

VECTOR SPACES OF SKEW-SYMMETRIC MATRICES OF CONSTANT RANK

MARIA LUCIA FANIA AND EMILIA MEZZETTI

ABSTRACT. We study the orbits of vector spaces of skew-symmetric matrices of constant rank $2r$ and type $(N + 1) \times (N + 1)$ under the natural action of $SL(N + 1)$, over an algebraically closed field of characteristic zero. We give a complete description of the orbits for vector spaces of dimension 2, relating them to some 1-generic matrices of linear forms. We also show that, for each rank two vector bundle on \mathbb{P}^2 defining a triple Veronese embedding of \mathbb{P}^2 in $\mathbb{G}(1, 7)$, there exists a vector space of 8×8 skew-symmetric matrices of constant rank 6 whose kernel bundle is the dual of the given rank two vector bundle.

INTRODUCTION

Vector spaces of skew-symmetric matrices of order $N+1$ and constant rank $2r$, with $N = 2r+1$, can be naturally interpreted as linear spaces contained in the $(r - 1)$ -th secant variety of the Grassmannian of lines $\mathbb{G}(1, N)$, not meeting the $(r - 2)$ -th secant variety, which is its singular locus. Therefore the special linear group $SL(N + 1)$ acts naturally on them and it is a natural problem to look for the maximal dimension of these spaces and to describe the orbits.

This problem has been considered both from the point of view of linear algebra and from that of algebraic geometry. An excellent survey of the results on the bounds, due to Ilic and Landsberg, is contained in [1], in the wider context of matrices, non-necessarily skew-symmetric ones. In particular, upper bounds are given for the dimensions of these spaces and a precise bound in a few cases.

As for the orbits of these vector spaces not much is known so far. In [2] the case of 6×6 skew-symmetric matrices of constant rank 4 was considered and the orbits were completely classified, up to the action of $SL(6)$. The point of view adopted by Manivel–Mezzetti is that of algebraic geometry: to a vector space M of matrices of constant rank $2r$ one can associate a vector bundle map

$$\phi_M : \mathcal{O}_{\mathbb{P}(M)}^{N+1} \longrightarrow \mathcal{O}_{\mathbb{P}(M)}(1)^{N+1}$$

on the projective space $\mathbb{P}(M)$. The kernel \mathcal{K} and the image \mathcal{E} are vector bundles of ranks respectively $N + 1 - 2r$ and $2r$, such that \mathcal{E} is generated by its global sections, $\mathcal{E} \simeq \mathcal{E}^*(1)$ and the splitting type of \mathcal{E} is $\mathcal{E}|_l = \mathcal{O}_l^r \oplus \mathcal{O}_l^r(1)$, for all lines $l \subset M$. In particular \mathcal{E} is uniform and $c_1(\mathcal{E}) = r = -c_1(\mathcal{K})$ (see Remark 1.2.)

The classification in [2] can be expressed in terms of vector bundles on \mathbb{P}^1 and \mathbb{P}^2 with $c_1 = 2$. Globally generated vector bundles on projective spaces with $c_1 = 2$ are completely described (see

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[3]), in particular on \mathbb{P}^2 there are 4 of such rank two bundles and for each of them there is an orbit of skew-symmetric matrices of constant rank 4 having it as dual of the kernel bundle \mathcal{K} .

In the present paper we continue this study pursuing two objectives: on one hand the classification of vector spaces M of dimension two, i.e. projective lines of skew-symmetric matrices of any order, on the other hand that of skew-symmetric matrices of rank 6, that correspond to vector bundles with $c_1 = 3$.

To classify the orbits of projective lines of skew-symmetric matrices of constant rank we rely on the fact that, in this case, congruence and strong equivalence of matrices are the same relation (see [4]). This allows us to restrict our attention to the “compression space” matrices introduced by Eisenbud–Harris in [5]. The classification of the orbits, given in Theorem 2.12, is similar to the one of the rational normal scrolls, and follows from a link we establish between our matrices and 1-generic matrices with two rows.

Globally generated vector bundles on projective spaces with $c_1 = 3$ which give a triple Veronese embedding of $\mathbb{P}(M)$ in $\mathbb{G}(1, N)$ have been studied in [6]. After refining such classification (see Theorem 4.1), we prove in Proposition 4.5 that the non split vector bundles given in Theorem 4.1 can all be expressed as quotient of vector bundles of higher rank of a very particular form. This turns out to be crucial in identifying some “building blocks” skew-symmetric matrices, that we use to construct matrices of constant rank 6 for each class of rank 2 bundles appearing in Theorem 4.1. This is done by suitably projecting some direct sum matrices constructed using the building blocks matrices of smaller rank (see Theorem 4.10). We note that the case of rank 6 is the first one in which infinitely many orbits appear.

As for 6×6 matrices, also in the case of 8×8 matrices there are no \mathbb{P}^3 's of matrices of rank two less than the order. The first example of such a situation was given by Westwick ([7]) and it is a \mathbb{P}^3 of 10×10 matrices of rank 8. We discuss this example in §5, where we also point out the applications to the classification of degenerations of an important class of projective varieties, known as Palatini scrolls (see [8], [9]).

1. DEFINITIONS AND PRELIMINARY RESULTS

Let V be a vector space of dimension $N+1$ over the field \mathbf{k} (algebraically closed of characteristic 0). We denote by $\mathbb{G}(1, N)$ the Grassmannian of the vector subspaces of V of dimension 2, i.e. the projective subspaces of $\mathbb{P}(V)$ of dimension 1. $\mathbb{G}(1, N)$ is embedded via the Plücker map in the projective space $\mathbb{P}(\wedge^2 V)$. The group $SL(N+1)$, as well as $PGL(N+1)$, acts naturally on $\mathbb{P}(V)$ and on $\mathbb{P}(\wedge^2 V)$. If we fix a basis on V then an element of $\wedge^2 V$ can be thought as a skew-symmetric matrix and the action of $SL(N+1)$ on $\mathbb{P}(\wedge^2 V)$ is the congruence. The orbits of the action on $\mathbb{P}(\wedge^2 V)$ are the Grassmannian and its secant varieties, and correspond respectively to the tensors of tensor rank $2, 4, \dots, \lfloor \frac{N+1}{2} \rfloor$. $SL(N+1)$ (as well as $PGL(N+1)$) acts also naturally on the Grassmannian of the subspaces of $\mathbb{P}(\wedge^2 V)$ of any fixed dimension $d \leq \binom{N+1}{2}$, or, in other words, on the skew-symmetric $(N+1) \times (N+1)$ matrices of linear forms. For this action, we are interested in describing the orbits of subspaces of constant rank, i.e. subspaces that are entirely contained in some orbit of the previous action on $\mathbb{P}(\wedge^2 V)$. Note that the orbits of the action given by $SL(N+1)$ coincide with the orbits of the action given by $PGL(N+1)$.

From now on, M will denote a vector space of skew-symmetric matrices of order $N+1$, dimension d and constant rank $\text{rk } M = 2r$.

If N is odd and $2r$ is maximal, that is, it is equal to $N+1$, then $d \leq 1$, because the matrices of submaximal rank form a hypersurface of degree $\frac{N+1}{2}$, the Pfaffian, in $\mathbb{P}(\wedge^2 V)$. So we will

assume that either N is odd and $2r$ is strictly less than $N + 1$, or N is even; in this last case the maximal rank is N and the matrices of rank $N - 2$ have codimension 3.

Given a vector space M of matrices of constant rank $2r$ we can associate a vector bundle map

$$(1.1) \quad \phi_M : V \otimes \mathcal{O}_{\mathbb{P}(M)} \longrightarrow V \otimes \mathcal{O}_{\mathbb{P}(M)}(1).$$

on the projective space $\mathbb{P}(M)$. Since M has constant rank then the kernel \mathcal{K} , the image \mathcal{E} and the cokernel \mathcal{N} of ϕ_M are vector bundles of ranks respectively $\text{rk } \mathcal{K} = \text{rk } \mathcal{N} = N + 1 - 2r$, $\text{rk } \mathcal{E} = 2r$ and determine short exact sequences:

$$(1.2) \quad 0 \rightarrow \mathcal{K} \rightarrow V \otimes \mathcal{O}_{\mathbb{P}(M)} \rightarrow \mathcal{E} \rightarrow 0$$

$$(1.3) \quad 0 \rightarrow \mathcal{E} \rightarrow V \otimes \mathcal{O}_{\mathbb{P}(M)}(1) \rightarrow \mathcal{N} \rightarrow 0$$

Proposition 1.1. ([5], [1])

- (1) \mathcal{E} is generated by its global sections;
- (2) $\mathcal{E} \simeq \mathcal{E}^*(1)$; $\mathcal{N} \simeq \mathcal{K}^*(1)$
- (3) the splitting type of \mathcal{E} is $\mathcal{E}|_l = \mathcal{O}_l^r \oplus \mathcal{O}_l^r(1)$, for all lines $l \subset M$. In particular \mathcal{E} is uniform.

Remark 1.2. Note that since the matrices are skew-symmetric the two short exact sequences (1.2) and (1.3) reduce to the single sequence

$$(1.4) \quad 0 \rightarrow \mathcal{K} \rightarrow V \otimes \mathcal{O}_{\mathbb{P}(M)} \rightarrow \mathcal{E} \rightarrow 0$$

with $\mathcal{E} \simeq \mathcal{E}^*(1)$. Moreover $c_1(\mathcal{E}) = r = -c_1(\mathcal{K})$ and \mathcal{K}^* is generated by global sections.

Remark 1.3. In the case $N = 2r + 1$, M is contained in $S_{r-1}\mathbb{G}(1, N)$, the top secant variety of $\mathbb{G}(1, N)$ strictly contained in $\mathbb{P}(\wedge^2 V)$. It is naturally isomorphic to $\overset{\vee}{\mathbb{G}}(1, N)$, the dual of $\mathbb{G}(1, N)$, which is the Pfaffian hypersurface. Hence the Gauss map

$$(1.5) \quad \gamma : \overset{\vee}{\mathbb{G}}(1, N) \dashrightarrow \mathbb{G}(1, N)$$

is defined by the partial derivatives of the Pfaffian, which are homogeneous polynomials of degree r .

Proposition 1.4. Let M be a vector space of dimension d of $(N + 1) \times (N + 1)$ matrices of constant rank $2r$, with $N = 2r + 1$. Let γ be the Gauss map in (1.5). Then $\gamma(\mathbb{P}(M))$ is a Veronese variety $v_r(\mathbb{P}^{d-1})$ contained in the Grassmannian $\mathbb{G}(1, N)$, or an isomorphic projection of it.

Proof. The restriction of γ to $\mathbb{P}(M)$ is regular, due to the hypothesis of constant rank, because $\mathbb{P}(M)$ does not intersect $S_{r-2}\mathbb{G}(1, N)$, which is the indeterminacy locus of γ . It remains to prove that $\gamma|_{\mathbb{P}(M)}$ is biregular onto its image, i.e. that $\mathbb{P}(M)$ intersects a general fibre of γ in only one point. Let l be a point of $\mathbb{G}(1, N)$; the fibre $\gamma^{-1}(l) \subset \overset{\vee}{\mathbb{G}}(1, N)$ can be interpreted as the set of hyperplanes containing the projective tangent space to the Grassmannian at l , \mathbb{T}_l ; hence $\gamma^{-1}(l) = \overset{\vee}{\mathbb{T}}_l$ is a linear space and $\gamma^{-1}(l) \cap \mathbb{P}(M)$ is also linear.

We can choose a basis e_0, \dots, e_N of V such that $l = \langle e_0, e_1 \rangle$; then the points in \mathbb{T}_l have Plücker coordinates p_{ij} , $0 \leq i < j \leq N$ such that $p_{ij} = 0$ for all $i \geq 2$. Therefore $\overset{\vee}{\mathbb{T}}_l$ is represented by matrices a_{ij} whose first two rows and columns are zero; it can be seen as the linear span of a subgrassmannian $\mathbb{G}(1, N - 2)$, and the matrices of rank $2r$ are an open set in it

whose complementar set is a hypersurface. If $\mathbb{P}(M)$ intersects \mathbb{T}_l in positive dimension, then its intersection with this hypersurface is non-empty, so the rank is non-constant, a contradiction. Hence the map $\gamma : \mathbb{P}(M) \rightarrow \mathbb{G}(1, N)$ is an embedding (γ denotes also $\gamma|_{\mathbb{P}(M)}$). \square

Facts 1.5. Let us recall some facts about embeddings in Grassmannians of lines (for details we refer to [10]). Let X be a smooth algebraic variety. To give a map $\varphi : X \rightarrow \mathbb{G}(1, N)$ is equivalent to give a rank 2 vector bundle \mathcal{F} on X and an epimorphism $V \otimes \mathcal{O}_X \rightarrow \mathcal{F} \rightarrow 0$, where V is an $(N + 1)$ -dimensional subspace of $H^0(X, \mathcal{F})$. The map φ is an embedding if any subscheme of X of length two imposes at least three conditions to V . Given an embedding in a Grassmannian $\varphi : X \rightarrow \mathbb{G}(1, N)$ there is a ruled variety obtained by taking the union of all lines defined by the points of X . Consider $Y = \mathbb{P}(\mathcal{F})$ the projective bundle associated to \mathcal{F} , let $\pi : Y \rightarrow X$ be the natural projection and let $\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$ be the tautological line bundle on Y , which has the property that $\pi_*(\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)) = \mathcal{F}$ and there is an epimorphism

$$\pi^*(\mathcal{F}) \rightarrow \mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$$

which induces an isomorphism $H^0(X, \mathcal{F}) \cong H^0(Y, \mathcal{O}_{\mathbb{P}(\mathcal{F})}(1))$. Hence an epimorphism $V \otimes \mathcal{O}_X \rightarrow \mathcal{F}$ induces an epimorphism $V \otimes \mathcal{O}_Y \rightarrow \pi^*(\mathcal{F}) \rightarrow \mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$. And viceversa, an epimorphism $V \otimes \mathcal{O}_Y \rightarrow \mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$ induces an epimorphism $V \otimes \mathcal{O}_X \rightarrow \pi_*(\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)) = \mathcal{F}$. Thus a map $\varphi : X \rightarrow \mathbb{G}(1, N)$ is equivalent to a map $\bar{\varphi} : Y \rightarrow \mathbb{P}^N$ of the corresponding ruled variety. In general $\bar{\varphi}$ is not an embedding.

If the variety X is two dimensional then $Y = \mathbb{P}(\mathcal{F})$ is a 3-fold and $\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)^3 = c_1(\mathcal{F})^2 - c_2(\mathcal{F})$ and hence we have

$$(1.6) \quad c_1(\mathcal{F})^2 - c_2(\mathcal{F}) = \deg(\bar{\varphi}) \cdot \deg \bar{\varphi}(Y)$$

Recall that it is also known the correspondence between globally generated vector bundle of rank k and maps to Grassmannians. In fact to give a regular map $\varphi_V : X \rightarrow \mathbb{G}(k - 1, N)$ is equivalent to give a globally generated vector bundle \mathcal{F} of rank k on X and an epimorphism $V \otimes \mathcal{O}_X \rightarrow \mathcal{F} \rightarrow 0$, where V is an $(N + 1)$ -dimensional subspace of $H^0(X, \mathcal{F})$. Moreover, for any epimorphism $V \otimes \mathcal{O}_X \rightarrow \mathcal{F} \rightarrow 0$ we get an exact sequence of vector bundles

$$(1.7) \quad 0 \rightarrow \mathcal{G} \rightarrow V \otimes \mathcal{O}_X \rightarrow \mathcal{F} \rightarrow 0$$

hence $V^* \otimes \mathcal{O}_X \rightarrow \mathcal{G}^* \rightarrow 0$ is also an epimorphism and thus the pair (\mathcal{G}^*, V^*) defines a map $\phi_{V^*} : X \rightarrow \mathbb{G}(N - k, N)$. The map ϕ_{V^*} is said the dual map of φ_V . It follows that $\varphi_V(X)$ and $\phi_{V^*}(X)$ are naturally isomorphic by the duality $\mathbb{G}(k - 1, N) \cong \mathbb{G}(N - k, N)$, see [3] for details.

Proposition 1.6. *If M is a vector space of $(N + 1) \times (N + 1)$ matrices of constant rank $2r$, with $2r = N - 1$, then \mathcal{K}^* is a vector bundle of rank 2 which defines an r -tuple embedding of $\mathbb{P}(M)$ in $\mathbb{G}(1, N)$.*

Proof. It is enough to note that the Gauss map (1.5) restricted to $\mathbb{P}(M)$ is given by the rank 2 bundle \mathcal{K}^* on $\mathbb{P}(M)$ and the epimorphism $V \otimes \mathcal{O}_{\mathbb{P}(M)} \rightarrow \mathcal{K}^*$ is obtained by dualizing (1.4). \square

2. VECTOR SPACES OF SKEW-SYMMETRIC MATRICES OF DIMENSION TWO

In this section we study the first non-trivial case of vector space of skew-symmetric matrices of constant rank $2r$ and of order $N + 1$, that is the case in which $\dim M = 2$. We will give a complete classification of the orbits and an explicit description of the corresponding matrices. This will be possible because the vector bundles on \mathbb{P}^1 are all decomposable.

We recall the notion of strictly equivalent pencils of matrices.

Definition 2.1. Two pencils of matrices $aA + bB$ and $aA_1 + bB_1$ are called strictly equivalent if there exist two non singular matrices P, Q with entries in \mathbf{k} such that

$$P(aA + bB)Q = aA_1 + bB_1$$

The following well known Theorem says, in particular, that for pencils of skew-symmetric matrices the notion of “strictly equivalent” coincides with the notion of “congruent”.

Theorem 2.2. ([4], Chap. XII, Theorem 6) *Two strictly equivalent pencils of complex symmetric (or skew-symmetric) matrices are always congruent.*

This theorem allows to use, in the case of $\dim M = 2$, the result obtained by Eisenbud-Harris in [5], where they consider the classification of vector spaces of matrices of linear forms for the relation of strict equivalence. We recall some terminology from [5].

Definition 2.3. Let M be a vector space of matrices of constant rank. M is *nondegenerate* if the kernels of the matrices of M intersect in the zero subspace and the images of the elements of M generate the vector space V .

This is equivalent to say that M is not $SL(N+1)$ -equivalent to a space of matrices with a row or a column of zeroes. In other words N is the minimum integer such that M can be embedded in $\langle \mathbb{G}(1, N) \rangle$.

From now on we will consider only nondegenerate vector spaces of $(N+1) \times (N+1)$ matrices of constant rank $2r$.

Definition 2.4. Let M be a vector space of matrices of constant rank $2r$. M is a *compression space* if there exist subspaces $V', W' \subset V$, such that every matrix in M maps V' into W' and $\text{rk } M = 2r = \text{codim } V' + \dim W'$.

It is easy to see, by an appropriate choice of basis of V , that M is a compression space if and only if it is $SL(N+1)$ -equivalent to a space of $(N+1) \times (N+1)$ matrices having a common block of zeroes of size $(N+1-h) \times (N+1-k)$ with $h+k=2r$.

Proposition 2.5. ([5], Corollary 2.2) *If $\dim M = 2$ then M is a compression space.*

The kernel bundle \mathcal{K} of the map ϕ_M (see (1.1)) is of the following form

$$(2.1) \quad \mathcal{K} = \mathcal{O}_{\mathbb{P}^1}^{m_0} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^{m_1} \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(-k)^{m_k}$$

where m_0, m_1, \dots, m_k are non-negative integers.

Proposition 2.6. *If M is nondegenerate, $\text{rk } M = 2r$ and $\dim M = 2$, then $2r \leq N \leq 3r - 1$.*

Proof. Comparing the ranks and the first Chern classes of the bundles appearing in (1.4), we get: $m_0 + m_1 + \cdots + m_k = N + 1 - 2r$ and $m_1 + 2m_2 + \cdots + km_k = r$. The assumption that M is nondegenerate implies moreover that $m_0 = 0$. The thesis follows by computing the dimensions of the cohomology groups of (1.4), taking into account that $\text{rk } \mathcal{K} \geq 1$. \square

The splitting type of \mathcal{K}^* is a partition of r of the form $r = r_1 + r_2 + \cdots + r_h$. From Proposition 2.6, it follows that the length h of the partition is $h = N + 1 - 2r$.

Conversely, for each $r \geq 1$ and each partition of r , we shall exhibit a pencil of matrices having it as splitting type of the associated bundle \mathcal{K}^* . All the corresponding matrices will have a $(N+1-r) \times (N+1-r)$ block of zeroes.

Let us start with a few examples. We fix a basis e_0, \dots, e_N for the vector space V and let M be a pencil of skew-symmetric matrices. Then a general matrix in M is of the form $aA + bB$, with A, B skew-symmetric matrices with constant entries and $a, b \in \mathbf{k}$.

Examples 2.7.

We will always write our skew-symmetric matrices indicating only the entries on the strict upper triangular part.

- $r = 1$

In this case $\text{rk } M = 2$, there is only one orbit, M is contained in $\mathbb{G}(1, 2)$ and corresponds to a pencil of lines in the plane. An element of the orbit is the following 3×3 matrix:

$$\begin{pmatrix} a & b \\ & 0 \end{pmatrix}$$

- $r = 2$

In this case $\text{rk } M = 4$, there are two orbits, corresponding to the bundles $\mathcal{O}_{\mathbb{P}^1}(2)$ and $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$, formed by matrices of order 5×5 , resp. 6×6 (see [2], §3):

$$\begin{pmatrix} 0 & a & b & 0 \\ & 0 & a & b \\ & & 0 & 0 \\ & & & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & a & b & 0 & 0 \\ & 0 & 0 & a & b \\ & & 0 & 0 & 0 \\ & & & 0 & 0 \\ & & & & 0 \end{pmatrix}$$

- $r = 3$

In this case $\text{rk } M = 6$. We give three examples which we denote by M_7, M_8, M_9 , respectively. They correspond to the bundles $\mathcal{O}_{\mathbb{P}^1}(3)$, $\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$, $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$; the orders of the matrices are 7×7 , 8×8 and 9×9 .

$$M_7 = \begin{pmatrix} 0 & 0 & a & b & 0 & 0 \\ & 0 & 0 & a & b & 0 \\ & & 0 & 0 & a & b \\ & & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix}, M_8 = \begin{pmatrix} 0 & 0 & a & b & 0 & 0 & 0 \\ & 0 & 0 & a & b & 0 & 0 \\ & & 0 & 0 & 0 & a & b \\ & & & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 \\ & & & & & 0 & 0 \\ & & & & & & 0 \end{pmatrix}, M_9 = \begin{pmatrix} 0 & 0 & a & b & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & a & b & 0 & 0 \\ & & 0 & 0 & 0 & 0 & a & b \\ & & & 0 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & & 0 & 0 & 0 \\ & & & & & & 0 & 0 \\ & & & & & & & 0 \\ & & & & & & & 0 \end{pmatrix}$$

Theorem 2.8. *Let $r = r_1 + r_2 + \dots + r_h$ be a partition of r , $r_1 \geq r_2 \geq \dots \geq r_h$. There exists an orbit of pencils of skew-symmetric matrices of constant rank $2r$ and order $N + 1$ with $N = 2r + h - 1$, whose associated bundle \mathcal{K}^* has splitting type (r_1, \dots, r_h) .*

Proof. In the orbit there is a matrix F of the following type.

$$(2.2) \quad F = \begin{pmatrix} 0_r & \overline{F} \\ -\overline{F}^t & 0_{N+1-r} \end{pmatrix}$$

where 0_k denotes the zero matrix of order k and \overline{F} is a block matrix of the form

$$(2.3) \quad \overline{F} = \begin{pmatrix} U_{r_1} & & & 0 \\ & U_{r_2} & & \\ & & \ddots & \\ 0 & & & U_{r_h} \end{pmatrix}.$$

For every $i = 1, \dots, h$, U_{r_i} is of type $r_i \times (r_i + 1)$ and

$$(2.4) \quad U_{r_i} = \begin{pmatrix} a & b & 0 & 0 & \cdots & . & . & 0 \\ 0 & a & b & 0 & \cdots & . & . & 0 \\ 0 & 0 & a & b & \cdots & . & . & 0 \\ & & & & \ddots & & & \\ . & . & . & . & . & a & b & 0 \\ . & . & . & . & . & . & a & b \end{pmatrix}$$

The resulting matrix F has clearly constant rank $2r$ and the associated bundle \mathcal{K}^* is as required. \square

We recall few facts about 1-generic matrices which will be used in classifying pencils of skew-symmetric matrices of constant rank. We refer to [11] for the definition and properties of 1-generic matrices.

Definition 2.9. Let Ω be a matrix of linear forms on \mathbb{P}^n . We say that Ω is 1-generic if no matrix of linear forms conjugate to Ω has a zero entry.

Proposition 2.10. ([11, Prop. 9.12 and its generalization])

- (1) Any 1-generic $2 \times (n - 1)$ matrix Ω of linear forms on \mathbb{P}^n is conjugate for some ℓ to the matrix

$$\Omega_0 = \begin{pmatrix} z_0 & \cdots & z_{\ell-1} & z_{\ell+1} & \cdots & z_{n-1} \\ z_1 & \cdots & z_\ell & z_{\ell+2} & \cdots & z_n \end{pmatrix}$$

where z_0, \dots, z_n are homogeneous coordinates on \mathbb{P}^n .

- (2) Let Ω be a 1-generic $2 \times k$ matrix of linear forms on \mathbb{P}^n , $k \leq n - 1$, whose entries span V^* . Then for some sequence of integers a_1, \dots, a_ℓ ($\ell = n - k$), Ω is conjugate to the matrix

$$\Omega_a = \left(\begin{array}{ccc|ccc|ccc|ccc} z_0 & \cdots & z_{a_1-1} & z_{a_1+1} & \cdots & z_{a_2-1} & z_{a_2+1} & \cdots & z_{a_\ell-1} & z_{a_\ell+1} & \cdots & z_{a_n-1} \\ z_1 & \cdots & z_{a_1} & z_{a_1+2} & \cdots & z_{a_2} & z_{a_2+2} & \cdots & z_{a_\ell} & z_{a_\ell+2} & \cdots & z_{a_n} \end{array} \right),$$

that is to a matrix consisting of $\ell + 1$ blocks of size $2 \times a_1, \dots, 2 \times (n - a_\ell - 1)$ with each block a catalecticant, that is a matrix in which $a_{i,j+1} = a_{i+1,j}$ for all i, j .

Remark 2.11. Computing the kernel of the matrix F in (2.2), one gets the family of $(h - 1)$ -spaces of a rational normal scroll of type (r_1, \dots, r_h) .

In fact if we write the vector $X = (x_0, \dots, x_{r-1}, x_r, \dots, x_N)$ as $X = (X^r, \overline{X})$, where $X^r = (x_0, \dots, x_{r-1})$, $\overline{X} = (x_r, \dots, x_N)$, then

$$\begin{aligned} \ker(F) &= \{X = (x_0, \dots, x_{r-1}, x_r, \dots, x_N) \mid F \cdot X = 0\} \\ &= \{X = (X^r, \overline{X}) \mid \overline{F} \cdot \overline{X} = 0 \text{ and } \overline{F}^t \cdot X^r = 0\} \end{aligned}$$

From $\overline{F}^t \cdot X^r = 0$ we get that $x_0 = \cdots = x_{r-1} = 0$.

Moreover, from (2.3) we deduce that $\overline{F} \cdot \overline{X} = 0$ is equivalent to

$$(2.5) \quad U_{r_1} X_{r_1} = 0, \dots, U_{r_h} X_{r_h} = 0,$$

where $X_{r_1} = (x_r, \dots, x_{r+r_1})$, \dots , $X_{r_h} = (x_{r+r_1+r_2+\dots+r_{h-1}+h-1}, \dots, x_{r+r_1+r_2+\dots+r_h+h-1}) = (x_{r+r_1+r_2+\dots+r_{h-1}+h-1}, \dots, x_N)$. Hence the conditions defining $\ker(F)$ are equivalent to the fact that the following matrix has rank 1:

$$\Omega(\overline{X}) = \left(\begin{array}{cccc} x_r & x_{r+1} & \cdots & x_{r+r_1-1} \\ x_{r+1} & x_{r+2} & \cdots & x_{r+r_1} \end{array} \left| \begin{array}{ccc} x_{r+r_1+1} & x_{r+r_1+2} & \cdots \\ x_{r+r_1+2} & x_{r+r_1+3} & \cdots \end{array} \right| \begin{array}{c} \cdots \\ \cdots \end{array} \right).$$

The matrix $\Omega(\overline{X})$ consists of h blocks of size $2 \times r_1, \dots, 2 \times r_h$, respectively, with each block catalecticant. If we denote such blocks by $\Omega_i(X)$, with $i = 1, \dots, h$, each block gives a rational normal scroll, that is the determinantal variety defined by $\text{rank } \Omega_i(X) = 1$.

Theorem 2.12. *Let M be a nondegenerate vector space of dimension 2 of matrices of constant rank $2r$ and order $N + 1$. Then M is $SL(N + 1)$ -equivalent by congruence and strict equivalence to one of the matrices of Theorem 2.8.*

Proof. If we think of M as a subspace of $\wedge^2 V$, we can write it as $M = \langle \omega, \omega' \rangle$, where ω, ω' are tensors of tensor rank r whose linear combinations have all rank equal to r . There exist expressions

$$(2.6) \quad \omega = u_0 \wedge v_0 + \cdots + u_{r-1} \wedge v_{r-1}, \quad \omega' = z_0 \wedge w_0 + \cdots + z_{r-1} \wedge w_{r-1}.$$

Let L and L' be the subspaces of V generated by the vectors $u_0, v_0, \dots, u_{r-1}, v_{r-1}$ and $z_0, w_0, \dots, z_{r-1}, w_{r-1}$, respectively. Note that $\dim L = \dim L' = 2r$, because $\omega \in \wedge^2 L$, $\omega' \in \wedge^2 L'$ and $2r$ is the minimal dimension of a vector space such that there exist skew-symmetric tensors of tensor rank r . So the given generators are linearly independent.

Since M is a compression space by Proposition 2.5, there exist vector subspaces V', W' of V such that every matrix in M maps V' into W' and $\text{rk } M = 2r = \text{codim } V' + \dim W'$.

The strategy of the proof is to first analyze the case $\text{codim } V' = \dim W' = r$: we will show that M is in the orbit of one of the examples of Theorem 2.8. Then we will consider the other possibilities for $\text{codim } V'$ and $\dim W'$ and will prove that they cannot occur.

- Assume $\text{codim } V' = \dim W' = r$.

We can choose a basis of V such that $V' = \langle e_r, \dots, e_N \rangle$ and $W' = \langle e_0, \dots, e_{r-1} \rangle$. This means that the submatrix of the last $N + 1 - r$ rows and columns is the zero matrix. Therefore in the expression (2.6) we have $u_0, \dots, u_{r-1}, z_0, \dots, z_{r-1} \in W'$. Possibly changing basis in W' , we can assume that

$$(2.7) \quad \omega = e_0 \wedge v_0 + \cdots + e_{r-1} \wedge v_{r-1}.$$

Therefore $z_j = \sum_{i=0}^{r-1} \lambda_{ji} e_i$, for all j and suitable scalars λ_{ji} . Using bilinearity, we can assume that ω' has the form

$$(2.8) \quad \omega' = e_0 \wedge w_0 + \cdots + e_{r-1} \wedge w_{r-1}.$$

Let \overline{M} be a matrix in M , it can be written as $aQ + bP$, where Q and P are the matrices representing ω and ω' respectively. Let us denote its general element by $aq_{ij} + bp_{ij}$. Moreover \overline{M} has the following form:

$$(2.9) \quad \begin{pmatrix} M' & \tilde{M} \\ -\tilde{M}^t & 0 \end{pmatrix}$$

where M' and \tilde{M} are matrices of linear forms in a, b of size respectively $r \times r$ and $r \times (r + h)$. Note that \tilde{M} has maximal rank r for every pair (a, b) . For each (a, b) , the vectors of $\ker \tilde{M}$ have the form $(0, \dots, 0, x_r, \dots, x_N)$, where (x_r, \dots, x_N) belongs to the kernel of \tilde{M} , which is a vector space of dimension $N - 2r + 1 = h$. Letting (a, b) vary, we obtain a variety Y of dimension h in \mathbb{P}^{N-r} covered by linear spaces (see Facts 1.5). The equations of Y are the 2×2 minors of the $2 \times r$ matrix

$$(2.10) \quad \Pi = \begin{pmatrix} \tilde{Q}_1 X' & \cdots & \tilde{Q}_r X' \\ \tilde{P}_1 X' & \cdots & \tilde{P}_r X' \end{pmatrix}$$

where \tilde{Q} and \tilde{P} are for \tilde{M} the analogous of Q and P for \overline{M} , and X' is the column matrix with entries x_r, \dots, x_N ; moreover $\tilde{Q}_1, \dots, \tilde{Q}_r$ are the rows of \tilde{Q} and similarly for \tilde{P} . The matrix Π is 1-generic because \tilde{M} has constant rank r and its entries generate $(\mathbf{k}^{N-r-1})^*$ because \overline{M} is nondegenerate. Hence Y is a rational normal scroll in \mathbb{P}^{N-r} .

By Proposition 2.10 it follows that Π is conjugate to a matrix Π' with h blocks:

$$(2.11) \quad \Pi' = (\Pi_1 \quad \Pi_2 \quad \cdots \quad \Pi_h)$$

where each Π_i is a catalecticant matrix. This means that Π' is obtained from Π by suitably multiplying it at the left and at the right by invertible scalar matrices.

By direct computations, one checks that left multiplication of Π by a 2×2 matrix corresponds to changing generators for the pencil $aQ + bP$, and right multiplication by a $r \times r$ matrix corresponds to changing the last $N - r + 1$ vectors of the basis of V . This shows that the matrix \overline{M} is equivalent to a matrix of the desired form.

- Assume $\text{codim } V' = r - k, \text{dim } W' = r + k, k > 0$.

We choose a basis (e_0, \dots, e_N) of V such that $V' = \langle e_{r-k}, \dots, e_N \rangle$ and $W' = \langle e_0, \dots, e_{r+k-1} \rangle$. In view of skew-symmetry, the matrix \overline{M} is concentrated in the first $r - k$ rows and columns, except for a $2k \times 2k$ skew-symmetric submatrix D in the rows and columns of indices $r - k, \dots, r + k - 1$, as sketched in (2.12):

$$(2.12) \quad \overline{M} = \begin{pmatrix} A & B & C \\ -B^t & D & 0 \\ -C^t & 0 & 0 \end{pmatrix}$$

Note that, since $\text{rk } \overline{M} = 2r$, necessarily $\text{rk } C = r - k$ and $\text{rk } D = 2k$. But D is a skew-symmetric matrix of order $2k$ whose entries are linear forms in a, b , hence its Pfaffian vanishes for some pair $(a, b) \neq (0, 0)$ (because the base field \mathbf{k} is algebraically closed). This contradicts the assumption that the rank of \overline{M} is constant and equal to $2r$. \square

Corollary 2.13. *The orbits of vector spaces of dimension two of matrices of constant rank $2r$ and order $N + 1$ are the ones of Theorem 2.12 and those of nondegenerate spaces of lower order with a suitable number of zero rows and columns added.*

2.1. An algorithm to compute the dimension of the orbits. It would be interesting to compute the dimension of the orbits. In the case of lines of skew-symmetric matrices of rank 4, that is $r = 2$, this has been done in [2].

To compute the dimension for $r \geq 3$, we use the computer algebra system Macaulay2 ([12]) with the script suggested to us by Giorgio Ottaviani, and we thank him for this.

We consider here the examples of the orbits with $r = 3$. We write down the case M_7 . For $M_7', M_7'', M_8, M_8', M_9$ one makes the appropriate changes. For $i = 7, 8$, with M_i' we denote the

matrix obtained from M_i by adding one row and one column (the last ones) of zeroes and with M_i'' we denote the matrix obtained from M_i by adding two rows and two columns (the last ones) of zeroes.

```

R=QQ[a,b]
-M is our matrix
N=map(R^7,R^7,{(0,3)=>a,(1,4)=>a,(2,5)=>a,(0,4)=>b,(1,5)=>b,(2,6)=>b})
M=N-transpose(N)
P=(M){0}_{1..6}
for s from 1 to 5 do P=P|(M){s}_{(s+1)..6}
- we create P with 21 components which represents the matrix M
for i from 0 to 6 do
for j from 0 to 6 do
E_{(i,j)}=map(R^7,R^7,{(i,j)=>1_R})
- E_{(i,j)} are the elementary matrices
W=(transpose(E_{(0,0)})*M+M*E_{(0,0)}){0}_{1..6}
for s from 1 to 5 do W=W|(transpose(E_{(0,0)})*M+M*E_{(0,0)}){s}_{(s+1)..6}
WW1=sub(P,{a=>1_R,b=>0_R})||sub(W,{a=>0_R,b=>1_R})
WW2=sub(W,{a=>1_R,b=>0_R})||sub(P,{a=>0_R,b=>1_R})
Z=exteriorPower(2,WW1)+exteriorPower(2,WW2)
- now Z has 210 components and represents the derivative of the action of E_{(0,0)}
- with the following commands we repeat the above for 49 times obtaining a matrix 50 x 210,
where the first two rows are equal
for i from 0 to 6 do for j from 0 to 6 do
{W=(transpose(E_{(i,j)})*M+M*E_{(i,j)}){0}_{1..6},
for s from 1 to 5 do W=W|(transpose(E_{(i,j)})*M+M*E_{(i,j)}){s}_{(s+1)..6},
WW1=sub(P,{a=>1_R,b=>0_R})||sub(W,{a=>0_R,b=>1_R}),
WW2=sub(W,{a=>1_R,b=>0_R})||sub(P,{a=>0_R,b=>1_R}),
Z=Z||exteriorPower(2,WW1)+exteriorPower(2,WW2) }
Z;
rank(Z)
- rank(Z) represents the affine dimension of the orbit.
In this case we get rank(Z)=39.

```

Recall that $N = \dim \mathbb{P}(V)$.

If $N = 6$ there is only one orbit. We get that the orbit \mathcal{O}_7 of M_7 has $\dim \mathcal{O}_7 = 38$. Hence it is open in $\mathbb{G}(1, \mathbb{P}(\wedge^2(\mathbb{C}^7)))$ and its complementary is formed by the lines which intersect $S_1(\mathbb{G}(1, 6))$.

If $N = 7$ there are two orbits. One, \mathcal{O}'_7 , corresponding to M'_7 . The other, \mathcal{O}_8 , corresponding to M_8 .

We get that $\dim \mathcal{O}_8 = 47$. This is the expected dimension. Indeed $\dim \mathbb{G}(1, \mathbb{P}(\wedge^2(\mathbb{C}^8))) = 52$ and $\deg S_2 \mathbb{G}(1, 7) = 4$, because its equation is the Pfaffian of a 8×8 matrix of linear forms. As for the dimension of \mathcal{O}'_7 , we have that $\dim \mathcal{O}'_7 = \dim \mathcal{O}_7 + \dim \check{\mathbb{P}}^7 = 38 + 7 = 45$, because a matrix of \mathcal{O}'_7 determines in a unique way a hyperplane in \mathbb{P}^7 .

If $N = 8$ there are three orbits, \mathcal{O}'_7 , \mathcal{O}'_8 , \mathcal{O}_9 , whose dimensions are respectively $\dim \mathcal{O}'_7 = \dim \mathcal{O}_7 + \dim \mathbb{G}(6, 8) = 38 + 14 = 52$, because a matrix of \mathcal{O}'_7 determines in a unique way a codimension two subspace of \mathbb{P}^8 , $\dim \mathcal{O}'_8 = \dim \mathcal{O}_8 + \dim \check{\mathbb{P}}^8 = 47 + 8 = 55$, $\dim \mathcal{O}_9 = 56$.

3. BUILDING BLOCKS MATRICES

We turn now to vector spaces M of skew-symmetric matrices of dimension at least 3, where the situation is much more complex. We will consider therefore mainly the cases of low rank, and precisely those of rank $2r \leq 6$, because the vector bundles on the projective spaces which are globally generated are classified for $c_1 \leq 2$ ([3]) as well as for $c_1 = 3$ and rank 2 ([6]).

3.1. Rank 2. For $r = 1$, we get the classification of the linear spaces contained in a Grassmannian of lines $\mathbb{G}(1, N)$. It is well known that the maximal ones belong to one of the following two types:

- (i) the lines contained in a fixed \mathbb{P}^2 ;
- (ii) the lines passing through a fixed point in \mathbb{P}^N .

In case (i) the corresponding exact sequence of bundles is

$$(3.1) \quad 0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(-1) \rightarrow \mathcal{O}_{\mathbb{P}^2}^3 \rightarrow T_{\mathbb{P}^2}(-1) \rightarrow 0$$

and a matrix in the orbit is

$$(3.2) \quad \begin{pmatrix} a & b \\ & c \end{pmatrix}.$$

In case (ii), we get a $\mathbb{P}^{N-1} \subset \mathbb{G}(1, N)$, the exact sequence is

$$(3.3) \quad 0 \rightarrow \Omega_{\mathbb{P}^{N-1}}(1) \rightarrow \mathcal{O}_{\mathbb{P}^{N-1}}^{N+1} \rightarrow \mathcal{O}_{\mathbb{P}^{N-1}} \oplus \mathcal{O}_{\mathbb{P}^{N-1}}(1) \rightarrow 0$$

and a representative matrix is

$$(3.4) \quad \begin{pmatrix} a_1 & \dots & a_N \\ & \dots & 0 \\ & & \vdots \\ & & 0 \end{pmatrix}.$$

3.2. Rank 4. For $r = 2$, a classification of the orbits for matrices of order at most 6×6 has been given in [2]. The result is that there are no vector spaces of dimension 4 of such matrices (see also [7]), while the orbits of vector spaces of dimension 3 are completely described. In the case of 5×5 matrices, there is only one orbit, i.e. the open subset of $\mathbb{G}(2, 9)$ complementary to the irreducible subvariety of codimension 1 representing 2-planes meeting $\mathbb{G}(1, 4)$. The exact sequence (1.2) in this case is

$$(3.5) \quad 0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(-2) \rightarrow \mathcal{O}_{\mathbb{P}^2}^5 \rightarrow \mathcal{E} \rightarrow 0$$

where \mathcal{E} is an indecomposable uniform bundle of rank 4. A representative matrix in M is

$$(3.6) \quad \begin{pmatrix} 0 & 0 & a & b \\ & a & b & c \\ & & c & 0 \\ & & & 0 \end{pmatrix}.$$

As for 6×6 matrices, there are 4 orbits, one for each of the globally generated rank 2 bundles on \mathbb{P}^2 with $c_1 = 2$ that are: $\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2)$, $\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1)$, the restricted null-correlation

bundle, which is a quotient of $\mathcal{O}_{\mathbb{P}^2}(1) \oplus T_{\mathbb{P}^2}(-1)$, and the Steiner bundle, which is a quotient of $T_{\mathbb{P}^2}(-1) \oplus T_{\mathbb{P}^2}(-1)$. The corresponding image bundles \mathcal{E} are respectively: \mathcal{E} appearing in 3.5, $T_{\mathbb{P}^2}(-1) \oplus T_{\mathbb{P}^2}(-1)$, $\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1) \oplus T_{\mathbb{P}^2}(-1)$ and $\mathcal{O}_{\mathbb{P}^2}^2 \oplus \mathcal{O}_{\mathbb{P}^2}(1)^2$. Representative matrices in the last 3 cases are for instance:

$$(3.7) \quad \begin{pmatrix} a & b & 0 & 0 & 0 \\ & c & 0 & 0 & 0 \\ & & 0 & 0 & 0 \\ & & & a & b \\ & & & & c \end{pmatrix}, \begin{pmatrix} 0 & 0 & a & b & c \\ & a & b & 0 & 0 \\ & & c & 0 & 0 \\ & & & 0 & 0 \\ & & & & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & a & b & c \\ & a & b & c & 0 \\ & & 0 & 0 & 0 \\ & & & 0 & 0 \\ & & & & 0 \end{pmatrix}.$$

For matrices of order at least 7, the possible image bundles \mathcal{E} remain the same, whereas the dual of the kernel, \mathcal{K}^* , can be either $\mathcal{O}_{\mathbb{P}^2}(1) \oplus T_{\mathbb{P}^2}(-1)$ or $T_{\mathbb{P}^2}(-1) \oplus T_{\mathbb{P}^2}(-1)$ or a quotient of it of rank 3 (up to trivial direct summands). Examples of matrices are the following:

$$(3.8) \quad \begin{pmatrix} 0 & 0 & 0 & a & b & c \\ & a & b & 0 & 0 & 0 \\ & & c & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & a & b & c \\ & a & b & c & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 \\ & & & & & 0 & 0 \\ & & & & & & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & a & b & c \\ & a & b & c & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix}.$$

It appears that all these examples are constructed using the “building blocks” coming from the matrices (3.2) and (3.4). Other similar examples of spaces of dimension ≥ 4 can be constructed using (3.4).

Remark 3.1. Kernel of rank 1.

Note that for all $r \geq 3$, there are examples of 3-dimensional vector spaces of matrices of order $2r + 1$ and rank $2r$, corresponding to $\mathcal{K} = \mathcal{O}_{\mathbb{P}^2}(-r)$, generalizing (3.6).

But $\dim SL(2r + 1) < \dim \mathbb{G}(2, \mathbb{P}(\Lambda^2 \mathbf{k}^{2r+1}))$. Hence there are infinitely many orbits corresponding to the same kernel bundle.

4. VECTOR SPACES OF SKEW-SYMMETRIC MATRICES OF DIMENSION THREE

In this section we consider 3-dimensional vector spaces M of skew-symmetric matrices of order 8 and constant rank 6. Since M has constant rank 6 then the vector bundle \mathcal{K} in (1.4) has rank 2, $c_1(\mathcal{K}^*) = 3$ and \mathcal{K}^* gives a 3-Veronese embedding of $\mathbb{P}(M)$ in $\mathbb{G}(1, 7)$, see Remark 1.6. Triple Veronese embeddings of \mathbb{P}^n in Grassmannians can be classified.

Theorem 4.1. *Let $X \subset \mathbb{G}(1, N)$ be a triple Veronese embedding of \mathbb{P}^n given by a vector bundle E of rank 2 on \mathbb{P}^n together with an epimorphism $\mathcal{O}_{\mathbb{P}^n}^{N+1} \rightarrow E$. Then one of the following holds:*

- (1) $E \cong \mathcal{O}_{\mathbb{P}^n}(a) \oplus \mathcal{O}_{\mathbb{P}^n}(3-a)$, $a = 0, 1$; $c_2(E) = 0$ if $a = 0$ and $c_2(E) = 2$ if $a = 1$;
- (2) $n = 2$ and $E \cong \Omega_{\mathbb{P}^2}(3) \cong T_{\mathbb{P}^2}$, $c_2(E) = 3$;
- (3) $n = 2$ and E admits a resolution,

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(2) \rightarrow E \rightarrow \mathcal{I}_p(1) \rightarrow 0$$

where \mathcal{I}_p is the ideal sheaf of a point $p \in \mathbb{P}^2$; $c_2(E) = 3$;

- (4) $n = 2$ and E is a stable vector bundle of rank 2 on \mathbb{P}^2 admitting one of the following resolutions

- (a) $0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 3} \rightarrow \mathcal{O}_{\mathbb{P}^2}^{\oplus 5} \rightarrow E \rightarrow 0$

$$(b) \ 0 \rightarrow \Omega_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2} \rightarrow \mathcal{O}_{\mathbb{P}^2}^{\oplus 6} \rightarrow E \rightarrow 0$$

$$(c) \ 0 \rightarrow \Omega_{\mathbb{P}^2}(1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \rightarrow \mathcal{O}_{\mathbb{P}^2}^{\oplus 7} \rightarrow E \rightarrow 0$$

In these last three cases $c_2(E) = 6$ in (a), $c_2(E) = 5$ in (b), $c_2(E) = 4$ in (c).

This result is due to S. Huh, [6, Theorem 1.1], but in his theorem appears also another globally generated vector bundle over \mathbb{P}^2 which does not give an embedding of \mathbb{P}^2 in $\mathbb{G}(1, N)$, as J. C. Sierra has pointed out to us. More precisely the following lemma holds.

Lemma 4.2. *Let E be a stable vector bundle on \mathbb{P}^2 admitting the resolution $0 \rightarrow \Omega_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2) \rightarrow \mathcal{O}_{\mathbb{P}^2}^{\oplus 5} \rightarrow E \rightarrow 0$, which corresponds to the case (4), (b) in [6]. E does not give an embedding of \mathbb{P}^2 in $\mathbb{G}(1, 4)$.*

Proof. We can write the resolution of E in the form

$$(4.1) \quad 0 \rightarrow \Omega_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2) \rightarrow V \otimes \mathcal{O}_{\mathbb{P}^2} \rightarrow E \rightarrow 0$$

where $\dim V = 5$. The epimorphism $V \otimes \mathcal{O}_{\mathbb{P}^2} \rightarrow E \rightarrow 0$ determines a regular morphism $\varphi_V : \mathbb{P}^2 \rightarrow \mathbb{G}(1, 4)$. Dualizing (4.1) we get

$$0 \rightarrow E^* \rightarrow V^* \otimes \mathcal{O}_{\mathbb{P}^2} \rightarrow T_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(2) \rightarrow 0.$$

The pair $(T_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(2), V^*)$ gives a triple Veronese embedding $\phi_{V^*} : \mathbb{P}^2 \rightarrow \mathbb{G}(2, 4)$ and $\varphi_V(\mathbb{P}^2) \cong \phi_{V^*}(\mathbb{P}^2)$ (see [3, §4]). Let \bar{Y} be the 3-fold in \mathbb{P}^4 union of the lines of $\varphi_V(\mathbb{P}^2)$ (see Facts 1.5). Note also that the vector bundle $T_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(2)$ and $V' = H^0(T_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(2))$ give a triple Veronese embedding $\varphi'_{V'} : \mathbb{P}^2 \rightarrow \mathbb{G}(2, 8)$, since $h^0(T_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(2)) = 9$. Let

$$0 \rightarrow \mathcal{K}' \rightarrow V' \otimes \mathcal{O}_{\mathbb{P}^2} \rightarrow T_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(2) \rightarrow 0$$

be the exact sequence of vector bundles associated to the epimorphism $V' \otimes \mathcal{O}_{\mathbb{P}^2} \rightarrow T_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(2) \rightarrow 0$, so $V'^* \otimes \mathcal{O}_{\mathbb{P}^2} \rightarrow \mathcal{K}'^* \rightarrow 0$ is also an epimorphism and hence the pair (\mathcal{K}'^*, V'^*) defines a map $\phi'_{V'^*} : \mathbb{P}^2 \rightarrow \mathbb{G}(5, 8)$ and $\phi'_{V'^*}(\mathbb{P}^2) \cong \varphi'_{V'}(\mathbb{P}^2)$ in the duality between $\mathbb{G}(5, 8)$ and $\mathbb{G}(2, 8)$. Let \bar{Y}' , \bar{Z}' be the subvarieties of \mathbb{P}^8 associated to $\phi'_{V'^*}(\mathbb{P}^2)$ and $\varphi'_{V'}(\mathbb{P}^2)$, respectively. The 3-fold $\bar{Y} \subset \mathbb{P}^4$ is obtained after slicing $\bar{Y}' \subset \mathbb{P}^8$ with 4 hyperplanes; this passes from $\phi'_{V'^*}(\mathbb{P}^2) \subset \mathbb{G}(5, 8)$ to $\varphi_V(\mathbb{P}^2) \subset \mathbb{G}(1, 4)$. By duality this is equivalent to projecting $\varphi'_{V'}(\mathbb{P}^2) \subset \mathbb{G}(2, 8)$ in $\mathbb{G}(2, 4)$ and successively dualizing to $\mathbb{G}(1, 4)$. Since $\varphi'_{V'}$ is given by $(T_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(2), V')$, the variety \bar{Z}' corresponding to $\varphi'_{V'}(\mathbb{P}^2)$ is constructed as follows: fix in \mathbb{P}^8 a \mathbb{P}^2 and a \mathbb{P}^5 complementary to each other and a $v_2(\mathbb{P}^2)$ in \mathbb{P}^5 , fix an isomorphism ψ between \mathbb{P}^{2*} and $v_2(\mathbb{P}^2)$ and consider the family of the 2-planes joining a line and a point corresponding to each other in ψ . When we project in \mathbb{P}^4 , we get a $\mathbb{P}^2 = \alpha$ and a projected Veronese surface, of degree 4, intersecting α in 4 points. Hence 4 planes of the family come together to coincide with α . This gives rise to a point of multiplicity 4 of $\phi_{V^*}(\mathbb{P}^2)$. Hence also $\varphi_V(\mathbb{P}^2)$ is singular. \square

Remark 4.3. To explain how the classification in Theorem 4.1 is organized, we note that the bundles in (1), (2) are uniform, while those in (3), (4) are not. Moreover the bundle appearing in (3) is unstable, while those in (2) and (4) are stable. The corresponding moduli spaces $\mathcal{M}(3, c_2)$ have dimension $4c_2 - 12$ (see [13, Ch. 2, §4]).

We will see that the non split vector bundles given in Theorem 4.1 can be seen as quotient of vector bundles of higher rank of a very particular form. This fact turns out to be crucial in constructing skew-symmetric matrices of constant rank 6.

Definition 4.4. ([3]) We say that a vector bundle \mathcal{F} on \mathbb{P}^n is a quotient of \mathcal{E} if there exists an exact sequence $0 \rightarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus s} \rightarrow \mathcal{E} \rightarrow \mathcal{F} \rightarrow 0$, corresponding to s sections of \mathcal{E} .

Proposition 4.5. Let E be a vector bundle of rank 2 on \mathbb{P}^2 defining a triple Veronese embedding of \mathbb{P}^2 in a Grassmannian $\mathbb{G}(1, N)$, as in Theorem 4.1.

- (i) If E is as in (2) then E is a quotient of $\mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 3}$;
- (ii) If E is as in (3) then E is a quotient of $T_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(2)$;
- (iii) If E is as in (4), (a) then E is a quotient of $T_{\mathbb{P}^2}(-1)^{\oplus 3}$;
- (iv) If E is as in (4), (b) then E is a quotient of $T_{\mathbb{P}^2}(-1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(1)$;
- (v) If E is as in (4), (c) then E is a quotient of $T_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2}$.

Proof. The case (i) follows from the Euler exact sequence.

In the case (ii) the vector bundle E is unstable because $h^0(E_{norm}) = h^0(E(-2)) \neq 0$. Let $\mathcal{G} = T_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(2)$, then $c_1(\mathcal{G}) = c_2(\mathcal{G}) = 3$, moreover $h^0(\mathcal{G}) \neq 0$ hence there exists an exact sequence

$$(4.2) \quad 0 \rightarrow \mathcal{O}_{\mathbb{P}^2} \rightarrow \mathcal{G} \rightarrow \mathcal{Q} \rightarrow 0$$

corresponding to a section of \mathcal{G} . Note that \mathcal{Q} normalized, $\mathcal{Q}_{norm} = \mathcal{Q}(-2)$. Twisting (4.2) with $\mathcal{O}_{\mathbb{P}^2}(-2)$ and considering its associated cohomology exact sequence it follows that $h^0(\mathcal{Q}(-2)) = 1$ and thus \mathcal{Q} cannot be stable, see [13, Lemma 1.2.5, pg 165]. Moreover, because $c_1(\mathcal{Q}) = 3$ and $rk(\mathcal{Q}) = 2$, then by [13, Remark 1.2.3 pg 163] it follows that \mathcal{Q} is stable if and only if is semistable. Hence, being $h^0(\mathcal{Q}(-2)) = 1$, we can conclude that \mathcal{Q} is unstable and thus \mathcal{Q} has to be the vector bundle E in (2), since $c_2(\mathcal{Q}) = 3$.

In the case (iii) the resolution of E along with the Euler exact sequence yields the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \downarrow & & \downarrow & \\
 & & & \mathcal{O}_{\mathbb{P}^2}^{\oplus 4} & \xlongequal{\quad} & \mathcal{O}_{\mathbb{P}^2}^{\oplus 4} & \\
 & & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 3} & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}^{\oplus 9} & \longrightarrow & T_{\mathbb{P}^2}(-1)^{\oplus 3} \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 3} & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}^{\oplus 5} & \longrightarrow & E \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow & \\
 & & & & 0 & & 0 &
 \end{array}$$

and we get that E is a quotient of $T_{\mathbb{P}^2}(-1)^{\oplus 3}$.

The proof of (iv) and (v) runs along the same lines of (iii), hence we omit it. \square

Corollary 4.6. Let E be a rank 2 vector bundle on \mathbb{P}^2 with $c_1(E) = 3$. If E gives an embedding of \mathbb{P}^2 in $\mathbb{G}(1, N)$ then either E splits or E is a quotient of a vector bundle of higher rank which is a direct sum of $T_{\mathbb{P}^2}(-1)$ (or more copies of it) and $\mathcal{O}_{\mathbb{P}^2}(k)$ (or more copies of it) for some positive integer k .

Remark 4.7. Let M be a vector space of skew-symmetric matrices of constant rank $2r$ and order $N + 1$. We can interpret it as a linear space contained in $S_{r-1}\mathbb{G}(1, N) \setminus S_{r-2}\mathbb{G}(1, N)$. Observe that the exact sequence (1.2), after taking a quotient \mathcal{Q} of \mathcal{K} , gives rise to another exact sequence in which a new matrix comes up and one wants to know if its rank is constant. Taking a quotient \mathcal{Q} of \mathcal{K} corresponds to projecting \mathbb{P}^N to \mathbb{P}^{N-1} from a point O . The projection π_O of centre O induces a projection π_{Λ_O} from $\mathbb{P}(\Lambda^2\mathbf{k}^{N+1})$ to $\mathbb{P}(\Lambda^2\mathbf{k}^N)$, whose centre is the subspace $\Lambda_O \subset \mathbb{G}(1, N)$, representing the lines through O . How should one choose the centre of projection in order that the rank of M remains constant under this projection? The answer is given by the following Proposition.

We recall that a point ω in $S_{r-1}\mathbb{G}(1, N) \setminus S_{r-2}\mathbb{G}(1, N)$ can be written in the form $[v_1 \wedge w_1 + \dots + v_r \wedge w_r]$, where $v_1, \dots, v_r, w_1, \dots, w_r$ are linearly independent vectors; the corresponding points generate a subspace L_ω of \mathbb{P}^n of dimension $2r - 1$. Then the entry locus of ω is the subgrassmannian $\mathbb{G}(1, L_\omega)$, namely a point of $\mathbb{G}(1, N)$ belongs to some $(r - 1)$ -plane r -secant to $\mathbb{G}(1, N)$ and containing ω if and only if it belongs to $\mathbb{G}(1, L_\omega)$.

Proposition 4.8. *Let $\mathbb{P}(M) \subset S_{r-1}\mathbb{G}(1, N)$ be a vector space of matrices of constant rank $2r$. Let $O \in \mathbb{P}^N$ be a point such that $\mathbb{P}(M) \cap \Lambda_O = \emptyset$. Then the matrices of $\pi_{\Lambda_O}(\mathbb{P}(M))$ have constant rank $2r$ if and only if O does not belong to the union of the spaces L_ω , as ω varies in $\mathbb{P}(M)$.*

Proof. Let $\omega = [v_1 \wedge w_1 + \dots + v_r \wedge w_r]$ be a point of $\mathbb{P}(M)$. Then $\pi_{\Lambda_O}(\mathbb{P}(M))(\omega) = [Av_1 \wedge Aw_1 + \dots + Av_r \wedge Aw_r]$, where A is a matrix representing π_O , and its rank is strictly less than r if and only if $v_1, \dots, v_r, w_1, \dots, w_r$ can be chosen so that some summand $Av_i \wedge Aw_i$ vanishes. But this means precisely that O belongs to L_ω . \square

Corollary 4.9. *Let $\mathbb{P}(M)$ be a linear space of matrices of constant rank $2r$ and dimension d . Then $\mathbb{P}(M)$ can be isomorphically projected to $S_{r-1}\mathbb{G}(1, 2r + d - 1)$ so that its rank remains constant and equal to $2r$.*

Proof. It is enough to note that $\dim \bigcup_{\omega \in \mathbb{P}(M)} L_\omega \leq \dim \mathbb{P}(M) + 2r - 1$. \square

In particular a projective 2-plane of matrices of constant rank 6 can be projected in $S_2\mathbb{G}(1, 7)$ maintaining constant rank 6.

We can now state the main result of this section, which gives a reverse statement to Proposition 1.6.

Theorem 4.10. *Let E be a rank two vector bundle on \mathbb{P}^2 defining a triple Veronese embedding of \mathbb{P}^2 in $\mathbb{G}(1, 7)$. Then there exists a vector space of 8×8 matrices of constant rank 6 whose associated bundle \mathcal{K} is such that $E \simeq \mathcal{K}^*$.*

Proof. By Corollary 4.6 E is a direct sum of copies of $\mathcal{O}_{\mathbb{P}^2}$, $\mathcal{O}_{\mathbb{P}^2}(1)$, $\mathcal{O}_{\mathbb{P}^2}(2)$, $\mathcal{O}_{\mathbb{P}^2}(3)$, $T_{\mathbb{P}^2}(-1)$, or a quotient of it. From the results of Section 3, each of these bundles is the dual of a bundle appearing as kernel in an exact sequence of the form (1.4). Taking a direct sum of matrices corresponding to the direct summands, we construct a matrix of constant rank 6 and order possibly bigger than 8. Finally, by Corollary 4.9, with a suitable projection we get a 8×8 matrix of the desired form. \square

4.1. Examples. For each class of rank two bundles appearing in Theorem 4.1, we will give now one or more examples of linear systems of matrices of constant rank 6. Unfortunately we are not able to give a complete classification of the orbits for the action of $SL(8)$. As we have already

noted in Subsection 3.1, for some bundles there are infinitely many orbits. On the other hand, for the bundles in Theorem 4.1, (4), there is a moduli space of positive dimension.

Example 1. Split bundles.

Let π_1 be the plane

$$\begin{pmatrix} 0 & 0 & 0 & 0 & a & b & 0 \\ & 0 & 0 & a & b & c & 0 \\ & & a & b & c & 0 & 0 \\ & & & c & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 \\ & & & & & 0 & 0 \\ & & & & & & 0 \end{pmatrix}.$$

The associated rank 2 vector bundle is $\mathcal{K}^* = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(3)$, $c_2(\mathcal{K}^*) = 0$.

Let π_2 be the plane

$$\begin{pmatrix} 0 & 0 & a & b & 0 & 0 & 0 \\ & a & b & c & 0 & 0 & 0 \\ & & c & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 \\ & & & & & a & b \\ & & & & & & c \end{pmatrix}.$$

The associated rank 2 vector bundle is $\mathcal{K}^* = \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(2)$, $c_2(\mathcal{K}^*) = 2$.

Example 2. Steiner bundles

Steiner bundles are quotients of $T_{\mathbb{P}^2}(-1)^3$, have $c_2 = 6$ and move in a moduli space of dimension 12. An example of matrix is

$$\begin{pmatrix} 0 & 0 & a & b & c & 0 & 0 \\ & 0 & 0 & a & b & c & 0 \\ & & 0 & 0 & a & b & c \\ & & & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 \\ & & & & & 0 & 0 \\ & & & & & & 0 \end{pmatrix}$$

In this case the associated bundle \mathcal{K}^* is a Schwarzenberger bundle, having a conic of jumping lines. General Steiner bundles have 6 jumping lines. Following the construction of Dolgachev-Kapranov (see [14]) and choosing as follows the equations of the jumping lines:

$$x_0 = 0;$$

$$x_1 = 0;$$

$$x_2 = 0;$$

$$\lambda_0 x_0 + \lambda_1 x_1 + \lambda_2 x_2 = 0;$$

$$\mu_0 x_0 + \mu_1 x_1 + \mu_2 x_2 = 0;$$

$$\nu_0 x_0 + \nu_1 x_1 + \nu_2 x_2 = 0,$$

we get a matrix M of the form:

$$(4.3) \quad M = \begin{pmatrix} 0 & B \\ -B^t & 0 \end{pmatrix}$$

where

$$(4.4) \quad B = \begin{pmatrix} \lambda_0 a + \mu_0 b + \nu_0 c & 0 & 0 & \lambda_0 a & \mu_0 b \\ 0 & \lambda_1 a + \mu_1 b + \nu_1 c & 0 & \lambda_1 a & \mu_1 b \\ 0 & 0 & \lambda_2 a + \mu_2 b + \nu_2 c & \lambda_2 a & \mu_2 b \end{pmatrix}.$$

Hence we have a family of examples, depending on the parameters λ, μ, ν . Such parameters have to be chosen so that the six jumping lines are in general position.

Example 3. Unstable bundle

Let π_3 be the plane

$$\begin{pmatrix} 0 & 0 & 0 & 0 & a & b & c \\ & 0 & 0 & a & b & 0 & 0 \\ & & a & b & c & 0 & 0 \\ & & & c & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 \\ & & & & & 0 & 0 \\ & & & & & & 0 \end{pmatrix}$$

In this case the associated rank 2 vector bundle \mathcal{K}^* is the unstable one, quotient of $T_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(2)$, $c_2(\mathcal{K}^*) = 3$ (case (3) of Theorem 4.1).

Example 4. Let π_4 be the plane

$$\begin{pmatrix} 0 & 0 & 0 & 0 & a & b & c \\ & 0 & 0 & a & b & c & 0 \\ & & a & b & 0 & 0 & 0 \\ & & & c & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 \\ & & & & & 0 & 0 \\ & & & & & & 0 \end{pmatrix}$$

In this case the associated rank 2 vector bundle \mathcal{K}^* is a quotient of $T_{\mathbb{P}^2}(-1) \oplus T_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(1)$, $c_2(\mathcal{K}^*) = 5$ (case (4)(b) of Theorem 4.1).

The expression of the matrices in the following Examples 5 and 6 is not so evident a priori. The matrices in these examples correspond, respectively, to the bundle (4)(c) of Theorem 4.1 and to the tangent bundle $T_{\mathbb{P}^2}$.

Example 5. A quotient of $T_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1)$.

Projecting from $[1, 0, 0, 0, 1, 0, 0, 0, 0, 1]$ and subsequently from $[0, 0, 1, 1, 0, 0, 0, 1, 0]$ the direct sum matrix naturally associated to this bundle, we get the following plane π_5 :

$$\begin{pmatrix} c & a & 0 & 0 & 0 & 0 & a \\ & b & 0 & 0 & 0 & 0 & b \\ & & c-b & 0 & 0 & a & 0 \\ & & & 0 & 0 & b & 0 \\ & & & & a & b & c \\ & & & & & 0 & 0 \\ & & & & & & 0 \end{pmatrix}$$

Example 6. The tangent bundle.

Let π_6 be the plane

$$\begin{pmatrix} c & 0 & a & 0 & 0 & 0 & a \\ & 0 & b & 0 & 0 & 0 & b \\ & & a & b & 0 & 0 & 0 \\ & & & c & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 \\ & & & & & a & b \\ & & & & & & c \end{pmatrix}$$

It has been constructed projecting from $[1, 0, 0, 0, 1, 0, 0, 0, 1]$ the direct sum matrix, coming from the expression of $\mathcal{K}^* = T_{\mathbb{P}^2}$, as a quotient of $\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1)$.

4.2. Triple Veronese embeddings. Let γ be the Gauss map from $S_2\mathbb{G}(1, 7)$ to $\mathbb{G}(1, 7)$. We will shortly give now the geometrical interpretation of the varieties $\gamma(\mathbb{P}(M))$ for each of the above examples. They are all (projections of) the Veronese variety $v_3(\mathbb{P}^2)$.

- $\gamma(\pi_1)$ is contained in $\mathbb{G}(1, 6)$ and represents the lines of a cone with vertex one point over $v_3(\mathbb{P}^2)$ projected from \mathbb{P}^9 to \mathbb{P}^6 . Since varying the centre of projection we get varieties isomorphic but not always projectively equivalent, this explains the presence of infinitely many orbits for these planes.
- $\gamma(\pi_2) \subset \mathbb{G}(1, 7)$ represents the lines joining the corresponding points in an isomorphism between a fixed 2-plane and a projected 2-Veronese surface in a fixed \mathbb{P}^4 .
- If M is one of the planes of Example 2, then $\gamma(\mathbb{P}(M))$ is contained in a subgrassmannian $\mathbb{G}(1, 4)$, where the \mathbb{P}^4 is defined by the equations $x_0 = x_1 = x_2 = 0$. It represents the lines of a cubic threefold, whose equation is the determinant of a 1-generic matrix Ω obtained as follows. Write $B = aB_1 + bB_2 + cB_3$, and let B_i^j denote the j -th row of B_i . Let X' be the transposed of $(x_3 \dots x_7)$. Then

$$(4.5) \quad \Omega = \begin{pmatrix} B_1^1 X' & B_1^2 X' & B_1^3 X' \\ B_2^1 X' & B_2^2 X' & B_2^3 X' \\ B_3^1 X' & B_3^2 X' & B_3^3 X' \end{pmatrix}$$

In the special case of the Schwarzenberger bundle, we get the cubic threefold of the secant lines of a quartic rational normal curve.

- The lines of $\gamma(\pi_3)$ are obtained as follows. Note that, since $\dim H^0(\mathbb{P}^2, T_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(2)) = 9$, this bundle gives a triple Veronese embedding of \mathbb{P}^2 in $\mathbb{G}(2, 8)$. Geometrically we fix an isomorphism between $v_2(\mathbb{P}^2)$ and \mathbb{P}^2 and we get a family of \mathbb{P}^2 's spanned by a point in $v_2(\mathbb{P}^2)$ and the corresponding line in \mathbb{P}^2 . Taking a quotient of $T_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(2)$ is the same as cutting this family with a hyperplane, and this gives $\gamma(\pi_3) \subset \mathbb{G}(1, 7)$.
- The description of $\gamma(\pi_4)$ and $\gamma(\pi_5)$ is similar to the previous one. Since $\dim H^0(\mathbb{P}^2, T_{\mathbb{P}^2}(-1) \oplus T_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(1)) = 9$ we have a triple Veronese embedding of \mathbb{P}^2 in $\mathbb{G}(4, 8)$. Geometrically we fix three planes and we have a correspondence between the first plane \mathbb{P}^2 and the dual of the other two \mathbb{P}^2 's. We get a family of \mathbb{P}^4 's spanned by a point in \mathbb{P}^2 and the two corresponding lines in the two \mathbb{P}^2 . Cutting this family with three hyperplane we get $\gamma(\pi_4)$. If we consider instead the bundle $\mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 3}$ this gives an

triple Veronese embedding of \mathbb{P}^2 in $\mathbb{G}(2, 8)$. We consider again three disjoint \mathbb{P}^2 's and we get the family of \mathbb{P}^2 's spanned by three corresponding points in fixed isomorphisms among them. Cutting this family with a hyperplane we get $\gamma(\pi_5)$.

Remark 4.11. The algorithm in Section 2.1 can be suitably modified to compute the dimensions of the orbits of the matrices constructed in this Section. One obtains that the dimension of the orbits is 54 (respectively 60) in Example 1; 52 (respectively 56) in Example 2; 58 in Example 3 and in Example 4; 59 in Example 5 and 60 in Example 6.

5. WESTWICK EXAMPLE RIVISITED

We start this final section with the following:

Remark 5.1. There do not exist vector spaces of dimension 4 of 8×8 matrices of constant rank 6. This follows from a computation on the Chern classes, see for instance [1, Example 2.12].

The first possibility for a \mathbb{P}^3 of skew-symmetric matrices of order $2r + 2$ and constant rank $2r$ is for $r = 4$. The only known example has been given by Westwick ([7]) and is the following:

$$(5.1) \quad \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & a & b & 0 \\ & 0 & 0 & 0 & 0 & a & b & 0 & c \\ & & 0 & 0 & -a & b & 0 & c & d \\ & & & a & b & 0 & c & d & 0 \\ & & & & 0 & c & -d & 0 & 0 \\ & & & & & d & 0 & 0 & 0 \\ & & & & & & 0 & 0 & 0 \\ & & & & & & & 0 & 0 \\ & & & & & & & & 0 \end{pmatrix}$$

We will say something about the vector bundles associated to such $\mathbb{P}^3 = \mathbb{P}(M)$ of skew-symmetric matrices of constant rank 8 and order 10.

With the notation in Section 1, one can easily see, using for instance [1, Example 2.12], that $c_1(\mathcal{K}^*) = 4$ and $c_2(\mathcal{K}^*) = 6$. Let $s \in H^0(\mathbb{P}^3, \mathcal{K}^*)$ be a generic section and let Y be its scheme of zeros. Because \mathcal{K}^* is spanned by global sections then Y is smooth of codimension $2 = rk(\mathcal{K}^*)$. The section s defines an exact sequence

$$(5.2) \quad 0 \rightarrow \mathcal{O}_{\mathbb{P}^3} \rightarrow \mathcal{K}^* \rightarrow \mathcal{J}_Y(c_1(\mathcal{K}^*)) \rightarrow 0$$

We see that $deg(Y) = c_2(\mathcal{K}^*) = 6$, $N_{Y/\mathbb{P}^3} = \mathcal{K}^*_{|Y}$. We get, by adjunction, that $K_Y = \mathcal{O}_Y$ and thus $g(Y) = 1$. Twisting the exact sequence (5.2) with $\mathcal{O}_{\mathbb{P}^3}(-2)$ and recalling that $c_1(\mathcal{K}^*) = 4$ we get

$$(5.3) \quad 0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-2) \rightarrow \mathcal{K}^*(-2) \rightarrow \mathcal{J}_Y(2) \rightarrow 0$$

From the cohomology sequence associated to (5.3), using the fact that $H^0(\mathbb{P}^3, \mathcal{J}_Y(2)) = 0$ because Y cannot be contained in any quadric surface, it follows that $H^0(\mathbb{P}^3, \mathcal{K}^*(-2)) = H^0(\mathbb{P}^3, \mathcal{K}^*_{norm}) = 0$ and thus \mathcal{K}^* is a stable vector bundle.

By Proposition 1.4 we see that $\gamma(\mathbb{P}(M))$ is a 4-tuple Veronese embedding of \mathbb{P}^3 in $\mathbb{G}(1, 9)$. Note that this embedding is given by a proper subspace of $H^0(\mathbb{P}^3, \mathcal{K}^*)$. In fact, using results contained in [15], one can show that $h^0(\mathbb{P}^3, \mathcal{K}^*) = 12$.

Thus on \mathbb{P}^3 such vector bundles \mathcal{K}^* are the only ones for which it can exist a 10×10 skew-symmetric matrix of constant rank 8. R. Hartshorne in [15, Corollary 9.8] has proved that the variety of moduli of these bundles is an irreducible nonsingular variety of dimension 13.

Remark 5.2. From (5.2) one computes also that $h^1(\mathbb{P}^3, \mathcal{K}^*) = 0$. Hence \mathcal{K}^* is not a quotient of any bundle of higher rank. Similarly, the restriction $\mathcal{K}^*|_H$ of \mathcal{K}^* to a general plane H is a stable bundle with $h^0(H, \mathcal{K}^*|_H) = 10$ and $h^1(\mathcal{K}^*|_H) = 0$. Therefore the matrix obtained by restricting (5.1) to a general plane can be thought of as a new building block for constructing vector spaces of dimension 3 of matrices of constant rank ≥ 8 and order at least 10.

Remark 5.3. The study of linear spaces of skew-symmetric matrices of constant rank is related to the study of possible degenerations of a class of projective varieties called *Palatini scrolls*, that is those varieties X in \mathbb{P}^N , with N odd, which are degeneracy loci of general morphisms $\phi : \mathcal{O}_{\mathbb{P}^N}^m \rightarrow \Omega_{\mathbb{P}^N}(2)$. Such X is smooth if $m < \frac{N+4}{2}$, [8]. As it is well known a morphism $\phi : \mathcal{O}_{\mathbb{P}^N}^m \rightarrow \Omega_{\mathbb{P}^N}(2)$ gives a $(N+1) \times (N+1)$ skew-symmetric matrix of linear forms M_ϕ on \mathbb{P}^{m-1} .

For instance, if $N = 4$ and $m = 3$, then X is a projected Veronese surface, if $N = 5$ and $m = 4$, then X is a Palatini threefold: its degenerations have been studied in [16] relying on the classification given in [2].

If $m = 5$ and $N = 7$ then X is a smooth fourfold in \mathbb{P}^7 with base of the scroll the quartic 3-fold Y in $\mathbb{P}^{m-1} = \mathbb{P}^4$, defined by $\text{Pf}(M_\phi)$. The fact that there do not exist vector spaces of dimension 4 of 8×8 matrices of constant rank 6 says that there cannot be a degeneration of X obtained by degenerating the base Y so that it acquires a \mathbb{P}^3 as irreducible component.

The situation is different for $m = 5$ and $N = 9$. In this case the base Y of the scroll is a quintic 3-fold defined by $\text{Pf}(M_\phi)$, with M_ϕ a 10×10 matrix of linear forms on \mathbb{P}^4 . From the example (5.1) it follows that the base Y can degenerate so that it contains a \mathbb{P}^3 as irreducible component.

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E-mail address: fania@univaq.it

DIPARTIMENTO DI MATEMATICA PURA E APPLICATA, UNIVERSITÀ DEGLI STUDI DELL’AQUILA,, VIA VETOIO
LOC. COPPITO, 67100 L’AQUILA, ITALY

E-mail address: mezzette@units.it

DIPARTIMENTO DI MATEMATICA E INFORMATICA, UNIVERSITÀ DEGLI STUDI DI TRIESTE, VIA VALERIO 12/1,,
34127 TRIESTE, ITALY