Witt groups of complex cellular varieties

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Abstract

We show that the Grothendieck-Witt and Witt groups of a smooth complex cellular variety can be identified with its topological KO-groups. As an application, we deduce the values of the Witt groups of several families of projective homogeneous varieties, including smooth complex quadrics, spinor varieties and symplectic Grassmannians.

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Introduction

The purpose of this paper is to demonstrate that Witt groups of complex homogeneous varieties can be computed in a purely topological way. That is, we will show in Theorem 2.6 below that the Witt groups of such varieties can be identified with certain topological KO-groups, and we will illustrate this with a series of known and new examples.

Our theorem holds more generally for any smooth complex cellular variety. By this we mean a smooth complex variety X with a filtration by closed subvarieties $\emptyset = Z_0 \subset Z_1 \subset Z_2 \cdots \subset Z_N = X$ such that the complement of Z_k in Z_{k+1} is an open "cell" isomorphic to $\mathbb{A}^{n_k}$ for some n_k. Let us put our result into perspective. It is well-known that for such cellular X we have an isomorphism

$$\mathrm{K}_0(X) \xrightarrow{\cong} \mathrm{K}^0(X(\mathbb{C}))$$

between the algebraic K-group of X and the complex K-group of the underlying topological space $X(\mathbb{C})$. In fact, both sides are easy to compute: they decompose as direct sums of the K-groups of the cells, each of which is isomorphic to $\mathbb{Z}$. Such decompositions are characteristic of oriented cohomology theories. Witt groups, however, are strictly non-oriented, and this makes computations much harder. It is true that the Witt groups of complex varieties are isomorphic to direct sums of copies of $\mathbb{Z}/2$, the Witt group of $\mathbb{C}$, but even in the cellular case there is no general understanding of how many copies to expect.

Nonetheless, we can prove our theorem by an induction over the number of cells of X. The main issue is to define the map from Witt groups to the relevant KO-groups in such a way that it respects various exact sequences. The basic idea is clear: the Witt group $\mathrm{W}^0(X)$ classifies vector bundles equipped with non-degenerate symmetric forms, and in topology symmetric complex vector bundles are in one-to-one correspondence with real vector bundles, classified by $\mathrm{KO}^0(X)$. More precisely, we have two natural maps:

$$\mathrm{GW}^0(X) \to \mathrm{KO}^0(X(\mathbb{C}))$$
$$\mathrm{W}^0(X) \to \tfrac{\mathrm{KO}^0(X(\mathbb{C}))}{\mathrm{K}^0(X(\mathbb{C}))}$$

Here, $\mathrm{GW}^0(X)$ is the Grothendieck-Witt group of X, and in the second line $\mathrm{K}^0(X)$ is mapped to $\mathrm{KO}^0(X)$ by sending a complex vector bundle to the underlying real bundle. It is possible to extend these maps to higher (Grothendieck-)Witt groups in a concrete and "elementary" way, and this was the approach taken in [Zib09]. It remained unclear in general, however, whether these maps commuted with the boundary morphisms in localization sequences, and the fact that it was nevertheless possible to prove Theorem 2.6 seemed somewhat coincidental.

The approach taken here will instead rely on a recent result in $\mathbb{A}^1$-homotopy theory: the representability of hermitian K-theory by a spectrum whose complex realization is the usual topological KO-spectrum. Unfortunately, the only written reference we have for this result at the moment is a draft paper of Morel [Mor06]. However, the author remains convinced that this approach is ultimately the most natural one to take, and hopeful that a published account of all relevant results will become available in due course.

The structure of this document is as follows: In the first section we assemble some of the basic definitions. Along the way, we give some naïve motivation why KO-theory should be viewed as a topological analogue of hermitian K-theory before finally stating in 1.9 the results in $\mathbb{A}^1$-homotopy theory that we will ultimately take as our starting point. Our main result, Theorem 2.6, is stated and proved in the second section. Section 3 reviews mostly well-known facts about the Atiyah-Hirzebruch spectral sequence, on which the computations of examples in the final section will rely.

1 Preliminaries

1.1 Witt groups and hermitian K-theory

From a modern point of view, the theory of Witt groups represents a K-theoretic approach to the study of quadratic forms. We briefly run through some of the basic definitions.

Recall that the algebraic K-group $\mathrm{K}_0(X)$ of a scheme X can be defined as the free abelian group on isomorphism classes of vector bundles over X modulo the following relation: for any short exact sequence of vector bundles

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$$

on X we have $[\mathcal{F}] = [\mathcal{E}] + [\mathcal{G}]$ in $\mathrm{K}_0(X)$. In particular, as far as $\mathrm{K}_0(X)$ is concerned, we may pretend that all exact sequences of vector bundles over X split.

Now let $(\mathcal{E}, \epsilon)$ be a symmetric vector bundle, by which we mean a vector bundle $\mathcal{E}$ equipped with a non-degenerate symmetric bilinear form ϵ. We may view ϵ as an isomorphism from $\mathcal{E}$ to its dual bundle $\mathcal{E}^\vee$, in which case its symmetry may be expressed by saying that ϵ and $\epsilon^\vee$ agree under the canonical identification of the double-dual $(\mathcal{E}^\vee)^\vee$ with $\mathcal{E}$.

$$\begin{tikzcd} \mathcal{E} \arrow[dr, "\epsilon"] & \\ & \mathcal{E}^\vee \\ (\mathcal{E}^\vee)^\vee \arrow[uu, "\cong"] \arrow[ur, "\epsilon^\vee"'] & \end{tikzcd}$$

Two symmetric vector bundles $(\mathcal{E}, \epsilon)$ and $(\mathcal{F}, \phi)$ are isometric if there is an isomorphism of vector bundles $i\colon \mathcal{E} \to \mathcal{F}$ compatible with the symmetries, i. e. such that $i^\vee \epsilon = \phi i$. The orthogonal sum of two symmetric bundles has the obvious definition $(\mathcal{E}, \epsilon) \perp (\mathcal{F}, \phi) := (\mathcal{E} \oplus \mathcal{F}, \epsilon \oplus \phi)$.

Any vector bundle $\mathcal{E}$ gives rise to a symmetric bundle $H(\mathcal{E}) := (\mathcal{E} \oplus \mathcal{E}^\vee, (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}))$, the hyperbolic bundle associated with $\mathcal{E}$. These hyperbolic bundles are the simplest members of a wider class of so-called metabolic bundles: symmetric bundles $(\mathcal{M}, \mu)$ which contain a subbundle $j\colon \mathcal{L} \to \mathcal{M}$ of half their own rank on which μ vanishes. In other words, $(\mathcal{M}, \mu)$ is metabolic if it fits into a short exact sequence of the form

$$0 \to \mathcal{L} \xrightarrow{j} \mathcal{M} \xrightarrow{j^\vee \mu} \mathcal{L}^\vee \to 0$$

The subbundle $\mathcal{L}$ is then called a Lagrangian of $\mathcal{M}$. If the sequence splits then $(\mathcal{M}, \mu)$ is isometric to $H(\mathcal{L})$, at least in any characteristic other than two. This motivates the definition of the Grothendieck-Witt group.

Definition 1.1. The Grothendieck-Witt group $\mathrm{GW}^0(X)$ of a scheme X is the free abelian group on isometry classes of symmetric vector bundles over X modulo the following two relations:

- $[(\mathcal{E}, \epsilon) \perp (\mathcal{G}, \gamma)] = [(\mathcal{E}, \epsilon)] + [(\mathcal{G}, \gamma)]$
- $[(M, \mu)] = [H(\mathcal{L})]$ for any metabolic bundle (M, μ) with Lagrangian $\mathcal{L}$

The Witt group of $\mathrm{W}^0(X)$ is defined similarly, except that the second relation reads $[(M, \mu)] = 0$. Equivalently, we may define $\mathrm{W}^0(X)$ by the exact sequence

$$\mathrm{K}_0(X) \xrightarrow{H} \mathrm{GW}^0(X) \longrightarrow \mathrm{W}^0(X) \to 0$$

Higher Witt groups. The groups above can be defined more generally in the context of exact or triangulated categories with dualities. The previous definitions are then recovered by considering the category of vector bundles over X or its bounded derived category. However, the abstract point of view allows for greater flexibility. In particular, a number of useful variants of Witt groups can be introduced by passing to related categories or dualities. For example, if we take a line bundle $\mathcal{L}$ on X and replace the usual duality $\mathcal{E}^\vee := \mathcal{H}om(\mathcal{E}, \mathcal{O}_X)$ on vector bundles by $\mathcal{H}om(-, \mathcal{L})$ we obtain "twisted" Witt groups $\mathrm{W}^0(X; \mathcal{L})$. On the bounded derived category, we can consider dualities that involve shifting complexes, leading to the definition of higher Witt groups $\mathrm{W}^i(X)$. This approach, pioneered by Paul Balmer in [Bal00, Bal01a], elevates the theory of Witt groups into the realm of cohomology theories. We illustrate the meaning and significance of these remarks with a few of the key properties of the theory, concentrating as before on the case of smooth schemes. The interested but unacquainted reader may prefer to consult [Bal01b] or [Bal05].

- For any smooth scheme X, any line bundle $\mathcal{L}$ on X and any integer i we have a Witt group
$$\mathrm{W}^i(X; \mathcal{L})$$
This is the i^{th} Witt group of X "with coefficients in $\mathcal{L}$", or "twisted by $\mathcal{L}$". When $\mathcal{L}$ is trivial it is frequently dropped from the notation.
- The Witt groups are four-periodic in i and "two-periodic in $\mathcal{L}$" in the sense that for any X, any i and any line bundles $\mathcal{L}$ and $\mathcal{M}$ on X we have canonical isomorphisms
$$\mathrm{W}^i(X; \mathcal{L}) \cong \mathrm{W}^{i+4}(X; \mathcal{L})$$
$$\mathrm{W}^i(X; \mathcal{L}) \cong \mathrm{W}^i(X; \mathcal{L} \otimes \mathcal{M}^{\otimes 2})$$
- More generally, for any closed subset Z of X we have Witt groups "with support on Z", written $\mathrm{W}^i_Z(X; \mathcal{L})$. For $Z = X$ these agree with $\mathrm{W}^i(X; \mathcal{L})$.
- We have long exact "localization sequences" relating the Witt groups of X and $X - Z$, which can be arranged as 12-term exact loops by periodicity.

Balmer's approach already works on the level of Grothendieck-Witt groups, as shown in [Wal03a]. In this context, the localization sequences take the form

$$\begin{aligned}&\mathrm{GW}^i_Z(X) \to \mathrm{GW}^i(X) \to \mathrm{GW}^i(X-Z)\\ &\qquad \to \mathrm{W}^{i+1}_Z(X) \to \mathrm{W}^{i+1}(X) \to \mathrm{W}^{i+1}(X-Z) \to \mathrm{W}^{i+2}_Z(X) \to \cdots\end{aligned}$$

continuing to the right with higher Witt groups of X, and similarly for arbitrary twists $\mathcal{L}$. However, if one wishes to continue the sequences to the left, one has to revert to the methods of higher algebraic K-theory.

Hermitian K-theory. Recall that the higher algebraic K-groups of a scheme X can be defined as the homotopy groups of a topological space $\mathrm{K}(X)$ associated with X. If one replaces $\mathrm{K}(X)$ by an appropriate spectrum one can similarly define groups $\mathrm{K}_n(X)$ in all degrees $n \in \mathbb{Z}$. On a smooth scheme X, however, the groups in negative degrees vanish.

An analogous construction of hermitian K-theory is developed in [Sch10]. Given a scheme X and a line bundle $\mathcal{L}$ on X, Schlichting constructs a family of spectra $\mathbb{G}\mathrm{W}^i(X;\mathcal{L})$ from which hermitian K-groups can be defined as

$$\mathrm{GW}^i_n(X) := \pi_n(\mathbb{G}\mathrm{W}^i(X;\mathcal{L}))$$

In degree $n = 0$, one recovers Balmer and Walter's Grothendieck-Witt groups, and the Witt groups appear as hermitian K-groups in negative degrees. To be precise, for any smooth scheme X over a field of characteristic not equal to two one has the following natural identifications:

$$\mathrm{GW}^i_0(X;\mathcal{L}) \cong \mathrm{GW}^i(X;\mathcal{L}) \tag{1}$$

$$\mathrm{GW}^i_n(X;\mathcal{L}) \cong \mathrm{W}^{i-n}(X;\mathcal{L}) \text{ for } n < 0 \tag{2}$$

For $i = 0$ these can be found in [Hor05, Corollary 7.5], whereas the more general identifications quoted here are due to appear in [Sch]. They also hold more generally for hermitian K-groups with support $\mathrm{GW}^i_{n,Z}(X)$. We will be using these identifications implicitly throughout.

1.2 KO-theory

We now turn to the corresponding theories in topology. To ensure that the definitions given here are consistent with the literature we will restrict our attention to finite-dimensional CW complexes.[1] Since we are ultimately only interested in topological spaces that arise as complex varieties, this will not be a problem. The definitions of K_0 and GW^0 given above applied to complex vector bundles over such a finite-dimensional CW complex X yield its complex and real topological K-groups $\mathrm{K}^0(X)$ and $\mathrm{KO}^0(X)$. However, short exact sequence of complex vector bundles over CW complexes always split, allowing a slight simplification of the definitions.

[1] The key property we need is that for any vector bundle $\mathcal{E}$ over a finite-dimensional CW complex X there exists a stable inverse, i. e. some vector bundle $\mathcal{E}^\perp$ over X such that $\mathcal{E} \oplus \mathcal{E}^\perp$ is a trivial bundle. See [Hus94, Chapter 3, Proposition 5.8].

Definition 1.2. For a finite-dimensional CW complex X, the complex K-group $\mathrm{K}^0(X)$ is the free abelian group on isomorphism classes of complex vector bundles over X modulo the relation $[\mathcal{E} \oplus \mathcal{G}] = [\mathcal{E}] + [\mathcal{G}]$. Likewise, the KO-group $\mathrm{KO}^0(X)$ is the free abelian group on isometry classes of symmetric complex vector bundles over X modulo the relation $[(\mathcal{E},\epsilon) \perp (\mathcal{G},\gamma)] = [(\mathcal{E},\epsilon)] + [(\mathcal{G},\gamma)]$.

There is a more common description of $\mathrm{KO}^0(X)$ as the K-group of real vector bundles. The equivalence with the definition given here can be traced back to the fact that the orthogonal group $\mathrm{O}(n)$ is a maximal compact subgroup of both $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{O}_n(\mathbb{C})$, but also seen very concretely along the following lines. We say that a complex bilinear form ϵ on a real vector bundle $\mathcal{F}$ is real if $\epsilon\colon \mathcal{F} \otimes \mathcal{F} \to \mathbb{C}$ factors through $\mathbb{R}$.

Lemma 1.3. *Let $(\mathcal{E},\epsilon)$ be a finite-dimensional symmetric complex vector bundle. There exists a unique real subbundle $\Re(\mathcal{E},\epsilon) \subset \mathcal{E}$ such that $\Re(\mathcal{E},\epsilon) \otimes_{\mathbb{R}} \mathbb{C} = \mathcal{E}$ and such that the restriction of ϵ to $\Re(\mathcal{E},\epsilon)$ is real and positive definite. Concretely, a fibre of $\Re(\mathcal{E},\epsilon)$ is given by the real span of any orthonormal basis of the corresponding fibre of $\mathcal{E}$.*

Corollary 1.4. *For any CW complex X, the monoid of isomorphism classes of real vector bundles over X is isomorphic to the monoid of isometry classes of symmetric complex vector bundles over X (with respect to the operations $\oplus$ and $\perp$, respectively).*

Proof of Lemma 1.3. In the case of a vector bundle over a point this is straight-forward linear algebra. We may assume without loss of generality that

$$(\mathcal{E},\epsilon) = \left(\mathbb{C}^r, \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}\right)$$

Clearly, the subspace $\mathbb{R}^r \subset \mathbb{C}^r$ has the required properties, and we only need to show uniqueness. So suppose W is another r-dimensional real subspace of $\mathbb{C}^r$ such that $\epsilon|_W$ is real and positive definite. Pick an orthonormal basis of W with respect to $\epsilon|_W$,

$$e_1 + if_1,\ e_2 + if_2,\ \ldots,\ e_r + if_r$$

with e_i, f_i in $\mathbb{R}^r$. Then we see that

$$\epsilon(e_j, f_k) = 0 \qquad \text{for all } j,\ k \tag{3}$$
$$\epsilon(e_j, e_k) = \epsilon(f_j, f_k) \qquad \text{for all } j \neq k \tag{4}$$
$$\epsilon(e_k, e_k) = \epsilon(f_k, f_k) + 1 \qquad \text{for all } k \tag{5}$$

It follows from (3) that the real spans of $e_1, \ldots, e_r$ and $f_1, \ldots, f_r$ in $\mathbb{R}^r$ are orthogonal. On the other hand, it follows from (4) and (5) that $e_1, \ldots, e_r$ are linearly independent: if $\sum_j \lambda_j e_j = 0$ for certain $\lambda_j \in \mathbb{R}$ then

$$0 = \epsilon\left(\sum \lambda_j e_j, \sum \lambda_j e_j\right) = \epsilon\left(\sum \lambda_j f_j, \sum \lambda_j f_j\right) + \sum \lambda_j^2$$

So all λ_j must be zero. It follows that $f_1 = \cdots = f_r = 0$, so $W = \mathbb{R}^r$.

Now if $(\mathcal{E},\epsilon)$ is an arbitrary symmetric complex vector bundle over a space X, then any point of X has some neighbourhood over which $(\mathcal{E},\epsilon)$ can be trivialized in the form above. We know how to define $\Re(\mathcal{E},\epsilon)$ over each such neighbourhood, and by uniqueness these local bundles can be glued together. □

Proof of Corollary 1.4. A map in one direction is given by sending a complex symmetric bundle $(\mathcal{E}, \epsilon)$ to $\Re(\mathcal{E}, \epsilon)$. Conversely, with any real vector bundle $\mathcal{E}$ on X we may associate a complex symmetric bundle $(\mathcal{E} \otimes_{\mathbb{R}} \mathbb{C}, \sigma_{\mathbb{C}})$, where $\sigma_{\mathbb{C}}$ is the $\mathbb{C}$-linear extension of some inner product σ on $\mathcal{E}$. Since σ is defined uniquely up to isometry, so is $(\mathcal{E} \otimes_{\mathbb{R}} \mathbb{C}, \sigma_{\mathbb{C}})$. See [MH73, Chapter V, § 2] for a proof that avoids the uniqueness part of the preceding lemma. □

Representing topological K-groups. A standard construction of the cohomology theories associated with K^0 and KO^0 is based on the fact that these functors are representable in the homotopy category of topological spaces. The starting point is the homotopy classification of vector bundles: Let $\mathrm{Gr}_{r,n}$ be the Grassmannian of complex r-bundles in $\mathbb{C}^{r+n}$, and let Gr_r be the union of $\mathrm{Gr}_{r,n} \subset \mathrm{Gr}_{r,n+1} \subset \cdots$ under the obvious inclusions. Denote by $\mathcal{U}_{r,n}$ and $\mathcal{U}_r$ the universal r-bundles over these spaces. For any paracompact Hausdorff space X we have a one-to-one correspondence between isomorphism classes of rank-r complex vector bundles and homotopy classes of maps from X to Gr_r under which a homotopy class $[f]$ in $\mathcal{H}(X, \mathrm{Gr}_r)$ corresponds to the pullback of $\mathcal{U}_r$ along f.

Let Gr denote the union of the Gr_r, where Gr_r is embedded into Gr_{r+1} by sending a complex r-plane W to $\mathbb{C} \oplus W$. The product $\mathbb{Z} \times \mathrm{Gr}$ can be viewed as the colimit of the inductive system

$$\coprod_{d\geq 0}\{d\} \times \mathrm{Gr}_d \hookrightarrow \coprod_{d\geq -1}\{d\} \times \mathrm{Gr}_{d+1} \hookrightarrow \coprod_{d\geq -2}\{d\} \times \mathrm{Gr}_{d+2} \hookrightarrow \cdots \subset \mathbb{Z} \times \mathrm{Gr}$$

Theorem 1.5. *For finite-dimensional CW complexes X we have natural isomorphisms*

$$\mathrm{K}^0(X) \cong \mathcal{H}(X, \mathbb{Z} \times \mathrm{Gr}) \tag{6}$$

such that, for $X = \mathrm{Gr}_{r,n}$, the class $[\mathcal{U}_{r,n}] + (d - r)[\mathbb{C}]$ in $\mathrm{K}^0(\mathrm{Gr}_{r,n})$ corresponds to the inclusion $\mathrm{Gr}_{r,n} \hookrightarrow \{d\} \times \mathrm{Gr}_{r,n} \hookrightarrow \mathbb{Z} \times \mathrm{Gr}$.

If we replace the complex Grassmannians by real Grassmannians $\mathbb{R}\mathrm{Gr}_{r,n}$, we obtain the analogous statement that KO^0 can be represented by $\mathbb{Z} \times \mathbb{R}\mathrm{Gr}$. Equivalently, but more in the spirit of Definition 1.2, we can work with the following spaces:

Definition 1.6. Let (V, ν) be a symmetric complex vector space, and let $\mathrm{Gr}(r, V)$ be the Grassmannian of complex k-planes in V. The "non-degenerate Grassmannian"

$$\mathrm{Gr}^{\mathrm{nd}}(r, (V, \nu))$$

is the open subspace of $\mathrm{Gr}(r, V)$ given by r-planes T for which the restriction $\nu|_T$ is non-degenerate. We have a universal symmetric complex vector bundle over $\mathrm{Gr}^{\mathrm{nd}}(r, (V, \nu))$, denoted $\mathcal{U}(r, (V, \nu))$.

The equivalence of these spaces with real Grassmannians will be shown in the next lemma. We will use $\mathrm{Gr}^{\mathrm{nd}}_{r,n}$ to abbreviate $\mathrm{Gr}^{\mathrm{nd}}(r, \mathbb{H}^{r+n})$, where $\mathbb{H}$ is the hyperbolic plane $(\mathbb{C}^2, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right))$, and similarly for the universal bundles. Colimits $\mathrm{Gr}^{\mathrm{nd}}_r$ and $\mathrm{Gr}^{\mathrm{nd}}$ can be defined in the same way as for the usual Grassmannians. Then, for finite-dimensional CW complexes X, we have natural isomorphisms

$$\mathrm{KO}^0(X) \cong \mathcal{H}(X, \mathbb{Z} \times \mathrm{Gr}^{\mathrm{nd}}) \tag{7}$$

under which the inclusion $\mathrm{Gr}^{\mathrm{nd}}_{r,n} \hookrightarrow \{d\} \times \mathrm{Gr}^{\mathrm{nd}}_{r,n} \hookrightarrow \mathbb{Z} \times \mathrm{Gr}^{\mathrm{nd}}$ corresponds to the class of $[\mathcal{U}^{\mathrm{nd}}_{r,n}] + \frac{d-r}{2}[\mathbb{H}]$ in $\mathrm{GW}^0(\mathrm{Gr}^{\mathrm{nd}}_{r,n})$, for even $(d-r)$.

Lemma 1.7. *Let (V,ν) be a symmetric complex vector space. The inclusion*

$$\begin{aligned} \mathbb{R}\mathrm{Gr}(k, \Re(V,\nu)) &\overset{j}{\hookrightarrow} \mathrm{Gr}^{nd}(k,(V,\nu)) \\ U &\mapsto U \otimes_{\mathbb{R}} \mathbb{C} \end{aligned}$$

is a homotopy equivalence.

Proof. Consider the projection $\pi\colon V = \Re(V,\nu) \oplus i\Re(V,\nu) \twoheadrightarrow \Re(V,\nu)$. We define a retract r of j by sending a complex k-plane $T \in \mathrm{Gr}^{\mathrm{nd}}(k,(V,\nu))$ to

$$\pi(\Re(T,\nu|_T)) \subset \Re(V,\nu)$$

This is indeed a linear subspace of real dimension k: since ν is postive definite on $\Re(T,\nu|_T)$ but negative definite on $i\Re(V,\nu)$, the intersection $\Re(T,\nu|_T) \cap i\Re(V,\nu)$ is trivial.

More generally, we can define a family of endomorphisms of V parametrized by $t \in [0,1]$ by

$$\begin{aligned} \pi_t\colon \Re(V,\nu) \oplus i\Re(V,\nu) &\twoheadrightarrow \Re(V,\nu) \oplus i\Re(V,\nu) \\ (x,y) &\mapsto (x,ty) \end{aligned}$$

This family interpolates between the identity π_1 and the projection π_0, which we can identify with π. We claim that

$$\pi_t(\Re(T,\nu|_T)) \subset V$$

is a real linear subspace of dimension k on which ν is real and positive definite. The claim concerning the dimension has already been verified in the case $t=0$ and follows for non-zero t from the fact that π_t is an isomorphism. Now take a non-zero vector $v \in \pi_t(\Re(T,\nu|_T))$ and write it as $v = x + tiy$, where $x, y \in \Re(V,\nu)$ and $x + iy \in \Re(T,\nu|_T)$. Since $\nu(x,x)$, $\nu(y,y)$ and $\nu(x+iy, x+iy)$ are all real we deduce that $\nu(x,y) = 0$; it follows that $\nu(v,v)$ is real as well. Moreover, since $\nu(x+iy,x+iy)$ is positive we have $\nu(x,x) > \nu(y,y)$, so that $\nu(v,v) > (1-t^2)\nu(y,y)$. In particular, $\nu(v,v) > 0$ for all $t \in [0,1]$, as claimed.

It follows that $T \mapsto \pi_t(\Re(T,\nu|_T)) \otimes_{\mathbb{R}} \mathbb{C}$ defines a homotopy from $j \circ r$ to the identity on $\mathrm{Gr}^{\mathrm{nd}}(k,(V,\nu))$. □

K-spectra and cohomology theories. The infinite Grassmannian Gr can be identified with the classifying space BU of the infinite unitary group. Consequently, K^0 can be represtend by $\mathbb{Z} \times \mathrm{BU}$, which is equivalent to its own two-fold loopspace $\Omega^2(\mathbb{Z} \times \mathrm{BU})$. This can be used to construct a 2-periodic Ω-spectrum $\mathbb{K}^{\mathrm{top}}$ in the stable homotopy category $\mathcal{SH}$ whose even terms are all given by $\mathbb{Z} \times \mathrm{BU}$. Similarly, $\mathbb{R}\mathrm{Gr}$ is equivalent to the classifying space BO of the infinite orthogonal group, and Bott periodicity in this case says that $\mathbb{Z} \times \mathrm{BO}$ is equivalent to $\Omega^8(\mathbb{Z} \times \mathrm{BO})$. Thus, one obtains a spectrum $\mathbb{K}\mathbf{O}^{\mathrm{top}}$ in $\mathcal{SH}$ which is 8-periodic. The associated cohomology theories are given by

$$\mathrm{K}^i(X) := \mathcal{SH}(\Sigma^\infty(X_+), S^i \wedge \mathbb{K}^{\mathrm{top}}) \tag{8}$$

$$\mathrm{KO}^i(X) := \mathcal{SH}(\Sigma^\infty(X_+), S^i \wedge \mathbb{K}\mathbf{O}^{\mathrm{top}}) \tag{9}$$

where X_+ denotes the union of X and a disjoint base point, and Σ^∞ is the functor assigning to a pointed space its suspension spectrum.

For convenience and later reference, we include here the values of the theories on a point. Since we are in fact dealing with multiplicative theories, these can be summarized in the form of coefficient rings. For K-theory we have

$$\mathrm{K}^*(\text{point}) = \mathbb{Z}[g, g^{-1}] \tag{10}$$

where g is a generator of degree -2. The coefficient ring of KO-theory is given by

$$\mathrm{KO}^*(\text{point}) = \frac{\mathbb{Z}[\eta, \alpha, \lambda, \lambda^{-1}]}{(2\eta,\ \eta^3,\ \alpha^2 - 4\lambda)} \tag{11}$$

with η, α and λ of degrees -1, -4 and -8, respectively [Bot69].

1.3 Some comparison maps

Now suppose X is a smooth complex variety $\mathbb{C}$. We write $X(\mathbb{C})$ for the set of complex points of X equipped with the analytical topology. If $\mathcal{E}$ is a vector bundle over X then $\mathcal{E}(\mathbb{C})$ has the structure of a complex vector bundle over $X(\mathbb{C})$, so that we obtain natural maps

$$\mathrm{K}_0(X) \to \mathrm{K}^0(X(\mathbb{C})) \tag{12}$$

$$\mathrm{GW}^0(X) \to \mathrm{KO}^0(X(\mathbb{C})) \tag{13}$$

and an induced map

$$\mathrm{W}^0(X) \to \frac{\mathrm{KO}^0(X(\mathbb{C}))}{\mathrm{K}^0(X(\mathbb{C}))} \tag{14}$$

We now wish to extend these definitions to obtain maps defined on $\mathrm{GW}^i(X)$ and $\mathrm{W}^i(X)$ for arbitrary i, and also on groups with support and twisted groups. Let us comment on some "elementary" constructions that are possible before outlining the approach that we will ultimately rely on here.

Firstly, a way to extend the maps to the groups $\mathrm{GW}^i(X)$ and $\mathrm{W}^i(X)$ would be to use the multiplicative structure of the theories together with Walter's results on projective bundles [Wal03b]. Namely, for any variety X one has isomorphisms

$$\mathrm{GW}^i(X \times \mathbb{P}^1) \cong \mathrm{GW}^i(X) \oplus \mathrm{GW}^{i-1}(X)$$
$$\mathrm{KO}^{2i}(X(\mathbb{C}) \times S^2) \cong \mathrm{KO}^{2i}(X(\mathbb{C})) \oplus \mathrm{KO}^{2i-2}(X(\mathbb{C}))$$

This allows an inductive definition of comparison maps, at least for all negative i. Basic properties of these maps, for example compatibility with the periodicities of Grothendieck-Witt and KO-groups, can be checked by direct calulations.

It is less clear how to obtain maps on Witt groups with restricted supports. One possibility, pursued in [Zib09], is to work on the level of complexes of vector bundles and adapt a construction of classes in relative K-groups described in [Seg68] to the case of KO-theory. However, it remains unclear to the author how to see in this approach that the resulting maps are compatible with the boundary morphisms in long exact localization sequences.

Theorem 2.6 below could in fact be proved without knowing that the comparison maps respect the boundary morphisms in localization sequences in general. However, recent results in $\mathbb{A}^1$-homtopy theory provide an alternative construction of a comparison map for which this property immediately follows from the construction, and which in any case is so compellingly beautiful that it would be difficult to argue in favour of any other approach.

1.4 $\mathbb{A}^1$-homotopy theory

Theorem 1.5 describing K^0 in terms of homotopy classes of maps to Grassmannians has an analogue in algebraic geometry, in the context of $\mathbb{A}^1$-homotopy theory. Developed mainly by Morel and Voevodsky, the theory provides a general framework for a homotopy theory of schemes emulating the situation for topological spaces. The authorative reference is [MV99]; closely related texts by the same authors are [Voe98], [Mor99] and [Mor04]. See [DLØ$^+$07] for a textbook introduction and [Dug01] for an enlightening perspective on one of the main ideas.

Let us summarize the main points relevant for us in just a few sentences: Consider the category Sm_k of smooth schemes over a field k. It can be embedded into some larger category of "spaces" Spc_k which is closed under all small limits and colimits, and which can be equipped with a model structure. The $\mathbb{A}^1$-homotopy category $\mathcal{H}(k)$ over k is the homotopy category associated with this model category.

There are several possible choices for Spc_k and many possible model structures yielding the same homotopy category $\mathcal{H}(k)$. One possibility is to consider the category of simplicial presheaves over Sm_k, or the category of simplicial sheaves with respect to the Nisnevich topology. Both categories contain Sm_k as full subcategories via the Yoneda embedding, and they also contain simplicial sets viewed as constant (pre)sheaves. One then applies a general recipe for equipping the category of simplicial (pre)sheaves over a site with a model structure (see [Jar87]). In a crucial last step, one forces the affine line $\mathbb{A}^1$ to become contractible by localizing with respect to the set of all projections $\mathbb{A}^1 \times X \twoheadrightarrow X$.

As in topology, we also have a pointed version $\mathcal{H}_\bullet(k)$ of $\mathcal{H}(k)$. Remarkably, these categories contain several distinct "circles": the simplicial circle S^1, the "Tate circle" $\mathbb{G}_m = \mathbb{A}^1 - 0$ (pointed at 1) and the projective line $\mathbb{P}^1$ (pointed at ∞). These are related by the intriguing formula $\mathbb{P}^1 = S^1 \wedge \mathbb{G}_m$. A common notational convention which we will follow is to define

$$S^{p,q} := (S^1)^{\wedge(p-q)} \wedge \mathbb{G}_m^{\wedge q}$$

for any $p \geq q$. In particular, we then have $S^1 = S^{1,0}$, $\mathbb{G}_m = S^{1,1}$ and $\mathbb{P}^1 = S^{2,1}$.

One can take the theory one step further by passing to the stable homotopy category $\mathcal{SH}(k)$, a triangulated category in which the suspension functors $S^{p,q} \wedge -$ become invertible. This category is usually constructed using $\mathbb{P}^1$-spectra. The triangulated shift functor is given by suspension with the simplicial sphere $S^{1,0}$.

Finally and crucially, the analogy with topology can be made precise: when we take our ground field k to be the complex numbers, or more general any subfield of $\mathbb{C}$, we have a complex realization functor from the $\mathbb{A}^1$-homotopy category over k to the usual homotopy category:

$$\mathcal{H}(k) \to \mathcal{H} \tag{15}$$

A smooth scheme X is simply taken to its set of complex points $X(\mathbb{C})$ equipped with the analytic topology. There is also a pointed realization functor and, moreover, a triangulated functor of the stable homotopy categories which takes $\Sigma^\infty(X_+)$ to $\Sigma^\infty(X(\mathbb{C})_+)$ for any smooth scheme X [Rio06, Rio07a]:

$$\mathcal{SH}(k) \to \mathcal{SH} \tag{16}$$

1.5 Representing algebraic and hermitian K-theory

We now come to the analogue of Theorem 1.5. For any field k we can consider the Grassmannian of r-planes in k^{n+r}; this is a smooth projective variety over k. Viewing these Grassmannians as objects in Spc_k, we can form the colimits Gr_r and Gr in the same way as before. Whether we take Spc_k to mean Nisnevich sheaves or simplicial (pre)sheaves on Sm_k does not matter here since colimits are preserved both by sheafification and by the canonical inclusion of sheaves of sets into simplicial sheaves. The following theorem is established in [MV99]; see also [Mor07] and [Rio06].

Theorem 1.8. *For smooth schemes X over k we have natural isomorphisms*

$$\mathrm{K}_0(X) \cong \mathcal{H}(k)(X, \mathbb{Z} \times \mathrm{Gr}) \tag{17}$$

such that the inclusion of $\mathrm{Gr}_{r,n} \hookrightarrow \{d\} \times \mathrm{Gr}_{r,n} \hookrightarrow \mathbb{Z} \times \mathrm{Gr}$ corresponds to the class $[\mathcal{U}_{r,n}] + (d-r)[\mathcal{O}]$ in $\mathrm{K}_0(\mathrm{Gr}_{r,n})$.

The following analogous result for hermitian K-theory was recently announced by Schlichting[2]: Let $\mathrm{Gr}^{\mathrm{nd}}_{r,n}$ denote the "non-degenerate Grassmannians" defined as open subvarieties of $\mathrm{Gr}_{r,r+2n}$ as above, and let $\mathrm{Gr}^{\mathrm{nd}}_r$ and $\mathrm{Gr}^{\mathrm{nd}}$ be the respective colimits. Then for smooth schemes over k we have natural isomorphisms

$$\mathrm{GW}^0(X) \cong \mathcal{H}(k)(X, \mathbb{Z} \times \mathrm{Gr}^{\mathrm{nd}}) \tag{18}$$

It follows from the construction that, when $(d-r)$ is even, the inclusion of $\mathrm{Gr}^{\mathrm{nd}}_{r,n} \hookrightarrow \{d\} \times \mathrm{Gr}^{\mathrm{nd}}_{r,n} \hookrightarrow \mathbb{Z} \times \mathrm{Gr}^{\mathrm{nd}}$ corresponds to the class of $[\mathcal{U}^{\mathrm{nd}}_{r,n}] + \frac{d-r}{2}[\mathbb{H}]$ in $\mathrm{GW}^0(\mathrm{Gr}^{\mathrm{nd}}_{r,n})$, where $\mathcal{U}^{\mathrm{nd}}_{r,n}$ is the universal symmetric bundle over $\mathrm{Gr}^{\mathrm{nd}}_{r,n}$.

The fact that hermitian K-theory is representable in $\mathcal{H}(k)$ has been known for longer, see [Hor05]. One of the advantages of having a geometric description of a representing space, however, is that one can easily see what its complex realization is. In particular, this gives us an alternative way to define the comparison maps. For any smooth complex scheme

[2] Talk "Geometric representation of hermitian K-theory in $\mathbb{A}^1$-homotopy theory" at the Workshop "Geometric Aspects of Motivic Homotopy Theory", 6.–10. September 2010 at the Hausdorff Center for Mathematics, Bonn

X we have the following commutative squares, in which the left vertical arrows are the comparison maps (12) and (13), the right vertical arrows are induced by the complex realization functor (15), and the horizontal isomorphisms are the ones described in Theorem 1.5 and directly above.

$$\begin{tikzcd} \mathrm{K}_0(X) \arrow[r, phantom, "\cong"] \arrow[d] & \mathcal{H}(\mathbb{C})(X, \mathbb{Z} \times \mathrm{Gr}) \arrow[d] \\ \mathrm{K}^0(X(\mathbb{C})) \arrow[r, phantom, "\cong"] & \mathcal{H}(X(\mathbb{C}), \mathbb{Z} \times \mathrm{Gr}) \end{tikzcd} \qquad \begin{tikzcd} \mathrm{GW}^0(X) \arrow[r, phantom, "\cong"] \arrow[d] & \mathcal{H}(\mathbb{C})(X, \mathbb{Z} \times \mathrm{Gr}^{\mathrm{nd}}) \arrow[d] \\ \mathrm{KO}^0(X(\mathbb{C})) \arrow[r, phantom, "\cong"] & \mathcal{H}(X(\mathbb{C}), \mathbb{Z} \times \mathrm{Gr}^{\mathrm{nd}}) \end{tikzcd}$$

The advantage of this alternative view of the comparison maps comes from the fact that the results quoted here are known in a much greater generality. Firstly, we can obtain higher algebraic and hermitian K-groups of X by passing to suspensions of X in (17) and (18). Even better, algebraic and hermitian K-theory are representable in the stable $\mathbb{A}^1$-homotopy category $\mathcal{SH}(k)$. Let us make the statement a little more precise by fixing some notation. Given a spectrum $\mathbb{E}$ in $\mathcal{SH}(k)$, we obtain a bigraded reduced cohomology theory $\widetilde{E}^{*,*}$ on $\mathcal{H}_\bullet(k)$ and a corresponding unreduced theory $E^{*,*}$ on $\mathcal{H}(k)$ by setting

$$\begin{aligned} \widetilde{E}^{p,q}(\mathcal{X}) &:= \mathcal{SH}(k)(\Sigma^\infty \mathcal{X}, S^{p,q} \wedge \mathbb{E}) && \text{for } \mathcal{X} \in \mathcal{H}_\bullet(k) \\ E^{p,q}(X) &:= E^{p,q}(X_+) && \text{for } X \in \mathcal{H}(k) \end{aligned}$$

A spectrum $\mathbb{K}$ representing algebraic K-theory was first constructed in [Voe98]; see [Rio06] or [Rio07b] for some further discussion. It is (2,1)-periodic, meaning that in $\mathcal{SH}(k)$ we have an isomorphism

$$S^{2,1} \wedge \mathbb{K} \xrightarrow{\cong} \mathbb{K}$$

Thus, the bigrading of the corresponding cohomology theory $\mathrm{K}^{p,q}$ is slightly artificial. The identification with the usual notation for algebraic K-theory is given by

$$\mathrm{K}^{p,q}(X) = \mathrm{K}_{2q-p}(X) \tag{19}$$

For hermitian K-theory we have an (8,4)-periodic spectrum $\mathbb{K}\mathbf{O}$, and the corresponding cohomology groups $\mathrm{KO}^{p,q}$ are honestly bigraded. The translation into the notation used for hermitian K-groups on page 5 is given by

$$\mathrm{KO}^{p,q}(X) = \mathrm{GW}^q_{2q-p}(X) \tag{20}$$

Thus, we will refer to the number $2q - p$ as the degree of the group $\mathrm{KO}^{p,q}(X)$. The relation with Balmer's Witt groups obtained by combining (20) and (1) is illustrated by the following table.

$\mathbf{KO^{p,q}}$	$p = 0$	1	2	3	4	5	6	7
$q = 0$	$\mathbf{GW^0}$	$\mathbf{W^1}$	$\mathbf{W^2}$	$\mathbf{W^3}$	$\mathbf{W^0}$	$\mathbf{W^1}$	$\mathbf{W^2}$	$\mathbf{W^3}$
$q = 1$	GW^1_2	GW^1_1	$\mathbf{GW^1}$	$\mathbf{W^2}$	$\mathbf{W^3}$	$\mathbf{W^0}$	$\mathbf{W^1}$	$\mathbf{W^2}$
$q = 2$	GW^2_4	GW^2_3	GW^2_2	GW^2_1	$\mathbf{GW^2}$	$\mathbf{W^3}$	$\mathbf{W^0}$	$\mathbf{W^1}$
$q = 3$	GW^3_6	GW^3_5	GW^3_4	GW^3_3	GW^3_2	GW^3_1	$\mathbf{GW^3}$	$\mathbf{W^0}$

As for the representing spaces in the unstable homotopy category, it is known that the complex realizations of $\mathbb{K}\mathbf{O}$ and $\mathbb{K}$ represent real and complex topological K-theory. This is well-documented in the latter case, but for $\mathbb{K}\mathbf{O}$ our references are slightly thin. Since the emphasis in this article is on showing how such a result in $\mathbb{A}^1$-homotopy theory can be used for some concrete computations, we will at this point succumb to an "axiomatic" approach – the key statements we will be using are as follows:

Standing assumptions 1.9. *There exist spectra $\mathbb{K}$ and $\mathbb{K}\mathbf{O}$ in $\mathcal{SH}(\mathbb{C})$ representing algebraic K-theory and hermitian K-theory in the sense described above, such that*

- *(a) The complex realization functor* (16) *takes $\mathbb{K}$ to $\mathbb{K}^{top}$ and $\mathbb{K}\mathbf{O}$ to $\mathbb{K}\mathbf{O}^{top}$*
- *(b) We have an exact triangle in $\mathcal{SH}(\mathbb{C})$ of the form*

$$\mathbb{K}\mathbf{O} \wedge S^{1,1} \xrightarrow{\eta} \mathbb{K}\mathbf{O} \to \mathbb{K} \to S^{1,0} \wedge \dots \tag{21}$$

  which corresponds to the usual triangle in $\mathcal{SH}$ (see page 14).

These results are announced in [Mor06]. Independent constructions of spectra representing hermitian K-theory can be found in [Hor05] and in a recent preprint of Panin and Walter [PW10].

2 The comparison maps

It follows immediately from 1.9 that complex realization induces comparison maps

$$\begin{aligned} \widetilde{k}^{p,q} &: \widetilde{\mathrm{K}}^{p,q}(\mathcal{X}) \to \widetilde{\mathrm{K}}^{p}(\mathcal{X}(\mathbb{C})) \\ \widetilde{k}_h^{p,q} &: \widetilde{\mathrm{KO}}^{p,q}(\mathcal{X}) \to \widetilde{\mathrm{KO}}^{p}(\mathcal{X}(\mathbb{C})) \end{aligned}$$

and hence comparison maps $k^{p,q}$ and $k_h^{p,q}$ on K- and hermitian K-groups. In particular, in degrees 0 and -1 we have maps

$$\begin{aligned} k^{0,0} &: \mathrm{K}_0(X) \to \mathrm{K}^0(X(\mathbb{C})) \\ \mathrm{gw}^q := k_h^{2q,q} &: \mathrm{GW}^q(X) \to \mathrm{KO}^{2q}(X(\mathbb{C})) \\ \mathrm{w}^q := k_h^{2q-1,q-1} &: \mathrm{W}^q(X) \to \mathrm{KO}^{2q-1}(X(\mathbb{C})) \end{aligned}$$

for any smooth complex scheme X. Moreover, many good properties of these maps follow directly from the construction:

- They commute with pullbacks along morphisms of smooth schemes.
- They are compatible with suspension isomorphisms.
- They are compatible with the periodicity isomorphisms, so we can identify $k_h^{p,q}$ with $k_h^{p+8,q+4}$ (and hence w^q with w^{q+4} and gw^q with gw^{q+4}).

It is also clear that they are compatible with long exact sequences arising from exact triangles in $\mathcal{SH}(\mathbb{C})$. This will be particularly useful in the following two cases.

Localization sequences. Given a smooth closed subscheme Z of a smooth scheme X, we have an exact triangle

$$\Sigma^\infty(X-Z)_+ \to \Sigma^\infty X_+ \to \Sigma^\infty\left(\tfrac{X}{X-Z}\right) \to S^{1,0}\wedge\dots$$

in $\mathcal{SH}(\mathbb{C})$. It induces long exact "localization sequences" for cohomology theories. For example, for hermitian K-theory we obtain sequences of the form

$$\dots\to \widetilde{\mathrm{KO}}^{p,q}\left(\tfrac{X}{X-Z}\right) \to \mathrm{KO}^{p,q}(X) \to \mathrm{KO}^{p,q}(X-Z) \to \widetilde{\mathrm{KO}}^{p+1,q}\left(\tfrac{X}{X-Z}\right) \to \dots$$

The comparison maps commute with all maps appearing in this sequence and the corresponding sequence of topological KO-groups.

In topology, the groups $\widetilde{\mathrm{KO}}^p(X/(X-Z))$ are usually written as relative groups $\mathrm{KO}(X, X-Z)$. The groups $\widetilde{\mathrm{KO}}^{p,q}(X/(X-Z))$, on the other hand, correspond to hermitian K-groups of X supported on Z. This should be viewed as part of any representability statement, see for example [PW10, Theorem 6.5]. Alternatively, a formal identification on the basis of representability of groups with full support can be achieved by considering the localization sequences for the inclusion of Z into $X\times\mathbb{A}^1$, since these sequences split.

A useful way of rewriting these localization sequences comes from the identification of the homotopy quotient $X/(X-Z)$ with the Thom space of the normal bundle $\mathcal{N}$ of Z in X. In topology, this identification follows easily from the existence of tubular neighbourhoods. In $\mathbb{A}^1$-homotopy theory, it is a theorem due to Morel and Voevodsky, based on a geometric construction known as "deformation to the normal bundle" [MV99, Chapter 3, Theorem 2.23].

Karoubi/Bott sequences. The KO- and K-groups of a topological space X fit into a long exact sequence known as the Bott sequence [Bot69], of the form

$$\begin{aligned}\dots &\to \mathrm{KO}^{2i-1}X \to \mathrm{KO}^{2i-2}X \to \mathrm{K}^0X \to \mathrm{KO}^{2i}X \to \mathrm{KO}^{2i-1}X \to \mathrm{K}^1X\\ &\to \mathrm{KO}^{2i+1}X \to \mathrm{KO}^{2i}X \to \mathrm{K}^0X \to \mathrm{KO}^{2i+2}X \to \mathrm{KO}^{2i+1}X \to \dots\end{aligned} \tag{22}$$

Here, the maps from KO- to K-groups are essentially given by complexification (or, depending on our choice of description of KO-groups, by forgetting the symmetric structure of a complex symmetric bundle), and the maps from K- to KO-groups are given by sending a complex vector bundle to its underlying real bundle (or to the associated hyperbolic bundle). The maps between KO-groups are given by multiplication with the generator η of $\mathrm{KO}^{-1}(\text{point})$ (see (11)).

This long exact sequence is induced by the triangle described in 1.9. The sequence arising from the corresponding triangle (21) in the stable $\mathbb{A}^1$-homotopy category is known as the Karoubi sequence. The comparison maps induce a commutative ladder diagram that allows us to compare the two. Near degree zero, this takes the following form:

$$\begin{tikzcd}
\dots \arrow[r] & \mathrm{KO}^{2i-1,i}X \arrow[r]\arrow[d] & \mathrm{GW}^{i-1}X \arrow[r]\arrow[d] & \mathrm{K}_0X \arrow[r]\arrow[d] & \mathrm{GW}^iX \arrow[r]\arrow[d] & \mathrm{W}^iX \arrow[r]\arrow[d] & 0 \arrow[r]\arrow[d] & \dots\\
\dots \arrow[r] & \mathrm{KO}^{2i-1}X \arrow[r] & \mathrm{KO}^{2i-2}X \arrow[r] & \mathrm{K}^0X \arrow[r] & \mathrm{KO}^{2i}X \arrow[r] & \mathrm{KO}^{2i-1}X \arrow[r] & \mathrm{K}^1X \arrow[r] & \dots
\end{tikzcd}$$

Remark 2.1. One of the consequences here is that the comparison maps w^i factor as

$$\mathrm{W}^i(X) \to \frac{\mathrm{KO}^{2i}(X)}{\mathrm{K}^0(X)} \to \mathrm{KO}^{2i-1}(X)$$

For cellular varieties, or more generally for spaces for which the odd topological K-groups vanish, the Bott sequence moreover implies that the second map in this factorisation is an isomorphism.

2.1 Twisting by line bundles

As described in section 1.1, there is a natural notion of Witt groups twisted by line bundles. In the homotopy theoretic approach, such a twist can be encoded by passing to the Thom space of the bundle.

Definition 2.2. For a vector bundle $\mathcal{E}$ of rank r over a smooth scheme X, we define the hermitian K-group of X with coefficients in $\mathcal{E}$ to be

$$\mathrm{KO}^{p,q}(X;\mathcal{E}) := \widetilde{\mathrm{KO}}^{p+2r,q+r}(\mathrm{Thom}\,\mathcal{E})$$

Likewise, for any complex vector bundle of rank r on a topological space X we define

$$\mathrm{KO}^{p}(X;\mathcal{E}) := \widetilde{\mathrm{KO}}^{p+2r}(\mathrm{Thom}\,\mathcal{E})$$

When $\mathcal{E}$ is a trivial bundle, these definitions give nothing new, so we can drop trivial twists from the notation:

$$\begin{aligned}
\mathrm{KO}^{p,q}(X;\mathcal{O}_X^{\oplus r}) &= \widetilde{\mathrm{KO}}^{p+2r,q+r}(S^{2r,r}\wedge X_+) = \mathrm{KO}^{p,q}(X)\\
\mathrm{KO}^{p}(X;\mathbb{C}^r) &= \mathrm{KO}^{p+2r}(S^{2r}\wedge X_+) = \mathrm{KO}^{p}(X)
\end{aligned}$$

Lemma 2.3. *For any vector bundle $\mathcal{E}$ on a smooth scheme X we have isomorphisms*

$$\begin{aligned}
\mathrm{KO}^{2q,q}(X;\mathcal{E}) &\cong \mathrm{GW}^{q}(X;\det\mathcal{E})\\
\mathrm{KO}^{p,q}(X;\mathcal{E}) &\cong \mathrm{W}^{p-q}(X;\det\mathcal{E}) \quad \textit{for} \;\; 2q-p<0
\end{aligned}$$

Proof. This follows from Nenashev's Thom isomorphisms for Witt groups [Nen07]: for any vector bundle $\mathcal{E}$ of rank r there is a canonical Thom class in $\mathrm{W}^r_X(\mathcal{E})$ which induces an isomorphism $\mathrm{W}^i(X;\det\mathcal{E}) \cong \mathrm{W}^{i+r}_X(\mathcal{E})$ by multiplication. This Thom class actually comes from a class in $\mathrm{GW}^r_X(\mathcal{E})$. Using the Karoubi-sequence in the triangulated setting (as described in [Wal03a]) one sees that multiplication with this class also induces an isomorphism of the corresponding Grothendieck-Witt groups. □

Similarly, it follows from Thom isomorphisms in topology that the groups $\mathrm{KO}(X;\mathcal{E})$ only depend on the determinant line bundle of $\mathcal{E}$:

Lemma 2.4. *For any two complex vector bundles $\mathcal{E}$ and $\mathcal{F}$ on a topological space X with identical first Chern class modulo 2 we have*

$$\mathrm{KO}^{p}(X;\mathcal{E}) \cong \mathrm{KO}^{p}(X;\mathcal{F})$$

Proof. A complex vector bundle $\mathcal{E}$ whose first Chern class vanishes modulo 2 has a spin structure and is therefore oriented with respect to KO-theory [ABS64]. That is, we have a Thom isomorphism

$$\mathrm{KO}^p X \xrightarrow{\cong} \widetilde{\mathrm{KO}}^{p+2r}(\mathrm{Thom}\,\mathcal{E})$$

Now suppose $c_1(\mathcal{E}) \equiv c_1(\mathcal{F}) \bmod 2$. We may view $\mathcal{E} \oplus \mathcal{E} \oplus \mathcal{F}$ both as a vector bundle on $\mathcal{E}$ and as a vector bundle on $\mathcal{F}$, and by assumption it is oriented with respect to KO-theory in both cases. Thus, both groups in the lemma can be identified with $\mathrm{KO}^p(X; \mathcal{E} \oplus \mathcal{E} \oplus \mathcal{F})$. □

Remark 2.5. In general the identifications of Lemma 2.4 are non-canonical. Given a spin structure on a real vector bundle, the constructions in [ABS64] do yield a canonical Thom class, but there may be several different spin structures on the same bundle. Still, canonical identifications exist in many cases. For example, there is a canonical spin structure on the square of any complex line bundle, yielding canonical identifications

$$\mathrm{KO}^p(X; \mathcal{L}) \cong \mathrm{KO}^p(X; \mathcal{L} \otimes \mathcal{M}^{\otimes 2})$$

for any two complex line bundles $\mathcal{L}$ and $\mathcal{M}$ on X. Moreover, as spin structures on a spin bundle over X are classified by $H^1(X; \mathbb{Z}/2)$, all spin structures arising in the context of cellular varieties below will be unique.

2.2 The comparison for cellular varieties

Theorem 2.6. *For a smooth cellular complex variety X, the following comparison maps are isomorphisms:*

$$\begin{aligned} \mathrm{K}_0(X) &\xrightarrow{\cong} \mathrm{K}^0(X(\mathbb{C})) \\ \mathrm{gw}^q \colon \mathrm{GW}^q(X) &\xrightarrow{\cong} \mathrm{KO}^{2q}(X(\mathbb{C})) \\ \mathrm{w}^q \colon \mathrm{W}^q(X) &\xrightarrow{\cong} \mathrm{KO}^{2q-1}(X(\mathbb{C})) \end{aligned}$$

This remains true for twisted groups (see section 2.1).

As indicated in the introduction, the first of these three isomorphisms is well-known and in fact almost obvious, given that both $\mathrm{K}_0(X)$ and $\mathrm{K}^0(X(\mathbb{C}))$ are free of rank equal to the number of cells of X. In particular, both the algebraic group $K_0(\mathbb{C})$ and the topological K-group $\mathrm{K}^0(\text{point})$ are isomorphic to the integers ($n \in \mathbb{Z}$ corresponds to the virtual trivial bundle of rank n), and it is clear in this case that the comparison map is an isomorphism. Let us begin the proof of the theorem by also considering the other two maps first in the case when X is just a point. We can easily see that the corresponding groups are isomorphic by direct comparison:

$\mathbf{KO}^{p,q}(\mathbb{C})$	$p=0$	1	2	3	4	5	6	7
$q=0$	$\mathbf{Z}$	$\mathbf{0}$	0	0	$\mathbb{Z}/2$	0	0	0
$q=1$	...	...	$\mathbf{0}$	$\mathbf{0}$	0	$\mathbb{Z}/2$	0	0
$q=2$	...	...	...	...	$\mathbf{Z}$	$\mathbf{0}$	$\mathbb{Z}/2$	0
$q=3$	...	...	...	...	...	...	$\mathbf{Z/2}$	$\mathbf{Z/2}$
$\mathbf{KO}^{p}(\text{point})$	$\mathbf{Z}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{Z}$	$\mathbf{0}$	$\mathbf{Z/2}$	$\mathbf{Z/2}$

To show that the isomorphisms are given by our comparison maps, we can use the exact sequences arising from triangle (21). First, it follows from the following diagram that gw^0 and w^0 are isomorphisms on a point:

$$\begin{tikzcd}
\dots \arrow[r] & \mathrm{K}_0 \arrow[r] \arrow[d, "\cong"] & \mathrm{GW}^0 \arrow[r] \arrow[d, "\mathrm{gw}^0"] & \mathrm{W}^0 \arrow[r] \arrow[d, "\mathrm{w}^0"] & 0 \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & \mathrm{K}^0 \arrow[r] & \mathrm{KO}^0 \arrow[r] & \mathrm{KO}^{-1} \arrow[r] & \mathrm{K}^{-1} \arrow[r] & \dots
\end{tikzcd}$$

$$=$$

$$\begin{tikzcd}
\dots \arrow[r] & \mathbb{Z} \arrow[r, "\cdot 2"] \arrow[d, "\cong"] & \mathbb{Z} \arrow[r] \arrow[d, "\mathrm{gw}^0"] & \mathbb{Z}/2 \arrow[r] \arrow[d, "\mathrm{w}^0"] & 0 \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & \mathbb{Z} \arrow[r, "\cdot 2"] & \mathbb{Z} \arrow[r] & \mathbb{Z}/2 \arrow[r] & 0 \arrow[r] & \dots
\end{tikzcd}$$

In this diagram and in the next, it is understood that all groups are computed on a point. As $\mathrm{W}^0(\mathbb{C})$ is the only non-trivial Witt group of a point, it follows that w^q is an isomorphism on a point in general, and we have diagrams

$$\begin{tikzcd}
\dots \arrow[r] & \mathrm{GW}^{q-1} \arrow[r] \arrow[d, "\mathrm{gw}^{q-1}"] & \mathrm{K}_0 \arrow[r] \arrow[d, "\cong"] & \mathrm{GW}^q \arrow[r] \arrow[d, "\mathrm{gw}^q"] & \mathrm{W}^q \arrow[r] \arrow[d, "\cong"] & 0 \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & \mathrm{KO}^{2q-2} \arrow[r] & \mathrm{K}^0 \arrow[r] & \mathrm{KO}^{2q} \arrow[r] & \mathrm{KO}^{2q-1} \arrow[r] & 0 \arrow[r] & \dots
\end{tikzcd}$$

Given the periodicity of the Grothendieck-Witt groups, repeated applications of the Five Lemma now show that gw^q is an isomorphism on a point for all values of q. (This strategy of proof is known as "Karoubi induction".)

We now treat the hermitian case in general. The case of algebraic/complex K-theory could be dealt with similarly, or deduced from the hermitian case using triangle (21). It will be helpful to consider not only the maps $\mathrm{gw}^q = k_h^{2q,q}$ and $\mathrm{w}^{q+1} = k_h^{2q+1,q}$ in degrees 0 and -1, respectively, but also the maps $k_h^{2q-1,q}$ in degree 1 and the maps $k_h^{2q+2,q}$ in degree -2. In fact, we will prove the following extended statement:

Theorem 2.7. *For a smooth cellular variety X, the hermitian comparison maps in degrees 1, 0, -1 and -2 have the properties indicated:*

$$\begin{aligned}
\mathrm{KO}^{2q-1,q}(X) &\twoheadrightarrow \mathrm{KO}^{2q-1}(X(\mathbb{C}))\\
\mathrm{KO}^{2q,q}(X) &\xrightarrow{\cong} \mathrm{KO}^{2q}(X(\mathbb{C}))\\
\mathrm{KO}^{2q+1,q}(X) &\xrightarrow{\cong} \mathrm{KO}^{2q+1}(X(\mathbb{C}))\\
\mathrm{KO}^{2q+2,q}(X) &\rightarrowtail \mathrm{KO}^{2q+2}(X(\mathbb{C}))
\end{aligned}$$

The analogous statements for twisted groups are also true.

The proof will proceed by induction over the number of cells of X and occupy the remainder of this section. To begin the induction, we need to consider the case of only one cell, which immediately reduces to the case of a point by homotopy invariance. In this case, degrees 0 and -1 have already been dealt with above. In degrees 1 and -2, on the other hand, most of the statements are trivial, and we only need to look at a few particular cases, which we postpone to the end of the proof.

Spheres. Assuming the theorem to be true for a point, the compatibility of the comparison maps with suspensions immediately shows that the theorem is also true for the reduced cohomology of the spheres $(\mathbb{P}^1)^{\wedge d} = S^{2d,d}$. To be precise, the following maps in degrees 1, 0, -1 and -2 have the properties indicated:

$$\begin{aligned}
\widetilde{\mathrm{KO}}^{2q-1,q}(S^{2d,d}) &\twoheadrightarrow \widetilde{\mathrm{KO}}^{2q-1}(S^{2d})\\
\widetilde{\mathrm{KO}}^{2q,q}(S^{2d,d}) &\xrightarrow{\cong} \widetilde{\mathrm{KO}}^{2q}(S^{2d})\\
\widetilde{\mathrm{KO}}^{2q+1,q}(S^{2d,d}) &\xrightarrow{\cong} \widetilde{\mathrm{KO}}^{2q+1}(S^{2d})\\
\widetilde{\mathrm{KO}}^{2q+2,q}(S^{2d,d}) &\rightarrowtail \widetilde{\mathrm{KO}}^{2q+2}(S^{2d})
\end{aligned}$$

Cellular Varieties. Now let X be a smooth cellular variety. By definition, X has a filtration by closed subvarieties $\emptyset = Z_0 \subset Z_1 \subset Z_2 \cdots \subset Z_N = X$ such that the open complement of Z_k in Z_{k+1} is isomorphic to $\mathbb{A}^{n_k}$ for some n_k. In general, the subvarieties Z_k will not be smooth. Their complements $U_k := X - Z_k$ in X, however, are always smooth as they are open in X. So we obtain an alternative filtration $X = U_0 \supset U_1 \supset U_2 \cdots \supset U_N = \emptyset$ of X by smooth open subvarieties U_k. Each U_k contains a closed cell $C_k \cong \mathbb{A}^{n_k}$ with open complement U_{k+1}.

Our inductive hypothesis is that we have already proved the theorem for U_{k+1}, and we now want to prove it for U_k. We can use the following exact triangle in $\mathcal{SH}(\mathbb{C})$:

$$\Sigma^\infty(U_{k+1})_+ \to \Sigma^\infty(U_k)_+ \to \Sigma^\infty\mathrm{Thom}(\mathcal{N}_{C_k\setminus U_k}) \to S^{1,0}\wedge\dots$$

As C_k is a cell, the Quillen-Suslin theorem tells us that the normal bundle $\mathcal{N}_{C_k\setminus U_k}$ of C_k in U_k has to be a trivial. Thus, $\mathrm{Thom}(\mathcal{N}_{C_k\setminus U_k})$ is $\mathbb{A}^1$-weakly equivalent to $S^{2d,d}$, where d is the codimension of C_k in U_k. Figure 1 displays the comparison between the long exact cohomology sequences induced by this triangle. The inductive step is completed by applying the Five Lemma to each dotted map in the diagram.

$$
\begin{tikzcd}
{} \arrow[d] \arrow[r, phantom, "\cdots"] & {} \arrow[d] \\
\widetilde{\mathrm{KO}}^{2q-1,q}(S^{2d,d}) \arrow[r, two heads] \arrow[d] & \widetilde{\mathrm{KO}}^{2q-1}(S^{2d}) \arrow[d] \\
\mathrm{KO}^{2q-1,q}(U_k) \arrow[r, dashed, "k_h^{2q-1,q}"] \arrow[d] & \mathrm{KO}^{2q-1}(U_k) \arrow[d] \\
\mathrm{KO}^{2q-1,q}(U_{k+1}) \arrow[r, two heads] \arrow[d] & \mathrm{KO}^{2q-1}(U_{k+1}) \arrow[d] \\
\widetilde{\mathrm{KO}}^{2q,q}(S^{2d,d}) \arrow[r, "\cong"] \arrow[d] & \widetilde{\mathrm{KO}}^{2q}(S^{2d}) \arrow[d] \\
\mathrm{KO}^{2q,q}(U_k) \arrow[r, dashed, "k_h^{2q,q}"] \arrow[d] & \mathrm{KO}^{2q}(U_k) \arrow[d] \\
\mathrm{KO}^{2q,q}(U_{k+1}) \arrow[r, "\cong"] \arrow[d] & \mathrm{KO}^{2q}(U_{k+1}) \arrow[d] \\
\widetilde{\mathrm{KO}}^{2q+1,q}(S^{2d,d}) \arrow[r, "\cong"] \arrow[d] & \widetilde{\mathrm{KO}}^{2q+1}(S^{2d}) \arrow[d] \\
\mathrm{KO}^{2q+1,q}(U_k) \arrow[r, dashed, "k_h^{2q+1,q}"] \arrow[d] & \mathrm{KO}^{2q+1}(U_k) \arrow[d] \\
\mathrm{KO}^{2q+1,q}(U_{k+1}) \arrow[r, "\cong"] \arrow[d] & \mathrm{KO}^{2q+1}(U_{k+1}) \arrow[d] \\
\widetilde{\mathrm{KO}}^{2q+2,q}(S^{2d,d}) \arrow[r, tail] \arrow[d] & \widetilde{\mathrm{KO}}^{2q+2}(S^{2d}) \arrow[d] \\
\mathrm{KO}^{2q+2,q}(U_k) \arrow[r, dashed, "k_h^{2q+2,q}"] \arrow[d] & \mathrm{KO}^{2q+2}(U_k) \arrow[d] \\
\mathrm{KO}^{2q+2,q}(U_{k+1}) \arrow[r, tail] \arrow[d] & \mathrm{KO}^{2q+2}(U_{k+1}) \arrow[d] \\
{} \arrow[r, phantom, "\cdots"] & {}
\end{tikzcd}
$$

Figure 1: The inductive step.

The twisted case. To obtain the theorem in the case of coefficients in a vector bundle $\mathcal{E}$ over X, we replace the exact triangle above by the triangle

$$\Sigma^\infty \operatorname{Thom}(\mathcal{E}_{|U_{k+1}}) \to \Sigma^\infty \operatorname{Thom}(\mathcal{E}_{|U_k}) \to \Sigma^\infty \operatorname{Thom}(\mathcal{E}_{|C_k} \oplus \mathcal{N}_{C_k \setminus U_k}) \to \dots$$

The existence of this exact triangle is shown in the next lemma. The Thom space on the right is again just a sphere, so we can proceed as in the untwisted case.

Lemma 2.8. *Given a smooth subvariety Z of a smooth variety X with complement U, and given any vector bundle $\mathcal{E}$ over X, we have an exact triangle*

$$\Sigma^\infty \operatorname{Thom}(\mathcal{E}_{|U}) \to \Sigma^\infty \operatorname{Thom} \mathcal{E} \to \Sigma^\infty \operatorname{Thom}(\mathcal{E}_{|Z} \oplus \mathcal{N}_{Z \setminus X}) \to \dots$$

Proof. From the Thom isomorphism theorem we know that the Thom space of a vector bundle over a smooth base is $\mathbb{A}^1$-weakly equivalent to the quotient of the vector bundle by the complement of the zero section. Consider the closed embeddings

$$\begin{aligned} U &\hookrightarrow (\mathcal{E} - Z) \\ X &\hookrightarrow \mathcal{E} \\ Z &\hookrightarrow \mathcal{E} \end{aligned}$$

Computing the normal bundles, we get

$$\begin{aligned} (\mathcal{E} - Z)/(\mathcal{E} - X) &\cong \operatorname{Thom}_U(\mathcal{E}_{|U}) \\ \mathcal{E}/(\mathcal{E} - X) &\cong \operatorname{Thom}_X \mathcal{E} \\ \mathcal{E}/(\mathcal{E} - Z) &\cong \operatorname{Thom}_Z(\mathcal{E}_{|Z} \oplus \mathcal{N}_{Z \setminus X}) \end{aligned}$$

The claim follows by passing to the stable homotopy category and applying the octahedral axiom to the composition of the embeddings

$$(\mathcal{E} - X) \subseteq (\mathcal{E} - Z) \subseteq \mathcal{E}$$

$\square$

Remaining details concerning a point. To finish the proof of Theorem 2.7, we now return to the maps of degrees 1 and -2 in the case of a point, which we skipped above. First, let us deal with degree 1. The odd KO-groups of a point are all trivial except for KO^{-1}, so $k_h^{2q-1,q}$ is trivially a surjection unless $q \equiv 0$ modulo 4. In that special case, surjectivity of $k_h^{-1,0}$ is clear from the following diagram:

$$\begin{tikzcd} \dots \arrow[r] & \mathrm{KO}^{-1,0} \arrow[r] \arrow[d, "k_h^{-1,0}"] & \mathrm{GW}^{-1} \arrow[r] \arrow[d, "\cong"] & \mathrm{K}_0 \arrow[r] \arrow[d, "\cong"] & \dots \\ \dots \arrow[r] & \mathrm{KO}^{-1} \arrow[r] & \mathrm{KO}^{-2} \arrow[r] & \mathrm{K}^0 \arrow[r] & \dots \end{tikzcd}$$

$$=$$

$$\begin{tikzcd} \dots \arrow[r] & \mathrm{KO}^{-1,0} \arrow[r, two heads] \arrow[d, "k_h^{-1,0}"] & \mathbb{Z}/2 \arrow[r, "0"] \arrow[d, "\cong"] & \mathbb{Z} \arrow[r] \arrow[d, "\cong"] & \dots \\ \dots \arrow[r] & \mathbb{Z}/2 \arrow[r, "\cong"] & \mathbb{Z}/2 \arrow[r, "0"] & \mathbb{Z} \arrow[r] & \dots \end{tikzcd}$$

Lastly, we consider what happens in degree -2. Again, three out of four cases are trivial as $\mathrm{KO}^{2q+2,q} = \mathrm{W}^{q+2}$ is zero unless $q \equiv 2$ modulo 2. For the non-trivial case, consider the map η appearing in triangle (21). As the negative algebraic K-groups of $\mathbb{C}$ are zero, η yields automorphisms of W^{p-q} in negative degrees. In topology, the corresponding maps are given by multiplication with a generator η of KO^{-1}, and η^2 generates KO^{-2}. So the commutative square

$$\begin{tikzcd} \mathrm{W}^0 \arrow[r, "\cong"] \arrow[d, "\cong"] & \mathrm{W}^0 \arrow[d, "k_h^{0,-2}"] \\ \mathrm{KO}^{-1} \arrow[r, "\eta", "\cong"'] & \mathrm{KO}^{-2} \end{tikzcd}$$

shows that $k_h^{0,-2}$ is an injection (in fact, an isomorphism), as claimed. This completes the proof of Theorem 2.7.

Note that a similar argument as the one directly above shows that nothing interesting can happen in degrees below -2: While $\eta\colon \mathrm{W}^{p-q}(X) \to \mathrm{W}^{p-q}(X)$ is an isomorphism in all negative degrees, the topological η is nilpotent: $\eta^3 = 0$, as $\mathrm{KO}^{-3} = 0$. This completely different behaviour forces $k_h^{p,q}$ to be zero in degrees below -2.

Remark 2.9. We indicate briefly how Theorem 2.6 can alternatively be obtained by working only with the maps in degrees 0 and -1 that can be defined by more elementary means. The basic strategy — comparing the localization sequences arising from the inclusion of a closed cell C_k into the union of "higher" cells U_k — still works. But the sections of the sequences that one can actually compare are now too short to allow one to deduce that the comparison maps are isomorphisms on U_k from the fact that they are isomorphisms on U_{k+1}. One can, however, still deduce that the maps in degree 0 with domains the Grothendieck-Witt groups of U_k are surjective, and that the maps in degree -1 with domains the Witt groups of U_k are injective. The inductive step can then be completed with the help of the Bott/Karoubi-sequence. As hinted earlier, this argument works even if it is not assumed that the comparison maps are compatible with the boundary maps in localization sequences in general: in the relevant cases the cohomology groups involved are so simple that this property can be checked by hand.

3 The Atiyah-Hirzebruch spectral sequence

We now aim to prepare the ground for the discussion of the KO-theory of some examples in the next section. The main computational tool for KO-theory is the Atiyah-Hirzebruch spectral sequence, which in topology exists for any generalized cohomology theory. It has the form

$$E_2^{p,q} = H^p(X; \mathrm{KO}^q(\text{point})) \Rightarrow \mathrm{KO}^{p+q}(X)$$

with differential d_r of bidegree $(r, -r+1)$. The E_2-page is thus concentrated in the half-plane $p \geq 0$ and 8-periodic in q; we have the integral cohomology of X in rows $q \equiv 0$ and $q \equiv -4 \bmod 8$ and its cohomology with $\mathbb{Z}/2$-coefficients in rows $q \equiv -1$ and $q \equiv -2$. All other rows are zero. The differential d_2 is given by $\mathrm{Sq}^2 \circ \pi_2$ and Sq^2 on rows $q \equiv 0$ and $q \equiv -1$, respectively, where

$$\mathrm{Sq}^2 \colon H^*(X; \mathbb{Z}/2) \to H^{*+2}(X; \mathbb{Z}/2)$$

is the second Steenrod square and π_2 is mod-2 reduction [Fuj67, 1.3].

3.1 The Atiyah-Hirzebruch spectral sequence for cellular varieties

For cellular varieties, or more generally for CW complexes with only even-dimensional cells, the spectral sequence becomes simple enough to make some general deductions. We summarize some results of Hoggar and Kono and Hara.

Lemma 3.1. [Hog69, 2.1 and 2.2] *Let X be a CW-complex with only even-dimensional cells. Then:*

- *The ranks of the free parts of $\mathrm{KO}^0 X$ and $\mathrm{KO}^4 X$ are equal to the number t_0 of cells of X of dimension a multiple of 4.*
- *The ranks of the free parts of $\mathrm{KO}^2 X$ and $\mathrm{KO}^6 X$ are equal to the number t_1 of cells of X of dimension 2 modulo 4.*
- *The groups of odd degrees are two-torsion, i. e. $\mathrm{KO}^{2i+1} X = (\mathbb{Z}/2)^{s_i}$ for some s_i.*
- *$\mathrm{KO}^{2i} X$ is isomorphic to the direct sum of its free part and $\mathrm{KO}^{2i+1} X$.*

See the table on page 28 for a summary of these statements.

Proof. The cohomology of X is free on generators given by the cells and concentrated in even degrees. The first two statements thus follow easily from the Atiyah-Hirzebruch spectral sequence for KO-theory (e. g. after tensoring with $\mathbb{Q}$). On the other hand, we see from the Atiyah-Hirzebruch spectral sequence for complex K-theory that $\mathrm{K}^0(X)$ is a free abelian group on the cells while $\mathrm{K}^1(X)$ is zero. The last two statements thus become consequences of the Bott sequence (22). □

The free part of KO^* is thus very simple. In good cases, the spectral sequence also provides a nice description of the 2-torsion. To see this, note that $\mathrm{Sq}^2 \mathrm{Sq}^2 = \mathrm{Sq}^3 \mathrm{Sq}^1$ must vanish when the cohomology of X with $\mathbb{Z}/2$-coefficients is concentrated in even degrees. So we can view $(H^*(X;\mathbb{Z}/2), \mathrm{Sq}^2)$ as a differential graded algebra over $\mathbb{Z}/2$. To lighten notation, we will write

$$H^*(X, \mathrm{Sq}^2) := H^*(H^*(X;\mathbb{Z}/2), \mathrm{Sq}^2)$$

for the cohomology of this algebra. We keep the same grading as before, so that it is concentrated in even degrees. The row $q \equiv -1$ on the E_3-page is given by $H^*(X, \mathrm{Sq}^2) \cdot \eta$. Since it is the only row that contributes to KO^* in odd degrees we arrive at

Lemma 3.2. *Let X be as above. If the Atiyah-Hirzebruch spectral sequence of $\mathrm{KO}^*(X)$ degenerates on the E_3-page then*

$$\mathrm{KO}^{2i-1}(X) \cong \bigoplus_k H^{2i+8k}(X, \mathrm{Sq}^2)$$

In all the examples we consider below, the spectral sequence does indeed degenerate at this stage. However, showing that it does can be tricky. One step in the right direction is the following observation of Kono and Hara [KH91].

Lemma 3.3. *Let X be as above. The first differential d_r after d_3 that is non-trivial can only appear on a page E_r with number $r \equiv 2 \mod 8$.*

Such a differential can only be non-zero on rows $q \equiv 0$ and $q \equiv -1$ mod 8. If it is non-zero on some x in row $q \equiv 0$, then it is also non-zero on ηx in row $q \equiv -1$, where η is the generator of $\mathrm{KO}^{-1}(\mathrm{point})$. Conversely, if it is non-zero on some y in row $q \equiv -1$, there exists some x in row $q \equiv 0$ such that $y = x\eta$, and d_r is non-zero on x.

Proof. We see from the spectral sequence of a point that $d_r \eta = 0$ for all differentials. Thus multiplication by η gives a map of bidegree $(0, -1)$ on the spectral sequence that commutes with the differentials. On the E_2-page this map is mod-2 reduction from row $q \equiv 0$ to row $q \equiv -1$ and the identity between rows $q \equiv -1$ and $q \equiv -2$. Thus, on the E_3-page multiplication by η induces a surjection from row $q \equiv 0$ to row $q \equiv -1$ and an injection of row $q \equiv -1$ into row $q \equiv -2$. This implies all statements above. □

From here, we can derive a corollary that we will use in many examples to deduce that the spectral sequences of certain Thom spaces collapse:

Corollary 3.4. *Suppose we have a continuous map $p\colon X \to T$ of CW-complexes with only even-dimensional cells. Suppose further that the Atiyah-Hirzebruch spectral sequence for $\mathrm{KO}^*(X)$ collapses on the E_3-page, and that p^* induces an injection in row $q \equiv -1$:*

$$p^*\colon H^*(T, \mathrm{Sq}^2) \hookrightarrow H^*(X, \mathrm{Sq}^2)$$

Then the spectral sequence for $\mathrm{KO}^*(T)$ also collapses at this stage.

Proof. Write d_r for the first non-trivial higher differential, so $r \equiv 2 \mod 8$. Then, for any element x in row $q \equiv 0$, we have $p^*(d_r x) = d_r p^*(x) = 0$ since the spectral sequence for X collapses. From our assumption on p^* we can deduce that $d_r x = 0$. By the preceding lemma, this is all we need to show. □

3.2 The Atiyah-Hirzebruch spectral sequence for Thom spaces

If $\mathcal{E}$ is a complex vector bundle over X, then the Thom isomorphism tells us that the reduced cohomology of $\mathrm{Thom}\,\mathcal{E}$ is additively isomorphic to the cohomology of X. Thus, the entries on the E_2-page of the spectral sequence for $\widetilde{\mathrm{KO}}^*(\mathrm{Thom}\,\mathcal{E})$ are identical to those on the E_2-page for $\mathrm{KO}^*(X)$, except that they are all shifted to the right according to the rank of the bundle. However, the Steenrod square Sq^2 on $H^*(\mathrm{Thom}\,\mathcal{E};\mathbb{Z}/2)$, which determines the differential d_2, differs from the Steenrod square for X by the first Chern class of $\mathcal{E}$:

Lemma 3.5. *Let $\mathcal{E} \xrightarrow{\pi} X$ be a rank r complex vector bundle over a connected space X, with Thom class θ in $H^{2r}(\mathrm{Thom}_X\mathcal{E})$. The second Steenrod square on $\widetilde{H}^*(\mathrm{Thom}_X\mathcal{E};\mathbb{Z}/2)$ is given by "$\mathrm{Sq}^2 + c_1(\mathcal{E})$", where $c_1(\mathcal{E})$ is the first Chern class of $\mathcal{E}$. That is,*

$$\mathrm{Sq}^2(\pi^*x\cdot\theta) = \pi^*\left(\mathrm{Sq}^2(x) + c_1(\mathcal{E})x\right)\cdot\theta$$

for any $x \in H^*(X;\mathbb{Z}/2)$.

Proof. Consider the long exact cohomology sequence obtained by viewing $\mathrm{Thom}_X\mathcal{E}$ as the quotient of $\mathbb{P}(\mathcal{E}\oplus\mathcal{O})$ by $\mathbb{P}\mathcal{E}$. Recall that

$$H^*(\mathbb{P}\mathcal{E}) = \bigoplus_{i=0}^{r-1}\pi^*H^*(X)\cdot u^i$$

where $u = c_1(\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)) \in H^2(\mathbb{P}\mathcal{E})$. The multiplicative structure is determined by

$$u^r + c_1(\mathcal{E})u^{r-1} + c_2(\mathcal{E})u^{r-2} + \cdots + c_r(\mathcal{E}) = 0$$

Note that $\mathcal{O}_{\mathbb{P}(\mathcal{E}\oplus\mathcal{O})}(1)$ restricts to $\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)$ on $\mathbb{P}\mathcal{E}$ and that the Chern classes of $\mathcal{E}\oplus\mathcal{O}$ agree with those of $\mathcal{E}$. Since the restriction map from $H^*(\mathbb{P}(\mathcal{E}\oplus\mathcal{O}))$ to $H^*(\mathbb{P}\mathcal{E})$ is surjective, the long exact cohomology sequence can be broken up and reassembled into a short exact sequence of $H^*(X)$-modules:

$$0 \to \widetilde{H}^*(\mathrm{Thom}_X\mathcal{E}) \xrightarrow{p^*} H^*(\mathbb{P}(\mathcal{E}\oplus\mathcal{O})) \to H^*(\mathbb{P}\mathcal{E}) \to 0$$

All of this is equally true with $\mathbb{Z}/2$-coefficients, to which we now pass. We claim that

$$p^*\theta = \sum_{i=0}^{r} c_{r-i}(\mathcal{E})u^i$$

Indeed, p^* is an isomorphism from the free $H^*(X;\mathbb{Z}/2)$-module over θ to the kernel of the restriction map, which is a free $H^*(X;\mathbb{Z}/2)$-module generated by the element on the right-hand side of this equation. The two sides can thus only differ by an invertible element of $H^*(X;\mathbb{Z}/2)$. Since p^* preserves degrees, this element must be the unit $1 \in H^0(X;\mathbb{Z}/2)$.

Let us compute $\mathrm{Sq}^2(p^*\theta)$. Using the splitting principle we see that

$$\mathrm{Sq}^2(c_i) \equiv (i+1)c_{i+1} + c_1c_i \mod 2 \tag{23}$$

for the Chern classes c_i of any complex vector bundle. We thus have:

$$\begin{aligned}
\mathrm{Sq}^2(p^*\theta) &= \mathrm{Sq}^2\left(\sum_{i=0}^{r} c_{r-i}(\mathcal{E})u^i\right) \\
&= \sum_{i=0}^{r} ic_{r-i}(\mathcal{E})u^{i+1} + \sum_{i=0}^{r}(r-i+1)c_{r-i+1}(\mathcal{E})u^i + \sum_{i=0}^{r} c_1(\mathcal{E})c_{r-i}(\mathcal{E})u^i \\
&= r\sum_{i=0}^{r+1} u^i c_{r+1-i}(\mathcal{E}) + c_1(\mathcal{E})\cdot p^*\theta
\end{aligned}$$

The second factor of the first term vanishes in $H^*(\mathbb{P}(\mathcal{E}\oplus\mathcal{O}))$, so that $\mathrm{Sq}^2(p^*\theta) = c_1(\mathcal{E})\cdot p^*\theta$. This implies the lemma. □

Now suppose that X is a CW-complex with cells only in even dimensions. Then $\mathrm{Sq}^2 + c_1$ can be viewed as a differential on $H^*(X;\mathbb{Z}/2)$ for any c_1 in $H^2(X;\mathbb{Z}/2)$. Let us extend our previous notation and denote cohomology with respect to this differential by

$$H^*(X, \mathrm{Sq}^2 + c_1) := H^*(H^*(X;\mathbb{Z}/2), \mathrm{Sq}^2 + c_1) \tag{24}$$

Lemma 3.2 shows the following:

Corollary 3.6. *If the Atiyah-Hirzebruch spectral sequence of $\widetilde{\mathrm{KO}}^*(\mathrm{Thom}\,\mathcal{E})$ degenerates on the E_3-page then*

$$\mathrm{KO}^{2i-1}(X;\mathcal{E}) = \bigoplus_k H^{2i+8k}(X, \mathrm{Sq}^2 + c_1\mathcal{E})$$

It is true more generally that the differentials in the spectral sequence for $\widetilde{\mathrm{KO}}^*(\mathrm{Thom}\,\mathcal{E})$ depend only on the first Chern class of $\mathcal{E}$ modulo 2. This follows from the observation that the Atiyah-Hirzebruch spectral sequence is compatible with Thom isomorphisms, as is made more precise by the next proposition:

Fix a vector bundle $\mathcal{E}$ of rank r over a connected finite CW-complex X. Suppose $\mathcal{E}$ is oriented with respect to ordinary cohomology (which is always the case for complex vector bundles) and let $\theta \in \widetilde{H}^*(\mathrm{Thom}\,\mathcal{E})$ be its Thom class.

Lemma 3.7. *If $\mathcal{E}$ is oriented with respect to KO^* then θ survives to the E_∞-page of the Atiyah-Hirzebruch spectral sequence for $\widetilde{\mathrm{KO}}^*(\mathrm{Thom}\,\mathcal{E})$, and the Thom isomorphism for H^* extends to an isomorphism of spectral sequences. That is, for each page right multiplication with the class of θ in $\widetilde{E}_s^{r,0}(\mathrm{Thom}\,\mathcal{E})$ gives an isomorphism of $E_s^{*,*}(X)$-modules*

$$E_s^{*,*}(X) \xrightarrow[\cong]{\cdot\theta} \widetilde{E}_s^{*+r,*}(\mathrm{Thom}\,\mathcal{E})$$

Moreover, any lift of $\theta \in \widetilde{E}_\infty^{r,0}(\mathrm{Thom}\,\mathcal{E})$ to $\widetilde{\mathrm{KO}}^r(\mathrm{Thom}\,\mathcal{E})$ defines a Thom class of $\mathcal{E}$ with respect to KO^*. The isomorphism of the E_∞-pages of the spectral sequences is induced by the Thom isomorphism given by multiplication with any such class.

Proof. Fix a point x on X. The inclusion of the fibre over x into $\mathcal{E}$ induces a map $i_x\colon S^r \hookrightarrow \mathrm{Thom}\,\mathcal{E}$. By assumption, the pullback i_x^* on ordinary cohomology maps θ to a generator of $\widetilde{H}^r(S^r)$, and the pullback on $\widetilde{\mathrm{KO}}^*$ gives a surjection

$$\widetilde{\mathrm{KO}}^*(\mathrm{Thom}\,\mathcal{E}) \overset{i_x^*}{\twoheadrightarrow} \widetilde{\mathrm{KO}}^r(S^r)$$

Consider the pullback along i_x on the E_∞-pages of the spectral sequences for S^r and $\mathrm{Thom}\,\mathcal{E}$. Since we can identify $\widetilde{E}_\infty^{r,0}(\mathrm{Thom}\,\mathcal{E})$ with a quotient of $\widetilde{\mathrm{KO}}^r(\mathrm{Thom}\,\mathcal{E})$ and $\widetilde{E}_\infty^{r,0}(S^r)$ with $\widetilde{\mathrm{KO}}^r(S^r)$, we must have a surjection

$$i_x^* \colon \widetilde{E}_\infty^{r,0}(\mathrm{Thom}\,\mathcal{E}) \twoheadrightarrow \widetilde{E}_\infty^{r,0}(S^r)$$

On the other hand, the behaviour of i_x^* on $\widetilde{E}_\infty^{r,0}$ is determined by its behaviour on $\widetilde{H}^r$, whence we can only have such a surjection if θ survives to the $\widetilde{E}_\infty$-page of $\mathrm{Thom}\,\mathcal{E}$. Thus, all differentials vanish on θ, and if multiplication by θ induces an isomorphism from $E_s^{*,*}(X)$ to $\widetilde{E}_s^{*+r,*}$ on page s, it also induces an isomorphism on the next page. Lastly, consider any lift of θ to an element Θ of $\widetilde{\mathrm{KO}}^r(\mathrm{Thom}\,\mathcal{E})$. It is clear by construction that right multiplication with Θ gives an isomorphism from $E_\infty(X)$ to $\widetilde{E}_\infty(\mathrm{Thom}\,E)$, and thus it also gives an isomorphism from $\mathrm{KO}^*(X)$ to $\widetilde{\mathrm{KO}}^*(\mathrm{Thom}\,\mathcal{E})$. Thus, Θ is a Thom class for $\mathcal{E}$ with respect to KO^*. Alternatively, we could deduce this last claim from the fact that Θ restricts to a generator $\widetilde{\mathrm{KO}}^r(S^r)$ via i_x^* for any x in X. $\square$

Lemma 3.7 allows the following strengthening of Lemma 2.4.

Corollary 3.8. *If $\mathcal{E}$ and $\mathcal{F}$ are complex vector bundles on X with the same first Chern class modulo 2 then the spectral sequences for $\widetilde{\mathrm{KO}}^*(\mathrm{Thom}\,\mathcal{E})$ and $\widetilde{\mathrm{KO}}^*(\mathrm{Thom}\,\mathcal{F})$ can be identified up to a possible shift of columns when $\mathcal{E}$ and $\mathcal{F}$ have different ranks.*

4 Examples

We now run through a list of examples for which the untwisted KO-groups are already known, and for which we can supply computations of their KO-groups twisted by a line bundle. In most cases, we – reassuringly – recover results for Witt groups that are already known. In a few other cases, we consider our results as new.

All of our examples will fall into the class of projective homogeneous varieties associated with the classical complex algebraic groups $\mathrm{GL}_n(\mathbb{C})$, $\mathrm{Sp}_{2n}(\mathbb{C})$ and $\mathrm{SO}_n(\mathbb{C})$. In general, any compact homogeneous variety for a linear algebraic group over an algebraically closed field has a cell decomposition (Bruhat decomposition), so our comparison theorem applies.

The homogeneous spaces can be realized as quotients of the groups above by certain parabolic subgroups, or, as far as we are only interested in their topology, as quotients of the corresponding compact groups $\mathrm{U}(n)$, $\mathrm{Sp}(n)$ and $\mathrm{SO}(n)$ by subgroups of maximal ranks. The KO-theory of such quotients has been intensively studied. In particular, we can quote all results on the untwisted KO-theory of our examples from papers of Fujii and Kono and Hara [Fuj67, KH91, KH92].

There is also a more recent paper by Kishimoto, Kono and Ohsita which computes the untwisted KO-theory of complete flag varieties in all three classical cases, and to which our comparison isomorphism likewise applies. We do not reproduce the result here but instead refer the reader directly to [KKO04]. By a recent result of Calmès and Fasel, all KO-groups with non-trivial twists turn out to vanish for these varieties.[3]

4.1 Notation

Topologically, a cellular variety is a CW complex with cells only in even (real) dimensions. For such a CW complex X the KO-groups can be written in the form displayed in Table 4.1 below. This was shown in section 3.1 in the case when the twist $\mathcal{L}$ is trivial, and the general case follows: if X is a CW-complex with only even-dimensional cells, so is the Thom space of any complex vector bundle over X.

In the following examples, results on KO^* will be displayed by listing the values of the t_i and s_i. Since the t_i are just given by counting cells, and since the number of odd- and even- dimensional cells of a Thom space $\mathrm{Thom}_X\mathcal{E}$ only depend on X and the rank of $\mathcal{E}$, the t_i are in fact independent of $\mathcal{L}$. The s_i, on the other hand, certainly will depend on the twist, and we will sometimes acknowledge this by using the notation $s_i(\mathcal{L})$.

[3] The result of Calmès and Fasel, announced at the workshop "Geometric Aspects of Motivic Homotopy Theory" in September 2010 in Bonn, gives a more general sufficient condition for the vanishing of twisted Witt groups of split projective homogeneous varieties. However, the condition is never satisfied for "Grassmannians", i. e. for quotients of semi-simple algebraic groups by maximal parabolic subgroups. In particular, the vanishing of the twisted Witt groups of the spinor varieties discussed in section 4.6 cannot be deduced from this result.

$$
\begin{aligned}
\mathrm{KO}^6(X;\mathcal{L}) &= \mathbb{Z}^{t_1} \oplus (\mathbb{Z}/2)^{s_0} &&= \mathrm{GW}^3(X;\mathcal{L})\\
\mathrm{KO}^7(X;\mathcal{L}) &= (\mathbb{Z}/2)^{s_0} &&= \mathrm{W}^0(X;\mathcal{L})\\
\mathrm{KO}^0(X;\mathcal{L}) &= \mathbb{Z}^{t_0} \oplus (\mathbb{Z}/2)^{s_1} &&= \mathrm{GW}^0(X;\mathcal{L})\\
\mathrm{KO}^1(X;\mathcal{L}) &= (\mathbb{Z}/2)^{s_1} &&= \mathrm{W}^1(X;\mathcal{L})\\
\mathrm{KO}^2(X;\mathcal{L}) &= \mathbb{Z}^{t_1} \oplus (\mathbb{Z}/2)^{s_2} &&= \mathrm{GW}^1(X;\mathcal{L})\\
\mathrm{KO}^3(X;\mathcal{L}) &= (\mathbb{Z}/2)^{s_2} &&= \mathrm{W}^2(X;\mathcal{L})\\
\mathrm{KO}^4(X;\mathcal{L}) &= \mathbb{Z}^{t_0} \oplus (\mathbb{Z}/2)^{s_3} &&= \mathrm{GW}^2(X;\mathcal{L})\\
\mathrm{KO}^5(X;\mathcal{L}) &= (\mathbb{Z}/2)^{s_3} &&= \mathrm{W}^3(X;\mathcal{L})
\end{aligned}
$$

Table 1: Notational conventions in the examples. Only the s_i depend on $\mathcal{L}$.

4.2 Projective spaces

Complex projective spaces are perhaps the simplest examples for which Theorem 2.6 asserts something non-trivial. The computations of the Witt groups of projective spaces were certainly landmark events in the history of the theory. In 1980, Arason was able to show that the Witt group $\mathrm{W}^0(\mathbb{P}^n)$ of $\mathbb{P}^n$ over a field k agrees with the Witt group of the field [Ara80]. The higher Witt groups of projective spaces, and more generally of arbitrary projective bundles, were first computed by Walter in [Wal03b]. Quite recently, Nenashev deduced the same results via different methods [Nen09].

In the topological world, complete computations of $\mathrm{KO}^i(\mathbb{CP}^n)$ were first published in a 1967 paper by Fujii [Fuj67]. It is not difficult to deduce the values of the twisted groups $\mathrm{KO}^i(\mathbb{CP}^n;\mathcal{O}(1))$ from these: the Thom space $\mathrm{Thom}(\mathcal{O}_{\mathbb{CP}^n}(1))$ can be identified with $\mathbb{CP}^{n+1}$, so

$$
\begin{aligned}
\mathrm{KO}^i(\mathbb{CP}^n;\mathcal{O}(1)) &= \widetilde{\mathrm{KO}}^{i+2}(\mathrm{Thom}(\mathcal{O}(1)))\\
&= \widetilde{\mathrm{KO}}^{i+2}(\mathbb{CP}^{n+1}).
\end{aligned}
$$

Alternatively, we could do all required computations directly following the methods outlined in section 3. The result, in any case, is as follows, coinciding with the known results for the (Grothendieck-)Witt groups.

$\mathrm{KO}^*(\mathbb{CP}^n;\mathcal{L})$			$\mathcal{L}\equiv\mathcal{O}$				$\mathcal{L}\equiv\mathcal{O}(1)$			
	t_0	t_1	s_0	s_1	s_2	s_3	s_0	s_1	s_2	s_3
$n\equiv 0 \mod 4$	$(n/2)+1$	$n/2$	1	0	0	0	1	0	0	0
$n\equiv 1$	$(n+1)/2$	$(n+1)/2$	1	1	0	0	0	0	0	0
$n\equiv 2$	$(n/2)+1$	$n/2$	1	0	0	0	0	0	1	0
$n\equiv 3$	$(n+1)/2$	$(n+1)/2$	1	0	0	1	0	0	0	0

4.3 Grassmannians

Our second example is really a generalization of the previous one: we consider the Grassmannians of complex m-planes in $\mathbb{C}^{m+n}$, which we will denote by $\mathrm{Gr}_{m,n}$ for brevity. Again both the usual KO-theory and the

Witt groups are already known: the former by Kono and Hara [KH91], the latter by the work of Balmer and Calmès [BC08]. In fact, a careful comparison of the two sets of results has also been carried out already by Yagita [Yag09]. All we will do here, therefore, is to provide a post-mortem topological computation of the twisted KO-groups.

Balmer and Calmès state their result by describing an additive basis of the total Witt group of $\mathrm{Gr}_{m,n}$ in terms of certain "even Young diagrams". This is probably the most elegant approach, but needs some space to explain. We will stick instead to the tabular exposition used in the other examples. Let $\mathcal{O}(1)$ be a generator of $\mathrm{Pic}(\mathrm{Gr}_{m,n})$, say the dual of the determinant line bundle of the universal m-bundle over $\mathrm{Gr}_{m,n}$.

$\mathrm{KO}^*(\mathrm{Gr}_{m,n};\mathcal{L})$			$\mathcal{L}\equiv\mathcal{O}$				$\mathcal{L}\equiv\mathcal{O}(1)$			
	t_0	t_1	s_0	s_1	s_2	s_3	s_0	s_1	s_2	s_3
m and n odd s.t. $m\equiv n \mod 4$	$\frac{a}{2}$	$\frac{a}{2}$	b	b	0	0	0	0	0	0
m and n odd s.t. $m\not\equiv n \mod 4$	$\frac{a}{2}$	$\frac{a}{2}$	b	0	0	b	0	0	0	0
$m\equiv n\equiv 0 \mod 4$, or: $m\equiv 0$ and n odd, or: $n\equiv 0$ and m odd	$\frac{a+b}{2}$	$\frac{a-b}{2}$	b	0	0	0	b	0	0	0
$m\equiv n\equiv 2 \mod 4$, or: $m\equiv 2$ and n odd, or: $n\equiv 2$ and m odd	$\frac{a+b}{2}$	$\frac{a-b}{2}$	b	0	0	0	0	0	b	0
$m\equiv 0$ and $n\equiv 2$ mod 4	$\frac{a+b}{2}$	$\frac{a-b}{2}$	b	0	0	0	b_1	0	b_2	0
$m\equiv 2$ and $n\equiv 0$ mod 4	$\frac{a+b}{2}$	$\frac{a-b}{2}$	b	0	0	0	b_2	0	b_1	0

Here, the values of a, b and b_1, b_2 are as follows: Put $k := \lfloor m/2 \rfloor$ and $l := \lfloor n/2 \rfloor$.

$$a := \binom{m+n}{m} \qquad b := \binom{k+l}{k} \qquad b_1 := \binom{k+l-1}{k} \qquad b_2 := \binom{k+l-1}{k-1}$$

In particular, $b = b_1 + b_2$.

Our computation of the twisted KO-theory of $\mathrm{Gr}_{m,n}$ will be based on the following geometric observation: Let $\mathcal{U}_{m,n}$ and $\mathcal{U}_{m,n}^{\perp}$ be the universal m-bundle and the orthogonal n-bundle on $\mathrm{Gr}_{m,n}$, so that $\mathcal{U}\oplus\mathcal{U}^{\perp} = \mathcal{O}^{\oplus(m+n)}$. We have various natural inclusions between the Grassmannians of different dimensions, of which we want to fix two: we will embed

$$\mathrm{Gr}_{m,n-1} \hookrightarrow \mathrm{Gr}_{m,n}$$

by embedding $\mathbb{C}^{m+n-1}$ into $\mathbb{C}^{m+n}$ via the natural inclusion of the first $m+n-1$ coordinates, and we will embed

$$\mathrm{Gr}_{m-1,n} \hookrightarrow \mathrm{Gr}_{m,n}$$

by sending an $(m-1)$-plane Λ to the m-plane $\Lambda \oplus \langle e_{m+n}\rangle$, where $e_1, e_2, \dots, e_{m+n}$ are the canonical basis vectors of $\mathbb{C}^{m+n}$.

Lemma 4.1. *The normal bundle of* $\mathrm{Gr}_{m,n-1}$ *in* $\mathrm{Gr}_{m,n}$ *is the dual of its universal* m*-bundle,* $U^*_{m,n-1}$*. Similarly, the normal bundle of* $\mathrm{Gr}_{m-1,n}$ *in* $\mathrm{Gr}_{m,n}$ *is given by* $U^{\perp}_{m-1,n}$*. In both cases, the embeddings of the subspaces extend to embeddings of their normal bundles, such that one subspace is the closed complement of the normal bundle of the other.*

This gives us two cofibration sequences:

$$\mathrm{Gr}_{m-1,n_+} \overset{i}{\hookrightarrow} \mathrm{Gr}_{m,n_+} \overset{p}{\twoheadrightarrow} \mathrm{Thom}(\mathcal{U}^*_{m,n-1}) \qquad (25)$$

$$\mathrm{Gr}_{m,n-1_+} \overset{i}{\hookrightarrow} \mathrm{Gr}_{m,n_+} \overset{p}{\twoheadrightarrow} \mathrm{Thom}(\mathcal{U}^{\perp}_{m-1,n}) \qquad (26)$$

These sequences are the key to relating the untwisted KO-groups to the twisted ones. We will try to follow the notation used in [KH91] as closely as possible. To start with, we write $A_{m,n}$ for the cohomology of $\mathrm{Gr}_{m,n}$ with $\mathbb{Z}/2$-coefficients.

$$A_{m,n} = \frac{\mathbb{Z}/2\,[a_1, a_2, \ldots, a_m, b_1, b_2, \ldots b_n]}{a \cdot b = 1}$$

Here, a_i and b_i are the Chern classes of $\mathcal{U}$ and $\mathcal{U}^{\perp}$, respectively, and we have written a and b for the sums $1 + a_1 + \cdots + a_m$ and $1 + b_1 + \cdots + b_n$.

We will write d for the differential given by the second Steenrod square Sq^2, and d' for $\mathrm{Sq}^2 + a_1$. To describe the cohomology of $A_{m,n}$ with respect to these differentials, it is convenient to introduce the algebra

$$B_{k,l} = \frac{\mathbb{Z}/2\,\left[a_2^2, a_4^2, \ldots, a_{2k}^2, b_2^2, b_4^2, \ldots, b_{2l}^2\right]}{(1 + a_2^2 + \cdots + a_{2k}^2)(1 + b_2^2 + \cdots + b_{2l}^2) = 1}$$

Note that this subquotient of $A_{2k,2l}$ is isomorphic to $A_{k,l}$ up to a “dilatation” in grading. Proposition 2 in [KH91] says that

$$H^*(A_{m,n}, d) = \begin{cases} B_{k,l} & \text{if } (m,n) = (2k,2l),\ (2k+1,2l) \text{ or } (2k,2l+1) \\ B_{k,l} \oplus B_{k,l} \cdot a_m b_{n-1} & \text{if } (m,n) = (2k+1,2l+1) \end{cases}$$

Here, the algebra structure in the case where both m and n are odd is determined by $(a_m b_{n-1})^2 = 0$.

Lemma 4.2. *The cohomology of* $A_{m,n}$ *with respect to the twisted differential* d' *is as follows:*

$$H^*(A_{m,n}, d') = \begin{cases} B_{k,l-1} \cdot a_m \oplus B_{k-1,l} \cdot b_n & \textit{if } (m,n) = (2k,2l) \\ B_{k,l} \cdot a_m & \textit{if } (m,n) = (2k,2l+1) \\ B_{k,l} \cdot b_n & \textit{if } (m,n) = (2k+1,2l) \\ 0 & \textit{if } (m,n) = (2k+1,2l+1) \end{cases}$$

Proof. Let us shift the dimensions in the cofibration sequences (25) and (26) above in such a way that we have the Thom spaces of $\mathcal{U}^*_{m,n}$ and $\mathcal{U}^{\perp}_{m,n}$ on the right. Writing θ^* and $\theta^{\perp}$ for their Thom classes, we obtain the following two short exact sequences of differential $(A_{m,n+1}, d)$ and $(A_{m+1,n}, d)$-modules, respectively:

$$0 \to (A_{m,n}, d') \cdot \theta^* \xrightarrow{p^*} (A_{m,n+1}, d) \xrightarrow{i^*} (A_{m-1,n+1}, d) \to 0 \qquad (27)$$

$$0 \to (A_{m,n}, d') \cdot \theta^{\perp} \xrightarrow{p^*} (A_{m+1,n}, d) \xrightarrow{i^*} (A_{m+1,n-1}, d) \to 0 \qquad (28)$$

The map i^* in the first row is the obvious quotient map annihilating a_m. Its kernel, the image of $A_{m,n}$ under multiplication by a_m, is generated as an $A_{m,n+1}$-module by its unique element in degree $2m$, and thus we must have $p^*(\theta^*) = a_m$. Likewise, in the second row we have $p^*(\theta^\perp) = b_n$.

The lemma can be deduce from here case by case. The last case is of course the easiest: we then have $H^*(A_{m,n+1}, d) = H^*(A_{m-1,n+1}, d) = B_{k,l+1}$ and i^* gives an isomorphism between these.

When $(m, n) = (2k + 1, 2l)$ we see that i^* is the surjection from $H^*(A_{m,n+1}, d) = B_{k,l} \oplus B_{k,l} \cdot a_m b_n$ onto its first summand $H^*(A_{m-1,n+1}, d) = B_{k,l}$. Its kernel thus has a unique element in degree $2(n+m)$ which generates it as a $H^*(A_{m,n+1})$-module. Since p^* maps $b_n\theta$ to this element, we obtain the result above. The case $(m, n) = (2k, 2l+1)$ follows by symmetry.

Lastly, consider the case when m and n are both even. In this case, i^* maps $H^*(A_{m,n+1}, d) = B_{k,l}$ to the first summand of $H^*(A_{m-1,n+1}, d) = B_{k-1,l} \oplus B_{k-1,l} \cdot a_{m-1} b_n$ by annihilating a_m^2. We know by comparison with the short exact sequences for the $A_{m,n}$ that the kernel of this map is $B_{k,l-1}$ mapping to $B_{k,l}$ under multiplication by a_m^2. Thus, we obtain a short exact sequence

$$0 \to B_{k-1,l} \cdot a_{m-1} b_n \xrightarrow{\partial} H^*(A_{m,n}, d') \cdot \theta^* \xrightarrow{p^*} B_{k,l-1} \cdot a_m^2 \to 0 \qquad (29)$$

We can check explicitly that $d'(a_m\theta^*) = 0$, e. g. by using equation (23) given in the proof of Lemma 3.5. Together with the fact that $H^*(A_{m,n}, d')$ is a module over $H^*(A_{m,n+1}, d)$ this shows that we can define a splitting of p^* by sending a_m^2 to $a_m\theta^*$. We see that $H^*(A_{m,n}, d')$ contains $B_{k,l-1} \cdot a_m$ as a direct summand. If instead of working with sequence (27) we work with sequence (28), we see that $H^*(A_{m,n}, d')$ also contains a direct summand $B_{k-1,l} \cdot b_n$. These two summands intersect trivially, and a dimension count using the short exact sequence directly above shows that together they encompass all of $H^*(A_{m,n}, d')$. This finishes the proof. One can also see explicitly that the boundary map ∂ above sends $a_{m-1}b_n$ to $b_n\theta$. □

Lemma 4.3. *The Atiyah-Hirzebruch spectral sequence for $\widetilde{\mathrm{KO}}^*(\mathrm{Thom}\,\mathcal{U}^*_{m,n})$ collapses at the $\widetilde{E}_3$-page.*

Proof. By [KH91] we know that the spectral sequence for $\mathrm{KO}^*(\mathrm{Gr}_{m,n})$ collapses as this stage, for any m and n. Now, if both m and n are even, we have

$$(B_{k,l-1} \cdot a_m \oplus B_{k-1,l} \cdot b_n) \cdot \theta$$

in the $(-1)^{\mathrm{st}}$ row of the E_3-pages of the spectral sequences for $\mathrm{Thom}\,\mathcal{U}^*$ and $\mathrm{Thom}\,\mathcal{U}^\perp$, where $\theta = \theta^*$ or $\theta^\perp$, respectively. In the case of $\mathcal{U}^*$ we see from (29) that p^* maps the second summand injectively to the E_3-page of the spectral sequence for $\mathrm{KO}^*(\mathrm{Gr}_{m,n+1})$. Similarly, in the case of $\mathcal{U}^\perp$, the first summand is mapped injectively to the E_3-page of $\mathrm{KO}^*(\mathrm{Gr}_{m+1,n})$. Since the spectral sequences for $\mathrm{Thom}\,\mathcal{U}^*$ and $\mathrm{Thom}\,\mathcal{U}^\perp$ can be identified via Corollary 3.8, we can argue as in Corollary 3.4 to see that they must collapse at this stage. The cases when at least one of m, n is odd are similar and more straightforward. □

4.4 Maximal symplectic Grassmannians

In this example we consider the Grassmannians X_n of isotropic n-planes in $\mathbb{C}^{2n}$ with respect to a non-degenerate skew-symmetric bilinear form. These are homogeneous spaces for the complex symplectic group $\mathrm{Sp}_{2n}(\mathbb{C})$. Topologically, we can alternatively view them as homogeneous spaces for the compact symplectic group $\mathrm{Sp}(n)$; we have

$$X_n = \mathrm{Sp}(n)/U(n)$$

The universal bundle $\mathcal{U}$ on the usual Grassmannian $\mathrm{Gr}(n, 2n)$ restricts to the universal bundle on X_n, and so does the orthogonal complement bundle $\mathcal{U}^\perp$. We will continue to denote these restrictions by the same letters. Their determinant line bundles give dual generators $\mathcal{O}(1)$ and $\mathcal{O}(-1)$ of the Picard group of X_n. Below we will show how to deduce the values of the KO-groups of X_n twisted by $\mathcal{O}(1)$ from the computations of the untwisted KO-theory by Kono and Hara [KH92]. The overall result is as follows:

Theorem 4.4. *The additive structure of* $\mathrm{KO}^*(X_n;\mathcal{L})$ *is as follows:*

	t_0	t_1	$s_i(\mathcal{O})$	$s_i(\mathcal{O}(1))$
n even	2^{n-1}	2^{n-1}	$\rho(\frac{n}{2}, i)$	$\rho(\frac{n}{2}, i-n)$
n odd	2^{n-1}	2^{n-1}	$\rho(\frac{n+1}{2}, i)$	0

Here for any $i \in \mathbb{Z}/4$ *we write* $\rho(n,i)$ *for the dimension of the* i*-graded piece of a* $\mathbb{Z}/4$*-graded exterior algebra* $\Lambda_{\mathbb{Z}/2}(g_1, g_2, \ldots, g_n)$ *on* n *homogeneous generators* g_1, g_2, $\ldots$, g_n *of degree* 1, *i. e.*

$$\rho(n,i) = \sum_{\substack{d\equiv i \\ \mathrm{mod}\ 4}} \binom{n}{d}$$

A table of the values of $\rho(n,i)$ *can be found in* [KH92].

It turns out to be convenient to work with the vector bundle $\mathcal{U}^\perp \oplus \mathcal{O}$ for the computation of the twisted groups $\mathrm{KO}^*(X_n;\mathcal{O}(1))$. Namely, we have the following analogue of Lemma 4.1.

Lemma 4.5. *The symplectic Grassmannian* X_n *embeds into the symplectic Grassmannian* X_{n+1} *with normal bundle* $\mathcal{U}^\perp \oplus \mathcal{O}$ *such that the embedding extends to an embedding of this bundle. The closed complement of* $\mathcal{U}^\perp \oplus \mathcal{O}$ *in* X_{n+1} *is again isomorphic to* X_n.

Proof. To fix notation, let e_1, e_2 be the first two canonical basis vectors of $\mathbb{C}^{2n+2}$, and let $\mathbb{C}^{2n}$ be embedded into $\mathbb{C}^{2n+2}$ via the remaining coordinates. Assuming X_n is defined in terms of a skew-symmetric form Q_{2n}, it will be convenient to define X_{n+1} with respect to the form

$$Q_{2n+2} := \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & Q_{2n} \end{pmatrix}$$

Then we have embeddings i_1 and i_2 of X_n in X_{n+1} sending an n-plane $\Lambda \subset \mathbb{C}^{2n}$ to $e_1 \oplus \Lambda$ or $e_2 \oplus \Lambda$ in $\mathbb{C}^{2n+2}$, respectively.

We extend i_1 to an embedding I of $\mathcal{U}^\perp \oplus \mathcal{O}$ in the following way: given an n-plane $\Lambda \in X_n$, a vector v in $\Lambda^\perp \subset \mathbb{C}^{2n}$ and a complex scalar z, we can consider the linear function given by

$$\begin{pmatrix} z & Q_{2n}(-,v) \\ v & 0 \end{pmatrix} : \langle e_1 \rangle \oplus \Lambda \to \langle e_2 \rangle \oplus \Lambda^\perp$$

The embedding I sends (Λ, v, z) to the graph $\Gamma_{\Lambda,v,z} \subset \mathbb{C}^{2n+2}$ of this function.

$$\begin{array}{ccccc} \mathcal{U}^\perp \oplus \mathcal{O} & \xhookrightarrow{I} & X_{n+1} & \xhookleftarrow{i_2} & X_n \\ (\Lambda, v, z) & \mapsto & \Gamma_{\Lambda,v,z} & & \\ & & \langle e_2 \rangle \oplus \Lambda & \leftarrow\!\shortmid & \Lambda \end{array}$$

To see that I and i_2 are complementary, take an arbitrary $(n+1)$-plane W in X_{n+1}. If $e_2 \in W$ then we can consider a basis

$$e_2, \begin{pmatrix} a_1 \\ 0 \\ v_1 \end{pmatrix}, \ldots, \begin{pmatrix} a_n \\ 0 \\ v_n \end{pmatrix}$$

of W and the fact that Q_{2n+2} vanishes on W implies that all a_i are zero. Thus W can be identified with $i_2(\langle v_1, \ldots, v_n \rangle)$.

If, on the other hand, e_2 is not contained in W then we must have a vector of the form

$$\begin{pmatrix} 1 \\ z' \\ v' \end{pmatrix}$$

in W, for some $z' \in \mathbb{C}$ and $v' \in \mathbb{C}^{2n}$. Consider a basis of W of the form

$$\begin{pmatrix} 1 \\ z' \\ v' \end{pmatrix}, \begin{pmatrix} 0 \\ b_1 \\ v_1 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ b_n \\ v_n \end{pmatrix}$$

and let $\Lambda := \langle v_1, \ldots, v_n \rangle$. The condition that Q_{2n+2} vanishes on W implies that Q vanishes on Λ and that $b_i = Q_{2n}(v_i, v')$ for each i. It follows in particular that Λ is n-dimensional. Moreover, we can replace the first vector of our basis by a vector

$$\begin{pmatrix} 1 \\ z \\ v \end{pmatrix}$$

with $v \in \Lambda^\perp$, by subtracting appropriate multiples of the remaining basis vectors. Since Q vanishes on Λ we have $Q_{2n}(v_i, v') = Q_{2n}(v_i, v)$ and our new basis has the form

$$\begin{pmatrix} 1 \\ z \\ v \end{pmatrix}, \begin{pmatrix} 0 \\ Q(v_1, v) \\ v_1 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ Q(v_n, v) \\ v_n \end{pmatrix}$$

This shows that $W = \Gamma_{\Lambda,v,z}$. $\square$

Corollary 4.6. *We have a cofibration sequence*

$$X_{n+} \overset{i}{\hookrightarrow} X_{n+1_+} \overset{p}{\twoheadrightarrow} \mathrm{Thom}_{X_n}(\mathcal{U}^\perp \oplus \mathcal{O})$$

The associated long exact cohomology sequence splits into short exact sequences since all cohomology here is concentrated in even degrees. We thus get a short exact sequence of $H^*(X_{n+1})$-modules:

$$0 \to \widetilde{H}^*(\mathrm{Thom}_{X_n}(\mathcal{U}^\perp \oplus \mathcal{O})) \overset{p^*}{\to} H^*(X_{n+1}) \overset{i^*}{\to} H^*(X_n) \to 0 \qquad (30)$$

Recall from (24) that we write $H^*(X_n, d)$ for the cohomology of $H^*(X_n; \mathbb{Z}/2)$ with respect to a differential d. From Lemma 3.5 we know that the cohomology of $\widetilde{H}^*(\mathrm{Thom}_{X_n}(\mathcal{U}^\perp \oplus \mathcal{O}); \mathbb{Z}/2)$ with respect to Sq^2 is given by $H^*(X_n, \mathrm{Sq}^2 + c_1\mathcal{U}) \cdot \theta$, where θ is the Thom class of $\mathcal{U}^\perp \oplus \mathcal{O}$.

Lemma 4.7. *Let c_i denote the i^{th} Chern classes of $\mathcal{U}$ on X_n. We have*

$$H^*(X_n, \mathrm{Sq}^2) = \begin{cases} \Lambda(a_1, a_5, a_9, \ldots, a_{4m-3}) & \text{if } n = 2m \\ \Lambda(a_1, a_5, a_9, \ldots, a_{4m-3}, a_{4m+1}) & \text{if } n = 2m+1 \end{cases}$$

$$H^*(X_n, \mathrm{Sq}^2 + c_1) = \begin{cases} \Lambda(a_1, a_5, \ldots, a_{4m-3}) \cdot c_{2m} & \text{if } n = 2m \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

for certain generators a_i of degree $2i$.

Proof. Consider the short exact sequence (30) above. The mod-2 cohomology of X_n is an exterior algebra on the Chern classes c_i of $\mathcal{U}$,

$$H^*(X_n; \mathbb{Z}/2) = \Lambda(c_1, c_2, \ldots, c_n)$$

and i^* is given by sending c_{n+1} to zero. Thus, p^* is the unique morphism of $H^*(X_{n+1}; \mathbb{Z}/2)$-modules that sends the Thom class θ of $\mathcal{U}^\perp \oplus \mathcal{O}$ to c_{n+1}.

This short exact sequence induces a long exact sequence of cohomology groups with respect to the Steenrod square Sq^2. Since $H^*(X_n, \mathrm{Sq}^2)$ was computed in [KH92], with the result displayed above, we already know two thirds of this long exact sequence. Explicitly, we have $a_{4i+1} = c_{2i}c_{2i+1}$,[4] so i^* is the obvious surjection sending a_i to a_i (or to zero). Thus, the long exact sequence once again splits.

If $n = 2m$ we obtain a short exact sequence

$$0 \to H^*(X_{2m}, \mathrm{Sq}^2 + c_1) \cdot \theta \xrightarrow{p^*} \Lambda(a_1, \ldots, a_{4m-3}, a_{4m+1})$$
$$\xrightarrow{i^*} \Lambda(a_1, \ldots, a_{4m-3}) \to 0$$

We see that $H^*(X_{2m}, \mathrm{Sq}^2 + c_1) \cdot \theta$ is isomorphic to $\Lambda(a_1, \ldots, a_{4m-3}) \cdot a_{4m+1}$ as a $\Lambda(a_1, \ldots, a_{4m+1})$-module. It is thus generated by a single element, which is the unique element of degree $8m+2$. Since $p^*(c_{2m}\theta) = a_{4m+1}$, the class of $c_{2m}\theta$ is the element we are looking for, and the result displayed above follows.

If, on the other hand, n is odd, then i^* is an isomorphism and $H^*(X_{2m+1}, \mathrm{Sq}^2 + c_1)$ must be trivial. □

We see from the proof that p^* induces an injection of $H^*(X_n, \mathrm{Sq}^2 + c_1) \cdot \theta$ into $H^*(X_n, \mathrm{Sq}^2)$. Since we already know from [KH92] that the Atiyah-Hirzebruch spectral sequence for $\mathrm{KO}^*(X_n)$ collapses, we can apply Corollary 3.4 to deduce that the spectral sequence for $\widetilde{\mathrm{KO}}^*(\mathrm{Thom}_{X_n}(\mathcal{U}^\perp \oplus \mathcal{O}))$ also collapses at the $\widetilde{E}_3$-page. This completes the proof of Theorem 4.4.

4.5 Quadrics

We next consider smooth complex quadrics Q^n in $\mathbb{P}^{n+1}$. As far as we are aware, the first complete results on (higher) Witt groups of split quadrics were due to Walter: the computations for quadrics are announced together with the computations for projective bundles in [Wal03a] as the

[4] In [KH92] the generators are written as $c_{2i}c'_{2i+1}$ with $c'_{2i+1} = c_{2i+1} + c_1c_{2i}$.

main applications of that paper. Unfortunately, in the case of quadrics these seem to have remained unpublished. A partial description of the Witt groups is included in Yagita's preprint [Yag04] (see Corollary 8.3). Almost complete results for split quadrics, obtained by considering the localization sequences arising from the inclusion of a linear subspace of maximal dimension, were recently published by Nenashev in [Nen09]. Calmès informs me that the geometric description of the boundary map given in [BC09] can be used to show that these localization seqeunces split in general, yielding a complete computation. The calculation described here is independent of these results.

For all $n > 2$ the Picard group of Q^n is free on a single generator given by the restriction of the universal line bundle $\mathcal{O}(1)$ on $\mathbb{P}^{n+1}$. We will use the same notation $\mathcal{O}(1)$ for this restriction.

Theorem 4.8. *The KO-theory of a smooth complex quadric Q^n of dimension $n > 2$ can be described as follows:*

$\mathrm{KO}^*(Q^n;\mathcal{L})$			$\mathcal{L} \equiv \mathcal{O}$				$\mathcal{L} \equiv \mathcal{O}(1)$			
	t_0	t_1	s_0	s_1	s_2	s_3	s_0	s_1	s_2	s_3
$n \equiv 0 \mod 8$	$(n/2)+2$	$n/2$	2	0	0	0	2	0	0	0
$n \equiv 1$	$(n+1)/2$	$(n+1)/2$	1	1	0	0	1	1	0	0
$n \equiv 2$	$(n/2)+1$	$(n/2)+1$	1	2	1	0	0	0	0	0
$n \equiv 3$	$(n+1)/2$	$(n+1)/2$	1	1	0	0	0	0	1	1
$n \equiv 4$	$(n/2)+2$	$(n/2)$	2	0	0	0	0	0	2	0
$n \equiv 5$	$(n+1)/2$	$(n+1)/2$	1	0	0	1	0	1	1	0
$n \equiv 6$	$(n/2)+1$	$(n/2)+1$	1	0	1	2	0	0	0	0
$n \equiv 7$	$(n+1)/2$	$(n+1)/2$	1	0	0	1	1	0	0	1

The untwisted groups $\mathrm{KO}^*(Q^n)$ were computed by Kono and Hara in [KH92]. To retrieve the relevant result from their paper, note that Q^n can be identified with the quotient

$$Q^n = \frac{\mathrm{SO}(n+2)}{\mathrm{SO}(n) \times \mathrm{SO}(2)}$$

We review the first part of their computation in some detail before turning to $\mathrm{KO}^*(Q^n;\mathcal{O}(1))$.

Untwisted KO-groups. The integral cohomology of Q^n is well-known:

If n is even, write $n = 2m$. We have a class x in $H^2(Q^n)$ given by a hyperplane section, and two classes a and b in $H^n(Q^n)$ represented by linear subspaces of Q of maximal dimension. These three classes generate the cohomology multiplicatively, modulo the relations

$$x^m = a + b \qquad x^{m+1} = 2ax$$

$$ab = \begin{cases} 0 & \text{if } n \equiv 0 \\ ax^m & \text{if } n \equiv 2 \end{cases} \qquad a^2 = b^2 = \begin{cases} ax^m & \text{if } n \equiv 0 \mod 4 \\ 0 & \text{if } n \equiv 2 \mod 4 \end{cases}$$

Additive generators can thus be given as follows:

d	0	2	4	$\dots$	$n-2$	n	$n+2$	$n+4$	$\dots$	$2n$
$H^d(Q^n)$	1	x	x^2	$\dots$	x^{m-1}	a,b	ax	ax^2	$\dots$	ax^m

If n is odd, write $n = 2m+1$. Then similarly multiplicative generators are given by the class of a hyperplane section x in $H^2(Q^n)$ and the class of a linear subspace a in $H^{n+1}(Q^n)$ modulo the relations $x^{m+1} = 2a$ and $a^2 = 0$.

d	0	2	4	$\dots$	$n-1$	$n+1$	$n+3$	$n+5$	$\dots$	$2n$
$H^d(Q^n)$	1	x	x^2	$\dots$	x^m	a	ax	ax^2	$\dots$	ax^m

The action of the Steenrod square on $H^*(Q^n;\mathbb{Z}/2)$ is also well-known. See for example [Ish92] or [EKM08, § 78].

$$\operatorname{Sq}^2(x) = x^2$$

$$\operatorname{Sq}^2(a) = \operatorname{Sq}^2(b) = \begin{cases} ax & \text{if } n \equiv 0 \mod 4 \\ 0 & \text{if } n \equiv 2 \mod 4 \end{cases}$$

$$\operatorname{Sq}^2(a) = \begin{cases} 0 & \text{if } n \equiv 1 \mod 4 \\ ax & \text{if } n \equiv 3 \mod 4 \end{cases}$$

As before, we write $H^*(Q^n, \operatorname{Sq}^2)$ for the cohomology of $H^*(Q^n;\mathbb{Z}/2)$ with respect to the differential Sq^2.

Lemma 4.9. *Let Q^n be as above, and write $n = 2m$ or $n = 2m+1$. The following table gives a complete list of the additive generators of $H^*(Q^n, \operatorname{Sq}^2)$.*

d	0	$\dots$	$n-1$	n	$n+1$	$\dots$	$2n$	
$H^d(Q^n, \operatorname{Sq}^2)$	1						ax^m	*if* $n \equiv 0 \mod 4$
	1				a			*if* $n \equiv 1$
	1			a,b			ab	*if* $n \equiv 2$
	1		x^m					*if* $n \equiv 3$

The results of Kono and Hara follow from here provided there are no non-trivial higher differentials in the Atiyah-Hirzebruch spectral sequence for $\mathrm{KO}^*(Q^n)$. This is fairly clear in all cases except for the case $n \equiv 2 \mod 4$. In that case, the class $a+b = x^m$ can be pulled back from Q^{n+1}, and therefore all higher differentials must vanish on $a+b$. But one has to work harder to see that all higher differentials vanish on a (or b). Kono and Hara proceed by relating the KO-theory of Q^n to that of the spinor variety $\mathrm{Gr}_{\mathrm{SO}}(^{n+2}/_2, n+2)$ discussed in section 4.6.

Twisted KO-groups. We now compute

$$\mathrm{KO}^*(Q^n;\mathcal{O}(1)) = \widetilde{\mathrm{KO}}^{*+2}(\mathrm{Thom}_{Q^n}\mathcal{O}(1))$$

Let $\theta \in H^2(\mathrm{Thom}_{Q^n}\mathcal{O}(1))$ be the Thom class of $\mathcal{O}(1)$, so that multiplication by θ maps the cohomology of Q^n isomorphically to the reduced cohomology of $\mathrm{Thom}_{Q^n}\mathcal{O}(1)$. The Steenrod square on $\widetilde{H}^*(\mathrm{Thom}_{Q^n}\mathcal{O}(1);\mathbb{Z}/2)$ is determined by Lemma 3.5: for any $y \in H^*(Q^n;\mathbb{Z}/2)$ we have $\operatorname{Sq}^2(y\cdot\theta) = (\operatorname{Sq}^2 y + xy)\cdot\theta$. We thus arrive at

Lemma 4.10. *The following table gives a complete list of the additive generators of* $\widetilde{H}^*(\mathrm{Thom}_{Q^n}\mathcal{O}(1), \mathrm{Sq}^2)$.

d	$\dots$	$n+1$	$n+2$	$n+3$	$\dots$	$2n+2$	
$\widetilde{H}^d(\dots)$			$a\theta, b\theta$				*if* $n \equiv 0 \mod 4$
		$x^m\theta$				$ax^m\theta$	*if* $n \equiv 1$
							if $n \equiv 2$
				$a\theta$		$ax^m\theta$	*if* $n \equiv 3$

We claim that all higher differentials in the Atiyah-Hirzebruch spectral sequence for $\widetilde{\mathrm{KO}}^*(\mathrm{Thom}_{Q^n}\mathcal{O}(1))$ vanish. For even n this is clear. But for $n = 8k+1$ the differential d_{8k+2} might a priori take $x^m\theta$ to $ax^m\theta$, and for $n = 8k+3$ the differential d_{8k+2} might take $a\theta$ to $ax^m\theta$.

We therefore need some geometric considerations. Namely, the Thom space $\mathrm{Thom}_{Q^n}\mathcal{O}(1)$ can be identified with the projective cone over Q^n embedded in $\mathbb{P}^{n+2}$. This projective cone can be realized as the intersection of a smooth quadric $Q^{n+2} \subset \mathbb{P}^{n+3}$ with its projective tangent space at the vertex of the cone [Har92, p. 283]. Thus, we can consider the following inclusions:

$$Q^n \overset{j}{\hookrightarrow} \mathrm{Thom}_{Q^n}\mathcal{O}(1) \overset{i}{\hookrightarrow} Q^{n+2}$$

Note that the composition is the inclusion of the intersection of Q^{n+2} with two transversal hyperplanes.

Lemma 4.11. *All higher differentials (d_k with $k > 2$) in the Atiyah-Hirzebruch spectral sequence for* $\mathrm{KO}^*(\mathrm{Thom}_{Q^n}\mathcal{O}(1))$ *vanish.*

Proof. We need only consider the cases when n is odd. Write $n = 2m+1$.

Suppose first that $n \equiv 1 \mod 4$, so $n+2 \equiv 3$. Consider the two elements $x^m\theta \in H^{n+1}(\mathrm{Thom}_{Q^n}\mathcal{O}(1), \mathrm{Sq}^2)$ and $x^{m+1} \in H^{n+1}(Q^{n+2}, \mathrm{Sq}^2)$.

We claim that i^* maps the second element to the first. Indeed, j^*i^* maps the class of the hyperplane section $x \in H^2(Q^{n+2})$ to the class of the hyperplane section $x \in H^2(Q^n)$. So $i^*x \in H^2(\mathrm{Thom}_{Q^n}\mathcal{O}(1))$ must be non-zero, hence equal to θ modulo 2. It follows that $i^*(x^{m+1}) = \theta^{m+1}$. Since $\theta^2 = \mathrm{Sq}^2(\theta) = x\theta$, we have $\theta^{m+1} = x^m\theta$, proving the claim. As we already know that all higher differentials vanish on $H^*(Q^{n+2}, \mathrm{Sq}^2)$, we may now deduce that they also vanish on $H^*(\mathrm{Thom}_{Q^n}\mathcal{O}(1), \mathrm{Sq}^2)$.

Secondly, consider the case $n \equiv 3 \mod 4$, so $n+2 \equiv 1$. In the same spirit as above, consider $a\theta \in H^{n+3}(\mathrm{Thom}_{Q^n}\mathcal{O}(1), \mathrm{Sq}^2)$ and $a \in H^{n+3}(Q^{n+2}, \mathrm{Sq}^2)$.

We claim that $i^*(a) = a\theta$. Indeed, a represents a linear subspace of codimension $m+2$ in Q^{n+2} and is thus mapped to the class of a linear subspace of the same codimension in Q^n: $j^*i^*(a) = ax \in H^{n+3}(Q^n)$. Thus, $i^*(a)$ is non-zero in $H^{n+3}(\mathrm{Thom}_{Q^n}\mathcal{O}(1))$, equal to $a\theta$ modulo 2. Again, this implies that all higher differentials vanish on $H^*(\mathrm{Thom}_{Q^n}\mathcal{O}(1), \mathrm{Sq}^2)$ since they vanish on $H^*(Q^{n+2}, \mathrm{Sq}^2)$. □

The additive structure of $\mathrm{KO}^*(Q^n; \mathcal{O}(1))$ thus follows directly from the result for $H^d(Q^n, \mathrm{Sq}^2 + x) = \widetilde{H}^{d+2}(\mathrm{Thom}_{Q^n}\mathcal{O}(1))$ displayed in Corollary 4.10 via Corollary 3.6.

4.6 Spinor varieties

Lastly, let $\mathrm{Gr}_{\mathrm{SO}}(n,N)$ be the Grassmannian of n-planes in $\mathbb{C}^N$ isotropic with respect to a fixed non-degenerate symmetric bilinear form, or, equivalently, the Fano variety of projective $(n-1)$-planes contained in the quadric Q^{N-2}. For all $n < N/2$ these are irreducible homogeneous varieties for $\mathrm{SO}(N)$. In particular, when N is odd and n is maximal, we obtain the spinor varieties $\mathrm{Gr}_{\mathrm{SO}}(n-1,2n-1)$. These are the varieties we will study in this section. To simplify notation we will write $\mathrm{Gr}_{\mathrm{SO}}(n-1,2n-1)$ as S_n.

The case of maximal isotropic planes in an even-dimensional ambient space will also be covered by the result below since $\mathrm{Gr}_{\mathrm{SO}}(n,2n)$ falls apart into two connected components both of which are isomorphic to S_n. This is reflected by the fact that we can equivalently identify S_n with $\mathrm{SO}(2n-1)/U(n-1)$ or $\mathrm{SO}(2n)/U(n)$.

As for all Grassmannians, the Picard group of S_n is isomorphic to $\mathbb{Z}$; let us fix a line bundle $\mathcal{S}$ which generates it. Once again, the usual KO-theory of these spaces has been computed in [KH92]. We will show here that the KO-theory with a non-trivial twist vanishes.

Theorem 4.12. *For all $n \geq 2$ the additive structure of $\mathrm{KO}^*(S_n;\mathcal{L})$ is as follows:*

	t_0	t_1	$s_i(\mathcal{O})$	$s_i(\mathcal{S})$
$n \equiv 2 \mod 4$	2^{n-2}	2^{n-2}	$\rho(\frac{n}{2}, 1-i)$	0
otherwise	2^{n-2}	2^{n-2}	$\rho(\lfloor\frac{n}{2}\rfloor, -i)$	0

Here, the values $\rho(n,i)$ are defined as in Theorem 4.4.

Proof. The cohomology of S_n with $\mathbb{Z}/2$-coefficients has simple generators $e_2, e_4, \ldots, e_{2n-2}$, i.e. it is additively generated by products of distinct elements of this list. Its multiplicative structure is determined by the rule $e_{2i}^2 = e_{4i}$, and the second Steenrod square is given by $\mathrm{Sq}^2(e_{2i}) = ie_{2i+2}$. In both formulae it is of course understood that $e_{2j} = 0$ for $j \geq n$. What we need to show is that for all $n \geq 2$ we have

$$H^*(S_n, \mathrm{Sq}^2 + e_2) = 0$$

Let us abbreviate $H^*(S_n, \mathrm{Sq}^2 + e_2)$ to (H_n, d'). We claim that we have the following short exact sequence of differential $\mathbb{Z}/2$-modules:

$$0 \to (H_n, d') \overset{\cdot e_{2n}}{\to} (H_{n+1}, d') \to (H_n, d') \to 0 \tag{31}$$

This can be checked by a direct calculation. Alternatively, it can be deduced from the geometric considerations below. Namely, it follows from the cofibration sequence of Corollary 4.14 that we have such an exact sequence of $\mathbb{Z}/2$-modules with maps respecting the differentials given by Sq^2 on all three modules. Since they also commute with multiplication by e_2, they likewise respect the differential $d' = \mathrm{Sq}^2 + e_2$.

The long exact cohomology sequence associated with (31) allows us to argue by induction: if $H^*(H_n, d') = 0$ then also $H^*(H_{n+1}, d') = 0$. Since we can see by hand that $H^*(H_2, d') = 0$, this completes the proof. ∎

We close with a geometric interpretation of the exact sequence (31), via an analogue of Lemmas 4.1 and 4.5. Denote by $\mathcal{U}$ the universal bundle on S_n, i. e. the restriction of the universal bundle on $\mathrm{Gr}(n-1, 2n-1)$ to S_n. Let $\mathcal{U}^\perp$ denote the restriction of the orthogonal complement bundle, so that $\mathcal{U} \oplus \mathcal{U}^\perp$ is the trivial $(2n-1)$-bundle on S_n.

Lemma 4.13. *The spinor variety S_n embeds into the spinor variety S_{n+1} with normal bundle $\mathcal{U}^\perp$ such that the embedding extends to an embedding of this bundle. The closed complement of $\mathcal{U}^\perp$ in S_{n+1} is again isomorphic to S_n.*

Corollary 4.14. *We have a cofibration sequence*

$$S_{n+} \overset{i}{\hookrightarrow} S_{n+1_+} \overset{p}{\twoheadrightarrow} \mathrm{Thom}_{S_n} \mathcal{U}^\perp$$

Note however that, unlike in the symplectic case, the first Chern classes of $\mathcal{U}$ and $\mathcal{U}^\perp$ pull back to *twice* a generator of the Picard group of S_n. For example, the embedding of S_2 into $\mathrm{Gr}(1,3)$ can be identified with the embedding of the one-dimensional smooth quadric into the projective plane, of degree 2, and the higher dimensional cases can be reduced to this example. Thus, $c_1(\mathcal{U})$ and $c_1(\mathcal{U}^\perp)$ are trivial in $\mathrm{Pic}(S_n)/2$.

proof of Lemma 4.13. The argument is similar to that used in the proof of Lemma 4.5. Let e_1, e_2 be the first two canonical basis vectors of $\mathbb{C}^{2n+1}$, and let $\mathbb{C}^{2n-1}$ be embedded into $\mathbb{C}^{2n+1}$ via the remaining coordinates. Let S_n be defined in terms of a symmetric form Q on $\mathbb{C}^{2n-1}$, and define S_{n+1} in terms of

$$Q_{2n+1} := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & Q \end{pmatrix}$$

Let i_1 and i_2 be the embeddings of S_n in S_{n+1} sending an $(n-1)$-plane $\Lambda \subset \mathbb{C}^{2n-1}$ to $e_1 \oplus \Lambda$ or $e_2 \oplus \Lambda$ in $\mathbb{C}^{2n+1}$, respectively.

We extend i_1 to an embedding I of $\mathcal{U}^\perp$: given an $(n-1)$-plane $\Lambda \in S_n$ together with a vector v in $\Lambda^\perp \subset \mathbb{C}^{2n-1}$, we consider the linear function

$$\begin{pmatrix} -\frac{1}{2}Q(v,v) & -Q(-,v) \\ v & 0 \end{pmatrix} : \langle e_1 \rangle \oplus \Lambda \to \langle e_2 \rangle \oplus \Lambda^\perp$$

The embedding I sends (Λ, v) to the graph $\Gamma_{\Lambda,v} \subset \mathbb{C}^{2n+1}$ of this function.

$$\begin{array}{ccccc} \mathcal{U}^\perp & \overset{I}{\hookrightarrow} & S_{n+1} & \overset{i_2}{\hookleftarrow} & S_n \\ (\Lambda, v) & \mapsto & \Gamma_{\Lambda,v} & & \\ & & \langle e_2 \rangle \oplus \Lambda & \leftarrow\!\shortmid & \Lambda \end{array}$$

To see that I and i_2 are complementary, take an arbitrary n-plane W in S_{n+1}. If $e_2 \in W$ then we can consider a basis

$$e_2, \begin{pmatrix} a_2 \\ 0 \\ v_2 \end{pmatrix}, \ldots, \begin{pmatrix} a_n \\ 0 \\ v_n \end{pmatrix}$$

of W. The fact that Q_{2n+1} vanishes on W implies that all a_i are zero. Moreover, $v_2, \ldots, v_n$ span an isotropic subspace of $\mathbb{C}^{2n-1}$ with respect to Q, and thus W can be identified with $i_2(\langle v_2, \ldots, v_n \rangle)$.

On the other hand, if e_2 is not contained in W then we must have a vector of the form

$$\begin{pmatrix} 1 \\ z' \\ v' \end{pmatrix}$$

in W, for some $z' \in \mathbb{C}$ and $v' \in \mathbb{C}^{2n-1}$. Since W is isotropic we must have $z' = -\frac{1}{2}Q(v', v')$. Let us extend this vector to a basis of W:

$$\begin{pmatrix} 1 \\ -\frac{1}{2}Q(v',v') \\ v' \end{pmatrix}, \begin{pmatrix} 0 \\ b_2 \\ v_2 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ b_n \\ v_n \end{pmatrix}$$

Put $\Lambda := \langle v_2, \dots, v_n \rangle$. The condition that Q_{2n+1} vanishes on W implies that Q vanishes on Λ and that $b_i = -Q(v_i, v')$ for $i = 2, \dots, n$. It follows in particular that Λ is $(n-1)$-dimensional. Moreover, we can replace the first vector of our basis by a vector

$$\begin{pmatrix} 1 \\ z \\ v \end{pmatrix}$$

with $v \in \Lambda^{\perp}$, by subtracting appropriate multiples of the remaining basis vectors. Since this vector must still be isotropic with respect to Q_{2n+1} we must have $z = -\frac{1}{2}Q(v, v)$. Also, $Q(v_i, v') = Q(v_i, v)$, so our new basis has the form

$$\begin{pmatrix} 1 \\ -\frac{1}{2}Q(v,v) \\ v \end{pmatrix}, \begin{pmatrix} 0 \\ -Q(v_2,v) \\ v_2 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ -Q(v_n,v) \\ v_n \end{pmatrix}$$

This shows that $W = \Gamma_{\Lambda,v}$. □

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