lower bounds for the representation dimension of various algebras. However, there does not exist a method for computing a good *upper* bound. The best upper bound available so far was proved by Auslander himself: the representation dimension of a selfinjective algebra is at most its Loewy length. For some selfinjective algebras, this bound equals the representation dimension, but there also exist algebras for which the difference between this bound and the precise value is arbitrarily large. The simplest example is the selfinjective algebra $k[x]/(x^n)$ for $n \geq 2$. This has finite representation type, and its representation dimension is therefore 2, whereas its Loewey length is n.

In this paper, we provide both an upper and a lower bound for the representation dimension of the Hecke algebra $\mathcal{H}_q(A_{n-1})$ of type A_{n-1}, where q is a primitive ℓth root of unity and the ground field is of characteristic zero. In particular, we show that

$$[n/\ell] + 1 \leq \operatorname{repdim} \mathcal{H}_q(A_{n-1}) \leq 2[n/\ell]$$

whenever $\mathcal{H}_q(A_{n-1})$ is not semisimple (where $[r]$ denotes the integer part of a rational number r). These bounds are obtained by passing to a maximal ℓ-parabolic subalgebra, which is just a tensor product of Brauer tree algebras and semisimple algebras. The proof carries over to some group algebras of symmetric groups, and consequently we obtain bounds also for such algebras. Namely, if k is a perfect field

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of positive characteristic p, and S_n is the nth symmetric group with $n < p^2$, then we show that the inequalities

$$[n/p] + 1 \leq \operatorname{repdim} kS_n \leq 2[n/p]$$

hold whenever kS_n is not semisimple (i.e. when $n \geq p$). For these algebras, the bounds are obtained by passing to a Sylow p-subgroup of S_n: when $n < p^2$, this is an elementary abelian p-group of rank $[n/p]$.

We also establish lower bounds for all Hecke algebras of types B and D, and these bounds are the same as for type A. Namely, if $\mathcal{H}$ is either a Hecke algebra $\mathcal{H}_{Q,q}(B_n)$ of type B_n, or a Hecke algebra $\mathcal{H}_q(D_n)$ of type D_n, then we show that the inequality

$$[n/\ell] + 1 \leq \operatorname{repdim} \mathcal{H}$$

holds whenever $[n/\ell] \geq 1$. Moreover, when a certain polynomial expression in the parameters is nonzero (in the ground field), and n is odd in type D, then we also provide an upper bound:

$$\operatorname{repdim} \mathcal{H} \leq 2[n/\ell].$$

As with the lower bound, this upper bound is the same as for type A.

2. Comparing global dimensions

In this section we prove two results that compare the global dimensions of endomorphism rings of modules over two different algebras. They both apply to Hecke algebras of type A and group algebras of symmetric groups. All modules are assumed to be finitely generated.

The first result considers the case of a subalgebra of an algebra.

Theorem 2.1. *Let Λ be a finite dimensional algebra, and suppose there exist a subalgebra Γ and a Γ-module M such that:*

(1) *the Γ-module $\Lambda \otimes_\Gamma M$ belongs to $\operatorname{add}_\Gamma M$,*
(2) *the restriction map*

$$\operatorname{Ext}^1_\Lambda(X, Y) \xrightarrow{\operatorname{res}^\Lambda_\Gamma} \operatorname{Ext}^1_\Gamma(X, Y)$$

is injective for all $X, Y \in \operatorname{add}_\Lambda(\Lambda \otimes_\Gamma M)$.

Then $\operatorname{gldim} \operatorname{End}_\Lambda(\Lambda \otimes_\Gamma M) \leq \operatorname{gldim} \operatorname{End}_\Gamma(M)$.

Proof. Suppose the global dimension of $\operatorname{End}_\Gamma(M)$ is finite, say $\operatorname{gldim} \operatorname{End}_\Gamma(M) = d$. Denote the induced Λ-module $\Lambda \otimes_\Gamma M$ by M^Λ, and let N_0 be an indecomposable summand of M^Λ. Then $\operatorname{Hom}_\Lambda(M^\Lambda, N_0)$ is an indecomposable projective $\operatorname{End}_\Lambda(M^\Lambda)$-module, and every indecomposable projective module for this endomorphism algebra is of this form. Denote by $S(N_0)$ the simple $\operatorname{End}_\Lambda(M^\Lambda)$-module corresponding to $\operatorname{Hom}_\Lambda(M^\Lambda, N_0)$, and recall that $\operatorname{Hom}_\Lambda(M^\Lambda, -)$ induces an equivalence between $\operatorname{add}_\Lambda M^\Lambda$ and the category of projective $\operatorname{End}_\Lambda(M^\Lambda)$-modules. Therefore there exists an exact sequence

$$\cdots \to N_2 \to N_1 \to N_0$$

of Λ-modules, in which each N_i belongs to $\operatorname{add}_\Lambda M^\Lambda$, and such that

$$\cdots \to \operatorname{Hom}_\Lambda(M^\Lambda, N_1) \to \operatorname{Hom}_\Lambda(M^\Lambda, N_0) \to S(N_0) \to 0$$

is a projective resolution of $S(N_0)$. For each $i \geq 1$, denote by K_i the image of the map $N_i \to N_{i-1}$.

By restricting to Γ and using adjointness, we obtain an exact sequence

$$\cdots \to \operatorname{Hom}_\Gamma(M, N_1) \to \operatorname{Hom}_\Gamma(M, N_0) \to L \to 0$$

of $\mathrm{End}_\Gamma(M)$-modules. As Γ-modules, each N_i belongs to $\mathrm{add}_\Gamma M$, since $\mathrm{add}_\Gamma M^\Lambda \subseteq \mathrm{add}_\Gamma M$ by assumption. Therefore each $\mathrm{Hom}_\Gamma(M, N_i)$ is a projective $\mathrm{End}_\Gamma(M)$-module, and the exact sequence is a projective resolution of L. Since the projective dimension of L is at most d, the image of the map

$$\mathrm{Hom}_\Gamma(M, N_d) \to \mathrm{Hom}_\Gamma(M, N_{d-1}),$$

namely $\mathrm{Hom}_\Gamma(M, K_d)$, is a projective $\mathrm{End}_\Gamma(M)$-module. Consequently the short exact sequence

$$0 \to \mathrm{Hom}_\Gamma(M, K_{d+1}) \to \mathrm{Hom}_\Gamma(M, N_d) \to \mathrm{Hom}_\Gamma(M, K_d) \to 0$$

of $\mathrm{End}_\Gamma(M)$-modules splits. The functor $\mathrm{Hom}_\Gamma(M, -)$ induces an equivalence between $\mathrm{add}_\Gamma M$ and the category of projective $\mathrm{End}_\Gamma(M)$-modules, hence the short exact sequence

$$0 \to K_{d+1} \to N_d \to K_d \to 0$$

splits in $\mathrm{mod}\,\Gamma$. By assumption, the restriction map

$$\mathrm{Ext}^1_\Lambda(K_d, K_{d+1}) \xrightarrow{\mathrm{res}^\Lambda_\Gamma} \mathrm{Ext}^1_\Gamma(K_d, K_{d+1})$$

is injective, and so the short exact sequence also splits in $\mathrm{mod}\,\Lambda$. Then K_d belongs to $\mathrm{add}_\Lambda M^\Lambda$, hence in the projective resolution of the $\mathrm{End}_\Lambda(M^\Lambda)$-module $S(N_0)$ the image of the map

$$\mathrm{Hom}_\Lambda(M^\Lambda, N_d) \to \mathrm{Hom}_\Lambda(M^\Lambda, N_{d-1})$$

is projective. This shows that the projective dimension of every simple $\mathrm{End}_\Lambda(M^\Lambda)$-module is at most d, thus proving the theorem. □

We shall also need the following result when we apply Theorem 2.1 to Hecke algebras and group algebras. It gives sufficient conditions for an induced module to be a generator, that is, contain all the indecomposable projective modules as direct summands.

Proposition 2.2. *Let Λ be a finite dimensional algebra, and suppose there exist a subalgebra Γ and a Γ-module M such that:*

(1) *Λ is projective as a left Γ-module,*
(2) *M is a generator in* $\mathrm{mod}\,\Gamma$.

Then the Λ-module $\Lambda \otimes_\Gamma M$ is a generator.

Proof. Since Λ is projective as a left Γ-module, it belongs to $\mathrm{add}_\Gamma M$, and hence $\Lambda \otimes_\Gamma \Lambda$ belongs to $\mathrm{add}_\Lambda(\Lambda \otimes_\Gamma M)$. The surjective multiplication map $\Lambda \otimes_\Gamma \Lambda \xrightarrow{\mu} \Lambda$ splits when viewed as a map of left Λ-modules, and so the left Λ-module Λ is a direct summand of $\Lambda \otimes_\Gamma \Lambda$. Therefore Λ must belong to $\mathrm{add}_\Lambda(\Lambda \otimes_\Gamma M)$, and consequently $\Lambda \otimes_\Gamma M$ is a generator in $\mathrm{mod}\,\Lambda$. □

We turn now to the second result comparing global dimensions of endomorphism rings of modules over two different algebras. Whereas one of the algebras in Theorem 2.1 was a subalgebra of the other, this is not necessarily the case in the following result. Still, the proofs are similar in nature.

Given two algebras Λ and Γ, we say that Λ *separably divides* Γ if there exist bimodules ${}_\Lambda X_\Gamma$ and ${}_\Gamma Y_\Lambda$, both projective on either side, such that the Λ-Λ-bimodule Λ is a direct summand of $X \otimes_\Gamma Y$. If, in addition, the Γ-Γ-bimodule Γ is a direct summand of $Y \otimes_\Lambda X$, then the algebras are *separably equivalent* (cf. [Li2]). Obviously, if Λ and Γ are separably equivalent, then each of them separably divides the other. However, the converse does not seem to hold automatically: the bimodules involved need not be the same. Note that a group algebra is separably divided by the group algebra of any subgroup. Namely, if k is a field, and G is a group with a subgroup H, then the bimodules ${}_{kG}X_{kH}$ and ${}_{kH}Y_{kG}$ defined by $X = kG = Y$ do the trick: the kH-kH bimodule kH is a summand of $Y \otimes_{kG} X$. If, in addition, the subgroup H is a Sylow subgroup, then kG and kH are separably equivalent, since in this case the kG-kG bimodule kG is a summand of $X \otimes_{kH} Y$. Similarly, a block algebra is separably equivalent to the group algebra of a defect group.

Theorem 2.3. *Let Λ and Γ be finite dimensional algebras, and suppose there exists a Γ-module M such that:*

(1) *Λ separably divides Γ through bimodules ${}_\Lambda X_\Gamma$ and ${}_\Gamma Y_\Lambda$,*
(2) $\operatorname{Hom}_\Lambda(X, X \otimes_\Gamma M) \in \operatorname{add}_\Gamma M$.

Then $\operatorname{gldim} \operatorname{End}_\Lambda(X \otimes_\Gamma M) \leq \operatorname{gldim} \operatorname{End}_\Gamma(M)$.

Proof. Since X is a projective left Λ-module, the linear map

$$\operatorname{Ext}^n_\Lambda(U, V) \xrightarrow{\operatorname{Hom}_\Lambda(X,-)} \operatorname{Ext}^n_\Gamma(\operatorname{Hom}_\Lambda(X, U), \operatorname{Hom}_\Lambda(X, V))$$

is well-defined for all n and all Λ-modules U, V. Suppose an element $\eta \in \operatorname{Ext}^n_\Lambda(U, V)$ maps to zero through this map, i.e. $\operatorname{Hom}_\Lambda(X, \eta) = 0$. Since Y is projective as a left Γ-module, the linear map

$$\operatorname{Ext}^n_\Gamma(U', V') \xrightarrow{\operatorname{Hom}_\Gamma(Y,-)} \operatorname{Ext}^n_\Lambda(\operatorname{Hom}_\Gamma(Y, U'), \operatorname{Hom}_\Gamma(Y, V'))$$

is well-defined for all n and all Γ-modules U', V'. Applying this map to the zero element $\operatorname{Hom}_\Lambda(X, \eta)$, and using adjunction, we obtain

$$0 = \operatorname{Hom}_\Gamma(Y, \operatorname{Hom}_\Lambda(X, \eta)) \simeq \operatorname{Hom}_\Lambda(X \otimes_\Gamma Y, \eta).$$

Now since the Λ-Λ-bimodule Λ is a direct summand of $X \otimes_\Gamma Y$, the extension η is a direct summand of the extension $\operatorname{Hom}_\Lambda(X \otimes_\Gamma Y, \eta)$, hence $\eta = 0$. Consequently, the linear map

$$\operatorname{Ext}^n_\Lambda(U, V) \xrightarrow{\operatorname{Hom}_\Lambda(X,-)} \operatorname{Ext}^n_\Gamma(\operatorname{Hom}_\Lambda(X, U), \operatorname{Hom}_\Lambda(X, V))$$

is injective for all n and all Λ-modules U, V.

Suppose the global dimension of $\operatorname{End}_\Gamma(M)$ is finite, say $\operatorname{gldim} \operatorname{End}_\Gamma(M) = d$. Denote the induced Λ-module $X \otimes_\Gamma M$ by M^Λ, and let S be a simple $\operatorname{End}_\Lambda(M^\Lambda)$-module. The arguments used in the proof of Theorem 2.1 show that there exists an exact sequence

$$\mathbb{S}\colon \cdots \to N_2 \to N_1 \to N_0$$

of Λ-modules, with the following properties:

(1) each N_i belongs to $\operatorname{add}_\Lambda M^\Lambda$,
(2) when applying $\operatorname{Hom}_\Lambda(M^\Lambda, -)$ to $\mathbb{S}$, we obtain a projective resolution

$$\cdots \to \operatorname{Hom}_\Lambda(M^\Lambda, N_1) \to \operatorname{Hom}_\Lambda(M^\Lambda, N_0) \to S \to 0$$

of S over $\operatorname{End}_\Lambda(M^\Lambda)$.

For each $i \geq 1$, denote by K_i the image of the map $N_i \to N_{i-1}$.

Using adjointness, we obtain an isomorphism

$$\begin{aligned} \operatorname{Hom}_\Lambda(M^\Lambda, \mathbb{S}) &= \operatorname{Hom}_\Lambda(X \otimes_\Gamma M, \mathbb{S}) \\ &\simeq \operatorname{Hom}_\Gamma(M, \operatorname{Hom}_\Lambda(X, \mathbb{S})). \end{aligned}$$

Consequently, the sequence $\operatorname{Hom}_\Lambda(M^\Lambda, \mathbb{S})$ gives rise to an exact sequence

$$\cdots \to \operatorname{Hom}_\Gamma(M, \operatorname{Hom}_\Lambda(X, N_1)) \to \operatorname{Hom}_\Gamma(M, \operatorname{Hom}_\Lambda(X, N_0)) \to L \to 0$$

of $\operatorname{End}_\Gamma(M)$-modules. Since each N_i belongs to $\operatorname{add}_\Lambda M^\Lambda$, and $\operatorname{Hom}_\Lambda(X, M^\Lambda) \in \operatorname{add}_\Gamma M$ by assumption, each $\operatorname{End}_\Gamma(M)$-module $\operatorname{Hom}_\Gamma(M, \operatorname{Hom}_\Lambda(X, N_i))$ is projective. Therefore the sequence $\operatorname{Hom}_\Gamma(M, \operatorname{Hom}_\Lambda(X, \mathbb{S}))$ is a projective resolution of L as a module over $\operatorname{End}_\Gamma(M)$.

Since the global dimension of $\mathrm{End}_\Gamma(M)$ is d, the image of the map

$$\mathrm{Hom}_\Gamma(M, \mathrm{Hom}_\Lambda(X, N_d)) \to \mathrm{Hom}_\Gamma(M, \mathrm{Hom}_\Lambda(X, N_{d-1})),$$

namely $\mathrm{Hom}_\Gamma(M, \mathrm{Hom}_\Lambda(X, K_d))$, is a projective $\mathrm{End}_\Gamma(M)$-module. Therefore, when we apply $\mathrm{Hom}_\Gamma(M, -)$ to the short exact sequence

$$(\dagger) \qquad 0 \to \mathrm{Hom}_\Lambda(X, K_{d+1}) \to \mathrm{Hom}_\Lambda(X, N_d) \to \mathrm{Hom}_\Lambda(X, K_d) \to 0$$

of Γ-modules, the result is a split exact sequence of projective $\mathrm{End}_\Gamma(M)$-modules. But the functor $\mathrm{mod}\,\Gamma \xrightarrow{\mathrm{Hom}_\Gamma(M,-)} \mathrm{mod}\,\mathrm{End}_\Gamma(M)$ induces an equivalence between $\mathrm{add}_\Gamma M$ and the category of projective $\mathrm{End}_\Gamma(M)$-modules, hence the sequence $(\dagger)$ splits itself. It follows from the beginning of this proof that the linear map

$$\mathrm{Ext}^1_\Lambda(K_d, K_{d+1}) \xrightarrow{\mathrm{Hom}_\Lambda(X,-)} \mathrm{Ext}^1_\Gamma\left(\mathrm{Hom}_\Lambda(X, K_d), \mathrm{Hom}_\Lambda(X, K_{d+1})\right)$$

is injective, and so the short exact sequence

$$0 \to K_{d+1} \to N_d \to K_d \to 0$$

splits in $\mathrm{mod}\,\Lambda$. The module K_d is then a summand of N_d, and since $N_d \in \mathrm{add}_\Lambda M^\Lambda$, we obtain $K_d \in \mathrm{add}_\Lambda M^\Lambda$. This implies that in the projective resolution of the $\mathrm{End}_\Lambda(M^\Lambda)$-module S, the image $\mathrm{Hom}_\Lambda(M^\Lambda, K_d)$ of the map

$$\mathrm{Hom}_\Lambda(M^\Lambda, N_d) \to \mathrm{Hom}_\Lambda(M^\Lambda, N_{d-1})$$

is projective, and consequently $\mathrm{pd}_{\mathrm{End}_\Lambda(M^\Lambda)} S \leq d$. Since S was an arbitrary simple module over $\mathrm{End}_\Lambda(M^\Lambda)$, the global dimension of this endomorphism algebra is at most d, and the proof is complete. □

We end this section with the counterpart to Proposition 2.2

Proposition 2.4. *Let Λ and Γ be finite dimensional algebras, and suppose there exists a Γ-module M such that:*

1. *Λ separably divides Γ through bimodules ${}_\Lambda X_\Gamma$ and ${}_\Gamma Y_\Lambda$,*
2. *M is a generator in $\mathrm{mod}\,\Gamma$.*

Then the Λ-module $X \otimes_\Gamma M$ is a generator.

Proof. Since Y is projective as a left Γ-module, it belongs to $\mathrm{add}_\Gamma M$, and hence the left Λ-module $X \otimes_\Gamma Y$ belongs to $\mathrm{add}_\Lambda(X \otimes_\Gamma M)$. But Λ is a direct summand of $X \otimes_\Gamma Y$ as a bimodule, and in particular as a left Λ-module. Therefore Λ belongs to $\mathrm{add}_\Lambda(X \otimes_\Gamma M)$. □

3. Symmetric algebras

In this section, we record some properties of symmetric algebras. All modules are assumed to be finitely generated left modules. Recall that a finite dimensional algebra Λ over a field k is *symmetric* if it is isomorphic as a bimodule to its k-dual $D(\Lambda) = \mathrm{Hom}_k(\Lambda, k)$. If this is the case, then fix such an isomorphism $\Lambda \xrightarrow{\phi} D(\Lambda)$, and denote by $s \in D(\Lambda)$ the element $\phi(1)$. It is not hard to see that

$$\begin{aligned} s(xy) &= s(yx) \\ \phi(x)(y) &= s(xy) \end{aligned}$$

for all $x, y \in \Lambda$. The map s is called a *symmetrizing form* for Λ. Now suppose that Γ is a *parabolic* subalgebra of Λ (cf. [Bro]), that is, the following conditions are satisfied:

1. Γ is symmetric,
2. the restriction of s to Γ is a symmetrizing form for Γ,
3. Λ is a finitely generated projective left (equivalently, right) Γ-module.

Condition (2) is equivalent to the condition that, as a bimodule over Γ, the algebra Λ is a direct sum $\Lambda = \Gamma \oplus B$, where $B \subseteq \operatorname{Ker} s$. In other words, the Γ-Γ-bimodule Γ has a complement in Λ contained in the kernel of the symmetrizing form.

The multiplication map $\Lambda \otimes_\Gamma \Lambda \xrightarrow{\mu} \Lambda$ and the bimodule isomorphism ϕ give rise to a Λ-Λ-bimodule homomorphism $\Lambda \to \Lambda \otimes_\Gamma \Lambda$, given as the composition

$$\Lambda \xrightarrow{\phi} D(\Lambda) \xrightarrow{D(\mu)} D(\Lambda \otimes_\Gamma \Lambda) \xrightarrow{\sim} D(\Lambda) \otimes_\Gamma D(\Lambda) \xrightarrow{\phi^{-1} \otimes \phi^{-1}} \Lambda \otimes_\Gamma \Lambda.$$

The image of the identity of Λ under this homomorphism is the *relative Casimir element*, and denoted by c_Γ^Λ (cf. [Bro] and [Li2]). This element satisfies $x \cdot c_\Gamma^\Lambda = c_\Gamma^\Lambda \cdot x$ for every $x \in \Lambda$, hence $\mu(c_\Gamma^\Lambda)$ belongs to the center of Λ. Write

$$c_\Gamma^\Lambda = \sum_{i=1}^{n} x_i \otimes y_i$$

for some elements $x_i, y_i \in \Lambda$, and let M and N be two Λ-modules. Moreover, for a Γ-homomorphism $M \xrightarrow{f} N$, consider the k-linear map $M \xrightarrow{\operatorname{tr}_\Gamma^\Lambda(f)} N$ defined by

$$\operatorname{tr}_\Gamma^\Lambda(f)(m) \stackrel{\text{def}}{=} \sum_{i=1}^{n} x_i f(y_i m)$$

for $m \in M$. It follows from [Bro, Section 6] that this map is Λ-linear, and so $\operatorname{tr}_\Gamma^\Lambda$ is a k-linear map

$$\operatorname{Hom}_\Gamma(M,N) \xrightarrow{\operatorname{tr}_\Gamma^\Lambda} \operatorname{Hom}_\Lambda(M,N)$$

called the *trace map*. Using this trace map, the following lemma shows that the restriction map

$$\operatorname{Hom}_\Lambda(M,N) \xrightarrow{\operatorname{res}_\Gamma^\Lambda} \operatorname{Hom}_\Gamma(M,N)$$

is injective whenever $\mu(c_\Gamma^\Lambda)$ is invertible in the center of Λ.

Lemma 3.1. *Let Λ be a finite dimensional symmetric algebra and Γ a parabolic subalgebra. If $\mu(c_\Gamma^\Lambda)$ is invertible in the center of Λ, then the restriction map*

$$\operatorname{Hom}_\Lambda(M,N) \xrightarrow{\operatorname{res}_\Gamma^\Lambda} \operatorname{Hom}_\Gamma(M,N)$$

is injective for all Λ-modules M and N.

Proof. Write $c_\Gamma^\Lambda = \sum_{i=1}^t x_i \otimes y_i$, and consider the composition

$$\operatorname{Hom}_\Lambda(M,N) \xrightarrow{\operatorname{tr}_\Gamma^\Lambda \circ \operatorname{res}_\Gamma^\Lambda} \operatorname{Hom}_\Lambda(M,N).$$

Then

$$(\operatorname{tr}_\Gamma^\Lambda \circ \operatorname{res}_\Gamma^\Lambda)(f)(m) = \sum_{i=1}^{t} x_i f(y_i m) = \left(\sum_{i=1}^{t} x_i y_i \right) f(m) = \mu(c_\Gamma^\Lambda) f(m)$$

for every $f \in \operatorname{Hom}_\Lambda(M,N)$ and every $m \in M$. Since $\mu(c_\Gamma^\Lambda)$ is invertible, the composition $\operatorname{tr}_\Gamma^\Lambda \circ \operatorname{res}_\Gamma^\Lambda$ is injective, hence $\operatorname{res}_\Gamma^\Lambda$ is injective. $\square$

Our aim in this section is to extend this result to cohomology. To do that, we need a lemma on the transitivity of trace maps. Suppose we have a parabolic chain $\Lambda \supseteq \Gamma \supseteq \Delta$ of symmetric algebras, that is, the algebra Γ is a parabolic subalgebra of Λ, and Δ is a parabolic subalgebra of Γ. Since Γ is a parabolic subalgebra of Λ, the Γ-Γ-bimodule Λ is a direct sum $\Lambda = \Gamma \oplus B_1$, where $B_1 \subseteq \operatorname{Ker} s$. Similarly, since Δ is a parabolic subalgebra of Γ, the Δ-Δ-bimodule Γ is a direct sum $\Gamma = \Delta \oplus B_2$, where $B_2 \subseteq \operatorname{Ker} s$. Therefore, as a bimodule over Δ, the algebra Λ is a direct sum $\Lambda = \Delta \oplus B_1 \oplus B_2$, with $B_1 \oplus B_2 \subseteq \operatorname{Ker} s$. Consequently, the algebra Δ is a parabolic subalgebra of Λ, and the following lemma shows the trace map is transitive in this case.

Lemma 3.2. *If $\Lambda \supseteq \Gamma \supseteq \Delta$ is a parabolic chain of symmetric algebras, then*

$$\mathrm{tr}_{\Gamma}^{\Lambda} \circ \mathrm{tr}_{\Delta}^{\Gamma} = \mathrm{tr}_{\Delta}^{\Lambda}.$$

Proof. This follows directly from [Li1, Proposition 2.11(ii)]. With the notation used in that proposition, take X and Y to be the bimodules ${}_{\Lambda}\Lambda_{\Gamma}$ and ${}_{\Gamma}\Gamma_{\Delta}$, respectively. □

The field k is obviously a parabolic subalgebra of Λ and Γ when the symmetrizing form s is nonzero. Moreover, the latter is automatic if $\mu(c_{\Gamma}^{\Lambda})$ is invertible in the center of Λ. Namely, the relative Casimir element is by definition the image of the unit 1_{Λ} in Λ under the composition

$$\Lambda \xrightarrow{\phi} D(\Lambda) \xrightarrow{D(\mu)} D(\Lambda \otimes_{\Gamma} \Lambda) \xrightarrow{\sim} D(\Lambda) \otimes_{\Gamma} D(\Lambda) \xrightarrow{\phi^{-1} \otimes \phi^{-1}} \Lambda \otimes_{\Gamma} \Lambda,$$

and the image of 1_{Λ} under the first map in this composition is precisely s. Therefore, if $\mu(c_{\Gamma}^{\Lambda})$ is invertible, then $\Lambda \supseteq \Gamma \supseteq k$ is a parabolic chain of symmetric algebras, and the above lemma applies. Using this, we end this section with a result which extends Lemma 3.1 to cohomology.

Proposition 3.3. *Let Λ be a finite dimensional symmetric algebra, and Γ a parabolic subalgebra such that $\mu(c_{\Gamma}^{\Lambda})$ is invertible in Λ. Then for every i the restriction map*

$$\mathrm{Ext}_{\Lambda}^{i}(M,N) \xrightarrow{\mathrm{res}_{\Gamma}^{\Lambda}} \mathrm{Ext}_{\Gamma}^{i}(M,N)$$

is injective for all Λ-modules M and N.

Proof. Given Λ-modules M and N, let $f \in \mathrm{Hom}_{\Lambda}(M,N)$ be a homomorphism factoring through a projective module. Since projective Λ-modules are projective also as Γ-modules, the restriction $\mathrm{res}_{\Gamma}^{\Lambda}(f)$ factors through a projective Γ-module. Thus restriction induces a k-linear map

$$\underline{\mathrm{Hom}}_{\Lambda}(M,N) \xrightarrow{\mathrm{res}_{\Gamma}^{\Lambda}} \underline{\mathrm{Hom}}_{\Gamma}(M,N)$$

for stable homomorphisms. Let g represent an element in $\underline{\mathrm{Hom}}_{\Lambda}(M,N)$, and suppose that $\mathrm{res}_{\Gamma}^{\Lambda}(g) = 0$ in $\underline{\mathrm{Hom}}_{\Gamma}(M,N)$. By [Bro, Lemma 3.15], there is a k-linear map $h \in \mathrm{Hom}_{k}(M,N)$ with the property that $\mathrm{res}_{\Gamma}^{\Lambda}(g) = \mathrm{tr}_{k}^{\Gamma}(h)$. Then

$$(\mathrm{tr}_{\Gamma}^{\Lambda} \circ \mathrm{res}_{\Gamma}^{\Lambda})(g) = (\mathrm{tr}_{\Gamma}^{\Lambda} \circ \mathrm{tr}_{k}^{\Gamma})(h) = \mathrm{tr}_{k}^{\Lambda}(h)$$

by Lemma 3.2, and so by [Bro, Lemma 3.15] again, the map $(\mathrm{tr}_{\Gamma}^{\Lambda} \circ \mathrm{res}_{\Gamma}^{\Lambda})(g)$ factors through a projective Λ-module. In the proof of Lemma 3.1 we saw that $(\mathrm{tr}_{\Gamma}^{\Lambda} \circ \mathrm{res}_{\Gamma}^{\Lambda})(g) = \mu(c_{\Gamma}^{\Lambda})g$, hence g itself must factor through a projective Λ-module.

We have just shown that the restriction map

$$\underline{\mathrm{Hom}}_{\Lambda}(M,N) \xrightarrow{\mathrm{res}_{\Gamma}^{\Lambda}} \underline{\mathrm{Hom}}_{\Gamma}(M,N)$$

is injective for all Λ-modules M and N. Since the cohomology group $\mathrm{Ext}_{\Lambda}^{i}(M,N)$ is isomorphic to $\underline{\mathrm{Hom}}_{\Lambda}(\Omega_{\Lambda}^{i}(M),N)$, where $\Omega_{\Lambda}^{i}(M)$ denotes the ith syzygy of M, the result follows. □

4. Hecke algebras of type A and symmetric groups

Let Λ be a finite dimensional algebra over a field k, and denote by $\operatorname{mod}\Lambda$ the category of finitely generated left Λ-modules. As in the previous sections, all modules are assumed to be finitely generated left modules. The *representation dimension* of Λ, denoted $\operatorname{repdim}\Lambda$, is defined as

$$\operatorname{repdim}\Lambda \stackrel{\text{def}}{=} \inf\{\operatorname{gldim}\operatorname{End}_{\Lambda}(M) \mid M \text{ generates and cogenerates } \operatorname{mod}\Lambda\},$$

where gldim denotes the global dimension of an algebra. To say that a module generates and cogenerates $\operatorname{mod}\Lambda$ means that it contains all the indecomposable projective and injective modules as direct summands. Of course, if Λ is selfinjective, these two notions coincide. It follows immediately from the definition that a semisimple algebra is of representation dimension zero. As mentioned, Auslander showed that the representation dimension of a non-semisimple algebra is two if its representation type is finite, and at least three whenever its representation type is infinite. Thus no algebra is of representation dimension one. Moreover, Auslander showed that the representation dimension of a selfinjective algebra is at most its Loewy length. Later, Iyama showed in [Iya] that the representation dimension is finite for every finite dimensional algebra.

The focus of this paper is the representation dimension of Hecke algebras and group algebras of symmetric groups. Let k be a field of characteristic zero, $\ell \geq 2$ an integer and $q \in k$ a primitive ℓth root of unity. Recall that the corresponding Hecke algebra $\mathcal{H}_q(A_{n-1})$ of type A_{n-1} is the k-algebra with generators $T_1, \dots, T_{n-1}$ satisfying the relations

$$\begin{aligned} (T_i+1)(T_i-q) &= 0 \quad \text{for} \quad 1 \leq i \leq n-1 \\ T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1} \quad \text{for} \quad 1 \leq i \leq n-2 \\ T_iT_j &= T_jT_i \quad \text{for} \quad |i-j| \geq 2. \end{aligned}$$

If $q=1$, this is just the group algebra of the symmetric group S_n, hence $\mathcal{H}_q(A_{n-1})$ is also referred to as the Hecke algebra of S_n. It is well-known that a Hecke algebra is symmetric, and that its representation type depends on the number $[n/\ell]$ (cf. [ErN]). Namely, write $n = \ell m + a$, where $0 \leq a \leq \ell - 1$ (hence $[n/\ell] = m$). Then $\mathcal{H}_q(A_{n-1})$ is semisimple if and only if $\ell = \infty$ or $m = 0$. It is non-semisimple of finite representation type if and only if $m = 1$, and of tame representation type if and only if $\ell = 2$ and n is either 4 or 5 (and then $m = 2$). In all other cases, the algebra $\mathcal{H}_q(A_{n-1})$ is of wild representation type.

It was shown in [BEM] and [Li2] there exists a maximal ℓ-parabolic subalgebra $\mathcal{B} \subseteq \mathcal{H}_q(A_{n-1})$ which is isomorphic to an m-fold tensor product

$$\mathcal{B} \simeq B_1 \otimes \cdots \otimes B_m,$$

where each B_i is either semisimple or a Brauer tree algebra. Thus each B_i has finite representation type. The same occurs for group algebras of certain symmetric groups. Suppose k is a field of positive characteristic p, let n be an integer with $n < p^2$, and S_n the nth symmetric group. If we write $m = [n/p]$, then a Sylow p-subgroup of S_n is a direct product $P = P_1 \times \cdots \times P_m$, where each P_i has order p, and these cyclic groups have disjoint supports. The group algebra kP is isomorphic to the tensor product $kP_1 \otimes \cdots \otimes kP_m$, and each kP_i is isomorphic to the local algebra $k[x]/(x^p)$. In particular, each algebra kP_i has finite representation type.

Note that in both these situations, there is a canonical generator for the subalgebra. Namely, for each $1 \leq i \leq m$, let M_i be the direct sum of a complete set of isomorphism classes of indecomposable B_i-modules (respectively, kP_i-modules). Then the $\mathcal{B}$-module (respectively, kP-module) $M_1 \otimes \cdots \otimes M_m$ is a generator. The following crucial lemmas show that when we induce and then restrict this module, the resulting module is contained in the additive closure of M. We include a proof only for the group algebra case; the proof of the Hecke algebra case is completely analogous.

Lemma 4.1. *Let k be a field of positive characteristic p, let n be an integer with $n < p^2$, and S_n the nth symmetric group. Furthermore, let $P = P_1 \times \cdots \times P_m$ be a Sylow p-subgroup, where each P_i has order p and $m = [n/p]$. Finally, for each $1 \leq i \leq m$, let M_i be the direct sum of a complete set of isomorphism classes of indecomposable kP_i-modules, and denote the kP-module $M_1 \otimes \cdots \otimes M_m$ by M. Then the kP-module $kS_n \otimes_{kP} M$ belongs to $\mathrm{add}_{kP} M$.*

Proof. By the Mackey formula, the restriction of $kS_n \otimes_{kP} M$ to kP is given by

$$kS_n \otimes_{kP} M = \bigoplus_x kP \otimes_{k(P \cap xPx^{-1})} (x \otimes M),$$

where the sum is taken over a system of double coset representatives. Our module M is an outer tensor product of the kP_i-modules M_i, so assume that each $P \cap xPx^{-1}$ in the formula is a direct product $P \cap xPx^{-1} = Q_1 \times \cdots \times Q_m$, with each Q_i a subgroup of P_i. Then the $k(P \cap xPx^{-1})$-module $x \otimes M$ is again an outer tensor product of modules over the algebras kQ_i, hence $kP \otimes_{k(P \cap xPx^{-1})} (x \otimes M)$ is also an outer tensor product $N_1 \otimes \cdots \otimes N_m$, with each N_i a module over kP_i. Thus, in this case, each kP-module $kP \otimes_{k(P \cap xPx^{-1})} (x \otimes M)$, and therefore also $kS_n \otimes_{kP} M$, belongs to $\mathrm{add}_{kP} M$.

We must therefore show that each $P \cap xPx^{-1}$ is a direct product $P \cap xPx^{-1} = Q_1 \times \cdots \times Q_m$, with each factor Q_i a subgroup of P_i. Write $n = mp + a$, where $0 \leq a < p$. The group $P \cap xPx^{-1}$ is a subgroup of $S_\lambda \cap xS_\lambda x^{-1}$, where S_λ is a Young subgroup for the partition $\lambda = (p^m, 1^a)$ of n, and $P \leq S_\lambda$. The intersection $S_\lambda \cap xS_\lambda x^{-1}$ is again a Young subgroup, for a partition which refines λ. The group $P \cap xPx^{-1}$ is a p-subgroup for this intersection, and by Sylow's theorem it is therefore contained in a Sylow p-subgroup R of $S_\lambda \cap xS_\lambda x^{-1}$. The group R is a direct product of the Sylow subgroups of the factors, and is therefore a direct product $R = R_1 \times \cdots \times R_m$. If R_i is nontrivial, then it is generated by a p-cycle, and distinct nontrivial R_i, R_j have disjoint supports. Therefore any subgroup of R, in particular $P \cap xPx^{-1}$, is a direct product of groups generated by p-cycles, with disjoint supports. □

As mentioned, we do not include the proof of the Hecke algebra version of Lemma 4.1, since it is completely analogous to the proof just given. For more background on the relevant machinery (for instance, the Mackey formula), see [DJ1].

Lemma 4.2. *Let $\mathcal{H}_q(A_{n-1})$ be the Hecke algebra of the symmetric group S_n, where q is a primitive ℓth root of unity and the ground field is of characteristic zero. Furthermore, let $\mathcal{B} \simeq B_1 \otimes \cdots \otimes B_m$ be a maximal ℓ-parabolic subalgebra, where $m = [n/\ell]$ and each B_i is either semisimple or a Brauer tree algebra. Finally, for each i, let M_i be the direct sum of a complete set of isomorphism classes of indecomposable B_i-modules, and denote the $\mathcal{B}$-module $M_1 \otimes \cdots \otimes M_m$ by M. Then the $\mathcal{B}$-module $\mathcal{H}_q(A_{n-1}) \otimes_{\mathcal{B}} M$ belongs to $\mathrm{add}_{\mathcal{B}} M$.*

We are now ready to prove our two main results. The first of these gives both an upper and a lower bound for the representation dimension of a non-semisimple Hecke algebra $\mathcal{H}_q(A_{n-1})$. The main ingredient in the proof of the upper bound is Theorem 2.1.

Theorem 4.3. *Let $\mathcal{H}_q(A_{n-1})$ be the Hecke algebra of type A_{n-1}, where q is a primitive ℓth root of unity and the ground field is of characteristic zero. If $\mathcal{H}_q(A_{n-1})$*

is not semisimple (i.e. if ℓ is finite and $[n/\ell] \geq 1$), then

$$[n/\ell] + 1 \leq \operatorname{repdim} \mathcal{H}_q(A_{n-1}) \leq 2[n/\ell].$$

Proof. Let k be the ground field, and write $n = \ell m + a$, where $0 \leq a \leq \ell - 1$. We must show that

$$m + 1 \leq \operatorname{repdim} \mathcal{H}_q(A_{n-1}) \leq 2m,$$

and we start with the lower bound. Throughout this proof, we denote our Hecke algebra $\mathcal{H}_q(A_{n-1})$ by Λ.

By [Li2, Theorem 1.1], the Hochschild cohomology ring

$$\operatorname{HH}^*(\Lambda) = \operatorname{Ext}^*_{\Lambda \otimes_k \Lambda^{\mathrm{op}}}(\Lambda, \Lambda)$$

of Λ is Noetherian, and $\operatorname{Ext}^*_{\Lambda \otimes_k \Lambda^{\mathrm{op}}}(\Lambda, X)$ is a finitely generated $\operatorname{HH}^*(\Lambda)$-module for every Λ-Λ-bimodule X. By [EHSST, Proposition 2.4], the latter is equivalent to $\operatorname{Ext}^*_\Lambda(\Lambda/\operatorname{rad}\Lambda, \Lambda/\operatorname{rad}\Lambda)$ being a finitely generated $\operatorname{HH}^*(\Lambda)$-module. Since the characteristic of k is zero, it is a perfect field, hence $(\Lambda/\operatorname{rad}\Lambda) \otimes_k (\Lambda/\operatorname{rad}\Lambda)$ is a semisimple algebra. Therefore, by [Ber, Corollary 3.6] (see also [BIKO, Corollary 5.12]), the inequality

$$\dim \operatorname{HH}^*(\Lambda) + 1 \leq \operatorname{repdim} \Lambda$$

holds, where $\dim \operatorname{HH}^*(\Lambda)$ is the Krull dimension of $\operatorname{HH}^*(\Lambda)$. By [Li2, Theorem 1.2], the Krull dimension of $\operatorname{HH}^*(\Lambda)$ is m, hence the lower bound follows.

To prove the upper bound, let $\mathcal{B}$ be a maximal ℓ-parabolic subalgebra of Λ. Then $\mathcal{B}$ is isomorphic to an m-fold tensor product $\mathcal{B} \simeq B_1 \otimes \cdots \otimes B_m$, where each B_i is either semisimple or a Brauer tree algebra. In any case, each B_i is of finite representation type. For each i, let M_i be the direct sum of a complete set of isomorphism classes of indecomposable B_i-modules, and consider the $\mathcal{B}$-module

$$M = M_1 \otimes \cdots \otimes M_m.$$

Then $\operatorname{gldim} \operatorname{End}_{B_i}(M_i) \leq 2$, and therefore

$$\operatorname{gldim} \operatorname{End}_{\mathcal{B}}(M) = \sum_{i=1}^{m} \operatorname{gldim} \operatorname{End}_{B_i}(M_i) \leq 2m$$

by [Xi, Corollary 3.3 and Lemma 3.4], since the ground field k is perfect. The module M is a generator in $\operatorname{mod} \mathcal{B}$, and the $\mathcal{B}$-module $\Lambda \otimes_{\mathcal{B}} M$ belongs to $\operatorname{add}_{\mathcal{B}} M$ by Lemma 4.2. Moreover, by [Du, Theorem 2.7] the element $\mu(c^\Lambda_{\mathcal{B}})$ is invertible in the center of Λ, hence by Proposition 3.3, the restriction map

$$\operatorname{Ext}^i_\Lambda(X, Y) \xrightarrow{\operatorname{res}^\Lambda_{\mathcal{B}}} \operatorname{Ext}^i_{\mathcal{B}}(X, Y)$$

is injective for all i and all Λ-modules X and Y. Then by Theorem 2.1 the inequality

$$\operatorname{gldim} \operatorname{End}_\Lambda(\Lambda \otimes_{\mathcal{B}} M) \leq \operatorname{gldim} \operatorname{End}_{\mathcal{B}}(M) \leq 2m$$

holds. Consequently, since the Λ-module $\Lambda \otimes_{\mathcal{B}} M$ is a generator by Proposition 2.2, the representation dimension of Λ is at most $2m$. □

Next, we prove the second of our main results, namely the analogue of Theorem 4.3 for group algebras of symmetric groups. Here we use Theorem 2.3 when we prove the upper bound.

Theorem 4.4. *Let k be a perfect field of positive characteristic p, let n be an integer with $n < p^2$, and S_n the nth symmetric group. If kS_n is not semisimple (i.e. if $p \leq n$), then*

$$[n/p] + 1 \leq \operatorname{repdim} kS_n \leq 2[n/p].$$

Proof. We start with the lower bound. Since $n < p^2$, any Sylow p-subgroup of S_n is elementary abelian of order $[n/p]$, hence this number is the p-rank of S_n. The lower bound now follows from [Op1, Corollary 19].

For the upper bound, let P be a Sylow p-subgroup of S_n. Then $P = P_1 \times \cdots \times P_m$, where each P_i has order p and $m = [n/p]$. For each $1 \le i \le m$, let M_i be the direct sum of a complete set of isomorphism classes of indecomposable kP_i-modules, and denote the kP-module $M_1 \otimes \cdots \otimes M_m$ by M. This module generates $\operatorname{mod} kP$, and as in the proof of Theorem 4.3, the global dimension of $\operatorname{End}_{kP}(M)$ is at most $2m$.

Define bimodules ${}_{kS_n}X_{kP}$ and ${}_{kP}Y_{kS_n}$ by $X = kS_n$ and $Y = kS_n$. Then the kS_n-kS_n-bimodule kS_n is a direct summand of $X \otimes_{kP} Y$, so kS_n separably divides kP. Moreover, the kP-module $\operatorname{Hom}_{kS_n}(X, X \otimes_{kP} M)$ is just $X \otimes_{kP} M$, and therefore belongs to $\operatorname{add}_{kP} M$ by Lemma 4.1. Theorem 2.3 now gives

$$\operatorname{gldim} \operatorname{End}_{kS_n}(X \otimes_{kP} M) \le \operatorname{gldim} \operatorname{End}_{kP}(M) \le 2m,$$

and since $X \otimes_{kP} M$ generates $\operatorname{mod} kS_n$ by Proposition 2.4, the result follows. □

We end this section with some remarks on our main results.

Remarks. (1) Instead of Theorem 2.1, we could have used Theorem 2.3 to prove the upper bound in Theorem 4.3. Indeed, if $\mathcal{B}$ is a maximal ℓ-parabolic subalgebra of $\mathcal{H}_q(A_{n-1})$, define bimodules ${}_{\mathcal{H}_q(A_{n-1})}X_{\mathcal{B}}$ and ${}_{\mathcal{B}}Y_{\mathcal{H}_q(A_{n-1})}$ by $X = Y = \mathcal{H}_q(A_{n-1})$. By [Li2, Proposition 5.1], the algebra $\mathcal{H}_q(A_{n-1})$ separably divides $\mathcal{B}$ through X and Y. Thus assumption (1) in Theorem 2.3 holds. Assumption (2) holds by Lemma 4.2 and the arguments used at the end of the proof of Theorem 4.4.

(2) Similarly, instead of Theorem 2.3, we could have used Theorem 2.1 to prove the upper bound in Theorem 4.4. First, note that when P is a Sylow p-subgroup of S_n, then kP is a parabolic subalgebra of kS_n. When $n < p^2$, the index $|S_n|/|P|$ is not divisible by p, hence $\mu(c_{kP}^{kS_n})$ is invertible in kS_n. Therefore, by Proposition 3.3, assumption (2) in Theorem 2.1 holds. Assumption (1) is just Lemma 4.1.

(3) As mentioned prior to Lemma 4.1, the Hecke algebra $\mathcal{H}_q(A_{n-1})$ is non-semisimple of finite representation type if and only if $[n/\ell] = 1$, and of tame representation type if and only if $\ell = 2$ and n is either 4 or 5 (and then $[n/\ell] = 2$). In all the other non-semisimple cases, the Hecke algebra has wild representation type. In the finite type case, Theorem 4.3 therefore gives $2 \le \operatorname{repdim} \mathcal{H}_q(A_{n-1}) \le 2$, i.e. $\operatorname{repdim} \mathcal{H}_q(A_{n-1}) = 2$. This was of course to be expected: every non-semisimple finite dimensional algebra of finite representation type has representation dimension 2. When $\mathcal{H}_q(A_{n-1})$ is tame, Theorem 4.3 gives $3 \le \operatorname{repdim} \mathcal{H}_q(A_{n-1}) \le 4$, that is, the representation dimension is either 3 or 4. It is known that the representation dimension is 3 in this case. Namely, by [ScS], there are two Morita equivalence classes of blocks, represented by $\mathcal{H}_q(A_3)$ and $\mathcal{H}_q(A_4)$. It is shown in [ErN] that these algebras are special biserial, and so by [EHIS, Corollary 1.3] they are both of representation dimension 3.

(4) The group algebra kS_n is non-semisimple when $p \le n$, and when $n < p^2$ it has finite representation type precisely when $[n/p] = 1$. For in this case, the Sylow p-subgroups of S_n are cyclic, and by a classical result of Higman this is equivalent to kS_n having finite type. As was the case for Hecke algebras, Theorem 4.4 gives $\operatorname{repdim} kS_n = 2$ in this case.

5. Hecke algebras of types B and D

In this final section, we consider Hecke algebras of types B_n ($n \ge 2$) and D_n ($n \ge 4$). These are associated to Coxeter groups $W(B_n) = \mathcal{S}_2 \wr \mathcal{S}_n$ and $W(D_n) = W(B_n) \cap \mathcal{A}_{2n}$, where $\mathcal{A}_{2n}$ is the alternating group of degree $2n$. As before, the ground field is assumed to be of characteristic zero.

For type B_n, the Hecke algebra involves two parameters q and Q, and we write $\mathcal{H}_{Q,q}(B_n)$. This is the k-algebra with generators $T_0, T_1, \dots, T_{n-1}$ satisfying the relations

$$\begin{aligned} (T_0+1)(T_0-Q) &= 0 \\ T_0T_1T_0T_1 &= T_1T_0T_1T_0 \\ (T_i+1)(T_i-q) &= 0 \quad \text{for} \quad 1 \le i \le n-1 \\ T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1} \quad \text{for} \quad 1 \le i \le n-2 \\ T_iT_j &= T_jT_i \quad \text{for} \quad |i-j| \ge 2. \end{aligned}$$

Note that there is a natural inclusion of the Hecke algebra $\mathcal{H}_q(A_{n-1})$ of type A_{n-1} into $\mathcal{H}_{Q,q}(B_n)$, taking the generator T_i of $\mathcal{H}_q(n)$ (for $1 \le i \le n-1$) to the generator T_i of $\mathcal{H}_{Q,q}(B_n)$, ignoring the element T_0. The algebra $\mathcal{H}_{Q,q}(B_n)$ is free as a left/right module over $\mathcal{H}_q(A_{n-1})$.

For Hecke algebras of type D_n, there is just one parameter q, and we write $\mathcal{H}_q(D_n)$. This is the k-algebra with generators $T_0, T_1, \dots, T_{n-1}$ satisfying the relations

$$\begin{aligned} T_0T_2T_0 &= T_2T_0T_2 \\ T_0T_i &= T_iT_0 \quad \text{for} \quad 1 \le i \le n-1, i \ne 2 \\ (T_i+1)(T_i-q) &= 0 \quad \text{for} \quad 0 \le i \le n-1 \\ T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1} \quad \text{for} \quad 1 \le i \le n-2 \\ T_iT_j &= T_jT_i \quad \text{for} \quad 1 \le i \le j-2 \le n-3. \end{aligned}$$

As with Hecke algebras of type B, there is a natural inclusion of $\mathcal{H}_q(A_{n-1})$ into $\mathcal{H}_q(D_n)$: the generator T_i of $\mathcal{H}_q(n)$ (for $1 \le i \le n-1$) maps to the generator T_i of $\mathcal{H}_q(D_n)$, ignoring the element T_0. Moreover, the algebra $\mathcal{H}_q(D_n)$ is free as a left/right module over $\mathcal{H}_q(A_{n-1})$.

We shall first establish lower bounds for the representation dimensions of Hecke algebras of types B and D. To do this, we compute lower bounds for the dimensions of the stable module categories involved, viewed as triangulated categories. Namely, if $\mathcal{H}$ is any Hecke algebra, then $\mathcal{H}$ is symmetric, in particular selfinjective. The stable module category $\underline{\mathrm{mod}}\,\mathcal{H}$ is then triangulated, with suspension functor $\Omega_{\mathcal{H}}^{-1} \colon \underline{\mathrm{mod}}\,\mathcal{H} \to \underline{\mathrm{mod}}\,\mathcal{H}$. Now let $\mathcal{T}$ be an arbitrary triangulated category with suspension functor $\Sigma \colon \mathcal{T} \to \mathcal{T}$, and $\mathcal{C}$ and $\mathcal{D}$ subcategories of $\mathcal{T}$. We denote by $\mathrm{thick}_{\mathcal{T}}^1(\mathcal{C})$ the full subcategory of $\mathcal{T}$ consisting of all the direct summands of finite direct sums of suspensions of objects in $\mathcal{C}$. Furthermore, we denote by $\mathcal{C} * \mathcal{D}$ the full subcategory of $\mathcal{T}$ consisting of objects M such that there exists a distinguished triangle

$$C \to M \to D \to \Sigma C$$

in $\mathcal{T}$, with $C \in \mathcal{C}$ and $D \in \mathcal{D}$. Now for each $n \ge 2$, define inductively $\mathrm{thick}_{\mathcal{T}}^n(\mathcal{C})$ by

$$\mathrm{thick}_{\mathcal{T}}^n(\mathcal{C}) \stackrel{\text{def}}{=} \mathrm{thick}_{\mathcal{T}}^1\left(\mathrm{thick}_{\mathcal{T}}^{n-1}(\mathcal{C}) * \mathrm{thick}_{\mathcal{T}}^1(\mathcal{C})\right).$$

Then the *dimension* of $\mathcal{T}$, denoted $\dim \mathcal{T}$, is defined as

$$\dim \mathcal{T} \stackrel{\text{def}}{=} \inf\{n \ge 0 \mid \exists \text{ an object } G \in \mathcal{T} \text{ such that } \mathcal{T} = \mathrm{thick}_{\mathcal{T}}^{n+1}(G)\}.$$

This notion was introduced by Rouquier in [Ro2], precisely in order to establish lower bounds for the representation dimension of certain algebras, namely exterior algebras.

Lemma 5.1. [Ro2, Proposition 3.9] *If Λ is a finite dimensional non-semisimple selfinjective algebra, then* $\dim(\underline{\mathrm{mod}}\,\Lambda) + 2 \le \mathrm{repdim}\,\Lambda$.

Thus, in order to compute lower bounds for the representation dimensions Hecke algebras of types B and D, we compute lower bounds for the dimensions of their stable module categories. We include also Hecke algebras of type A.

Lemma 5.2. *Let k be a field of characteristic zero, and $q \in k$ a primitive ℓth root of unity, where ℓ is finite. Furthermore, let n be an integer, and $\mathcal{H}$ either a Hecke algebra $\mathcal{H}_q(A_{n-1})$ of type A_{n-1}, a Hecke algebra $\mathcal{H}_{Q,q}(B_n)$ of type B_n, or a Hecke algebra $\mathcal{H}_q(D_n)$ of type D_n. Then* $\dim(\underline{\operatorname{mod}}\,\mathcal{H}) \geq [n/\ell] - 1$.

Proof. The Hecke algebra $\mathcal{H}_q(A_{n-1})$ of type A_{n-1} is a subalgebra of $\mathcal{H}$, but also a factor algebra in a natural way (factor out the generator T_0). Therefore, there is a diagram

$$\mathcal{H}_q(A_{n-1}) \xrightarrow{i} \mathcal{H} \xrightarrow{\pi} \mathcal{H}_q(A_{n-1})$$

of k-algebra homomorphisms, in which the composition $\pi \circ i$ is the identity on $\mathcal{H}_q(A_{n-1})$. Since $\mathcal{H}$ is projective as a left $\mathcal{H}_q(A_{n-1})$-module, the inequality

$$\dim(\underline{\operatorname{mod}}\,\mathcal{H}_q(A_{n-1})) \leq \dim(\underline{\operatorname{mod}}\,\mathcal{H})$$

holds by [BO1, Lemma 2.3]. It is therefore enough to prove the result for $\mathcal{H}_q(A_{n-1})$.

Denote our algebra $\mathcal{H}_q(A_{n-1})$ by Λ. Since the ground field k is of characteristic zero, it is a perfect field, hence the algebra $(\Lambda/\operatorname{rad}\Lambda) \otimes_k (\Lambda/\operatorname{rad}\Lambda)$ is semisimple. As we saw in the proof of Theorem 4.3, the Hochschild cohomology ring $\operatorname{HH}^*(\Lambda)$ is Noetherian of Krull dimension $[n/\ell]$, and $\operatorname{Ext}^*_\Lambda(\Lambda/\operatorname{rad}\Lambda, \Lambda/\operatorname{rad}\Lambda)$ is a finitely generated $\operatorname{HH}^*(\Lambda)$-module. Moreover, in the proof of [Ber, Corollary 3.6], it was shown that the Krull dimension of the Hochschild cohomology ring equals the complexity of the Λ-module $\Lambda/\operatorname{rad}\Lambda$. It then follows from [Ber, Theorem 3.2] that $\dim(\underline{\operatorname{mod}}\,\Lambda) \geq [n/\ell] - 1$. □

Combining this lemma with Lemma 5.1, we obtain the lower bounds for the representation dimensions of Hecke algebras of types B and D. The same bound holds for Hecke algebras of types A_{n-1}, B_n and D_n.

Theorem 5.3. *Let k be a field of characteristic zero, and $q \in k$ a primitive ℓth root of unity, where ℓ is finite. Furthermore, let n be an integer such that $[n/\ell] \geq 1$, and $\mathcal{H}$ either a Hecke algebra $\mathcal{H}_{Q,q}(B_n)$ of type B_n, or a Hecke algebra $\mathcal{H}_q(D_n)$ of type D_n. Then* $\operatorname{repdim}\mathcal{H} \geq [n/\ell] + 1$.

Remark. There is a common generalization of Hecke algebras of types A and B, namely the *Ariki-Koike* algebras. Let $q, Q_1, \ldots, Q_t$ be elements of k, and denote the sequence $(Q_1, \ldots, Q_t)$ by $\mathbf{Q}$. The corresponding Ariki-Koike algebra $\mathcal{H}_{\mathbf{Q},q}(n)$ is the k-algebra with generators $T_0, T_1, \ldots, T_{n-1}$ satisfying the relations

$$\begin{aligned} \prod_{i=1}^{t}(T_0 - Q_i) &= 0 \\ T_0T_1T_0T_1 &= T_1T_0T_1T_0 \\ (T_i + 1)(T_i - q) &= 0 \quad \text{for} \quad 1 \leq i \leq n-1 \\ T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1} \quad \text{for} \quad 1 \leq i \leq n-2 \\ T_iT_j &= T_jT_i \quad \text{for} \quad |i-j| \geq 2. \end{aligned}$$

When $\mathbf{Q} = (1)$, this is just the Hecke algebra $\mathcal{H}_q(A_{n-1})$ of type A_{n-1}, whereas when $\mathbf{Q} = (-1, Q)$ we retrieve the Hecke algebra $\mathcal{H}_{Q,q}(B_n)$ of type B_n. As with Hecke algebras of types B_n and D_n, the algebra $\mathcal{H}_q(A_{n-1})$ is a subalgebra of $\mathcal{H}_{\mathbf{Q},q}(n)$, and the latter is free over $\mathcal{H}_q(A_{n-1})$. Moreover, there is a diagram

$$\mathcal{H}_q(A_{n-1}) \xrightarrow{i} \mathcal{H}_{\mathbf{Q},q}(n) \xrightarrow{\pi} \mathcal{H}_q(A_{n-1})$$

of k-algebra homomorphisms, in which the composition $\pi \circ i$ is the identity on $\mathcal{H}_q(A_{n-1})$. As in the proof of Lemma 5.2, we obtain the inequalities

$$\dim(\underline{\mathrm{mod}}\, \mathcal{H}_{\mathbf{Q},q}(n)) \geq \dim(\underline{\mathrm{mod}}\, \mathcal{H}_q(A_{n-1})) \geq [n/\ell] - 1,$$

and so the representation dimension of $\mathcal{H}_{\mathbf{Q},q}(n)$ is also bounded below by $[n/\ell]+1$.

Finally, we turn to upper bounds for the representation dimensions of Hecke algebras of types B and D. The situation is more complicated than for type A, and our method depends on whether certain polynomial expressions in the parameters are nonzero. First we treat type B_n for $n \geq 2$: we set

$$f_n(Q,q) = \prod_{i=1-n}^{n-1} (Q + q^i).$$

By a result of Dipper and James, when $f_n(Q,q)$ is nonzero, then the algebra $\mathcal{H}_{Q,q}(B_n)$ is Morita equivalent to a product of tensor products of Hecke algebras of type A. Of course, the condition that $f_n(Q,q)$ be nonzero is equivalent to the condition $Q + q^i \neq 0$ for $1 - n \leq i \leq n - 1$.

Lemma 5.4. [DJ2, Theorem 4.17] *If $f_n(Q,q)$ is nonzero, then the Hecke algebra $\mathcal{H}_{Q,q}(B_n)$ is Morita equivalent to the algebra*

$$\prod_{j=0}^{n} \mathcal{H}_q(A_{j-1}) \otimes_k \mathcal{H}_q(A_{n-j-1}).$$

Using this result, we obtain the upper bound for the representation dimension in type B, provided the relevant polynomial is invertible.

Theorem 5.5. *Let k be a field of characteristic zero, and $q \in k$ a primitive ℓth root of unity, where ℓ is finite. Furthermore, let n be an integer, and $\mathcal{H}_{Q,q}(B_n)$ a Hecke algebra of type B_n. If $f_n(Q,q)$ is nonzero, then* $\operatorname{repdim} \mathcal{H}_{Q,q}(B_n) \leq 2[n/\ell]$.

Proof. In general, the representation dimension of a direct product of algebras equals the maximum of the representation dimensions of the factors. Therefore, by Lemma 5.4, the representation dimension of $\mathcal{H}_{Q,q}(B_n)$ equals

$$\max\{\operatorname{repdim}(\mathcal{H}_q(A_{j-1}) \otimes_k \mathcal{H}_q(A_{n-j-1})) \mid 0 \leq j \leq n\}.$$

Now by [Xi, Theorem 3.5], the representation dimension of $\mathcal{H}_q(A_{j-1}) \otimes_k \mathcal{H}_q(A_{n-j-1})$ is at most

$$\operatorname{repdim} \mathcal{H}_q(A_{j-1}) + \operatorname{repdim} \mathcal{H}_q(A_{n-j-1}),$$

and by Theorem 4.3 this sum is at most $2([j/\ell] + [(n-j)/\ell])$ (of course, the upper bound in Theorem 4.3 holds without the assumption that $[n/\ell] \geq 1$: if $[n/\ell] = 0$ then the Hecke algebra is semisimple). Since $[j/\ell] + [(n-j)/\ell] \leq [n/\ell]$, we are done. □

Next, we consider type D_n for $n \geq 4$, and here we set

$$g_n(q) = 2 \prod_{i=1}^{n-1} (1 + q^i).$$

The following theorem, which is analogous to the one for type B, is implicit in [Pal, Theorems 3.6 and 3.7], and is made explicit in [Hu].

Lemma 5.6. *If $g_n(q)$ is nonzero and n is odd, then $\mathcal{H}_q(D_n)$ is Morita equivalent to the algebra*

$$\prod_{j=(n+1)/2}^{n} \mathcal{H}_q(A_{j-1}) \otimes_k \mathcal{H}_q(A_{n-j-1}).$$

There is a corresponding theorem when n is even, a result which was proved in [Hu]. However, the Morita description in this case has a direct factor which is not a tensor product of type A Hecke algebras, and therefore we cannot deal with the case when n is even. But when n is odd we obtain an upper bound. We skip the proof since it is exactly the same as that of Theorem 5.5.

Theorem 5.7. *Let k be a field of characteristic zero, and $q \in k$ a primitive ℓth root of unity, where ℓ is finite. Furthermore, let n be an odd integer, and $\mathcal{H}_q(D_n)$ a Hecke algebra of type D_n. If $g_n(q)$ is nonzero, then* $\operatorname{repdim} \mathcal{H}_q(D_n) \leq 2[n/\ell]$.

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Petter Andreas Bergh
Institutt for matematiske fag, NTNU, N-7491 Trondheim, Norway
E-mail address: bergh@math.ntnu.no

Karin Erdmann, Mathematical Institute, 24-29 St. Giles, Oxford OX1 3LB, United Kingdom
E-mail address: erdmann@maths.ox.ac.uk