On the distribution of exponential functionals for Lévy processes with jumps of rational transform

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Abstract

We derive explicit formulas for the Mellin transform and the distribution of the exponential functional for Lévy processes with rational Laplace exponent. This extends recent results by Cai and Kou [3] on the processes with hyper-exponential jumps.

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1 Introduction

Assume that X is a Lévy process and $\mathrm{e}(q)$ is an independent random variable, which is exponentially distributed with parameter $q > 0$. The exponential functional of X is defined as

$$I_q = \int_0^{\mathrm{e}(q)} e^{X_s} \mathrm{d}s. \tag{1}$$

We can extend this definition to the case $q = 0$, if we interpret $\mathrm{e}(0) \equiv +\infty$ and assume that the process X drifts to $-\infty$. Our main object of interest is the probability density function of I_q, defined as

$$p(x) = \frac{\mathrm{d}}{\mathrm{d}x} \mathbb{P}(I_q \le x), \quad x > 0.$$

Exponential functional of a Lévy process is a very interesting object, which has many applications in such areas as self-similar Markov processes, branching processes and Mathematical Finance. See [1] for an overview of this topic. There exists an extensive literature covering the asymptotic behavior of $p(x)$ as $x \to +\infty$ (see [5, 6, 13, 17, 18] and the references therein) or as $x \to 0^+$ (see [14, 2]). At the same time, the distribution of the exponential functional is known explicitly only in a few special cases: when X is a standard Poisson process, Brownian motion with drift, a particular spectrally negative Lamperti-stable process (see for instance [5, 11, 15]), spectrally positive Lévy process satisfying the Cramér's condition (see for instance [16]). Note that in all above cases we have processes with one-sided jumps, and until very recent time there were no known examples of processes with double-sided jumps, for which the distribution of the exponential functional is known explicitly. However this situation has changed in the last several years.

First of all, in a recent paper [3] (which is the main inspiration for our current work), Cai and Kou have obtained an explicit formula for the Mellin transform of I_q in the case when X has hyper-exponential jumps. While the authors did not provide a formula for the probability density function $p(x)$, it does follow from their results rather easily with the help of the theory of Meijer's G-function. Second, in [10] Kuznetsov and Pardo have derived an explicit infinite series representation and complete asymptotic expansions for $p(x)$ in the case of hypergeometric Lévy processes.

In this paper we will pursue the following three goals. First of all, we will generalize the results of Cai and Kou [3] to the class of Lévy processes with jumps of rational transform, and we will also cover the cases when X has zero Gaussian component and/or zero linear drift. Second, we will show how to use the verification technique which was developed in [10]; we feel that this technique is of independent interest as it considerably simplifies the derivation of many results on exponential functionals. Finally, we will present explicit formulas and complete asymptotic expansions for $p(x)$, which allow very simple numerical evaluation of this function and other related quantities, such as the price of an Asian option with exponentially distributed maturity.

2 Results

We will work with a Lévy process X, for which the density of the Lévy measure is given by

$$\pi(x) = \mathbf{1}_{\{x>0\}} \sum_{j=1}^{J} \alpha_j x^{m_j - 1} e^{-\rho_j x} + \mathbf{1}_{\{x<0\}} \sum_{j=1}^{\hat{J}} \hat{\alpha}_j x^{\hat{m}_j - 1} e^{\hat{\rho}_j x}. \tag{2}$$

We assume that $m_j, \hat{m}_j \in \mathbb{N}$, $\mathrm{Re}(\rho_j) > 0$ and $\mathrm{Re}(\hat{\rho}_j) > 0$. It is clear that X has jumps of finite intensity λ, where

$$\lambda = \int_{\mathbb{R}} \pi(x)\mathrm{d}x = \sum_{j=1}^{J} \alpha_j (m_j - 1)! \rho_j^{-m_j} + \sum_{j=1}^{\hat{J}} \hat{\alpha}_j (\hat{m}_j - 1)! \hat{\rho}_j^{-\hat{m}_j},$$

in particular the process X can be identified as

$$X_t = \sigma W_t + \mu t + \sum_{i=1}^{N_{\lambda t}} \xi_i,$$

where N_t is the standard Poisson process and ξ_i are independent and have distribution $\mathbb{P}(\xi_i \in \mathrm{d}x) = \lambda^{-1}\pi(x)\mathrm{d}x$.

Formula (2) and the Lévy-Khinchine formula imply that the Laplace exponent $\psi(z) = \ln \mathbb{E}[\exp(zX_1)]$ is a rational function, which has the following partial fraction decomposition

$$\psi(z) = \frac{\sigma^2}{2} z^2 + \mu z + \sum_{j=1}^{J} \frac{\alpha_j (m_j - 1)!}{(\rho_j - z)^{m_j}} + \sum_{j=1}^{\hat{J}} \frac{\hat{\alpha}_j (\hat{m}_j - 1)!}{(\hat{\rho}_j + z)^{\hat{m}_j}} - \lambda. \tag{3}$$

We see that $\psi(z)$ has poles at $\{\rho_j\}_{1 \le j \le J}$ and $\{-\hat{\rho}_j\}_{1 \le j \le \hat{J}}$ with the corresponding multiplicities m_j and $\hat{m}_j$. Note that $\psi(z)$ has $M = \sum_{j=1}^{J} m_j$ poles (counting with multiplicity) in the half-plane $\mathrm{Re}(z) > 0$ and $\hat{M} = \sum_{j=1}^{\hat{J}} \hat{m}_j$ poles in the half-plane $\mathrm{Re}(z) < 0$.

The poles of $\psi(z)$ are important objects for our further results, but even more important are the zeros of $\psi(z) - q$, or, equivalently, the solutions to the equation $\psi(z) = q$, where $q \ge 0$. First of all, let us find the total number of these solutions. Since $\psi(z)$ is a rational function of the form (3), we can rewrite it as

$$\psi(z) = \frac{\mathcal{Q}(z)}{\mathcal{P}(z)}. \tag{4}$$

The degree of the numerator {denominator} will be denoted as $Q = \deg(\mathcal{Q})$ $\{P = \deg(\mathcal{P})\}$. Note that (3) implies that $P = M + \hat{M}$ and

$$\begin{cases} Q = P + 2 \text{ if } \sigma > 0, \\ Q = P + 1 \text{ if } \sigma = 0 \text{ and } \mu \neq 0, \\ Q = P \text{ if } \sigma = \mu = 0. \end{cases}$$

We claim that in all three cases the equation $\psi(z) = q$ has exactly Q solutions $\zeta = \zeta(q)$ in $\mathbb{C}$ (counting multiplicity). This is obvious in the case $\sigma > 0$ or $\sigma = 0$ and $\mu \neq 0$, as in this case the equation $\psi(z) = q$ is equivalent to the polynomial equation $\mathcal{Q}(z) - q\mathcal{P}(z) = 0$, which has degree Q. This statement is also true in the case $\sigma = \mu = 0$: here we have $\psi(z) = \mathcal{Q}(z)/\mathcal{P}(z) \to -\lambda$ as $z \to \infty$, therefore the leading term of the polynomial $\mathcal{Q}(z) - q\mathcal{P}(z)$ can not be zero, thus the degree of this polynomial is also equal to Q (note that the degree of this polynomial is equal to $Q - 1$ if $q = -\lambda < 0$).

Let us denote the zeros of $\psi(z) - q$ in the half-plane $\mathrm{Re}(z) > 0$ $\{\mathrm{Re}(z) < 0\}$ as $\zeta_1, \zeta_2, \dots, \zeta_K$ $\{-\hat{\zeta}_1, -\hat{\zeta}_2, \dots, -\hat{\zeta}_{\hat{K}}\}$. Note that by definition we have $Q = K + \hat{K}$. We also assume that these numbers are labelled in the order of increase of the real part, so that $\mathrm{Re}(\zeta_j) \le \mathrm{Re}(\zeta_{j+1})$ and $\mathrm{Re}(\hat{\zeta}_j) \le \mathrm{Re}(\hat{\zeta}_{j+1})$.

Let us summarize the definition of some important quantities, which will be used frequently later:

- M $\{\hat{M}\}$ is the number of poles of $\psi(z)$ in the half-plane $\text{Re}(z) > 0$ $\{\text{Re}(z) < 0\}$,
- K $\{\hat{K}\}$ is the number of zeros of $\psi(z) - q$ in the half-plane $\text{Re}(z) > 0$ $\{\text{Re}(z) < 0\}$,
- $P = M + \hat{M}$ $\{Q = K + \hat{K}\}$ is the total number of poles $\{$zeros$\}$ of $\psi(z) - q$.

We collect all important information about the solutions of the equation $\psi(z) = q$ in the next Proposition.

Proposition 1. *Assume that $q > 0$. Then*

(i) ζ_1 *and* $\hat{\zeta}_1$ *are real positive numbers. Moreover, for all* $j \neq 1$, $\zeta_1 < \text{Re}(\zeta_j)$ *and* $\hat{\zeta}_1 < \text{Re}(\hat{\zeta}_j)$.

(ii) *If* $\sigma > 0$ *then* $K = M + 1$ *and* $\hat{K} = \hat{M} + 1$.

(iii) *If* $\sigma = 0$, *and* $\mu > 0$ $\{\mu < 0\}$ *then* $K = M + 1$ *and* $\hat{K} = \hat{M}$ $\{$ $K = M$ *and* $\hat{K} = \hat{M} + 1\}$.

(iv) *If* $\sigma = \mu = 0$, *then* $K = M$ *and* $\hat{K} = \hat{M}$.

(v) *There exist at most* $2Q - 1$ *complex numbers* q *such that equation* $\psi(z) = q$ *has solutions of multiplicity greater than one.*

(vi) *As* $q \to 0^+$ *we have the following possibilities:*

$$\begin{cases} \text{if } \mathbb{E}[X_1] > 0, \text{ then } \zeta_1(0^+) = 0 \text{ and } \hat{\zeta}_1(0^+) > 0, \\ \text{if } \mathbb{E}[X_1] < 0, \text{ then } \zeta_1(0^+) > 0 \text{ and } \hat{\zeta}_1(0^+) = 0, \\ \text{if } \mathbb{E}[X_1] = 0, \text{ then } \zeta_1(0^+) = 0 \text{ and } \hat{\zeta}_1(0^+) = 0. \end{cases}$$

Proof. The proof of (i)-(iv) follows from Lemma 1.1 in [12] (a somewhat simpler proof of a similar result has appeared recently in [8]). The proof of (vi) follows easily from the fact that $\psi(z) \sim \mathbb{E}[X_1]z + O(z^2)$ as $z \to 0$.

Let us prove (v). Assume that equation $\psi(z) = q$ has a solution $z = z_0$ of multiplicity greater than one, therefore $\psi'(z_0) = 0$. Using (4) we find that $\psi'(z_0) = 0$ implies $\mathcal{Q}'(z_0)\mathcal{P}(z_0) - \mathcal{Q}(z_0)\mathcal{P}'(z_0) = 0$. The polynomial $H(z) = \mathcal{Q}'(z)\mathcal{P}(z) - \mathcal{Q}(z)\mathcal{P}'(z)$ has degree at most $Q + P - 1 \leq 2Q - 1$, thus there exist at most $2Q - 1$ distinct points z_k for which $\psi'(z_k) = 0$, which implies that there exist at most $2Q - 1$ values q, given by $q_k = \psi(z_k)$, for which the equation $\psi(z) = q$ has solutions of multiplicity greater than one. $\square$

Remark 1. *Proposition 1(v) states that in general it is very unlikely for equation* $\psi(z) = q$ *to have multiple solutions, unless* $q = 0$ *and* $\mathbb{E}[X_1] = 0$, *in which case we have a double root at zero. In particular, when doing numerical computations, we can safely assume that all solutions to* $\psi(z) = q$ *are simple.*

Our first goal is to identify the Mellin transform of I_q, that is $\mathbb{E}[I_q^{s-1}]$. Our main tool is the following Proposition, which we have borrowed from [10]. Let us introduce the main ingredients needed to state this result. We assume that Y is a Lévy process started from zero, and that $\psi_Y(z) = \ln \mathbb{E}[\exp(zY_1)]$ is its Laplace exponent. As usual, we define the exponential functional $I_q(Y) = \int_0^{\text{e}(q)} \exp(Y_t)\text{d}t$ and the Mellin transform $\mathcal{M}_Y(s) = \mathbb{E}[I_q(Y)^{s-1}]$.

Proposition 2. (Verification result) *Assume that Cramér's condition is satisfied: there exists $z_0 > 0$ such that $\psi_Y(z)$ is finite for all $z \in (0, z_0)$ and $\psi_Y(\theta) = q$ for some $\theta \in (0, z_0)$. If $f(s)$ satisfies the following three properties*

(i) $f(s)$ is analytic and zero-free in the strip $\mathrm{Re}(s) \in (0, 1+\theta)$,

(ii) $f(1) = 1$ and $f(s+1) = sf(s)/(q - \psi_Y(s))$ for all $s \in (0, \theta)$,

(iii) $|f(s)|^{-1} = o(\exp(2\pi|\mathrm{Im}(s)|))$ as $\mathrm{Im}(s) \to \infty$, $\mathrm{Re}(s) \in (0, 1+\theta)$,

then $\mathcal{M}_Y(s) \equiv f(s)$ for $\mathrm{Re}(s) \in (0, 1+\theta)$.

Proof. The proof can be found in [10], however for sake of completeness we will present the main steps of the proof here. First of all, the Cramér's condition and Lemma 2 in [19] imply that $\mathcal{M}_Y(s)$ can be extended to an analytic function in the vertical strip $\mathrm{Re}(s) \in (0, 1+\theta)$. Since $|\mathcal{M}_Y(s)| < \mathcal{M}_Y(\mathrm{Re}(s))$ we see that $\mathcal{M}_Y(s)$ is bounded in the strip $\mathrm{Re}(s) \in [\theta/2, 1+\theta/2]$. It is well-known (see Lemma 2.1 in [13] or Proposition 3.1 in [4]) that $\mathcal{M}_Y(s)$ satisfies the same functional equation as $f(s)$. Therefore the ratio $F(s) = \mathcal{M}_Y(s)/f(s)$ is a periodic function: $F(s+1) = F(s)$; and due to condition (i) $F(s)$ can be extended to an analytic function in the entire complex plane. Finally, condition (iii) and boundedness of $\mathcal{M}_Y(s)$ imply that $F(s) = o(\exp(2\pi|\mathrm{Im}(s)|)$ in the entire complex plane, and any function which is analytic, periodic with period equal to one, and which satisfies this upper bound must be identically equal to a constant. Since $F(1) = 1$ we conclude that $F(s) \equiv 1$, that is $\mathcal{M}_Y(s) \equiv f(s)$. □

Theorem 1. *For $\mathrm{Re}(s) \in (0, 1+\zeta_1)$ we have*

$$\mathbb{E}\left[I_q^{s-1}\right] = A^{1-s} \times \Gamma(s) \times \frac{\mathcal{G}(s)}{\mathcal{G}(1)}, \tag{5}$$

where

$$\mathcal{G}(s) = \frac{\prod_{j=1}^{K} \Gamma(1+\zeta_j - s)}{\prod_{j=1}^{J} \Gamma(1+\rho_j - s)^{m_j}} \times \frac{\prod_{j=1}^{\hat{J}} \Gamma(\hat{\rho}_j + s)^{\hat{m}_j}}{\prod_{j=1}^{\hat{K}} \Gamma(\hat{\zeta}_j + s)}, \tag{6}$$

and the constant A is defined as

$$\begin{cases} A = \frac{\sigma^2}{2}, \text{ if } \sigma > 0, \\ A = |\mu|, \text{ if } \sigma = 0 \text{ and } \mu \neq 0, \\ A = q + \lambda, \text{ if } \sigma = \mu = 0. \end{cases} \tag{7}$$

Proof. Let us prove this Theorem in the case $\sigma = \mu = 0$, the proof in other cases is identical. First of all, we check that the Cramér's condition in Proposition 2 is satisfied with $\theta = \zeta_1$. Let $f(s)$ be the function in the right-hand side of (5). Since $\Gamma(s)$ has no zeros in $\mathbb{C}$ and simple poles at $s \in \{0, -1, -2, \dots\}$, it is clear that $f(s)$ is analytic and zero-free in the vertical strip $\mathrm{Re}(s) \in (0, 1+\zeta_1)$.

Next, we factorize the rational function $s/(q-\psi(s))$ as follows

$$\frac{s}{q-\psi(s)} = \frac{s}{q+\lambda} \frac{\prod_{j=1}^{J}(s-\rho_j)^{m_j}}{\prod_{j=1}^{K}(s-\zeta_j)} \times \frac{\prod_{j=1}^{\hat{J}}(s+\hat{\rho}_j)^{\hat{m}_j}}{\prod_{j=1}^{\hat{K}}(s+\hat{\zeta}_j)}. \tag{8}$$

One can check that this is the correct factorization, since the rational functions in the right/left hand sides of this equation have identical roots/poles and the same asymptotic behavior as $s \to \infty$ (recall that $\psi(s) \to -\lambda$ when $s \to \infty$ in the case $\sigma = \mu = 0$). Using (8) and the functional identity $\Gamma(s+1) = s\Gamma(s)$ we check that $f(s)$ satisfies the functional equation $f(s+1) = sf(s)/(q-\psi(s))$ for $\mathrm{Re}(s) \in (0, 1+\zeta_1)$. Finally, we use the asymptotic expansion

$$|\Gamma(x+\mathrm{i}y)| = \exp\left(-\frac{\pi}{2}|y| + \left(x-\frac{1}{2}\right)\ln|y| + O(1)\right), \quad y \to \infty$$

(see formula 8.328.1 in [9]), the fact that

$$M = \sum_{j=1}^{J} m_j = K, \quad \hat{M} = \sum_{j=1}^{\hat{J}} \hat{m}_j = \hat{K}$$

(see Proposition 1(iv)) and conclude

$$|f(s)| = \exp\left(-\frac{\pi}{2}|\mathrm{Im}(s)| + O(\ln|s|)\right), \quad \mathrm{Im}(s) \to \infty$$

in the vertical strip $\mathrm{Re}(s) \in (0, 1+\zeta_1)$. Thus we see that $f(s)$ satisfies all three conditions of Proposition 2, therefore (5) is true for $\mathrm{Re}(s) \in (0, 1+\zeta_1)$. $\square$

Formula (5) gives us the Mellin transform of I_q, which uniquely characterizes the distribution of I_q. It turns out that with little extra work we can obtain an explicit formula for the joint transform of $(X_{\mathrm{e}(q)}, I_q)$.

Corollary 1. *Assume that $q > 0$. For $\mathrm{Re}(u) \in (-\hat{\zeta}_1, \zeta_1)$ and $\mathrm{Re}(s) \in (0, 1+\zeta_1-\mathrm{Re}(u))$ we have*

$$\mathbb{E}\left[e^{uX_{\mathrm{e}(q)}} I_q^{s-1}\right] = \frac{qA^{1-s}}{q-\psi(u)} \times \Gamma(s) \times \frac{\mathcal{G}(s+u)}{\mathcal{G}(1+u)}. \tag{9}$$

Proof. Assume that $u \in (-\hat{\zeta}_1, \zeta_1)$ and define a new measure $\tilde{\mathbb{P}}$ as the Escher transform of $\mathbb{P}$

$$\left.\frac{\mathrm{d}\tilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = e^{uX_t - t\psi(u)}.$$

Define $I(t) = \int_0^t \exp(X_s)\mathrm{d}s$. Then we have

$$\mathbb{E}\left[e^{uX_{\mathrm{e}(q)}} I_q^{s-1}\right] = q\int_0^{\infty} e^{-qt}\mathbb{E}\left[e^{uX_t} I(t)^{s-1}\right]\mathrm{d}t = q\int_0^{\infty} e^{-qt+t\psi(u)}\tilde{\mathbb{E}}\left[I(t)^{s-1}\right]\mathrm{d}t = \frac{q}{\tilde{q}}\tilde{\mathbb{E}}\left[I_{\tilde{q}}^{s-1}\right], \tag{10}$$

where we have denoted $\tilde{q} = q - \psi(u)$. Note that $\psi(u) < 0$ for $u \in (-\hat{\zeta}_1, \zeta_1)$, therefore $\tilde{q} > 0$. The Laplace exponent of X under the measure $\tilde{\mathbb{P}}$ is equal to $\tilde{\psi}(z) = \psi(z+u) - \psi(u)$, therefore the zeros and poles of the rational function $\tilde{\psi}(z) - \tilde{q}$ can be obtained from the corresponding zeros and poles of $\psi(z) - q$ by shifting by u units:

$$\zeta_j \mapsto \zeta_j - u, \quad \rho_j \mapsto \rho_j - u, \quad \hat{\zeta}_j \mapsto \hat{\zeta}_j + u, \quad \hat{\rho}_j \mapsto \hat{\rho}_j + u,$$

We apply results of Theorem 1 and conclude that

$$\tilde{\mathbb{E}}\left[I_{\tilde{q}}^{s-1}\right] = A^{1-s} \times \Gamma(s) \times \frac{\mathcal{G}(s+u)}{\mathcal{G}(1+u)}. \tag{11}$$

Identity (9) for real values of u follows from (10) and (11), the general case follows by analytic continuation. □

As we have mentioned above, formula (5) uniquely characterizes the distribution of I_q via the inverse Mellin transform

$$p(x) = \frac{1}{2\pi \mathrm{i}} \int\limits_{1+\mathrm{i}\mathbb{R}} \mathbb{E}\left[I_q^{s-1}\right] x^{-s} \mathrm{d}s. \tag{12}$$

In fact, $p(x)$ can be computed explicitly, and this can be achieved in a variety of ways. One approach (which is quite general) is to use the fact that the right-hand side of (5) is a meromorphic function in $\mathbb{C}$, then compute its residues, and then by shifting the contour of integration we will obtain either convergent series representations or complete asymptotic expansions for $p(x)$. This is the method that was used in [10] in the case of hypergeometric Lévy processes. The second approach, which we will follow in this paper, is to use the theory of the Meijer G-function.

First of all, we want to ensure that the Mellin transform of I_q given in (5) does not have multiple poles. One can still obtain explicit formula in the case when we do have multiple poles, however in this case we will have logarithmic terms and much more complicated expressions. We would like to avoid this situation, therefore from now on we will work under the following assumption, which ensures that $\Gamma(s)\mathcal{G}(s)$ does not have multiple poles.

Assumption A.

A.1 $\psi(z) - q$ *has no multiple poles in the half-plane* $\mathrm{Re}(z) < 0$.

A.2 For $1 \le i \le \hat{M}$ *we have* $\hat{\rho}_i \notin \mathbb{N}$.

A.3 For $1 \le i < j \le \hat{M}$ *we have* $\hat{\rho}_j - \hat{\rho}_i \notin \mathbb{N}$.

A.4 $\psi(z) - q$ *has no multiple zeros in the half-plane* $\mathrm{Re}(z) > 0$.

A.5 For $1 \le i < j \le K$ *we have* $\zeta_j - \zeta_i \notin \mathbb{N}$.

Note that Assumptions A.1, A.2 and A.3 can be easily verified, since usually we know explicitly the Lévy measure, and therefore the values of $\hat{\rho}_j$. Assumption A.1 is equivalent to requiring $\hat{m}_j = 1$ for $j = 1, 2, \ldots, \hat{J}$. Assumption A.4 can be verified numerically, however due to Proposition 1(v) we would expect that this assumption will be satisfied in all numerical experiments, unless we have a specifically engineered counter-example. Similarly, condition A.5 should be satisfied in all numerical experiments, since a "generic" polynomial equation normally does not have solutions which differ by an integer number. Thus Assumptions A.4 and A.5 are not very restrictive.

Next, let us introduce some notations. For any $w \in \mathbb{C}$ and $n \in \mathbb{N}$ we define the vector $[w]_n \in \mathbb{C}^n$ as

$$[w]_n = [w, w, \dots, w].$$

We define vectors $\mathbf{a} \in \mathbb{C}^{P+1}$ and $\mathbf{b} \in \mathbb{C}^Q$ as

$$\begin{aligned} \mathbf{a} &= [1, [1-\hat{\rho}_1]_{\hat{m}_1}, [1-\hat{\rho}_2]_{\hat{m}_2}, \dots, [1-\hat{\rho}_{\hat{J}}]_{\hat{m}_{\hat{J}}}, [1+\rho_1]_{m_1}, [1+\rho_2]_{m_2}, \dots, [1+\rho_J]_{m_J}], \\ \mathbf{b} &= [1+\zeta_1, 1+\zeta_2, \dots, 1+\zeta_K, 1-\hat{\zeta}_1, 1-\hat{\zeta}_2, \dots, 1-\hat{\zeta}_{\hat{K}}]. \end{aligned}$$

For any $\mathbf{x} \in \mathbb{C}^n$ and $j \in \{1, 2, \dots, n\}$ we will denote as $(\mathbf{x})^j \in \mathbb{C}^{n-1}$ the vector obtained from $\mathbf{x}$ by deleting the j-th coordinate, that is

$$(\mathbf{x})^j = [x_1, x_2, \dots, x_{j-1}, x_{j+1}, \dots, x_n].$$

Finally, for $\mathbf{x} \in \mathbb{C}^n$ and $w \in \mathbb{C}$, we define the vector $w + \mathbf{x} \in \mathbb{C}^n$ as follows

$$w + \mathbf{x} = [w + x_1, w + x_2, \dots, w + x_n].$$

For $\mathbf{u} \in \mathbb{C}^p$ and $\mathbf{v} \in \mathbb{C}^q$ the generalized hypergeometric function is defined as a formal power series

$${}_pF_q\left[\begin{array}{c} \mathbf{u} \\ \mathbf{v} \end{array}\middle| z\right] = \sum_{n\geq 0} \frac{(u_1)_n \dots (u_p)_n}{(v_1)_n \dots (v_q)_n} \frac{z^n}{n!}.$$

where $(w)_n = w(w+1)\dots(w+n-1)$ is the Pochhammer symbol. Using the ratio test, it is easy to see that the domain of convergence of this series is (a) $\mathbb{C}$ when $p \leq q$ (b) $\{|z| < 1\}$ when $p = q+1$ (c) $\{0\}$ when $p \geq q+2$. In the latter case we will interpret ${}_pF_q$ as an asymptotic expansion as $z \to 0$.

Finally, let us define the following two functions (or formal power series in the case if they do not converge):

$$G_1(x) = \sum_{j=1}^{K} \frac{\prod\limits_{\substack{1\leq i\leq K \\ i\neq j}} \Gamma(b_i - b_j) \prod\limits_{i=1}^{\hat{M}+1} \Gamma(1 + b_j - a_i)}{\prod\limits_{i=K+1}^{Q} \Gamma(1 + b_j - b_i) \prod\limits_{i=\hat{M}+2}^{P+1} \Gamma(a_i - b_j)} \times x^{-b_j} {}_{P+1}F_{Q-1}\left[\begin{array}{c} 1 + b_j - \mathbf{a} \\ 1 + b_j - (\mathbf{b})^j \end{array}\middle| (-1)^{M-K} x^{-1}\right] \tag{13}$$

$$G_2(x) = \sum_{j=1}^{\hat{M}+1} \frac{\prod\limits_{\substack{1\leq i\leq \hat{M}+1 \\ i\neq j}} \Gamma(a_j - a_i) \prod\limits_{i=1}^{K} \Gamma(1 + b_i - a_j)}{\prod\limits_{i=\hat{M}+2}^{P+1} \Gamma(1 + a_i - a_j) \prod\limits_{i=K+1}^{Q} \Gamma(a_j - b_i)} \times x^{1-a_j} {}_QF_P\left[\begin{array}{c} 1 - a_j + \mathbf{b} \\ 1 - a_j + (\mathbf{a})^j \end{array}\middle| (-1)^{\hat{M}-\hat{K}-1} x\right] \tag{14}$$

Our final result is the following proposition, which completely describes the distribution of the exponential functional.

Proposition 3.
(i) *If* $\sigma > 0$ *then*

$$\begin{aligned} p(x) &= \frac{\sigma^2}{2\mathcal{G}(1)} G_1\left(\frac{\sigma^2 x}{2}\right), \quad x > 0, \\ p(x) &\sim \frac{\sigma^2}{2\mathcal{G}(1)} G_2\left(\frac{\sigma^2 x}{2}\right), \quad x \to 0^+. \end{aligned}$$

(ii) *If* $\sigma = 0$ *and* $\mu \neq 0$ *then*

$$\begin{aligned} p(x) &= \frac{|\mu|}{\mathcal{G}(1)} G_1\left(|\mu| x\right), \quad x > |\mu|^{-1}, \\ p(x) &= \frac{|\mu|}{\mathcal{G}(1)} G_2\left(|\mu| x\right), \quad x < |\mu|^{-1}. \end{aligned}$$

(i) *If* $\sigma = \mu = 0$ *then*

$$\begin{aligned} p(x) &\sim \frac{q+\lambda}{\mathcal{G}(1)} G_1\left((q+\lambda) x\right), \quad x \to +\infty, \\ p(x) &= \frac{q+\lambda}{\mathcal{G}(1)} G_2\left((q+\lambda) x\right), \quad x > 0. \end{aligned}$$

Proof. We use the expression for $p(x)$ as the inverse Mellin transform (12), formula (5) and the definition of the Meijer's G-function (see formula 9.301 in [9]) to conclude that

$$p(x) = \frac{A}{\mathcal{G}(1)} \times G^{K, \hat{M}+1}_{P+1, Q}\left[(Ax)^{-1} \middle| \begin{array}{c} \mathbf{a} \\ \mathbf{b} \end{array} \right],$$

where G is the Meijer G-function. All the convergent series representations presented in Proposition 3 follow immediately from 9.303 and 9.304 in [9], while the asymptotic expansions follow from Theorem 1 in [7]. □

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