SYMMETRY OF EMBEDDED GENUS-ONE HELICOIDS

JACOB BERNSTEIN AND CHRISTINE BREINER

Abstract. In this note, we use the Lopez-Ros deformation introduced in [9] to show that any embedded genus-one helicoid must be symmetric with respect to rotation by 180° around a normal line. This partially answers a conjecture of Bobenko from [3]. We also show this symmetry holds for an embedded genus-k helicoid Σ, provided the underlying conformal structure of Σ is hyperelliptic.

In [3], Bobenko conjectures that any immersed genus-k helicoid (i.e. a minimally immersed, once punctured genus-k surface with "helicoid-like" behavior at the puncture) is symmetric with respect to rotation by 180° around a line perpendicular to the surface. This conjecture is motivated by the observation in [3] that the period problem for these surfaces is algebraically "well-posed" when there is such a symmetry, but is "over-determined" without it. In this note, we verify Bobenko's conjecture for *embedded* genus-one helicoids. That is:

Theorem 0.1. *Let Σ be an embedded genus-one helicoid. Then there is a line ℓ normal to Σ so that rotation by 180° about ℓ acts as an orientation preserving isometry on Σ.*

We define a genus-k helicoid to be a complete, minimal surface immersed in $\mathbb{R}^3$ which has genus k, one end, and is asymptotic to a helicoid. A consequence of Theorem 3 of [5] is that any (immersed) minimal surface which is conformally a once-punctured compact genus-k Riemann surface with "helicoid-like" Weierstrass data at the puncture is a genus-k helicoid in this sense. In particular, the above definition encompasses the surfaces studied by Bobenko. Importantly, by Theorem 1.1 of [2], any complete, *embedded* minimal surface in $\mathbb{R}^3$ with genus k and one end has "helicoid-like" Weierstrass data and hence is a genus-k helicoid. The space of such objects is not vacuous. Weber, Hoffman and Wolf [12] and Hoffman and White [7] have given (very different) constructions of embedded genus-one helicoids – at present it is unknown whether the two constructions give the same surface. Both constructions produce a genus-one helicoid that has, in addition to the orientation preserving symmetry of Theorem 0.1, two orientation reversing symmetries. Whether all genus-one helicoids possess these additional symmetries is also unknown.

We emphasize that our argument does not generalize to genus $k > 1$ because we crucially use the fact that every genus-one Riemann surface admits a large number of biholomorphic involutions – more precisely, that any once-punctured genus-one Riemann surface admits a non-trivial biholomorphic involution. This need not be true for higher genus. Indeed, *a priori* there may fail to be any non-trivial biholomorphic automorphisms. However, if we restrict attention to genus-k

2000 *Mathematics Subject Classification.* 53A10.
The authors were supported respectively by the NSF grants DMS-0902721 and DMS-0902718.

helicoids whose underlying Riemann surface structure is *hyperelliptic* − that is the surface admits a biholomorphic involution I with $2k+2$ fixed points, one of which is the puncture, then our arguments continue to hold. Consequently, we present the argument in this more general context. Finally, we note that Francisco Martin has pointed out to us that with only slight modifications, the argument also proves that embedded *periodic* genus-one helicoids admit such a symmetry.

Let us outline the proof of Theorem 0.1 for genus-one helicoids. By Theorem 1.1 of [2], Σ is conformally a once-punctured torus with "helicoid-like" Weierstrass data at the puncture. Thus, Σ admits a biholomorphic involution, I, which is compatible with this data. Indeed, if dh is the height differential and g is the stereographic projection of the Gauss map then $I^*dh = -dh$ and $g \circ I = Cg^{-1}$, for $C \in \mathbb{C}\setminus\{0\}$. If $|C| = 1$, a simple computation using the Weierstrass representation implies Theorem 0.1. On the other hand, if $|C| \neq 1$ then the interaction between the period conditions and the involution I imply Σ has vertical flux. In this case, following Perez and Ros [10], we may deform the Weierstrass data to obtain a smooth family of immersed minimal surfaces, Σ_λ. Here $\Sigma = \Sigma_1$ and Σ_λ is the Lopez-Ros deformation [9] of Σ. As in [10], for λ near 1, Σ_λ is embedded, while for $\lambda >> 1$, Σ_λ is not embedded, contradicting the maximum principle.

Acknowledgments. We would like to thank David Hoffman, Francisco Martin and the anonymous referees for their many constructive comments. We also thank Brian White for clarifying some results regarding mass minimizing currents.

1. Asymptotic properties of Σ and properties of the Involution

1.1. **The Weierstrass Representation and the Flux.** We recall the Weierstrass representation for immersed minimal surfaces in $\mathbb{R}^3$. Let M be a Riemann surface and suppose that g is a meromorphic function on M and dh a holomorophic one-form. Suppose, moreover, that the meromorphic one-forms gdh and $g^{-1}dh$ have no poles and do not simultaneously vanish. Then the map $\mathbf{F} : M \to \mathbb{R}^3$ given by

$$(x_1, x_2, x_3) = \mathbf{F} := \operatorname{Re} \int \left(\frac{1}{2}(g^{-1} - g), \frac{i}{2}(g^{-1} + g), 1 \right) dh \tag{1.1}$$

is a minimal immersion with the property that g is the stereographic projection of the Gauss map of the image of $\mathbf{F}$ and $\operatorname{Re} dh = \mathbf{F}^* dx_3$. Without further restrictions on the data, $\mathbf{F}$ is potentially only defined on $\tilde{M}$, the universal cover of M. These restrictions are known as the *period conditions*, which, when satisfied, ensure that $\mathbf{F}$ is well-defined on M. Explicitly they may be stated as:

$$\int_\gamma g dh = \overline{\int_\gamma g^{-1} dh} \text{ and } \operatorname{Re} \int_\gamma dh = 0 \tag{1.2}$$

for any closed curve, γ, on M. Conversely, given a minimal immersion $\mathbf{F} : M \to \mathbb{R}^3$, one obtains g and dh satisfying the period conditions and with $gdh, g^{-1}dh$ holomorphic and not vanishing simultaneously and so that the image of the map given by (1.1) coincides with the image of $\mathbf{F}$ (up to a translation).

We will also consider the *flux* of the immersion $\mathbf{F}$ along closed curves. For γ a closed curve on Σ, we denote by $\nu = -d\mathbf{F}(J\gamma')$ the conormal vector field along γ. Here J is the complex structure of Σ and γ' is the derivative of γ with respect to arc length. We define the flux of Σ along γ equivalently as:

$$Flux(\gamma) = \int_\gamma \nu \, ds = \mathrm{Im} \int_\gamma \left(\frac{1}{2}(g^{-1} - g), \frac{i}{2}(g^{-1} + g), 1 \right) dh. \tag{1.3}$$

The equivalence of the two definitions is a simply consequence of the Cauchy-Riemann equations and (1.1). Indeed, on an oriented Riemannian surface, every harmonic one-form ω has a harmonic conjugate $\omega^* = -\omega \circ J$ with $\omega + i\omega^*$ holomorphic. Conversely, any holomorphic one-form can be written as $\omega + i\omega^*$ with ω harmonic. As holomorphic one-forms are closed, Stokes' theorem implies the flux of a curve depends only on its homology class.

Of particular interest in this paper are surfaces with everywhere *vertical flux*, i.e. where the horizontal components of the flux are zero for all closed curves. A minimal surface with vertical flux can be smoothly deformed to give a smooth family of minimal immersions. Indeed, suppose Σ is a minimal surface with vertical flux and Weierstrass data (g, dh, M). Then for $\lambda \in \mathbb{R}^+$, it follows from (1.3) that the triple $(\lambda g, dh, M)$ satisfies the period conditions (1.2) and so (1.1) gives the desired family of immersions $\mathbf{F}_\lambda : M \to \mathbb{R}^3$. Such a deformation was introduced by Lopez and Ros in [9].

1.2. **The Involution of Σ.** We now consider Σ, an embedded genus-k helicoid with asymptotic helicoid H. Denote the Weierstrass data of Σ by (g, dh, M). By Theorem 1.1 of [2], Σ is conformal to a once-punctured compact genus-k surface and the one-forms dh and $\frac{dg}{g}$ both have double poles at this puncture with zero residue – this is the "helicoid-like" behavior alluded to in the introduction. More precisely, there is a compact Riemann surface M_k and a point $\infty \in M_k$ so that $M = M_k \backslash \{\infty\}$ with dh and $\frac{dg}{g}$ meromorphic one forms on M_k both with a double pole at ∞ and no residue there. We assume also that Σ is *hyperelliptic*. That is, there exists a non-trivial biholomorphic involution $I : M_k \to M_k$ with $2k + 2$ fixed points and so that $I(\infty) = \infty$. An important property of hyperelliptic involutions is that for $0 \neq [\gamma] \in H_1(M_k, \mathbb{Z})$ a non-trivial element of the first homology group of M_k, $I_*[\gamma] = -[\gamma]$ – see [4]. As inclusion of M in M_k induces an isomorphism between $H_1(M, \mathbb{Z})$ and $H_1(M_k, \mathbb{Z})$ this property also holds for Σ.

Hyperellipticity is a very strong condition when $k > 1$. However, genus-one helicoids are always hyperelliptic. Indeed, let $\Lambda_\tau = \{n + m\tau : n, m \in \mathbb{Z}\} \subset \mathbb{C}$ be the lattice so that $\mathbb{T}^{2*}_\tau = \mathbb{C}/\Lambda_\tau \backslash \{\bar{0}\}$ is conformally equivalent to Σ, where $\bar{0} = 0 + \Lambda_\tau$. As $-\Lambda_\tau = \Lambda_\tau$, the map $u \to -u$ induces a biholomorphic involution of $\mathbb{T}^{2*}_\tau$ and hence an involution $I : \Sigma \to \Sigma$. The half-period lattice of Λ_τ is fixed by $u \to -u$, and so ∞ and exactly 3 points of Σ are fixed by I.

Before we proceed we note the following simple lemma:

Lemma 1.1. *Suppose that p is a point in N, a Riemann surface with a non-trival involution $I : N \to N$, so that $I(p) = p$. Then there is a coordinate neighborhood U about p with coordinate u so that: $I(U) = U$ and $u \circ I = -u$. Moreover, suppose ω is a meromorphic one-form on U of the form:*

$$\omega = \left(\frac{a}{u^2} + \frac{b}{u} + H(u) \right) du$$

with H holomorphic. Then the one-form $\omega + I^*\omega$

(1) *Has a simple pole at p iff $b \neq 0$,*
(2) *Has a zero at p iff $b = 0$,*

(3) *Is identically zero on U iff $b = 0$ and H is even.*

By (3), if H has a simple zero at $u = 0$ then $\omega + I^*\omega$ can't vanish identically on U.

Proof. The existence of such a coordinate is a straightforward consequence of the inverse function theorem. One calculates that in U, $I^*\omega = -\left(\frac{a}{u^2} - \frac{b}{u} + H(-u)\right) du$ and so $\omega + I^*\omega = \left(\frac{2b}{u} + H(u) - H(-u)\right) du$. Clearly $H(u) - H(-u)$ is odd and so vanishes at $u = 0$; from this all three claims follow. □

We now analyze how I acts on the Weierstrass data:

Lemma 1.2. *$I^*dh = -dh$ and $I^*\frac{dg}{g} = -\frac{dg}{g}$*

Proof. Let us denote by $F \subset M_k$ the fixed point set of I. For a given meromorphic one-form α, let $Z[\alpha]$ and $P[\alpha]$ represent the sets, respectively, of zeros and poles of α. The Riemann-Roch theorem implies that for any non-vanishing meromorphic one-form α on M_k with $Z[\alpha] \neq \emptyset$ one has the following relation:

$$\#Z[\alpha] - \#P[\alpha] = 2k - 2 \tag{1.4}$$

where $\#Z[\alpha]$ and $\#P[\alpha]$ denote the number of, respectively, zeros and poles of α counting multiplicity. For an arbitrary set of points, X, we denote by $|X|$ the number of points of X. In general, $\#Z[\alpha] \geq |Z[\alpha]|$ and $\#P[\alpha] \geq |P[\alpha]|$.

Recall that dh has only one pole (at ∞) and it is a double pole with no residue. Given that ∞ is fixed by I, Lemma 1.1 implies that $I^*dh + dh$ has no poles and has zeros at each point of F. As $|F| = 2k+2$, (1.4) implies that $I^*dh + dh$ must vanish identically. This proves the first part of the lemma and that I preserves $Z[dh]$. Lemma 1.1, in particular (3), and (1.4) then imply that $|Z[dh]| \leq 2k - |F \cap Z[dh]|$.

Set $\omega = \frac{dg}{g}$ and $\tilde{\omega} = \omega + I^*\omega$. In Σ, the poles of ω are simple and, as gdh and $g^{-1}dh$ do not simultaneously vanish, these poles occur precisely at the points of $Z[dh]$. Thus, Lemma 1.1 implies that: $\tilde{\omega}$ has only simple poles; $P[\tilde{\omega}] \subset Z[dh]$; and $Z[\tilde{\omega}] \supset F \backslash (F \cap Z[dh])$. As the poles are simple:

$$\#P[\tilde{\omega}] \leq |Z[dh]| \leq 2k - |F \cap Z[dh]| \tag{1.5}$$

while

$$\#Z[\tilde{\omega}] \geq |F| - |F \cap Z[dh]| = 2k + 2 - |F \cap Z[dh]|. \tag{1.6}$$

If $\tilde{\omega}$ does not vanish identically then (1.4) implies $k \geq 2$. This proves the theorem for $k = 1$.

For $k > 1$ we must use further properties of genus-k helicoids. To begin the argument assume $\tilde{\omega}$ is not identically zero. Observe that in Σ, ω has only simple poles which occur at the zeros and poles of g. The residue of ω at such a zero or pole is exactly equal to $\pm m$ where m is the order of the zero or pole. Lemma A.3 proves that p is a pole of g if and only if $I(p)$ is a zero of the same order. Thus, the residues of ω and of $I^*\omega$ cancel at any pole of ω in Σ. Hence, $\#P[\tilde{\omega}] = 0$ and by (1.4) if $\tilde{\omega}$ doesn't vanish identically then $\#Z[\tilde{\omega}] = 2k - 2$.

Finally, Lemma A.3 implies that if p is a zero or pole of g, then $p \notin F$. As $P[\omega] = Z[dh]$, it follows that $F \cap Z[dh] = \emptyset$. Then by (1.6), $\#Z[\tilde{\omega}] \geq 2k + 2$. This gives the necessary contradiction and completes the proof. □

We next compute how the Gauss map is transformed under I. First, pick $p_0 \in \Sigma$ satisfying $I(p_0) = p_0$ so that $\frac{dg}{g}$ does not have a pole at p_0 – there are $2k+1$ points of Σ fixed by I and by (1.4) at most $2k$ poles of $\frac{dg}{g}$, so such a point p_0 exists. We determine the transformation of the Gauss map using its value at this fixed point:

Corollary 1.3. *$g \circ I = \frac{g(p_0)^2}{g}$ for $g(p_0) \in \mathbb{C} \setminus \{0\}$ and p_0 as determined above.*

Proof. By analytic continuation it suffices to consider U the neighborhood of p_0 from Lemma 1.1. We have: $\frac{g(p)}{g(p_0)} = \exp(\int_\gamma \frac{dg}{g})$ where γ is a path in U connecting p_0 to p. Then $\frac{g(I(p))}{g(p_0)} = \exp(\int_{I(\gamma)} \frac{dg}{g})$. However, $\int_{I(\gamma)} \frac{dg}{g} = \int_\gamma I^* \frac{dg}{g} = -\int_\gamma \frac{dg}{g}$ and so $g(I(p)) = \frac{g(p_0)^2}{g(p)}$. □

2. The Rotational Symmetry

Using the properties of the involution I, we can now prove that Σ has the claimed symmetry. Note that by rotating Σ about the x_3-axis and translating $\mathbb{R}^3$, we may assume that $g(p_0) > 0$ and $\mathbf{F}(p_0) = 0$; here p_0 is the point from Corollary 1.3. If $g(p_0) = 1$ then a simple computation using the Weierstrass representation gives that $(x_1, x_2, x_3) \circ I = (x_1, -x_2, -x_3)$, proving Theorem 0.1. Thus, we must rule out the possibility that $g(p_0) \neq 1$.

To that end, we use I to see that in this case Σ has vertical flux:

Lemma 2.1. *If $g(p_0) \neq 1$ then gdh and $\frac{1}{g}dh$ are exact forms on Σ.*

Proof. Recall gdh and $\frac{1}{g}dh$ are both holomorphic one-forms on Σ and are hence closed. As a consequence, it will suffice to show that over the $2k$ generators of the homology group $[\eta_i]$ that $\int_{\eta_i} gdh = \int_{\eta_i} \frac{1}{g}dh = 0$. Here η_i are simple closed curves and $[\eta_i]$ the corresponding homology classes. For simplicity, we treat only gdh. By the first equation in (1.2):

$$\int_{\eta_i} gdh = \overline{\int_{\eta_i} \frac{1}{g} dh}.$$

Recall, hyperelliptic involutions satisfy $I_*[\eta_i] = -[\eta_i]$. Hence, $\int_{\eta_i} gdh = -\int_{I(\eta_i)} gdh = g(p_0)^2 \int_{\eta_i} \frac{1}{g}dh = g(p_0)^2 \overline{\int_{\eta_i} gdh}$. Taking absolute values, if $g(p_0) \neq 1$, then $\int_{\eta_i} gdh = 0$. □

We will argue as in [10] to show that the existence of a Σ with vertical flux is precluded by the maximum principle. First we show:

Lemma 2.2. *Suppose Σ has vertical flux. Then, there is a smooth family of immersed minimal surfaces Σ_λ, $\lambda > 0$, with $\Sigma_1 = \Sigma$ and a fixed helicoid H so that:*

1. *Each Σ_λ is a genus-k helicoid and is asymptotic to H.*
2. *The set $E = \{\lambda \in \mathbb{R}^+ : \Sigma_\lambda \text{ is embedded}\}$ is open.*
3. *For λ sufficiently large Σ_λ is not embedded.*

Proof. As Σ has vertical flux, the triple $(\lambda g, dh, M)$, for $\lambda \in \mathbb{R}^+$, gives rise to a minimal immersion $\mathbf{F}_\lambda : M_k \to \mathbb{R}^3$. Let us denote by Σ_λ the image of $\mathbf{F}_\lambda$ and set $g_\lambda = \lambda g$. Notice that each Σ_λ is a complete, minimally immersed genus-k surface and $\Sigma_1 = \Sigma$. It remains only to verify that this family satisfies (1)-(3).

To that end, we note that by Corollary 1.2 of [2], M_k has a neighborhood of infinity, U, with holomorphic coordinate $z : U \setminus \{\infty\} \to \mathbb{C}$ so that (after possibly rescaling Σ) on $U \setminus \{\infty\}$, $g(p) = \exp(iz(p) + F(p))$, where F is holomorphic and has a zero at ∞. As a consequence, $g_\lambda(p) = \exp(iz(p) + \log \lambda + F(p))$. Hence, Theorem 3 of [5] implies that, outside of a ball of radius R_λ, Σ_λ is asymptotic to a scale 1 helicoid – i.e. a helicoid with Weierstrass data $(e^{iz}, dz, \mathbb{C})$. After a translation, this gives (1). To see (2), we note that for any λ_0 there is an $R > 1$ so that, for $\lambda \in [\lambda_0/2, 2\lambda_0]$, outside of B_R each Σ_λ is a normal exponential graph over the helicoid H with small L^∞ norm. In particular, the Σ_λ are embedded outside B_R. On the other hand as $\lambda \to \lambda_0$ the $B_R \cap \Sigma_\lambda$ converge smoothly to $B_R \cap \Sigma_{\lambda_0}$. By (1) this convergence must be with multiplicity 1 and so for λ close enough to λ_0, $B_R \cap \Sigma_\lambda$ can be written as the normal exponential graph with small L^∞ norm over $B_R \cap \Sigma_{\lambda_0}$. Thus, if $B_R \cap \Sigma_{\lambda_0}$ is embedded then, for λ sufficiently close to λ_0, so is $B_R \cap \Sigma_\lambda$. This gives (2). Finally, (1.4) implies that g must have at least one pole and one zero on Σ. Lemma 4 of [10] then gives (3). □

Proof. (of Theorem 0.1) By the above, it suffices to show that Σ does not have vertical flux. We proceed by contradiction. If Σ has vertical flux then Lemma 2.2 gives a family Σ_λ with the properties (1)-(3). By (2) and (3), there exists a $\lambda_0 > 1$ so that Σ_{λ_0} is not embedded, but for $\lambda \in [1, \lambda_0)$, Σ_λ is embedded. By (1), (3), and the fact that the Σ_λ smoothly depend on λ, there are points p^1_λ, p^2_λ with $|p^i_\lambda| \leq R < \infty$ so that $\lim_{\lambda \nearrow \lambda_0} |p^1_\lambda - p^2_\lambda| \to 0$ but $\mathrm{dist}_{\Sigma_\lambda}(p^1_\lambda, p^2_\lambda) \geq \delta > 0$. By the strong maximum principle, this is only possible if the Σ_λ converge to Σ_{λ_0} as $\lambda \nearrow \lambda_0$ with multiplicity greater than 1. This contradicts (1). □

3. Periodic Genus-One Helicoids

Francisco Martin has kindly pointed out to us that our argument can be readily adapted to embedded *periodic* genus-one helicoids. For the sake of completeness, we include here a sketch of his argument.

Roughly speaking, a periodic genus-one helicoid looks like a helicoid with an infinite number of handles placed periodically along the axis. More precisely, it is an infinite genus surface, Σ so that: Σ is asymptotic to a helicoid; Σ is invariant under a "screw-motion" $S_{\theta,t}$; and the quotient of Σ by the group generated by $S_{\theta,t}$ is a twice punctured torus. Here $S_{\theta,t}$ is the isometry of $\mathbb{R}^3$ given by rotating about the x_3-axis by θ followed by a translation by t in the x_3 direction. There exists a family of embedded examples. Indeed, in [12] an embedded genus-one helicoid is constructed as a limit of periodic genus-one helicoids. See also [6].

As discussed in [1], after a homothety, one may consider a periodic genus-one helicoid to be a triple of Weierstrass data: $(g, dh, \mathbb{T}^2 \setminus \{E_1, E_2\})$. Here E_1, E_2 are distinct points of $\mathbb{T}^2$ and g extends meromorphically to $\mathbb{T}^2$ with a zero at E_1 and a pole at E_2. Further, dh extends meromorphically to $\mathbb{T}^2$ with simple poles at E_1, E_2, and residue $-i$ at E_1 and i at E_2. Finally, the compatibility conditions of Section 1.1 hold except that the vertical periods around E_1 and E_2 do not close up; recall we are parameterizing a surface of infinite topology. For simplicity, take this as our definition and refer to [1, 8] for weaker characterizations. It is a straightforward computation to see that a periodic genus-one helicoid is asymptotic to a scale-one helicoid if and only if the one-form $\frac{dg}{g} - idh$ has no poles at E_1, E_2 – see [8, 10].

Lemma 3.1. *There is a non-trivial biholomorphic involution I of $\mathbb{T}^2$ so that $I(E_1) = E_2$. Moreover, $I^* dh = -dh$ and $I^* \frac{dg}{g} = -\frac{dg}{g}$.*

Proof. Identify $\mathbb{T}^2$ with the quotient $\mathbb{C}/\Lambda_\tau$ where $\Lambda_\tau = \{n + m\tau : n, m \in \mathbb{Z}\}$. As translation along 1 or τ in $\mathbb{C}$ induce biholomorphic automorphisms, we may represent E_i by points $p_i + \Lambda_\tau$ where the p_i are placed symmetrically with respect to 0. Hence, the map $u \to -u$ on $\mathbb{C}$ descends to an involution I of $\mathbb{T}^2$ that swaps E_1 and E_2. As I swaps the E_i and the residues of dh at E_1 and E_2 are of opposite sign, $I^* dh + dh$ has no poles. By Lemma 1.1, $I^* dh + dh$ has at least four zeros and so by (1.4) vanishes identically.

By construction, $\frac{dg}{g}$ has simple poles at E_1 and E_2 with residues of opposite sign. Thus, $I^* \frac{dg}{g} + \frac{dg}{g}$ has no residue at either E_1 or E_2 and hence no poles there. On the other hand, all other poles of $\frac{dg}{g}$ occur at the zeros of dh and these are involuted by I and so $I^* \frac{dg}{g} + \frac{dg}{g}$ has at most two poles. By Lemma 1.1, this form has at least three zeros and so by (1.4) must vanish identically. □

Corollary 3.2. *Let Σ be an embedded periodic genus-one helicoid. Then there is a line ℓ normal to Σ so that rotation by $180°$ about ℓ acts as an orientation preserving isometry on Σ.*

Proof. The corollary follows from Lemma 3.1 and the arguments of Section 2 as long as we can rule out the existence of an embedded periodic genus-one helicoid with vertical flux. Notice that, by construction, the periods around E_1 and E_2 always have vertical flux. Suppose Σ is a periodic genus-one helicoid with vertical flux, asymptotic to some helicoid H. Let Σ_λ denote the family of surfaces given by the Lopez-Ros deformation. Necessarily, this family remains in the class of periodic genus-one helicoids and all have the same asymptotic behavior. Thus, outside of a bounded cylinder, each Σ_λ is embedded and asymptotic to H. Due to the periodicity, the non-compactness of the cylinder does not introduce additional difficulties. Clearly, (1.4) implies g has a pole or zero in $\mathbb{T}^2 \setminus \{E_1, E_2\}$ and so for $\lambda >> 1$, Σ_λ fails to be embedded. Hence, the Σ_λ satisfy the conclusions of Lemma 2.2 and so one obtains a contradiction exactly as in the proof of Theorem 0.1. □

Appendix A. Hyperelliptic case

In this appendix we complete the proof of Lemma 1.2. The arguments are of a rather different flavor than the rest of the paper and are a refinement of those used in [2] to show that $\frac{dg}{g}$ had no residue at ∞. We first recall the following elementary facts about level sets of harmonic functions:

Lemma A.1. *Let V be an open set in a Riemannian surface Σ with f a harmonic function on V. If $p \in V$ is a critical point of f and $t_0 = f(p)$ then:*

1. *There is a simply-connected neighborhood $U(p)$ of p so that $\{f = t_0\} \cap U(p)$ consists of $m + 1$ smooth embedded curves σ_i with $\partial\sigma_i \subset \partial U(p)$; m is the order of vanishing of f at p. The σ_i meet only at p and do so transversally.*
2. *There is a decomposition $\{f > t_0\} \cap U(p) = \Sigma^+(p) = \Sigma^+_1(p) \cup \ldots \cup \Sigma^+_{m+1}(p)$ and $\{f < t_0\} \cap U(p) = \Sigma^-(p) = \Sigma^-_1(p) \cup \ldots \cup \Sigma^-_{m+1}(p)$ so for $i \neq j$ the closure of $\Sigma^\pm_i$ meets the closure of $\Sigma^\pm_j$ only at p.*
3. *For $t > t_0$, the set $\Sigma^{\geq t}(p) = \{f \geq t\} \cap U(p)$ consists of $m + 1$ components each in a different component of $\Sigma^+(p)$.*

(4) *For each i, there is a piecewise smooth parameterization $\gamma_i : (-1,1) \to M$ of $\partial \Sigma_i^{\pm}(p) \cap U(p)$ and a sequence of smooth injective maps $\gamma_i^j : (-1,1) \to \Sigma_i^{\pm}(p)$ so that: on $(-1,-1/2) \cup (1/2,1)$, $\gamma_i^j = \gamma_i$ and $\gamma_i^j \to \gamma_i$ in C^0 as $j \to \infty$.*

We also note the following facts regarding the asymptotic properties of level curves of the height function of a genus-k helicoid:

Lemma A.2. *Given Σ an embedded genus-k helicoid, there is a cylinder:*

$$C = C_{h,R} = \left\{ |x_3| \leq h, x_1^2 + x_2^2 \leq R^2 \right\}$$

and a component Σ' of $C \cap \Sigma$ so that:

(1) *all critical points of $x_3 : \Sigma \to \mathbb{R}$ lie on the interior of Σ'.*
(2) *$\partial \Sigma' = \gamma_t \cup \gamma_b \cup \gamma_u \cup \gamma_d$, four smooth curves, with $x_3 = h$ on γ_t, $x_3 = -h$ on γ_b and for $s \in (-h,h)$, $\{x_3 = s\}$ meets γ_u and γ_d each in one point.*

Proof. As Σ is properly embedded there exist h and R so all the zeros and poles of g lie in the interior of the cylinder C. Moreover, by increasing the size of the cylinder one can take a component, Σ', of $\Sigma \cap C$ so that all of the zeros and poles of g lie in Σ' and, as Σ has one end, so that $\Sigma \backslash \Sigma'$ is an annulus. Finally, as Σ is asymptotic to some helicoid H, by further enlarging the cylinder, we may take Σ' so $\gamma = \partial \Sigma'$ is the union of four smooth curves, two at the top and bottom, γ_t and γ_b, and two disjoint helix like curves γ_u, γ_d so $\frac{d}{dt} x_3(\gamma_u(t)) > 0$ and $\frac{d}{dt} x_3(\gamma_d(t)) < 0$. □

Lemma A.3. *A point $p \in \Sigma$ is a pole of g if and only if $I(p)$ is a zero of g of the same order.*

Proof. First note that Lemma 1.2 implies $I^* dh = -dh$; thus, if p is a zero of order m of dh so is $I(p)$ and, up to a vertical translation, $x_3 \circ I = -x_3$. Recall, $g dh$ and $g^{-1} dh$ are holomorphic in Σ and do not simultaneously vanish, hence the order of a pole of g or zero of g at p is equal to the order of the zero of dh at p. Thus, it suffices to show that if p is a zero of g then $I(p)$ is a pole.

Let R, h, C and Σ' be as in Lemma A.2. It is a standard topological fact that a closed, oriented and connected surface in $\mathbb{R}^3$ divides $\mathbb{R}^3$ into two components. Thus, $C \backslash \Sigma'$ consists of two components Ω^+ and Ω^-, labeled so that the normal, $\mathbf{n}$, to Σ points into Ω^+. Denote by σ_t the set $\Sigma' \cap \{x_3 = t\}$ and by $\Omega_t^{\pm}$ the set $\Omega^{\pm} \cap \{x_3 = t\}$. A fact we will use below is that the closed sets $\bar{\Omega}^{\pm}$ are the complements of the union of open sets with smooth boundaries. Indeed, let $U_1 = U_1^{\pm}$ be the component of $\mathbb{R}^3 \backslash \Sigma$ containing $\Omega^{\mp}$, $U_2 = \left\{ x_1^2 + x_2^2 > R^2 \right\}$, $U_3 = \{x_3 > h\}$ and $U_4 = \{x_3 < -h\}$. Then all the U_i are open with smooth boundary and $\bar{\Omega}^{\pm} = \mathbb{R}^3 \backslash \cup_{i=1}^4 U_i$.

Consider a critical point p of x_3 with $t_0 = x_3(p)$ and x_3 vanishing to order m at p. At p the normal, $\mathbf{n}(p)$, is vertical. As a consequence, for $\epsilon = \epsilon(p)$ sufficiently small, near p, Σ is the graph of a function over the disk $D_\epsilon(p) \subset \{x_3 = t_0\}$. Equivalently, let $C_\epsilon(p) = \left\{ (q,t) \in \mathbb{R}^3 : q \in D_\epsilon \right\}$ be the vertical cylinder over p and π_p be the natural projection $\pi_p : C_\epsilon(p) \to D_\epsilon(p)$. Then there is a neighborhood $U(p)$ of p in Σ so that π_p restricted to $U(p)$ is a diffeomorphism onto $D_\epsilon(p)$. For ϵ small enough, $U(p)$ behaves with respect to x_3 as in Lemma A.1.

Let $\Sigma^{\pm}(p), \Sigma^{\geq t}(p) \subset \Sigma$ denote the sets given by Lemma A.1 (2), (3). As $\pi_p(U(p) \cap \{x_3 = t_0\}) = D_\epsilon(p) \cap \sigma_{t_0}$, if we let $\Omega^{\pm}(p) = \Omega_{t_0}^{\pm} \cap D_\epsilon(p)$ then either $\pi_p(\Sigma^+(p)) = \Omega^+(p)$ or $\pi_p(\Sigma^-(p)) = \Omega^+(p)$. We claim the parity of the identification is determined by whether the normal points up or down at p. Indeed, if the normal to Σ points up at p then, for $t > t_0$, $\pi_p(\Sigma^{\geq t}(p)) = \pi_p(\Omega_t^-)$. Letting $t \to t_0$ gives $\pi_p(\Sigma^+(p)) = \Omega^-(p)$. Conversely, if the normal to Σ points down at p then, for $t > t_0$, $\pi_p(\Sigma^{\geq t}(p)) = \pi_p(\Omega_t^+)$. Letting $t \to t_0$ gives $\pi_p(\Sigma^+(p)) = \Omega^+(p)$. We further claim that the identification at p determines the identification at $q = I(p)$. Indeed, the identification is reversed and so the normals point in opposite directions – in particular $q \neq p$. Figure 1 illustrates this for a simple critical point.

To verify this we suppose, without loss of generality, that $\Sigma^+(p)$ is identified with $\Omega^+(p)$ – i.e. the normal at p is down. We will show that $\Sigma^-(q)$ is then identified with $\Omega^+(q)$ – i.e. the normal is up. To begin the argument, we find a planar domain $B(p)$, a connected component of either $\Omega_{t_0}^+$ or $\Omega_{t_0}^-$, so that

(1) $p \in \partial B(p)$,
(2) $\partial B(p) \subset \sigma_{t_0}$,
(3) $D_\epsilon(p) \cap B(p)$ consists of exactly one connected component, $B_0(p)$.

We justify the existence of $B(p)$ as follows: there are at least four curves emanating from p in $\Sigma' \cap \{x_3 = t_0\}$ while $\partial\sigma_{t_0}$ consists of only two points. Hence, there is a connected component, A_1, of $\{x_3 = t_0\} \setminus \sigma_{t_0}$ with $p \in \partial A_1$ and $\partial A_1 \subset \sigma_{t_0}$. As A_1 satisfies (1) and (2), if $A_1 \cap D_\epsilon(p)$ has one component we are done. If there is more than one component, they all lie in either $\Omega^+(p)$ or $\Omega^-(p)$ and so cannot be adjacent. Thus, there is a simple closed curve τ_1 through p lying in the closure of A_1 which bounds a (topological) disk $\tilde{A}_1 \subset \{x_3 = t_0\}$ so that $\tilde{A}_1$ meets both $\Omega^+(p)$ and $\Omega^-(p)$. In particular, $\{x_3 = t\} \setminus A_1$ has a connected component, $A_2 \subset \tilde{A}_1 \setminus \sigma_{t_0}$, so $p \in \partial A_2$ and $\partial A_2 \subset \sigma_{t_0}$. If A_2 is not the desired set, the same method produces a set A_3 disjoint from $A_1 \cup A_2$, so $p \in \partial A_3$ and $\partial A_3 \subset \sigma_{t_0}$. Proceeding in this fashion, because there are only finitely many components of $D_\epsilon(p) \setminus \sigma_{t_0}$, one must eventually find $A_{i_0} = B(p)$ satisfying (3).

Without loss of generality, we suppose $B(p) \subset \Omega_{t_0}^-$. Label the components of $\partial B(p)$ as $\alpha^1(p), \ldots, \alpha^k(p)$ so $p \in \alpha^1(p)$, and let $\alpha_0^1(p) = D_\epsilon(p) \cap \partial B_0(p)$. There exists a single connected component $\Sigma_0^B(p) \subset \Sigma^-(p)$ with $\partial\Sigma_0^B(p) = \alpha_0^1(p)$ and $\pi_p(\Sigma_0^B(p)) = B_0(p)$. Let $I(\alpha^i(p)) = \alpha^i(q)$. We will show that the $\alpha^i(q)$ are the boundary of a connected planar domain $B(q) \subset \Omega_{-t_0}^-$ that satisfies (1), (2) and (3) (with p and t_0 replaced by q and $-t_0$). Notice *a priori* there need not be any planar domain in $\Omega_{-t_0}^-$ with boundary the $\alpha^i(q)$. We will be able to construct such a domain using topological properties of I and by solving an appropriate Plateau problem. Our argument exploits the existence of nice curves $\alpha_j^i(p)$ that are smoothly embedded is Σ and pairwise disjoint (for fixed j) and that converge to $\alpha^i(p)$ in a C^0 sense and in the flat metric. In particular, this allows us to think of the $\alpha^i(p)$ as cycles in Σ. The existence of the $\alpha_j^i(p)$ follows from (4) of Lemma A.1. As I is a diffeomorphism we may then set $\alpha_j^i(q) = I(\alpha_j^i(p))$ and obtain corresponding curves approximating the $\alpha^i(q)$. Using the $\alpha_j^i(p)$ and $\alpha_j^i(q)$ together with the maximum principle and the classification of surfaces we conclude that no collection of either the $\alpha^i(p)$ or of the $\alpha^j(q)$ can be null-homologous in Σ.

Recall the hyperelliptic involution negates homology classes of Σ, thus

$$\sum_{i=1}^{k} [\alpha^i(q)] = -\sum_{i=1}^{k} [\alpha^i(p)] \tag{A.1}$$

where $[\alpha^i(p)]$ and $[\alpha^i(q)]$ denote the class in $H_1(\Sigma, \mathbb{Z})$ of the cycles $\alpha^i(p)$ and $\alpha^i(q)$. As $\Sigma \setminus \Sigma'$ is an annulus, the inclusion map induces an isomorphism between

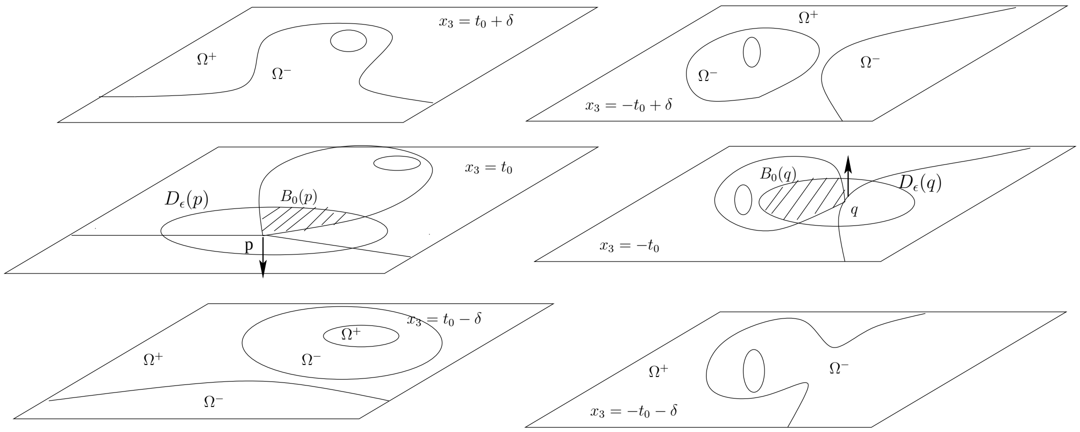

FIGURE 1. The left column shows level sets of x_3 near p. The right shows the same near $q = I(p)$. The shaded regions are $B_0(p)$ and $B_0(q)$.

$H_1(\Sigma', \mathbb{Z})$ and $H_1(\Sigma, \mathbb{Z})$. Inclusion also induces a map from $H_1(\Sigma', \mathbb{Z})$ to $H_1(\bar{\Omega}^-, \mathbb{Z})$. It follows that in $H_1(\bar{\Omega}^-, \mathbb{Z})$, $[\alpha^i(q)]$ and $[\alpha^i(p)]$ still satisfy (A.1). Thus, there is a chain C_0 in $\bar{\Omega}^-$ with $\partial C_0 = \sum_{i=1}^k \left(\alpha^i(p) + \alpha^i(q)\right)$. Notice $\overline{B(p)} \subset \bar{\Omega}^-$ may be thought of as a chain in $\bar{\Omega}^-$ with $\partial \overline{B(p)} = \sum_{i=1}^k \alpha^i(p)$. Hence, $C_0 - \overline{B(p)}$ is a chain in $\bar{\Omega}^-$ with $\partial(C_0 - \overline{B(p)}) = \sum_{i=1}^k \alpha^i(q)$. Solving a constrained Plateau problem gives a mass minimizing current B' in Ω^- with $\partial B' = \sum_{i=1}^k \alpha^i(q)$. By the convex hull property, $B(q) := spt(B') \subset \{x_3 = -t_0\}$ and is a union of connected planar domains. We claim $B(q)$ is our desired domain.

For the sake of completeness we first discuss why B' exists. As the $\alpha^i_j(q)$ converge in flat norm to $\alpha^i(q)$, for each j, the set of curves are null-homologous in $\bar{\Omega}^-$. Let B_j be the mass minimizing current in $\bar{\Omega}^-$ with $\partial B_j = \sum_i \alpha^i_j(q)$; such a B_j exists by direct methods. As $\bar{\Omega}^-$ is the complement of the union of open sets with smooth boundary, Proposition 6.1 and Theorem 6.2 of [13] imply that $spt(B_j) \backslash spt(\partial B_j)$ is a $C^{1,\alpha}$ surface in $\mathbb{R}^3$ for some $0 < \alpha < 1$. Thus, $spt(B_j)$ is disjoint from the singularities of $\partial \Omega^-$ and so we may apply the strong maximum principle of Solomon and White [11] to see that $spt(B_j) \backslash spt(\partial B_j)$ is either disjoint from $\partial \Omega^-$ or a subset of Σ. The latter case cannot occur, for if it did the $\alpha^i_j(q)$, and hence the $\alpha^i(q)$, would be null-homologous in Σ. Thus, $spt(B_j) \backslash spt(\partial B_j) \subset \Omega^-$. We recover B' from the B_j by letting $j \to \infty$ and using standard compactness theorems.

Let us now check that $B(q)$ is connected. Denote by $\hat{B}(q)$ the connected component of $B(q)$ with $\alpha^1(q) \subset \partial \hat{B}(q)$; if $B(q)$ is not connected $\hat{B}(q)$ is a proper subset of $B(q)$. In this case, up to a relabeling, $\partial \hat{B}(q) = \cup_{i=1}^{k'} \alpha^i(q)$ where $k' < k$. By the argument of the preceding two paragraphs, there is a a mass minimizing current B'' in Ω^- with $\partial B'' = \sum_{i=1}^{k'} \alpha^i(p)$. As above, the convex hull property implies $spt(B'') \subset \{x_3 = t_0\}$. This implies $B(p)$ is disconnected and so is impossible.

By construction, $q \in \alpha^1(q)$ and $\partial B(q) = \cup_i \alpha^i(q) \subset \sigma_{-t_0}$. Taking small enough values of ϵ (possibly differing at p and q) we may ensure $U(q) \subset I(U(p))$. Then $\alpha^1_0(q) = \alpha^1(q) \cap U(q) \subset I(\alpha^1(p) \cap U(p)) = I(\alpha^1_0(p))$. As any component of $B(q) \cap D_\epsilon(q)$ must have boundary containing $\alpha^1(q) \cap U(q)$ we conclude that $B(q) \cap D_\epsilon(q)$

has only one component, $B_0(q)$, satisfying $D_\epsilon(q) \cap \partial B_0(q) = \alpha_0^1(q)$. Hence, $B(q)$ is connected and satisfies (1), (2) and (3) as claimed.

Clearly,

$$U(q) \cap \partial I(\Sigma_0^B(p)) = U(q) \cap I(\alpha_0^1(p)) = \alpha_0^1(q)$$

and so

$$\pi_q(U(q) \cap I(\Sigma_0^B(p))) = B_0(q) \subset \Omega^-(q).$$

However, I flips the sign of x_3 and so

$$U(q) \cap I(\Sigma_0^B(p)) \subset \Sigma^+(q).$$

Hence the identification at q is reverse what it is at p; that is the normal points up rather than down. □

References

1. A. Alarcon, L. Ferrer, and F. Martin, *A uniqueness theorem for the singly periodic genus-one helicoid*, Trans. Amer. Math. Soc. **359** (2007), no. 6, 2819–2829.
2. J. Bernstein and C. Breiner, *Conformal structure of minimal surfaces with finite topology*, To Appear, Comment. Math. Helv. http://arxiv.org/abs/0810.4478.
3. A. I. Bobenko, *Helicoids with handles and Baker-Akhiezer spinors*, Math. Zeit. **229** (1998), no. 1, 9–29.
4. H. M. Farkas and I. Kra, *Riemann surfaces*, second ed., Graduate Texts in Mathematics, vol. 71, Springer-Verlag, New York, 1992.
5. L. Hauswirth, J. Perez, and P. Romon, *Embedded minimal ends of finite type*, Trans. Amer. Math. Soc. **353** (2001), no. 4, 1335–1370.
6. D. Hoffman, H. Karcher, and F. Wei, *The singly periodic genus-one helicoid*, Comment. Math. Hel. **74** (1999), 248–279.
7. D. Hoffman and B. White, *Genus-one helicoids from a variational point of view*, Comment. Math. Helv. **83** (2008), no. 4, 767–813.
8. W.H. Meeks III and H. Rosenberg, *The geometry of periodic minimal surfaces*, Comment. Math. Helv. **68** (1993), 538–578.
9. F. López and A. Ros, *On embedded complete minimal surfaces of genus zero*, J. Differential Geom. **33** (1991), no. 1, 293–300.
10. J. Pérez and A. Ros, *Some uniqueness and nonexistence theorems for embedded minimal surfaces*, Math. Ann. **295** (1993), no. 3, 513–525.
11. B. Solomon and B. White, *A strong maximum principle for varifolds that are stationary with respect to even parametric elliptic functionals,*, Indiana Univ. Math. J. **38** (1989), no. 3, 683–691.
12. M. Weber, D. Hoffman, and M. Wolf, *An embedded genus-one helicoid*, Ann. of Math. (2) **169** (2009), no. 2, 347–448.
13. B. White, *Existence of smooth embedded surfaces of prescribed genus that minimize parametric even elliptic functionals on* 3*-manifolds*, J. Differential Geom. **33** (1991), no. 2, 413–443.

Dept. of Math, Stanford University, Stanford, CA 94305, USA
E-mail address: jbern@math.stanford.edu

Dept. of Math, Massachusetts Institute of Technology, Cambridge, MA 02139, USA
E-mail address: breiner@math.mit.edu