A dimensionally continued Poisson summation formula

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Abstract

We generalize the standard Poisson summation formula for lattices so that it operates on the level of theta series, allowing us to introduce noninteger dimension parameters (using the dimensionally continued Fourier transform). When combined with one of the proofs of the Jacobi imaginary transformation of theta functions that does not use the Poisson summation formula, our proof of this generalized Poisson summation formula also provides a new proof of the standard Poisson summation formula for dimensions greater than 2 (with appropriate hypotheses on the function being summed). In general, our methods work to establish the (Voronoi) summation formulae associated with functions satisfying (modular) transformations of the Jacobi imaginary type by means of a density argument (as opposed to the usual Mellin transform approach). Additionally, our result relaxes several of the hypotheses in the standard statements of these summation formulae. The density result we prove for Gaussians in the Schwartz space may be of independent interest.

Keywords: Summation formulae, Voronoi summation, theta functions, modular transformation

1. Introduction

Consider a lattice $\Lambda \subset \mathbb{R}^n$ and a sufficiently well-behaved function $F : \mathbb{R}^n \to \mathbb{R}$. [Taking F to belong to the Schwartz space $\mathscr{S}(\mathbb{R}^n)$ is sufficient, and is what we shall do in our later generalization.] The standard Poisson summation formula then says that

$$\sum_{k\in\Lambda} F(k) = \frac{1}{\sqrt{\det\Lambda}} \sum_{p\in\Lambda^*} \tilde{F}(p). \tag{1}$$

Here Λ^* is the lattice dual to Λ, $\det\Lambda$ denotes the volume of a Voronoi cell of Λ, and

$$\tilde{F}(p) := \int_{\mathbb{R}^n} F(x) e^{-2\pi i x\cdot p} dx$$

denotes the Fourier transform of F. (We use a tilde here so that we can reserve the circumflex for our more general, dimensionally continued Fourier transform.) We wish to construct a dimensionally continued version of this result.

This problem was originally inspired by a condensed matter physics investigation involving the dimensional continuation of electrostatic lattice sums, computed using the Ewald method (see, e.g., [1] for a modern exposition of this method). However, the ensuing discussion is a purely mathematical offshoot of this investigation. For one thing, the results we were able to prove do not include the physically relevant case of a slowly decaying function, even though we have numerical evidence that the results still hold in this case. Nevertheless, the methods used here might also be applicable to the dimensional regularization of lattice sums: See [2] for an approach using zeta functions and the Mellin transform.

If we specialize to the case where F is a radial function, we can obtain a dimensionally continued Poisson summation formula in terms of the lattice's theta series—see Theorem 1 for the (particularly simple) version for $\mathbb{Z}^d$ ($d > 1$). This result even holds when one uses, instead of a theta series derived from a lattice (or lattice-like object), a function Υ that merely possesses the appropriate modular transformation properties (with sufficiently strong bounds on the

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growth of its power series coefficients). This more general result is given in Theorem 2. The following Proposition shows that a large family of Υs satisfying the hypotheses of Theorem 2 can be constructed from Jacobi theta functions. This family contains all the theta series given in Chap. 4 of Conway and Sloane [3] (except for the general forms of those for the root lattice A_d and its translates). The final Corollary states the resulting summation formula explicitly.

This relation between modular transformations and summation formulae is not new (though the dimensional continuation is). The relation has reached its most refined form in the association between automorphic forms and Voronoi summation formulae (see, e.g., Miller and Schmid [4] for a review of recent work). However, work on this correspondence first arose in the context of transformations related to the functional equation for the zeta function, inspired by a question by Voronoi on analogues of the Poisson summation formula—see, e.g., [5] and references therein. (We call particular attention to the work of Ferrar [6].) Additionally, Baake, Frettlöh, and Grimm [7] give a (distributional) radial Poisson summation formula in their Theorem 3 in a form that is very similar to our dimensionally continued form. However, they do not show how to dimensionally continue the lattice (or, indeed, mention theta functions explicitly), and their proof (which relies on the standard Poisson summation formula) only holds for integer dimensions. There are also discussions of similar formulae—these derived from modular transformations—at the beginning of Chapter 4 of Iwaniec and Kowalski [8], and in Sec. 10.2 of Huxley [9]—what Huxley terms the Wilton summation formula. These formulae are presented in what appears to be a dimensionally continued form, though their hypotheses assume integer dimensions. Regardless, the summation formula in Iwaniec and Kowalski and Huxley's Wilton summation formula are derived from cusp forms, while our result (in the language of modular forms) does not require that the constant term in the form's Fourier series vanish. (Indeed, we do not mention modular forms *qua* modular forms at all in the sequel.)

The work presented here gives a view of these matters that differs from all the other investigations we have seen. In particular, our method of proof bears no resemblance to any of the other proofs of which we are aware. The other proofs that begin with a modular transformation rely heavily on the Mellin transform (possibly supplemented with heavy specialized machinery), while the standard proof of the classical Poisson summation formula uses the lattice's periodicity and Fourier series. Our primary analytic tool is a density result for Gaussians in the Schwartz space (established by basic functional analytic means). The remainder of the proof uses basic real and complex analysis (primarily Taylor's theorem, the Lebesgue dominated convergence theorem, and Cauchy's integral formula) to show that the dimensionally continued Poisson summation formula holds for the Gaussians (in essence, by reversing the usual derivation of the Jacobi imaginary transformation using the Poisson summation formula), and that it remains true in the limit in the Schwartz space.

2. Ingredients

2.1. Theta series

Here we recall various facts about theta series and theta functions that we shall need for the rest of our discussion, following Chap. 4 of Conway and Sloane [3]. The theta series of a lattice Λ is defined by ,

$$\Theta_\Lambda(q) := \sum_{k\in\Lambda} q^{|k|^2}.$$

(This is often treated as a formal series, but converges for $q \in \mathbb{C}$, $|q| < 1$.) The utility of the theta series stems from the fact that the coefficient of q^l in the expansion of $\Theta_\Lambda(q)$ in powers of q gives the number of points in the intersection of the lattice and a sphere of radius $\sqrt{l}$ centred at the origin. Thus, if we write

$$\Theta_\Lambda(q) =: \sum_{l=0}^{\infty} N_l q^{A_l}, \tag{2}$$

then

$$\sum_{k\in\Lambda} f(|k|) = \sum_{l=0}^{\infty} N_l f(\sqrt{A_l}).$$

[*Nota bene*: We have written the radial function F as $f(|\cdot|)$, and shall only consider these radial parts in the sequel.] Examples of dimensionally continued theta series for families of lattices include the d-dimensional cubic lattice $\mathbb{Z}^d$, with $\Theta_{\mathbb{Z}^d}(q) = \vartheta_3^d(q)$, and the root lattice D^d, with $\Theta_{D^d}(q) = [\vartheta_3^d(q) + \vartheta_4^d(q)]/2$. (See Chap. 4 in Conway and Sloane [3] for further examples.) Here

$$\vartheta_2(q) := 2q^{1/4}\sum_{l=1}^{\infty} q^{l^2-l}, \qquad \vartheta_3(q) := 1 + 2\sum_{l=1}^{\infty} q^{l^2}, \qquad \vartheta_4(q) := 1 + 2\sum_{l=1}^{\infty} (-q)^{l^2} \tag{3}$$

are Jacobi theta functions (where ϑ_2 is defined for future use).

Nota bene: It is often customary to take theta functions and theta series to be functions of a complex variable z, instead of the nome $q = e^{i\pi z}$ that we have used here. We have chosen to regard the nome as fundamental since we are primarily interested in the expansions of these functions in powers of q. However, when discussing transformations of these functions, it is considerably more convenient to regard them as functions of z. On the few occasions where we do this, we shall use an overbar to denote the difference, e.g., $\bar{\Theta}_\Lambda(z) := \Theta_\Lambda(e^{i\pi z})$. (In the literature on summation formulae derived from automorphic forms, one thinks of our expansions of Θ_Λ in q as the Fourier coefficients of $\bar{\Theta}_\Lambda$.)

Since the Poisson summation formula involves the dual lattice, we need to know how to obtain its theta series. This is given by the Jacobi formula [Eq. (19) in Chap. 4 of Conway and Sloane [3]], which states that

$$\bar{\Theta}_{\Lambda^*}(z) = \sqrt{\det \Lambda}(i/z)^{d/2}\bar{\Theta}_\Lambda(-1/z), \tag{4}$$

where d is the dimension of the lattice. The Jacobi formula is typically proved using the Poisson summation formula [see, e.g., the discussion leading up to our Eq. (8)]. However, all we need in our discussion is the intimately related Jacobi imaginary transformation of the Jacobi theta functions (also known as the modular identity or reciprocity formula for the theta functions), i.e.,

$$\bar{\vartheta}_2(-1/z) = (z/i)^{1/2}\bar{\vartheta}_4(z), \qquad\qquad \bar{\vartheta}_3(-1/z) = (z/i)^{1/2}\bar{\vartheta}_3(z). \tag{5}$$

(The first of these is also true with the labels 2 and 4 switched.) The standard proof of these identities is a direct application of the Poisson summation formula, but there are alternative proofs that are independent of it. For instance, one such proof is given in Sec. 21.51 of Whittaker and Watson [10], while Bellman's text [11] discusses several others—see, in particular, Sec. 30 for Polya's derivation—in addition to the standard Poisson summation version (in Sec. 9). Our discussion will thus be independent of the standard Poisson summation formula (with the exception of a brief appeal to establish Theorem 1 for $d = 1$).

2.2. The dimensionally continued Fourier transform

We also need to dimensionally continue the Fourier transform. Stein and Weiss give a dimensionally continued version of the Fourier transform for radial functions in Theorem 3.3 of Chap. IV of [12], viz.,

$$\begin{aligned}\hat{f}(p) &:= 2\pi p^{-(d-2)/2}\int_0^\infty f(r)J_{(d-2)/2}(2\pi pr)r^{d/2}dr \\ &= \frac{2\pi^{d/2}}{\Gamma(d/2)}\int_0^\infty f(r)\,{}_0F_1(d/2;-\pi^2p^2r^2)r^{d-1}dr.\end{aligned} \tag{6}$$

(This reduces to the standard Fourier transform for a radial function when $d \in \mathbb{N}$.) Here the first equality gives the expression from Stein and Weiss (J_k is a Bessel function) and the second gives an equivalent (perhaps slightly neater) expression in terms of the confluent hypergeometric limit function ${}_0F_1$. The hypergeometric expression has the advantage of only involving one appearance of p (and being manifestly regular at $p = 0$ for all $d \geq 1$), in addition to showing the d-dimensional polar coordinate measure for radial functions explicitly. We shall thus use the hypergeometric expression exclusively in the sequel. (One can obtain the hypergeometric expression using the Stein and Weiss derivation—the only difference is that one uses a different special function to evaluate the final integral.[1])

For this expression to be well-defined, it is sufficient to take $d \geq 1$: One assumes $d > 1$ when using integral representations to express the result in terms of either of the two given special functions, and can also check that the

[1] The integral representation for ${}_0F_1$ we used is 07.17.07.0004.01 on the Wolfram Functions Site [13].

integral is convergent for all $p \in \mathbb{R}$ in that case, provided that $f \in L^1(\mathbb{R}_+)$. Additionally, Eq. (6) reduces to the expected expression for $d = 1$ [using $J_{-1/2}(z) = \sqrt{2/\pi z}\cos z$ or ${}_0F_1(1/2; -z) = \cos(2\sqrt{z})$].[2] (Stein and Weiss restrict to $d \geq 2$ so that the integral they use in their derivation is well-defined, since they are only considering integer dimensions.) This restriction to $d > 1$ is necessary for other parts of our discussion, though we have numerical evidence that it can be relaxed.

The following result is central to understanding why this dimensionally continued Fourier transform agrees with the dimensional continuation of the theta series.

Lemma 1. *For $d \geq 1$, the dimensionally continued Fourier transform (for radial functions) defined in Eq.* (6) *takes a Gaussian $\mathcal{G}_\alpha(r) := e^{-\alpha r^2}$, $\operatorname{Re}\alpha > 0$ to another Gaussian, $\widehat{\mathcal{G}_\alpha}(p) = (\pi/\alpha)^{d/2} e^{-\pi^2 p^2/\alpha}$.*

Remark. Intuitively, this result follows from dimensionally continuing the well-known integer-dimension result. We should get the same result from direct calculation using Eq. (6) since that expression was obtained using the same dimensional continuation procedure.

Proof. The case $d = 1$ is classical. For $d > 1$, we use ${}_0F_1$'s defining series,

$$
{}_0F_1(d/2; -\pi^2 p^2 r^2) = \sum_{n=0}^{\infty} \frac{(-\pi^2 p^2 r^2)^n}{(d/2)_n n!} \tag{7}
$$

[$(\cdot)_n$ denotes the Pochhammer symbol] and integrate term-by-term, evaluating each integral using the gamma function.[3] The resulting series is the Maclaurin series for the expression we gave for $\hat{\mathcal{G}}_\alpha$. The term-by-term integration is justified by the Lebesgue dominated convergence theorem. To see this, we use the same integral representation for ${}_0F_1$ used in the derivation of Eq. (6), which gives, for any $N \in \mathbb{N}$,

$$
\left| \sum_{n=0}^{N} \frac{(-\pi^2 p^2 r^2)^n}{(d/2)_n n!} \right| \leq {}_0F_1(d/2; \pi^2 p^2 r^2) \leq K \cosh(2\pi p r) \int_0^1 (1-t^2)^{(d-3)/2} dt,
$$

where $K > 0$ is a constant.[4] This allows us to apply the dominated convergence theorem, since the integral in the final term is finite for $d > 1$ and $\int_0^\infty \cosh(2\pi p r)|e^{-\alpha r^2}|dr$ is finite for $\operatorname{Re}\alpha > 0$. □

Remark. The importance of this result to our discussion comes in its use in obtaining the integer dimension Jacobi transformation formula (and thus also the Jacobi imaginary transformations of the Jacobi theta functions) via the standard Poisson summation formula: For a lattice Λ of dimension $n \in \mathbb{N}$, we have (taking $\operatorname{Im} z > 0$ so that everything converges)

$$
\bar{\Theta}_\Lambda(z) := \sum_{k\in\Lambda} e^{i\pi z|k|^2} = \frac{1}{\sqrt{\det\Lambda}} \left(\frac{i}{z}\right)^{n/2} \sum_{p\in\Lambda^*} e^{-i\pi|p|^2/z} = \frac{1}{\sqrt{\det\Lambda}} \left(\frac{i}{z}\right)^{n/2} \bar{\Theta}_{\Lambda^*}(-1/z),
$$

which can be written as

$$
\bar{\Theta}_{\Lambda^*}(z) = \sqrt{\det\Lambda}(i/z)^{n/2}\bar{\Theta}_\Lambda(-1/z), \tag{8}
$$

the Jacobi transformation formula. We thus expect that the dimensionally continued dual theta series that we obtain using this formula will agree with the dimensionally continued Fourier transform to give a dimensionally continued Poisson summation formula.

[2] These identities are 03.01.03.0004.01 and 07.17.03.0037.01, respectively, on the Wolfram Functions Site [13].

[3] The Maclaurin series for ${}_0F_1$ is 07.17.02.0001.01 on the Wolfram Functions Site [13].

[4] *Nota bene*: We denote the set of positive integers by $\mathbb{N}$, and the set of nonnegative integers by $\mathbb{N}_0$.

3. The dimensionally continued Poisson summation formula for $\mathbb{Z}^d$

With these results in hand, we can thus write Eq. (1) [for a radial function $F =: f(|\cdot|)$] as

$$\sum_{l=0}^{\infty} N_l f(\sqrt{A_l}) = \frac{1}{\sqrt{\det \Lambda}} \sum_{l=0}^{\infty} N_l^* \hat{f}(\sqrt{A_l^*}),$$

where the starred quantities come from writing the theta series of Λ^* in the power series form given by Eq. (2), and we calculate $\hat{f}$ by taking the dimension parameter d to be the dimension of the lattice. (As we shall see later, what is important is that the d one uses here is the same d that appears in the Jacobi transformation formula.) It is clear that this equality holds when $d \in \mathbb{N}$, by the standard Poisson summation formula. What is perhaps surprising is that the equality still holds for, e.g., $\Lambda = \mathbb{Z}^d$, with $d \in \mathbb{R}$ ($d \geq 1$). We shall first prove the result for this simple case ($\mathbb{Z}^d$ is self-dual, $\det \mathbb{Z}^d = 1$, and $A_l = l$), where it becomes

Theorem 1. *If $f \in \mathscr{S}^{\mathsf{E}}(\mathbb{R})$ (i.e., f is an even Schwartz function) and $d \geq 1$, then*

$$\sum_{l=0}^{\infty} N_l f(\sqrt{l}) = \sum_{l=0}^{\infty} N_l \hat{f}(\sqrt{l}), \tag{9}$$

where N_l is given by the power series expansion of the theta series of $\mathbb{Z}^d$, viz.,

$$\Theta(q) = \vartheta_3^d(q) = \left[1 + 2\sum_{k=1}^{\infty} q^{k^2}\right]^d =: \sum_{l=0}^{\infty} N_l q^l,$$

and $\hat{f}$ is computed using Eq. (6)*.*

However, the simplifications are primarily notational. As we shall see in the discussion in Sec. 6, the proof works with minimal modifications for a much larger class of Θ, including functions that cannot be the theta series of a lattice (even though they have an integer dimension parameter).

Remark. The restriction that f be an even function should not be surprising: In integer dimensions, it corresponds to the lack of a cusp at the origin for the full radial function $F = f(|\cdot|)$. Moreover, as Miller and Schmid note [4], the standard one-dimensional Poisson summation formula is a trivial $0 = 0$ for odd functions.

4. A Schwartz space density result

Since the proof proceeds by noting that the desired formula holds almost trivially for the Gaussians from Lemma 1, and then extends to an interesting set of functions [viz., $\mathscr{S}^{\mathsf{E}}(\mathbb{R})$] by density, we start by establishing the requisite density result.

Lemma 2. $\overline{\mathrm{Span}\{x \mapsto e^{-\alpha x^2} | \alpha > 0\}}^{\mathscr{S}(\mathbb{R})} = \mathscr{S}^{\mathsf{E}}(\mathbb{R})$ *[i.e., the Schwartz space closure of the given family of Gaussians is all the even Schwartz functions].*

Proof. We shall prove this by showing that

$$\mathcal{X} := \mathrm{Span}\{x \mapsto e^{-\alpha x^2} | \alpha > 0\} + \mathrm{Span}\{x \mapsto x e^{-\alpha x^2} | \alpha > 0\}$$

is dense in $\mathscr{S}(\mathbb{R})$, so its even part, $\mathrm{Span}\{x \mapsto e^{-\alpha x^2} | \alpha > 0\}$, is thus dense in $\mathscr{S}^{\mathsf{E}}(\mathbb{R})$. We shall use Corollary IV.3.14 from Conway's text [14], which states that a linear manifold (here $\mathcal{X}$) is dense in a locally convex topological vector space [here $\mathscr{S}(\mathbb{R})$] if and only if the only element of the dual of the topological vector space that vanishes on all elements of the linear manifold is the zero element.

It is most convenient to proceed by identifying $\mathscr{S}(\mathbb{R})$ with a sequence space, following Simon [15]. Here the sequence space is given by the coefficients of the Hermite function expansion of elements of $\mathscr{S}(\mathbb{R})$, and provides a particularly nice characterization of the tempered distributions [the elements of $\mathscr{S}'(\mathbb{R})$, the dual of $\mathscr{S}(\mathbb{R})$]. Namely, if

a_n are the Hermite coefficients of $f \in \mathscr{S}(\mathbb{R})$ [i.e., $a_n := \int_{\mathbb{R}} f(x)h_n(x)dx$, where h_n is the nth Hermite function], then $\varphi \in \mathscr{S}'(\mathbb{R})$ can be written as $\varphi(f) = \sum_{n=0}^{\infty} c_n a_n$, where c_n are the Hermite coefficients of φ, with $|c_n| \le C(1+n)^m$ for some $C, m > 0$. (This is Theorem 3 in Simon [15].) Note that Simon defines the Hermite functions to be L^2 normalized, so, we have, from the first equation in Sec. 2 of Simon,[5]

$$h_n(x) := \frac{e^{-x^2/2}}{\sqrt{\pi^{1/2}2^n n!}} H_n(x), \qquad H_n(x) := (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2},$$

where the H_n are the Hermite polynomials, with generating function[6]

$$\sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} = e^{2tx - t^2}.$$

We can now use this generating function to show that the Hermite coefficients of $x \mapsto e^{-\alpha x^2}$ are given by

$$a_n = \mathcal{N}_n \frac{d^n}{dt^n} \left[\int_{\mathbb{R}} e^{-(\alpha x^2 + x^2/2 - 2tx + t^2)} dx \right] \Big|_{t=0} = \mathcal{N}_n \frac{d^n}{dt^n} \left[\sqrt{\pi\beta} e^{(\beta-1)t^2} \right] \Big|_{t=0},$$

where $\mathcal{N}_n := (\pi^{1/2} 2^n n!)^{-1/2}$ is the Hermite functions' normalization factor and $\beta := 1/(\alpha + 1/2)$. We thus have $a_{2n} = \mathcal{N}_{2n}(\pi/\beta)^{1/2}(\beta - 1)^n/n!$, $a_{2n+1} = 0$, by the series expansion of the exponential. [We used Lemma 2.2 in Chap. 13 of Lang [16] to interchange differentiation and integration. We only need to consider the case where t lies in some neighbourhood of 0, so the t-derivatives of the integrand are each bounded by a polynomial in x times a Gaussian in x (for all t in the neighbourhood), and those functions of x are integrable over $\mathbb{R}$.] Similarly, the Hermite coefficients of $x \mapsto xe^{-\alpha x^2}$ are $b_{2n} = 0$ and $b_{2n+1} = \mathcal{N}_{2n+1}(\pi/\beta^3)^{1/2}(\beta - 1)^n/n!$. Thus, we consider

$$E_{\beta,\pm}(x) := (\beta/\pi)^{1/2} e^{-\alpha x^2} \pm (\beta^3/\pi)^{1/2} x e^{-\alpha x^2},$$

which has Hermite coefficients of $(\pm 1)^n \mathcal{N}_n (\beta - 1)^{\lfloor n/2 \rfloor}/\lfloor n/2 \rfloor!$, where $\lfloor \cdot \rfloor$ denotes the greatest integer less than or equal to its argument.

Now, for any $\varphi \in \mathscr{S}'(\mathbb{R})$, $\mathcal{E}_\pm(\beta) := \varphi(E_{\beta,\pm})$ is a holomorphic function of β. To see this, we note that

$$\mathcal{E}_\pm(\beta) = \sum_{n=0}^{\infty} (\pm 1)^n c_n \mathcal{N}_n \frac{(\beta - 1)^{\lfloor n/2 \rfloor}}{\lfloor n/2 \rfloor!} = \sum_{n=0}^{\infty} (\mathcal{N}_{2n} c_{2n} \pm \mathcal{N}_{2n+1} c_{2n+1}) \frac{(\beta - 1)^n}{n!}, \tag{10}$$

where c_n are the Hermite coefficients of φ. Since the c_n are bounded by a polynomial in n, the series converges for all $\beta \in \mathbb{C}$, giving holomorphy. Thus, if $\mathcal{E}_\pm(\beta) = 0$ for all β in an interval (as is the case here), then all of $\mathcal{E}_\pm$'s power series coefficients are zero. Applying this result to the two choices of sign, we obtain (since the $\mathcal{N}_n$ are never zero) $c_n = 0\ \forall\, n \in \mathbb{N}_0 \Rightarrow \varphi \equiv 0$, which thus proves the lemma. □

Remark. This result may be of wider applicability, particularly in harmonic analysis, due to the ubiquity of the Gaussian. We thus note that the proof of the lemma shows that α need merely belong to some subset of the right half-plane with an accumulation point to guarantee density. One could have also proved this result more abstractly (and without recourse to the Hermite expansion) by a slightly indirect application of the Stone-Weierstrass theorem, though the basic Hahn-Banach argument (contained in the Corollary from Conway we use) remains the same.[7]

[5] *Nota bene*: Simon defines the h_n without the factor of $(-1)^n$ (that here comes from our H_n). We have included the $(-1)^n$ for notational simplicity (since we use the standard convention for the Hermite polynomials). This does not have any effect on Simon's Theorem 3, since it simply amounts to a sign change of the odd Hermite coefficients.

[6] This is 05.01.11.0001.01 on the Wolfram Functions Site [13].

[7] Personal communication from John Roe.

5. Proof of Theorem 1

We first note that Eq. (9) is clearly true for $d = 1$ (indeed, $d \in \mathbb{N}$) by the standard Poisson summation formula for lattices (applied to $\mathbb{Z}^d$). To prove the result for $d > 1$, we shall first establish that it holds for the Gaussians from Lemma 1, and then show that the equality still holds in the limit in the Schwartz space topology. The control afforded by demanding convergence in the Schwartz space makes this quite straightforward. The primary result that needs to be shown is that two functions that are ϵ-close in the Schwartz space topology have dimensionally continued Fourier transforms that are $C\epsilon$-close in a given Schwartz space seminorm (where the constant C depends on the seminorm under consideration, as well as d).

To show that Eq. (9) holds when $f = \mathcal{G}_\alpha$, we first consider the left-hand side and note that

$$\sum_{l=0}^{\infty} N_l e^{-\alpha l} = \Theta(e^{-\alpha}). \tag{11}$$

Convergence is guaranteed because Θ is analytic inside the unit disk. [To see that Θ is analytic inside the unit disk, note that ϑ_3 is analytic there, and, moreover, nonzero, so its dth power is analytic, as well. It is easiest to see that ϑ_3 is nonzero inside the unit disk from its infinite product expansion, given in, e.g., Eq. (35) in Chap. 4 of Conway and Sloane [3].] Using Lemma 1, the right-hand side of Eq. (9) becomes

$$\left(\frac{\pi}{\alpha}\right)^{d/2} \sum_{l=0}^{\infty} N_l e^{-\pi^2 l/\alpha} = \left(\frac{\pi}{\alpha}\right)^{d/2} \Theta(e^{-\pi^2/\alpha}).$$

Now, the Jacobi imaginary transformation for ϑ_3 [Eq. (5)] implies that $(\pi/\alpha)^{d/2}\Theta(e^{-\pi^2/\alpha}) = \Theta(e^{-\alpha})$, so we have thus established the result for $\mathcal{G}_\alpha$.

We shall now show that this equality continues to hold in the limit. The equality is clearly true for any finite linear combination of the Gaussians from Lemma 1, so we use Lemma 2 to approximate an arbitrary $f \in \mathscr{S}^{\mathsf{E}}(\mathbb{R})$ by a finite linear combination of these Gaussians, g. Specifically, we have $\|f - g\|_{n,m} < \epsilon \ \forall\, n, m \in \mathbb{N}_0$, where $\|f\|_{n,m} := \sup_{x\in\mathbb{R}} |x^n f^{(m)}(x)|$ is the family of seminorms that gives the Schwartz space topology. (We denote the mth derivative of f by $f^{(m)}$.) We wish to bound the difference between the two sides of Eq. (9) by a constant times ϵ. We have

$$\left|\sum_{l=0}^{\infty} N_l f(\sqrt{l}) - \sum_{l=0}^{\infty} N_l \hat{f}(\sqrt{l})\right| \le \left|\sum_{l=0}^{\infty} N_l (f-g)(\sqrt{l})\right| + \left|\sum_{l=0}^{\infty} N_l (\hat{f}-\hat{g})(\sqrt{l})\right|, \tag{12}$$

where we used the fact that the dimensionally continued Poisson summation formula holds for g, along with the triangle inequality. We can bound the two sums on the right-hand side by constants times ϵ using the assumption about the closeness of f to g in the Schwartz space topology and the fact that N_l grows at most polynomially with l. The latter fact also shows that the two sums on the left converge for $f \in \mathscr{S}(\mathbb{R})$.

5.1. *Bounds on the growth of N_l and on the right-hand side of Eq.* (12)

We obtain the polynomial bound on N_l using Cauchy's integral formula with the contour C_R, a circle of radius $R \in (0, 1)$, centred at the origin (and oriented counterclockwise):

$$|N_l| = \left|\frac{1}{2\pi i}\int_{\mathsf{C}_R} \frac{\vartheta_3^d(z)}{z^{l+1}} dz\right| = \frac{1}{2\pi}\left|\int_0^{2\pi} \frac{\vartheta_3^d(Re^{i\theta})}{R^l e^{il\theta}} d\theta\right| \le \frac{2^d}{R^l(1-R)^d}.$$

Here we have used $|\vartheta_3(q)| \le 2/(1-|q|)$ (for $|q| < 1$, obtained using the geometric series). The right-hand side attains its minimum [for $R \in (0, 1)$] at $R = l/(l + d)$, so we have

$$|N_l| \le 2^d(1 + d/l)^l(1 + l/d)^d \le C_d l^d,$$

where $C_d > 0$ is some constant (and the second inequality only holds for $l \ge 1$). [We have used the fact that $(1 + 1/r)^r < e$ for $r > 0$.]

If we write $h := f - g$, then this bound implies that $|N_l h(\sqrt{l})| \le C_d l^d |h(\sqrt{l})| \le \epsilon C_d/l^2$ (for $l \ge 1$), where the second inequality follows from the fact that h is ϵ-close to 0 in the Schwartz space topology. [Explicitly, we have

$|x^{2d+4}h(x)| \leq \epsilon \;\forall\; x > 1 \Rightarrow l^d|h(\sqrt{l})| \leq \epsilon/l^2 \;\forall\; l \in \mathbb{N}$. The first inequality comes from noticing that for any $\gamma \geq 0$, we have $|x^\gamma h(x)| \leq |x^{\lceil\gamma\rceil}h(x)| \leq \epsilon$ for $x \geq 1$, where $\lceil\cdot\rceil$ denotes the smallest integer greater than or equal to its argument.] We shall show that $|p^{2n}\hat{h}(p)| \leq K_d\epsilon \;\forall\; n \in \mathbb{N}$, $p \in \mathbb{R}$ (where K_d is some n-dependent constant), so we have $|p^{2d+4}\hat{h}(p)| \leq K_d\epsilon \;\forall\; p \in \mathbb{R}$. We can thus apply the same argument to the second sum and hence bound both sums by constants times ϵ (since $\sum_{l=1}^\infty l^{-2}$ is finite), showing that the dimensionally continued Poisson summation formula is true in the limit [since we will have shown that the right-hand side of Eq. (12) is bounded by a constant times ϵ].

5.2. *Bound on* $|p^{2n}\hat{h}(p)|$

To prove the bound on $|p^{2n}\hat{h}(p)|$, we first dimensionally continue some standard Fourier results.

Lemma 3. *If we define the d-dimensional Laplacian for radial functions by*

$$\triangle_d f(r) := f''(r) + \frac{d-1}{r}f'(r), \tag{13}$$

then, for $d > 1$,

i) $\mathcal{F}_p(r) := {}_0F_1(d/2; -\pi^2p^2r^2)$ *satisfies* $\triangle_d\mathcal{F}_p = -4\pi^2p^2\mathcal{F}_p$, *so*

ii) $\widehat{\triangle_d^n f}(p) = (-1)^n(2\pi p)^{2n}\hat{f}(p)$ *for* $f \in \mathscr{S}(\mathbb{R})$.

Proof. Part *i* follows from the fact that $y_a(r) := {}_0F_1(a;r)$ satisfies $ry_a''(r) + ay_a'(r) = y_a(r)$.[8] [Alternatively, it can be obtained by direct calculation using Eq. (7), justifying term-by-term differentiation using analyticity.] Part *ii* is then obtained by induction, applying Eq. (6) to $\triangle_d^{n-1}f$ and integrating by parts twice. [The boundary terms at infinity vanish because $f \in \mathscr{S}(\mathbb{R})$; those at 0 vanish because $d > 1$ (or cancel amongst themselves).] $\square$

We can thus write $|p^{2n}\hat{h}(p)| = (2\pi)^{-2n}|\widehat{\triangle_d^n h}(p)|$. Then, since we shall show below that $|r^k\triangle_d^n h(r)| \leq \mathcal{D}\epsilon$, where $\mathcal{D}$ is some (n- and d-dependent constant), we obtain [using Eq. (6) and the fact that ${}_0F_1(a;r)$ is a bounded function of r, as was seen in the proof of Lemma 1]

$$\begin{aligned}
|p^{2n}\hat{h}(p)| &\leq \mathcal{C}\int_0^\infty |\triangle_d^n h(r)|r^{d-1}dr \\
&\leq \mathcal{C}\left[\int_0^1 |\triangle_d^n h(r)|dr + \int_1^\infty |\triangle_d^n h(r)|r^{d-1}dr\right] \\
&\leq \mathcal{C}\mathcal{D}\left[1 + \int_1^\infty r^{d-1-s}dr\right]\epsilon,
\end{aligned}$$

where $\mathcal{C} > 0$ is some (n- and d-dependent) constant and we used $|r^k\triangle_d^n h(r)| \leq \mathcal{D}\epsilon$ with $k = 0$ and $k = s$. We can choose $s > d$, so the integral in the final term is finite, thus giving the desired result.

5.3. *Bound on* $|r^k\triangle_d^n h(r)|$

To see that $|r^k\triangle_d^n h(r)|$ is bounded by some (n- and d-dependent) constant (called $\mathcal{D}$ above), we first note that we can use induction to write

$$\triangle_d^n h(r) = \sum_{j=1}^{2n} a_j\frac{h^{(j)}(r)}{r^{2n-j}} \tag{14}$$

for some (n- and d-dependent) constants a_j (and an arbitrary differentiable h). Thus, for $|r| \geq 1$, we have $|r^k\triangle_d^n h(r)| \leq \epsilon\sum_{j=1}^{2n} a_j$. For $|r| < 1$, matters are considerably more subtle, and we have to rely on the fact that h is even to see that $\triangle_d^n h$ remains bounded at the origin. The argument goes as follows: We write $h = \mathcal{P} + \mathcal{R}$, where $\mathcal{P}$ is h's

[8] This differential equation for ${}_0F_1$ is 07.17.13.0003.01 on the Wolfram Functions Site [13].

$(2n)$th-degree Maclaurin polynomial (necessarily even, since h is) and $\mathcal{R}$ is the associated remainder. We then have $|r^k \triangle_d^n h(r)| \leq |r^k \triangle_d^n \mathcal{P}(r)| + |r^k \triangle_d^n \mathcal{R}(r)|$. Since $\triangle_d$ maps even polynomials to even polynomials [as can be seen from Eq. (14)], $|r^k \triangle_d^n \mathcal{P}(r)|$ is bounded by a (k-, n-, and d-dependent) constant times ϵ for $|r| \leq 1$. [Since the coefficients of $\mathcal{P}$ are given by derivatives of h, they are bounded by constants times ϵ, by hypothesis.] To deal with $|r^k \triangle_d^n \mathcal{R}(r)|$, we first need to establish an identity for derivatives of $\mathcal{R}$, viz., (for $j \leq 2n$)

$$\mathcal{R}^{(j)}(r) = \frac{h^{(2n+1)}(\xi_j)}{(2n+1-j)!} r^{2n+1-j},$$

for some $\xi_j \in (0, r)$. This follows from differentiating $h = \mathcal{P} + \mathcal{R}$ j times, and comparing the resulting expression with the $(2n-j)$th order Maclaurin expansion (with Lagrange remainder) of $h^{(j)}$. The polynomial pieces are the same, while the remainder pieces give the two sides of the equality. Combining this identity with Eq. (14), we obtain

$$\triangle_d^n \mathcal{R}(r) = r \sum_{j=1}^{2n} b_j h^{(2n+1)}(\xi_j),$$

where the b_j are (n- and d-dependent) constants. This shows that $|r^k \triangle_d^n \mathcal{R}(r)|$ is bounded by an (n- and d-dependent) constant times ϵ for $|r| \leq 1$, so $|r^k \triangle_d^n h(r)|$ is, as well, proving the desired result, and hence the theorem. $\square$

Remark. The restrictions on f and d in the statement of the theorem are surely not optimal: There is numerical evidence that the given result holds for $d \in \mathbb{C}$, $\operatorname{Re} d > 0$ and less smooth f [e.g., $f(r) = e^{-|r|^3}$]. (The evidence also extends to the generalization given in Theorem 2 and is provided by a MATHEMATICA notebook, available online.[9]) While one could use a slightly larger function space than $\mathscr{S}^{\mathsf{E}}(\mathbb{R})$ without any change to the proof—the proof does not need control over $\|f-g\|_{n,m}$ for all n and m—we did not investigate this in any detail: The resulting function space would still require a fair amount of differentiability (while we have numerical evidence that the formula remains true for at least some functions with a cusp at the origin), and faster decay than the standard Poisson summation formula. Moreover, the closure of the family of Gaussians in this less restrictive topology would almost surely be more recondite than $\mathscr{S}^{\mathsf{E}}(\mathbb{R})$.

6. Generalization of Theorem 1

Since there are other families of lattices with dimensionally continued theta series besides $\mathbb{Z}^d$ (e.g., the root lattice D^d mentioned in Sec. 2.1), it is reasonable to expect that Theorem 1 can be generalized by replacing Θ with some more general function Υ. In fact, we have the following

Theorem 2. *Assume that we have a function Υ and $d > 1$ such that*

i) We can write

$$\Upsilon(q) = \sum_{l=0}^{\infty} N_l q^{A_l},$$

where

1. $A_{l+1} > A_l$, $A_0 \geq 0$.
2. $\sum_{l=1}^{\infty} A_l^{-m} < \infty$ *for some* $m \in \mathbb{N}$.
3. *There exists* $L \in \mathbb{N}$ *and* $C, n > 0$ *such that* $|N_l| \leq C A_l^n$ *for all* $l \geq L$,
4. *The* A_l *are such that the series converges inside the unit disk.*[10]

[9] The notebook is available at `http://gravity.psu.edu/~nathanjm/Dim_cont_PSF_test.nb`.

[10] The ratio test and the given bound on N_l provide sufficient conditions for convergence inside the unit disk, viz., $A_{l+1} - A_l \geq \delta$ for some $\delta > 0$ and $\lim_{l\to\infty} A_{l+1}/A_l = 1$. (These are satisfied by the specific examples we consider.) However, one could relax the requirement of convergence to some subset of the unit disk, as the remark following Lemma 2 reveals.

ii) If we define

$$\bar{\Upsilon}^*(z) := (i/z)^{d/2}\bar{\Upsilon}(-1/z)$$

(recall the overbar notation introduced in the nota bene *in Sec. 2.1), then we can write*

$$\Upsilon^*(q) = \sum_{l=0}^{\infty} N_l^* q^{A_l^*},$$

with the same hypotheses about the series as in part i.

Then, for any $f \in \mathscr{S}^{\mathsf{E}}(\mathbb{R})$, we have the summation formula

$$\sum_{l=0}^{\infty} N_l f(\sqrt{A_l}) = \sum_{l=0}^{\infty} N_l^* \hat{f}(\sqrt{A_l^*}),$$

where we compute $\hat{f}$ using Eq. (6) *(with the d used in part* ii*).*

Remark. The definition of Υ^* is just the dimensionally continued Jacobi transformation [Eq. (4)] of Υ with the factor of $\sqrt{\det \Upsilon}$ omitted. (We leave off this factor, since it would just cancel against the one present in the standard Poisson summation formula [cf. Eq. (1)].)

Proof. The proof is almost the same as that for Theorem 1 (replacing Θ by Υ, and noting that we can no longer appeal to the standard Poisson summation formula for $d = 1$, so we simply exclude that case). Our hypotheses are such that the only new part is checking that $\sum_{l=0}^{\infty} |N_l h(\sqrt{A_l})| \to 0$ as $\epsilon \to 0$ if h is ϵ-close to 0 in the Schwartz space topology [and similarly for $\sum_{l=0}^{\infty} |N_l^* h(\sqrt{A_l^*})|$]. To do this, we simply note that we have $|N_l h(\sqrt{A_l})| \leq C A_l^n |h(\sqrt{A_l})|$, by hypothesis, and that $x^{2(n+m)}|h(x)| \leq \epsilon \ \forall\ x \in \mathbb{R} \Rightarrow A_l^n |h(\sqrt{A_l})| \leq \epsilon / A_l^m$, from which the desired result follows immediately. (The same argument holds for the starred quantities, since the hypotheses are identical.) $\square$

Remark. This theorem can likely be interpreted as a trace formula for the dimensionally continued, spherically symmetric Laplacian [Eq. (13)], since the kernel of the dimensionally continued Fourier transform is an eigenfunction of this operator (see Lemma 3). See, e.g., Sec. 1.3 (particularly Theorem 1.3) of Uribe [17] for a presentation of the standard Poisson summation formula for an integer dimension lattice as a trace formula for the Laplacian.

It is not clear how to construct the most general Υ satisfying the hypotheses of Theorem 2.[11] Nevertheless, it is easy enough to write down a reasonably general, yet fairly simple family of functions that does satisfy these hypotheses, viz., finite linear combinations of products of the three basic theta functions given in Eq. (3). Specifically, we can consider products of the form

$$\Upsilon_d(q) := \prod_{m=1}^{\mathcal{M}} \vartheta_2^{\lambda_m}(q^{s_m})\vartheta_3^{\rho_m}(q^{t_m})\vartheta_4^{\sigma_m}(q^{u_m}), \tag{15}$$

where $\lambda_m, \rho_m, \sigma_m \geq 0$, $\sum_{m=1}^{\mathcal{M}}(\lambda_m + \rho_m + \sigma_m) = d$, and $s_m, t_m, u_m \in \mathbb{Q}_+$. The extension to finite linear combinations of Υ_ds with the same d is trivial, by linearity. One can write all the theta series given in Chap. 4 of Conway and Sloane [3] as such finite sums of Υ_ds, except for the general form of the theta series of the root lattice A^d and its translates. In fact, theorems in Conway and Sloane (Theorems 7, 15, and 17 in Chap. 7 and Theorem 5 in Chap. 8) show that the theta series of large classes of lattices can be written in such a form. However, the expression in terms of Υ_ds is considerably more general, since one only requires $\lambda_m, \rho_m, \sigma_m \in \mathbb{N}_0$ to reproduce the theta series in Conway and Sloane, while here they can be arbitrary nonnegative real numbers. One can also use the general template provided by Eq. (15) to construct other admissible Υs from, e.g., automorphic forms, or other such functions ψ that satisfy the relation $\bar{\psi}(-1/z) = (z/i)^{\eta}\bar{\phi}(z)$ for some $\eta \in \mathbb{R}$ (where ϕ and ψ are well-behaved enough that the power series of the Υ constructed using these functions satisfies the hypotheses of Theorem 2).

We now want to establish the following

[11] However, Ryavec characterizes all admissible Υs (under certain assumptions) for $d = 1$ in [18]. We also call attention to the work of Córdoba [19, 20], who shows that in integer dimensions, large classes of generalized Poisson summation formulae arise from the standard Poisson summation formula applied to the finite disjoint union of (integer dimensional) lattices.

Proposition. Υ_d *[defined in Eq.* (15)*] satisfies the requirements given in the statement of Theorem 2.*

Proof. First note that the Jacobi imaginary transformations of the theta functions [given in Eq. (5)] imply that

$$\Upsilon_d^*(q) = \prod_{m=1}^{\mathcal{M}} \frac{\vartheta_2^{\sigma_m}(q^{1/u_m})\vartheta_3^{\rho_m}(q^{1/t_m})\vartheta_4^{\lambda_m}(q^{1/s_m})}{\sqrt{s_m^{\lambda_m} t_m^{\rho_m} u_m^{\sigma_m}}}, \tag{16}$$

so the arguments we give for Υ_d can be applied to Υ_d^* immediately. Now, we have

$$A_l = (l + \mathcal{A})/V, \qquad \mathcal{A} := \sum_{m=1}^{\mathcal{M}} V\lambda_m s_m/4$$

where V is the least common denominator of s_m, t_m, and u_m (for all m). [We have the additive constant $\mathcal{A}$ due to the overall factor of $q^{1/4}$ in $\vartheta_2(q)$.] Thus the first required property (positivity and monotonicity of the A_l) is obviously true, and the second (convergence of the series whose terms are A_l^{-m}) is clearly true for $m = 2$ (and $L = 1$), as before.

For the third property (polynomial boundedness of the N_l), we use the same Cauchy's integral formula argument used in Sec. 5.1. Here N_l is given by the lth term in the Maclaurin expansion of $\Upsilon_d(q^V)/q^{\mathcal{A}}$, so we have

$$\begin{aligned} |N_l| &= \left| \frac{1}{2\pi i} \int_{\mathsf{C}_R} \prod_{m=1}^{\mathcal{M}} \frac{\vartheta_2^{\lambda_m}(z^{V s_m})\vartheta_3^{\rho_m}(z^{V t_m})\vartheta_4^{\sigma_m}(z^{V u_m})}{z^{V\lambda_m s_m/4} z^{l+1}} dz \right| \\ &\leq 2^d \prod_{m=1}^{\mathcal{M}} \frac{1}{R^l (1 - R^{V s_m})^{\lambda_m} (1 - R^{V t_m})^{\rho_m} (1 - R^{V u_m})^{\sigma_m}} \leq \frac{2^d}{R^l (1-R)^d}, \end{aligned}$$

where C_R is the same contour used previously. We have used the geometric series to obtain the bound $|\tau(q)| \leq 2/(1 - |q|)$, where $\tau(q)$ is any of $\vartheta_2(q)/q^{1/4}$, $\vartheta_3(q)$, or $\vartheta_4(q)$. Additionally, we have used the fact that $\kappa \geq 1$, where κ is any of $V s_m$, $V t_m$, or $V u_m$, so $|1 - R^\kappa| \geq 1 - R$, since $R \in (0,1)$. We also recalled that $\lambda_m, \rho_m, \sigma_m \geq 0$ and $\sum_{m=1}^{\mathcal{M}} (\lambda_m + \rho_m + \sigma_m) = d$. Since there is an $R \in (0,1)$ such that $2^d/[R^l(1-R)^d] \leq C_d l^d$ (for $l \geq 1$), as was shown in Sec. 5.1, we are done.

The fourth property (convergence of the q-series in the unit disk) follows from the analyticity and lack of zeros of the theta functions inside the unit disk, as in the argument given below Eq. (11). Specifically, $\Upsilon(q^V)/q^{\mathcal{A}}$ is an analytic function of q inside the unit disk; the lack of zeros can be seen from the infinite product representations of ϑ_2 and ϑ_4 given, e.g., in Eqs. (34) and (36) in Chap. 4 of Conway and Sloane [3]. $\square$

For clarity, we present the summation formula given by the Proposition and Theorem 2 explicitly (and without any reference to the transformation formula) as the following

Corollary. *Let* Φ *be a finite linear combination of* Υ_d*s (from Eq.* (15)*; all with the same* $d > 1$*) and let* Ψ *be the analogous linear combination of the* Υ_d^**s given in Eq.* (16) *(i.e., with the same parameters and coefficients as for the* Υ_d*s). If we write*

$$\Phi(q) =: \sum_{l=0}^{\infty} N_l q^{A_l}, \qquad \Psi(q) =: \sum_{l=0}^{\infty} P_l q^{B_l},$$

then, for any $f \in \mathscr{S}^{\mathsf{E}}(\mathbb{R})$*, we have*

$$\sum_{l=0}^{\infty} N_l f(\sqrt{A_l}) = \sum_{l=0}^{\infty} P_l \hat{f}(\sqrt{B_l}),$$

where $\hat{f}$ *is computed using Eq.* (6)*.*

Remark. This result shows that one can apply this extended Poisson summation formula to lattice-like objects whose theta series have coefficients of both signs, so they do not exist as a lattice, even though $d \in \mathbb{N}$: For a trivial example, consider $d = 2$ and $\Phi(q) = \vartheta_4^2(q) = 1 - 4q + 4q^2 + \cdots$. Of course, this is in some sense too trivial, since one can write $\vartheta_4^2 = 2\Theta_{D^2} - \Theta_{\mathbb{Z}^2}$, and then apply the standard Poisson summation formula to each of those lattices to establish the result in this case (cf. the discussion in Córdoba [20]). However, in more complicated higher-dimensional cases, it will likely not be clear how to construct the lattice(s) associated with the theta series (if they indeed exist).

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References

[1] R. E. Johnson, S. Ranganathan, Generalized approach to Ewald sums, Phys. Rev. E 75 (2007) 056706.

[2] R. Contino, A. Gambassi, On dimensional regularization of sums, J. Math. Phys. 44 (2003) 570–587.

[3] J. H. Conway, N. J. A. Sloane, Sphere Packings, Lattices and Groups, Springer-Verlag, New York, third edition, 1999.

[4] S. D. Miller, W. Schmid, Summation formulas, from Poisson and Voronoi to the present, in: P. Delorme, M. Vergne (Eds.), Noncommutative Harmonic Analysis: In honor of Jacques Carmona, Birkhäuser, Boston, 2004, pp. 419–440.

[5] A. Sklar, On Some Exact Formulae in Analytic Number Theory, in: D. C. B. Marsh (Ed.), Report of the Institute in the Theory of Numbers, American Mathematical Society, Boulder, CO, 1959, pp. 104–110.

[6] W. L. Ferrar, Summation formulae and their relation to Dirichlet's series II, Compos. Math. 4 (1937) 394–405.

[7] M. Baake, D. Frettlöh, U. Grimm, A radial analogue of Poisson's summation formula with applications to powder diffraction and pinwheel patterns, J. Geom. Phys. 57 (2007) 1331–1343.

[8] H. Iwaniec, E. Kowalski, Analytic Number Theory, American Mathematical Society, Providence, RI, 2004.

[9] M. N. Huxley, Area, Lattice Points, and Exponential Sums, Oxford University Press, New York, 1996.

[10] E. T. Whittaker, G. N. Watson, A Course of Modern Analysis, Cambridge University Press, Cambridge, 1944.

[11] R. Bellman, A Brief Introduction to Theta Functions, Holt, Rinehart and Winston, New York, 1961.

[12] E. M. Stein, G. Weiss, Introduction to Fourier Analysis on Euclidean Spaces, Princeton University Press, Princeton, 1971.

[13] Wolfram Research, *The Wolfram Functions Site*, 2010. http://functions.wolfram.com.

[14] J. B. Conway, A Course in Functional Analysis, Springer, New York, second edition, 1990.

[15] B. Simon, Distributions and Their Hermite Expansions, J. Math. Phys. 12 (1971) 140–148.

[16] S. Lang, Real Analysis, Addison-Wesley, Reading, MA, second edition, 1983.

[17] A. Uribe, Trace formulae, in: S. Pérez-Esteva, C. Villegas-Blas (Eds.), First Summer School in Analysis and Mathematical Physics, Cuernavaca, Mexico, American Mathematical Society, 2000, pp. 61–90.

[18] C. Ryavec, The Poisson summation formula, Aequ. Math. 21 (1980) 246–250.

[19] A. Córdoba, La formule sommatoire de Poisson, C. R. Acad. Sci. Paris Sér. I Math. 306 (1988) 373–376.

[20] A. Córdoba, Dirac Combs, Lett. Math. Phys. 17 (1989) 191–196.