TOTAL EMBEDDING DISTRIBUTIONS OF RINGEL LADDERS

Yichao Chen, Lu Ou, Qian Zou

College of mathematics and econometrics, Hunan University, 410082 Changsha, China

ycchen@hnu.edu.cn, 50371081@qq.com, Joe_king520@qq.com

[1]

ABSTRACT

The total embedding distributions of a graph is consisted of the orientable embeddings and non-orientable embeddings and have been know for few classes of graphs. The genus distribution of Ringel ladders is determined in [Discrete Mathematics 216 (2000) 235-252] by E.H. Tesar. In this paper, the explicit formula for non-orientable embeddings of Ringel ladders is obtained.

Key words: Graph embedding; Ringel ladders; Overlap matrix; Chebyshev polynomials;
2000 Mathematics Subject Classification: 05C10, 30B70, 42C05

1. Background

One enumerative aspect of topological graph theory is to count genus distributions of a graph. The history of genus distribution began with J. Gross in 1980s. Since then, it has been attracted a lot of attentions, for the details, we may refer to [1, 8, 9, 10, 11, 13, 16, 17, 19, 22, 26, 28, 31, 32, 33, 34, 35, 36, 37, 38] etc (We only list a few). However, for the total embedding distributions, only few classes are known. For example, Chen, Gross and Rieper [2] computed the total embedding distribution for necklaces of type $(r, 0)$, close-end ladders and cobblestone paths, Kwak and Shim [21] computed for bouquets of circles and dipoles. In [3], Chen, Liu and Wang calculated the total embedding distributions of all graphs with maximum genus 1. Furthermore, in [4], Chen, Mansour and Zou obtained explicit formula for total embedding distributions for the necklaces of type (r, s), closed-end ladders and cobblestone path.

It is assumed that the reader is somewhat familiar with the basics of topological graph theory as found in Gross and Tucker [12]. A graph $G = (V(G), E(G))$ is permitted to have both loops and multiple edges. A *surface* is a compact closed 2-dimensional manifold without boundary. In topology, surfaces are classified into O_m, the *orientable surface* with $m(m \geq 0)$ handles and N_n, the *nonorientable surface* with $n(n > 0)$ crosscaps. A graph embedding into a surface means a *cellular embedding*.

A *spanning tree* of a graph G is a tree on its edges has the same order as G. The number co-tree edges of a spanning tree of G is called the *Betti number*, $\beta(G)$, of G. A *rotation* at a vertex v of a graph G is a cyclic order of all edges incident with v. A *pure rotation system* P of a graph G is the collection of rotations at all vertices of G. A *general rotation system* is a pair (P, λ), where P is a pure rotation system and λ is a mapping $E(G) \rightarrow \{0, 1\}$. The edge e is said to be *twisted* (respectively, *untwisted*) if $\lambda(e) = 1$ (respectively, $\lambda(e) = 0$). It is well known that every orientable embedding of a graph G can be described by a general rotation

[1]The work was partially supported by NNSFC under Grant No. 10901048

system (P,λ) with $\lambda(e)=0$ for all $e\in E(G)$. By allowing λ to take the non-zero value, we can describe nonorientable embeddings of G, see [2, 30] for more details. A *T-rotation system* (P,λ) of G is a general rotation system (P,λ) such that $\lambda(e)=0$, for all $e\in E(T)$.

Theorem 1.1. *(see [2, 30]) Let T be a spanning tree of G and (P,λ) a general rotation system. Then there exists a general rotation system $(P^{'},\lambda^{'})$ such that*

1. *$(P^{'},\lambda^{'})$ yields the same embedding of G as (P,λ), and*
2. *$\lambda^{'}(e)=0$, for all $e\in E(T)$.*

Two embeddings are considered to be the *same* if their T-rotation systems are combinatorially equivalent. Fix a spanning tree T of a graph G. Let Φ_G^T be the set of all T-rotation systems of G. It is known that

$$|\Phi_G^T| = 2^{\beta(G)}\prod_{v\in V(G)}(d_v-1)!.$$

Suppose that in these $|\Phi_G^T|$ embeddings of G, there are a_i, $i=0,1,\ldots$, embeddings into orientable surface O_i and b_j, $j=1,2,\ldots$, embeddings into nonorientable surface N_j. We call the polynomial

$$I_G^T(x,y)=\sum_{i=0}^{\infty}a_ix^i+\sum_{j=1}^{\infty}b_jy^j$$

the *T-distribution polynomial* of G. By the *total genus polynomial* of G, we shall mean the polynomial

$$I_G(x,y)=\sum_{i=0}^{\infty}g_ix^i+\sum_{i=1}^{\infty}f_iy^i,$$

where g_i is the number of embeddings (up to equivalence) of G into the orientable surface O_i and f_i is the number of embeddings (up to equivalence) of G into the nonorientable surface N_i. We call the first (respectively, second) part of $I_G(x,y)$ the *genus polynomial* (respectively, *crosscap number polynomial*) of G and denoted by $g_G(x)=\sum_{i=0}^{\infty}g_ix^i$ (respectively, $f_G(y)=\sum_{i=1}^{\infty}f_iy^i$). Clearly, $I_G(x,y)=g_G(x)+f_G(y)$. This means the number of orientable embeddings of G is $\prod_{v\in G}(d_v-1)!$, while the number of non-orientable embeddings of G is $(2^{\beta(G)}-1)\prod_{v\in G}(d_v-1)!$. Let T be a spanning tree of G and $(P^{'},\lambda^{'})$ be a T-rotation system. Let $e_1,e_2,\ldots,e_{\beta(G)}$ be the cotree edges of T. The *overlap matrix* of $(P^{'},\lambda^{'})$ is the $\beta\times\beta$ matrix $M=[m_{ij}]$ over $GF(2)$ such that $m_{ij}=1$ if and only if either $i\neq j$ and the restriction of the underlying pure rotation system to $T+e_i+e_j$ is nonplanar, or $i=j$ and e_i is twisted. The following theorem due to Mohar.

Theorem 1.2. *(see [24]) Let (P,λ) be a general rotation system for a graph, and let M be the overlap matrix. Then the rank of M equals twice the genus, if the corresponding embedding surface is orientable, and it equals the crosscap number otherwise. It is independent of the choice of a spanning tree.*

An *n-rung closed-end ladder* L_n can be obtained by taking the graphical cartesian product of an *n-vertex* path with the complete graph K_2, and then doubling both its end edges. Figure 1 presents a 4-rung closed-end ladder.

Ringel ladders, R_n, are the graphs used by Ringel and Youngs in their proof of the Heawood Map Coloring Theorem. In fact, A *Ringel ladder*, R_n, can be formed by subdividing the end-rungs of the closed-end ladder, L_n, and adding an edge between these two new vertices. Figure 2 shows the Ringel ladder R_4.

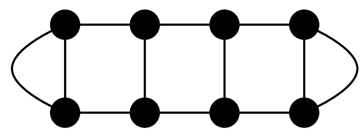

FIGURE 1. The 4-rung closed-end ladder L_4

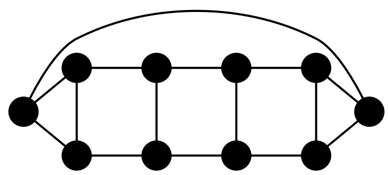

FIGURE 2. The Ringel ladder R_4

2. Homogeneous recurrence relation and Chebyshev polynomials

To begin with the discussion, we give some concepts of the n-th Chebyshev polynomials of the second kind which is related to the solution of the recurrence relation. Let the recurrence function $U_n(x)$ be

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x)$$

with the initial conditions $U_0(x) = 1$, $U_1(x) = 2x$, then we derived the *n-th Chebyshev polynomials with the second kind* $U_n(x)$ (see [27]). For instance, $U_2(x) = 4x^2 - 1$, $U_3(x) = 8x^3 - 4x$, $U_4(x) = 16x^4 - 12x^2 + 1$. Moreover, we have the identity that

$$U_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} (-1)^k (2x)^{n-2k}. \tag{1}$$

Now, we will build the relation between the recurrence relation and the Chebyshev polynomials with the second kind. Let $P_n(z) = \sum\limits_{m=0}^{n} C_n(m) z^m$, satisfy the following

$$P_n(z) = a_1(z)P_{n-1}(z) + a_2(z)P_{n-2}(z),$$

where $a_i(z) = \sum\limits_{k=0}^{q} a_{i,\ k} z^k$ for $i = 1, 2.$ and the initial conditions $P_0(z) = c_0$, and $P_1(z),\ P_2(z)$ can be derived by the initial values of $C_n(m)$.
Let $Q_n(z) = \frac{P_n(z)}{(\sqrt{a_2(z)}i)^n}$, then it is easy to verify that

$$Q_n(z) = \frac{a_1(z)}{\sqrt{a_2(z)}i} Q_{n-1}(z) - Q_{n-2}(z)$$

with the initial conditions $Q_0(z) = P_0(z) = c_0,\ Q_1(z) = \frac{P_1(z)}{\sqrt{a_2(z)}i}$ and $Q_2(z) = \frac{P_2(z)}{-a_2(z)}$. Using the fact that $U_0(x) = 1,\ U_1(x) = 2x,\ U_2(x) = 4x^2 - 1$, by induction on $n = 0, 1, 2$, we obtain that

$$Q_n(z) = AU_n(\frac{a_1(z)}{2\sqrt{a_2(z)}i}) + BU_{n-1}(\frac{a_1(z)}{2\sqrt{a_2(z)}i}) + CU_{n-2}(\frac{a_1(z)}{2\sqrt{a_2(z)}i}), \tag{2}$$

where A, B, and C are determined by the initial conditions.
Thus we have

$$P_n(z) = (\sqrt{a_2(z)}i)^n AU_n(\frac{a_1(z)}{2\sqrt{a_2(z)}i}) + BU_{n-1}(\frac{a_1(z)}{2\sqrt{a_2(z)}i}) + CU_{n-2}(\frac{a_1(z)}{2\sqrt{a_2(z)}i}). \tag{3}$$

Using the fact that

$$U_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} (-1)^k (2x)^{n-2k}.$$

We can derive that

$$(i\sqrt{a_2(z)})^n U_n(\frac{a_1(z)}{2\sqrt{a_2(z)}i}) = \sum_{j\geq 0} \binom{n-j}{j} (a_1(z))^{n-2j}. \tag{4}$$

Since $a_1(z)$ is a polynomial of degrees less than q, then $(a_1(z))^{n-2j}$ can be expressed as the type of power series. Plug the above formula into (3) and comparing the coefficient z^m in both sides and we can obtain the explicit formulae $C_n(m)$ for $0 \leq m \leq n$.

3. Total embedding distributions of Ringel ladders

3.1. **The rank-distribution polynomial of Closed-end ladders.** we adopt the notations of [4], the overlap of matrix of Closed-end ladders L_{n-1} has the following form $M_n^{X,Y}$(see [4] for more details).
Let $X = (x_1, x_2, \dots, x_n) \in (GF(2))^n$ and $Y = (y_1, y_2, \dots, y_{n-1}) \in (GF(2))^{n-1}$. We define the tridiagonal matrix $M_n^{X,Y}$ as

$$M_n^{X,Y} = \begin{pmatrix} x_1 & y_1 & & & & & \\ y_1 & x_2 & y_2 & & & \mathbf{0} & \\ & y_2 & x_3 & y_3 & & & \\ & & & & & & \\ & \mathbf{0} & & & y_{n-2} & x_{n-1} & y_{n-1} \\ & & & & & y_{n-1} & x_n \end{pmatrix}.$$

Furthermore, we define $\mathscr{L}_n = \{M_n^{X,Y} \mid X \in (GF(2))^n \text{ and } Y \in (GF(2))^{n-1}\}$, which is the set of all matrices over $GF(2)$ that are of the type $M_n^{X,Y}$. We define the *rank-distribution polynomial* to be the polynomial $\mathscr{L}_n(z) = \sum_{j=0}^n D_n(j)z^j$, where $D_n(j),\ j = 0, 1, \dots, n$, is the number of different assignment of the variables x_j, y_k, where $j = 1, 2, \cdots, n$ and $k = 1, 2, \cdots, n-1$, for which the matrix $M_n^{X,Y}$ in $\mathscr{L}_n$ has rank j. Similarly, Let $\mathscr{O}_n = \{M_n^{0,Y} \mid Y \in (GF(2))^{n-1}\}$, and $\mathscr{O}_n(z) = \sum_{j=0}^n O_n(j)z^j$ be the *rank-distribution polynomial* of $\mathscr{O}_n$,where $O_n(j),\ j = 0, 1, \dots, n$, is the number of different assignment of the variables y_k, where $k \in \{1, 2, \dots, n-1\}$, for which the matrix M_n^Y in $\mathscr{A}_n$ has rank j.

Lemma 3.1. *(see [4]) The polynomial $\mathscr{O}_n(z)$ satisfies the recurrence relation*

$$\mathscr{O}_n(z) = \mathscr{O}_{n-1}(z) + 2z^2\mathscr{O}_{n-2}(z)$$

with the initial conditions $\mathscr{O}_1(z) = 1$ and $\mathscr{O}_2(z) = z^2 + 1$.

Theorem 3.2. *(see [4]) For all $n \geq 1$,*

$$\mathscr{O}_n(z) = \sum_{j\geq 0} \binom{n-j}{j} 2^j \; z^{2j} - \sum_{j\geq 0} \binom{n-2-j}{j} 2^j \; z^{2j+2}.$$

Corollary 3.3. *For all $1 \geq m \leq [\frac{n}{2}]$.*

$$\mathscr{O}_n(2m+1) = 0,$$
$$\mathscr{O}_n(2m) = \binom{n-m}{m} \cdot 2^m - \binom{n-m-1}{m-1} \cdot 2^{m-1}.$$

Lemma 3.4. *(see [4]) The polynomial $\mathscr{L}_n(z)$ satisfies the recurrence relation*

$$\mathscr{L}_n(z) = (1+2z)\mathscr{L}_{n-1}(z) + 4z^2\mathscr{L}_{n-2}(z)$$

with the initial conditions $\mathscr{L}_1(z) = 1 + z$ and $\mathscr{L}_2(z) = 4z^2 + 3z + 1$.

Theorem 3.5. *(see [4]) For all $n \geq 1$,*

$$\mathscr{L}_n(z) = (2iz)^n \left[U_n\left(\frac{1+2z}{4iz}\right) + \frac{i}{2} U_{n-1}\left(\frac{1+2z}{4iz}\right) - \frac{1}{2} U_{n-2}\left(\frac{1+2z}{4iz}\right) \right],$$

where $U_s(t)$ is the s-th Chebyshev poynomial of the second kind and $i^2 = -1$.

Corollary 3.6. *(see [4]) For all $n \geq 1$ and $0 \leq m \leq n$,*

$$D_n(m) = 2^m \sum_{j=0}^{[m/2]} \binom{n-j}{j}\binom{n-2j}{n-m} - 2^{m-1} \sum_{j=0}^{[(m-1)/2]} \binom{n-1-j}{j}\binom{n-1-2j}{n-m}$$
$$+ 2^{m-1} \sum_{j=0}^{[(m-2)/2]} \binom{n-2-j}{j}\binom{n-2-2j}{n-m}.$$

3.2. **The overlap matrix of Ringel ladders.** We adopt the same notation used by Ringel [27, p.17]. A cubic graph at each vertex has two cyclic orderings of its neighbors. One of these two cyclic orderings is denoted as clockwise and the other *counterclockwise*. We color the vertex *black*, if that vertex has the *clockwise* ordering of its neighbors, otherwise, we will color the counterclockwise vertices *white*. This will bring convenient to embed a cubic graph into surfaces, as we can draw an imbedding on the plane and only need to color the vertices black and white.

Definition 3.7. *An edge is called matched if it has the same color at both ends, otherwise it is called unmatched.*

We fix a spanning tree T of R_{n-1} shown as the thicker lines in Figure 3, that is to say, the cotree edges are $e, a_1, a_2, \cdots, a_n$.

Property 3.8. *Two cotree edges e and a_i, for $i = 1, 2, \cdots, n$, overlap if and only if the edge c_i is unmatched.*

Property 3.9. *Two cotree edges a_i and a_{i+1},for $i = 1, 2, \cdots, n-1$, overlap if and only if the edge b_i is unmatched.*

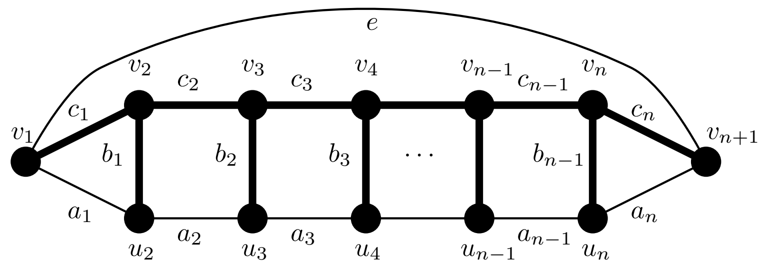

FIGURE 3.

It is easy to see that the overlap matrix of R_{n-1} has the following form.

$$M_{n+1}^{X,Y,Z} = \begin{pmatrix} x_0 & z_1 & z_2 & z_3 & \cdots & z_{n-1} & z_n \\ z_1 & x_1 & y_1 & & & & \\ z_2 & y_1 & x_2 & y_2 & & \mathbf{0} & \\ z_3 & & y_2 & x_3 & \ddots & & \\ \vdots & & & \ddots & \ddots & y_{n-2} & \\ z_{n-1} & & \mathbf{0} & & y_{n-2} & x_{n-1} & y_{n-1} \\ z_n & & & & & y_{n-1} & x_n \end{pmatrix},$$

where $X = (x_0, x_1, \ldots, x_n) \in (GF(2))^n$, $Y = (y_1, y_2, \ldots, y_{n-1}) \in (GF(2))^{n-1}$ and $Y = (z_1, z_2, \ldots, z_n) \in (GF(2))^{n-1}$. Note that $x_0 = 1$ if and only if the edge e is twisted, $x_i = 1$ if and only if the edge a_i is twisted, for all $i = 1, 2, \ldots, n$, $y_j = 1$ if and only if b_j is unmatched. for all $j = 1, 2, \ldots, n-1$, and $z_k = 1$ if and only if c_k is unmatched, for all $k = 1, 2, \ldots, n$.

Property 3.10. *For a fixed matrix of the form $M_{n+1}^{X,Y,Z}$, there are exactly 2 different T-rotation systems corresponding to that matrix.*

Proof. Given a matrix $M_{n+1}^{X,Y,Z}$, the values of $z_1, z_2, \cdots, z_n$ and $y_1, y_2, \cdots, y_{n-1}$ are determined.

- $z_1 = 0$. If we color the vertex v_1 black, by Property 3.8, the color of v_2 is black. Since the values of $z_2, \cdots, z_n$ and $y_1, y_2, \cdots, y_{n-1}$ are given, by Property 3.8 and Property 3.9, all the colors of $v_2, u_2, \cdots, v_n, u_n, v_{n+1}$ are determined. That is to say, all the rotations of vertices of R_n is determined. Otherwise the vertex v_1 is colored white, by Property 3.8, the color of v_2 is also white, by the values of $z_2, \cdots, z_n$ and $y_1, y_2, \cdots, y_{n-1}$ and by Property 3.8 and Property 3.9, the color all vertices of R_n is determined.
- $z_1 = 1$, Similar discuss like the case $z_1 = 0$, the details are omitted.

□

Now, we denote $\mathscr{R}_{n+1}$ be the set of all matrices over $GF(2)$ that are of the form $M_{n+1}^{X,Y,Z}$. The we calculate the rank distribution of the set $\mathscr{R}_{n+1}$.
Let $\mathscr{R}_{n+1}(z) = \sum_{j=0}^{n+1} C_{n+1}(j) z^j$ be the *rank-distribution polynomial* of the set $\mathscr{R}_{n+1}$. In other words, for $j = 0, 1, \ldots, n+1$, $C_{n+1}(j)$ is the number of different assignment of the variables x_i, $i = 0, 1, \cdots, n$, y_k, $k = 1, 2, \cdots, n-1$, and z_l, $l = 1, 2, \cdots, n$ for which the matrix $M_{n+1}^{X,Y,Z}$ in $\mathscr{R}_{n+1}$ has rank j.

Similarly, Let $\mathscr{P}_{n+1}$ be the set of all matrices over $GF(2)$ that are of the form $M_{n+1}^{O,Y,Z}$. The we calculate the rank distribution of the set $\mathscr{P}_{n+1}$. Let $\mathscr{P}_{n+1}(z) = \sum_{j=0}^{n+1} D_{n+1}(j)z^j$ be the *rank-distribution polynomial* of the set $\mathscr{O}_{n+1}$. In other words, for $j = 0, 1, \ldots, n+1$, $D_{n+1}(j)$ is the number of different assignment of the variables y_k, $k = 1, 2, \cdots, n-1$, and z_l, $l = 1, 2, \cdots, n$ for which the matrix $M_{n+1}^{O,Y,Z}$ in $\mathscr{P}_{n+1}$ has rank j.

Lemma 3.11. *The polynomial $\mathscr{P}_n(z)$ $(n \geq 3)$ satisfies the recurrence relation*

$$\mathscr{P}_{n+1}(z) = \mathscr{P}_n(z) + 8z^2\mathscr{P}_{n-1}(z) + 2^{n-1}z^2\mathscr{O}_{n-1}(z). \tag{5}$$

with the initial condition $\mathscr{P}_2(z) = z^2 + 1$, $\mathscr{P}_3(z) = 7z^2 + 1$ and $\mathscr{P}_4(z) = 12z^4 + 19z^2 + 1$ where $\mathscr{O}_{n-1}(z)$ is rank-distribution polynomial of closed-end ladders L_{n-2}.

Proof. To obtain the relation between $\mathscr{P}_{n+1}(z)$ and $\mathscr{P}_n(z)$, we consider the four different ways to assign the variables y_{n-1} and z_n in the matrix $M_{n+1}^{Y,Z}$.
Case 1: $y_{n-1} = 0$.

- **Subcase 1:** $z_n = 0$. Then the rank of $M_{n+1}^{Y,Z}$ is the same as the upper left $n \times n$ submatrix, which is a matrix of the form $M_n^{Y,Z}$. We conclude that this case contributes to the polynomial $\mathscr{P}_{n+1}(z)$ by a term $\mathscr{P}_n(z)$.
- **Subcase 2:** $z_n = 1$. It is easy to sea that, no matter what assignments of the variables $z_1, z_2, \cdots, z_{n-1}$, we can transform $M_{n+1}^{Y,Z}$ to the following form.

$$M_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & y_1 & & & & \\ 0 & y_1 & 0 & y_2 & & & \\ 0 & & y_2 & 0 & \ddots & & \\ \vdots & & & \ddots & \ddots & y_{n-2} & \\ 0 & & & & y_{n-2} & 0 & 0 \\ 1 & & & & & 0 & 0 \end{pmatrix},$$

We firstly delete the first column and the last column then delete the first row and the last row of M_1, then we obtain a matrix which is a overlap matrix of closed ladders L_{n-2}. Since there are 2^{n-1} different assignments of the variables $z_1, z_2, \cdots, z_{n-1}$, it contributes to the polynomial $\mathscr{P}_{n+1}(z)$ by a term $2^{n-1}z^2\mathscr{O}_{n-1}(z)$.
Case 2:$y_{n-1} = 1$. If $z_n = 1$, we first add the last row to the first low, then add the last column to the fist column. A similar discussion for y_{n-2} and z_{n-1}, we transform $M_{n+1}^{Y,Z}$ to the following form.

$$M = \begin{pmatrix} 0 & z_1 & z_2 & \cdots & z_{n-2} & 0 & 0 \\ z_1 & 0 & y_1 & & & & \\ z_2 & y_1 & 0 & \ddots & & & \\ \vdots & & \ddots & \ddots & y_{n-3} & & \\ z_{n-2} & & & y_{n-3} & 0 & 0 & \\ 0 & & & & 0 & 0 & 1 \\ 0 & & & & & 1 & 0 \end{pmatrix},$$

Note that the upper left $(n-1) \times (n-1)$ submatrix of M_2, which is a matrix of the form $M_{n-1}^{Y,Z}$. There are 2^3 different assignments of the variables y_{n-2}, z_{n-1} and z_n in the matrix $M_n^{Y,Z}$. In this case, it contributes to the polynomial $\mathscr{P}_{n+1}(z)$ by a term $8z^2\mathscr{P}_n(z)$. □

Lemma 3.12. *The polynomial* $\mathscr{R}_n(z)$ $(n \geq 3)$ *satisfies the recurrence relation*

$$\mathscr{R}_{n+1}(z) = (4z+1)\mathscr{R}_n(z) + 16z^2\mathscr{R}_{n-1}(z) + 2^n z^2 \mathscr{L}_{n-1}(z). \tag{6}$$

with the initial condition $\mathscr{R}_2(z) = 4z^2 + 3z + 1,\ \mathscr{R}_3(z) = 28z^3 + 28z^2 + 7z + 1,$ *where* $\mathscr{L}_{n-1}(z)$ *is rank-distribution polynomial of closed-end ladders* L_{n-2}.

Proof. To obtain the relation between $\mathscr{R}_{n+1}(z)$ and $\mathscr{R}_n(z)$, we consider the eight different ways to assign the variables x_n, y_{n-1} and z_n in the matrix $M_{n+1}^{X,Y,Z}$.

Case 1: $x_n = 0$.

- **Subcase 1:** $y_{n-1} = z_n = 0$. Then the rank of $M_{n+1}^{X,Y,Z}$ is the same as the upper left $n \times n$ submatrix, which is a matrix of the form $M_n^{X,Y,Z}$. We conclude that this case contributes to the polynomial $\mathscr{R}_{n+1}(z)$ by a term $\mathscr{R}_n(z)$.
- **Subcase 2:** $y_{n-1} = z_n = 1$. We first add the last row to the first low, then add the last column to the fist column. If $x_{n-1} = 1$, we add the last column to the n-th column. A similar discussion for y_{n-2} and z_{n-1}, we transform $M_{n+1}^{X,Y,Z}$ to the following form.

$$M = \begin{pmatrix} x_0 & z_1 & z_2 & \dots & z_{n-2} & 0 & 0 \\ z_1 & x_1 & y_1 & & & & \\ z_2 & y_1 & x_2 & \ddots & & & \\ \vdots & & \ddots & \ddots & y_{n-3} & & \\ z_{n-2} & & & y_{n-3} & x_{n-2} & 0 & \\ 0 & & & & 0 & 0 & 1 \\ 0 & & & & & 1 & 0 \end{pmatrix},$$

  Note that the upper left $(n-1) \times (n-1)$ submatrix, which is a matrix of the form $M_{n-1}^{X,Y,Z}$. There are 2^3 different assignments of the variables x_{n-1}, y_{n-2} and z_{n-1}, in these case it contributes to the polynomial $\mathscr{R}_{n+1}(z)$ by a term $8z^2\mathscr{R}_{n-1}(z)$.
- **Subcase 3:** $y_{n-1} = 1, z_n = 0$. Similarly discuss like subcase 2, it contributes to the polynomial $\mathscr{R}_{n+1}(z)$ by a term $8z^2\mathscr{R}_{n-1}(z)$.
- **Subcase 4:** $y_{n-1} = 0, z_n = 1$. It is easy to sea that, no matter what assignments of the variables x_0, $z_1, z_2, \cdots, z_{n-1}$, we can transform $M_{n+1}^{X,Y,Z}$ to the following form.

$$M_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & x_1 & y_1 & & & & \\ 0 & y_1 & x_2 & y_2 & & & \\ 0 & & y_2 & x_3 & \ddots & & \\ \vdots & & & \ddots & \ddots & y_{n-2} & \\ 0 & & & & y_{n-2} & x_{n-1} & 0 \\ 1 & & & & & 0 & 0 \end{pmatrix},$$

We firstly delete the first column and the last column then delete the first row and the last row of M_1, then we obtain a matrix which is a overlap matrix of closed ladders L_{n-2}. Since there are 2^n different assignments of the variables x_0, $z_1, z_2, \cdots, z_{n-1}$, it contributes to the polynomial $\mathscr{R}_{n+1}(z)$ by a term $2^n z^2 \mathscr{L}_{n-1}(z)$.

Case 2: $x_n = 1$. If $z_n = 1$, we first add the last column to the first column then add the last row to the first row. Similarly, if $y_{n-1} = 1$, we add the last column to the n-th column and add the last row to the n-th row. As last we can transfer the matrix $M_{n+1}^{X,Y,Z}$ to the matrix M_2 of following form.

$$M_2 = \begin{pmatrix} x_0 & z_1 & z_2 & z_3 & \cdots & z_{n-1} & 0 \\ z_1 & x_1 & y_1 & & & & \\ z_2 & y_1 & x_2 & y_2 & & & \\ z_3 & & y_2 & x_3 & \ddots & & \\ \vdots & & & \ddots & \ddots & y_{n-2} & 0 \\ z_{n-1} & & & & y_{n-2} & x_{n-1} & 0 \\ 0 & & & & & 0 & 1 \end{pmatrix},$$

Note that the upper left $n \times n$ submatrix of M_2, which is a matrix of the form $M_n^{X,Y,Z}$. There are 2^2 different assignments of the variables y_{n-1} and z_n in the matrix $M_n^{X,Y,Z}$. In this case, it contributes to the polynomial $\mathscr{R}_{n+1}(z)$ by a term $4z\mathscr{R}_n(z)$. □

Theorem 3.13. *For all $n \geq 2$,*

$$\begin{aligned}\mathscr{P}_n(z) =&(2\sqrt{2}iz)^n \left\{U_n(\frac{1}{4\sqrt{2}iz}) - \frac{z^2+1}{4\sqrt{2}iz}U_{n-1}(\frac{1}{4\sqrt{2}iz}) + \frac{17z^2-1}{16z^2}U_{n-2}(\frac{1}{4\sqrt{2}iz})\right\} \\ &+ 2^{n-1}z^2\mathscr{O}_{n-1}(z).\end{aligned}$$

where $U_s(t)$ is the s-th Chebyshev poynomial of the second kind, $i^2 = -1$ and $\mathscr{O}_{n-1}(z)$ is rank-distribution polynomial of closed-end ladders L_{n-2}.

Proof. Note that

$$\mathscr{P}_{n+1}(z) = \mathscr{P}_n(z) + 8z^2\mathscr{P}_{n-1}(z) + 2^{n-1}z^2\mathscr{O}_{n-1}(z). \tag{7}$$

We first consider the homogeneous recurrence relation part of (11).

$$\mathscr{P}_{n+1}(z) = \mathscr{P}_n(z) + 8z^2\mathscr{P}_{n-1}(z). \tag{8}$$

By the method of subsection 2, we have a solution of (12).

$$\mathscr{P}_n(z) = (\sqrt{a_2(z)}i)^n \left\{AU_n(\frac{a_1(z)}{2\sqrt{a_2(z)}i}) + BU_{n-1}(\frac{a_1(z)}{2\sqrt{a_2(z)}i}) + CU_{n-2}(\frac{a_1(z)}{2\sqrt{a_2(z)}i})\right\} \tag{9}$$

Now, let $Y_n(z) = 2^n f(z)\mathscr{O}_n(z)$ be one special solution of $\mathscr{P}_n(z)$, plug it into (11), using the relation

$$\mathscr{O}_n(z) = (1+2z)\mathscr{O}_{n-1}(z) + 4z^2\mathscr{O}_{n-2}(z),$$

it leads to

$$Y_n(z) = 2^{n-1}z^2\mathscr{O}_{n-1}(z) = \sum_{m\geq 0} 2^{n-1}O_{n-1}(m)z^{m+2}.$$

Thus,

(10)

$$\mathscr{P}_n(z) = (2\sqrt{2}zi)^n \left\{U_n(\frac{1}{2\sqrt{2}iz}) + BU_{n-1}(\frac{1}{2\sqrt{2}iz}) + CU_{n-2}(\frac{1}{2\sqrt{2}iz})\right\} + 2^{n-1}z^2\mathscr{O}_{n-1}(z).$$

Plug the initial values $\mathscr{P}_2(z), \mathscr{P}_3(z)$ into (14), it follows that

$$\begin{cases} -8z^2\left\{(2(\frac{1}{2\sqrt{2}iz})+B)\ \frac{1}{2\sqrt{2}iz}+(C-1)\right\}+2z^2=z^2+1 \\ -16\sqrt{2}iz^3\left\{(\frac{1}{2\sqrt{2}iz}+B)\ (-\frac{1}{8z^2}-1)+\frac{1}{2\sqrt{2}iz})(C-1)\right\}+4z^2(z^2+1)=7z^2+1. \end{cases}$$

By simple computation, we immediately obtain

$$B=\frac{-z^2-1}{4\sqrt{2}iz},\quad C=\frac{17z^2-1}{16z^2}.$$

□

Then according to the identity (1), the formula (14) is as follows

$$\mathscr{P}_n(z)=\sum_{j\geq 0}\binom{n-j}{j}(8z^2)^j-\frac{z^2+1}{2}\times\left\{\sum_{j\geq 0}\binom{n-1-j}{j}(8z^2)^j\right\}$$
$$-\frac{17z^2-1}{2}\left\{\sum_{j\geq 0}\binom{n-2-j}{j}(8z^2)^j\right\}+2^{n-1}z^2\mathscr{O}_{n-1}(z).$$

Comparing the coefficient of z^m in both sides, thus for all $n\geq 2$ and $0\leq m\leq n$, we have the following result.

Theorem 3.14. *For all $n\geq 2$,*

$$\mathscr{R}_n(z)=(4zi)^n\left\{U_n(\frac{1+4z}{8iz})-\frac{2z^2+7z+1}{8iz}U_{n-1}(\frac{1+4z}{8iz})+\frac{34z^2-z-1}{32z^2}U_{n-2}(\frac{1+4z}{8iz})\right\}$$
$$+2^nz^2\mathscr{L}_{n-1}(z).$$

where $U_s(t)$ is the s-th Chebyshev poynomial of the second kind, $i^2=-1$ and $\mathscr{L}_{n-1}(z)$ is rank-distribution polynomial of closed-end ladders L_{n-2}.

Proof. Note that

$$\mathscr{R}_n(z)=(4z+1)\mathscr{R}_{n-1}(z)+16z^2\mathscr{R}_{n-2}(z)+2^{n-1}z^2\mathscr{L}_{n-2}(z). \tag{11}$$

We first consider the homogeneous recurrence relation part of (11).

$$\mathscr{R}_n(z)=(4z+1)\mathscr{R}_{n-1}(z)+16z^2\mathscr{R}_{n-2}(z). \tag{12}$$

By the method of subsection 2, we have a solution of (12).

$$\mathscr{R}_n(z)=(\sqrt{a_2(z)}i)^n\left\{AU_n(\frac{a_1(z)}{2\sqrt{a_2(z)}i})+BU_{n-1}(\frac{a_1(z)}{2\sqrt{a_2(z)}i})+CU_{n-2}(\frac{a_1(z)}{2\sqrt{a_2(z)}i})\right\} \tag{13}$$

Now, let $Y_n(z)=2^nf(z)\mathscr{L}_n(z)$ be one special solution of $\mathscr{R}_n(z)$, plug it into (11), using the relation

$$\mathscr{L}_n(z)=(1+2z)\mathscr{L}_{n-1}(z)+4z^2\mathscr{L}_{n-2}(z),$$

it leads to $f(z)=\frac{z^2\mathscr{L}_{n-1}(z)}{\mathscr{L}_n(z)}$.

Thus we obtain a special solution of non-homogeneous recurrence (11)

$$Y_n(z)=2^nz^2\mathscr{L}_{n-1}(z)=\sum_{m\geq 0}2^nC_{n-1}(m)z^{m+2}.$$

Thus,

$$\mathscr{R}_n(z) = (4zi)^n \left\{ U_n(\frac{1+4z}{8iz}) + BU_{n-1}(\frac{1+4z}{8iz}) + CU_{n-2}(\frac{1+4z}{8iz}) \right\} + 2^n z^2 \mathscr{L}_{n-1}(z). \tag{14}$$

Plug the initial values $\mathscr{R}_2(z), \mathscr{R}_3(z)$ into (14), it follows that

$$\begin{cases} -16z^2 \left\{ (2(\frac{1+4z}{8iz}) + B)\ U_1(\frac{1+4z}{8iz}) + C - 1 \right\} + 4z^2(1+z) = 4z^2 + 3z + 1 \\ -64iz^3 \left\{ (2(\frac{1+4z}{8iz}) + B)\ U_2(\frac{1+4z}{8iz}) + U_1(\frac{1+4z}{8iz})(C-1) \right\} + 8z^2(4z^2+3z+1) \\ = 28z^3 + 28z^2 + 7z + 1. \end{cases}$$

By simple computation, we immediately obtain

$$B = -\frac{-2z^2 - 7z - 1}{8iz}, \quad C = \frac{34z^2 - z - 1}{32z^2}.$$

□

Then according to the identity (1), the formula (14) is as follows

$$\begin{aligned} \mathscr{R}_n(z) = &\sum_{j \geq 0} \binom{n-j}{j} (1+4z)^{n-2j} (4z)^{2j} - \frac{2z^2+7z+1}{2} \times \\ &\left\{ \sum_{j \geq 0} \binom{n-1-j}{j} (1+4z)^{n-1-2j} (4z)^{2j} \right\} \\ &+ \frac{34z^2 - z - 1}{2} \left\{ \sum_{j \geq 0} \binom{n-2-j}{j} (1+4z)^{n-2-2j} (4z)^{2j} \right\} \\ &+ 2^n z^2 \mathscr{L}_{n-1}(z). \end{aligned} \tag{15}$$

Comparing the coefficient of z^m in both sides of (15), thus for all $n \geq 2$ and $0 \leq m \leq n$, we have the following result.

Corollary 3.15. *For all $n \geq 2$ and $0 \leq m \leq n$,*

$$\begin{aligned} C_n(m) = &\sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \binom{n-j}{j} \binom{n-2j}{n-m} 4^m - \sum_{j=0}^{\lfloor \frac{m-2}{2} \rfloor} \binom{n-j-1}{j} \binom{n-1-2j}{n-m+1} 4^{m-2} \\ &- \frac{7}{2} \sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{n-j-1}{j} \binom{n-1-2j}{n-m} 4^{m-1} - \frac{1}{2} \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \binom{n-j-1}{j} \binom{n-1-2j}{n-m-1} 4^m \\ &- 17 \sum_{j=0}^{\lfloor \frac{m-2}{2} \rfloor} \binom{n-j-2}{j} \binom{n-2-2j}{n-m} 4^{m-2} + \frac{1}{2} \sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{n-j-2}{j} \binom{n-2-2j}{n-m-1} 4^{m-1} \\ &+ \frac{1}{2} \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \binom{n-j-2}{j} \binom{n-2-2j}{n-m-2} 4^m + 2^n D_{n-1}(m-2). \end{aligned}$$

where

$$D_n(m) = 2^m \sum_{j=0}^{[m/2]} \binom{n-j}{j}\binom{n-2j}{n-m} - 2^{m-1} \sum_{j=0}^{[(m-1)/2]} \binom{n-1-j}{j}\binom{n-1-2j}{n-m}$$
$$+ 2^{m-1} \sum_{j=0}^{[(m-2)/2]} \binom{n-2-j}{j}\binom{n-2-2j}{n-m}.$$

Theorem 3.16. *The total genus polynomial of Ringel ladders* R_{n-1} *is as follows:*

$$\mathbb{I}_{R_{n-1}}(x,y) = 2\sum_{j=0}^{n+1} C_{n+1}(j)y^j - \mathbb{I}_0(R_{n-1},\ y^2) + \mathbb{I}_0(R_{n-1},\ x)$$

where $\mathbb{I}_0(R_{n-1}, x)$ *is the genus polynomial of Ringel ladder* R_{n-1}*, which has been derived by E.H. Tesar* [35].

Proof. By Property 3.10, the theorem follows. □

For instance, the above theorem gives

$$\begin{aligned}
I_{R_1}(x,y) =& 2 + 14x + 14y + 42y^2 + 56y^3,\\
I_{R_2}(x,y) =& 2 + 38x + 24x^2 + 22y + 122y^2 + 424y^3 + 392y^4,\\
I_{R_3}(x,y) =& 2 + 70x + 184x^2 + 30y + 242y^2 + 1448y^3 + 3272y^4 + 2944y^5,\\
I_{R_4}(x,y) =& 2 + 118x + 648x^2 + 256x^3\\
&+ 38y + 410y^2 + 3496y^3 + 12952y^4 + 26880y^5 + 20736y^6,\\
I_{R_5}(x,y) =& 2 + 198x + 1656x^2 + 2240x^3\\
&+ 46y + 642y^2 + 7240y^3 + 36808y^4 + 120832y^5 + 207168y^6 + 147456y^7.
\end{aligned}$$

References

[1] D. Archdeacon, Calculations on the average genus and genus distribution of graphs, *Congr. Numer.* **67** (1988) 114–124.

[2] J. Chen, J. Gross and R. G. Rieper, Overlap matrices and total embeddings, *Discrete Math.* **128** (1994) 73–94.

[3] Y. Chen, Y. Liu and T. Wang, The total embedding distributions of cacti and necklaces, *Acta Mathematica Sinica* **22**(5) (2006) 1583–1590.

[4] Y. Chen, T. Mansour and Q. Zou, The total embedding distributions of some types of graphs, submitted for publication, 2009, 21pages.

[5] Y. Chen, T. Mansour, Lu Ou and Q. Zou, Genus distribution, homogeneous recurrence relation and chebyshev polynomial, Priprint, 2010, 25pages.

[6] Y. Chen, A note on a conjecture of S. Stahl, *Canad. J. Math.* **60**(4) (2008) 958–959.

[7] Y. Chen and Y. liu, On a conjecture of S. Stahl, *Canad. J. Math.***62**(5) (2010) 1058–1059.

[8] J. Edmonds, A combinatorial representation for polyhedral surfaces, *Notices Amer. Math. Soc.* **7** (1960), 646.

[9] J. L. Gross and M. L. Furst, Hierarchy for imbedding-distribution invariants of a graph, *J. Graph Theory* **11** (1987), 205–220.

[10] J. L. Gross, I. F. Khan, and M. I. Poshni, Genus distribution of graph amalga- mations: Pasting at root-vertices, *Ars Combinatoria* **94** (2010), 33–53.

[11] J. L. Gross, D. P. Robbins and T. W. Tucker, Genus distributions for bouquets of circles, *J. Combin. Theory* (B) **47** (1989), 292–306.

[12] J. L. Gross and T. W. Tucker, Topological Graph Theory, Dover, 2001; (original edn. Wiley, 1987).
[13] D. M. Jackson, Counting cycles in permutations by group characters with an application to a topological problem, *Trans. Amer. Math. Soc.* **299** (1987), 785–801.
[14] D. M. Jackson and T. I. Visentin, A character-theoretic approach to embeddings of rooted maps in an orientable surface of given genus, *Trans. Amer. Math. Soc.* **322** (1990), 343–363.
[15] D. M. Jackson and T. I. Visentin, An Atlas of the Smaller Maps in Orientable and Nonorientable Surfaces, Chapman and Hall/CRC Press, 2001.
[16] M. Furst, J. Gross and R. Statman, Genus distributions for two classes of graphs, *J. Combin. Ser. B* **46** (1989) 22–36.
[17] I. F. Khan, M. I. Poshni, and J. L. Gross, Genus distribution of graph amalgama- tions: pasting when one root has higher degree, *Ars Math. Contemporanea* (2010), to appear.
[18] V. P. Korzhik and H-J Voss, Exponential families of non-isomorphic non- triangular orientable genus embeddings of complete graphs, *J. Combin. Theory* (B) **86** (2002), 86–211.
[19] J. H. Kwak and J. Lee, Genus polynomials of dipoles, *Kyungpook Math. J.* **33** (1993), 115–125.
[20] J. H. Kwak and J. Lee, Enumeration of graph embeddings, *Discrete Math.* **135** (1994), 129–151.
[21] J. H. Kwak and S. H. Shim, Total embedding distributions for bouquets of circles, Discrete Math. 248 (2002), 93–108.
[22] L. A. McGeoch, Algorithms for two graph problems: computing maximum-genus imbedding and the two-server problem, PhD thesis, Carnegie-Mellon University, 1987.
[23] B. Mohar and C. Thomassen, Graphs on Surfaces, Johns Hopkins Press, 2001.
[24] B. Mohar, An obstruction to embedding graphs in surface, *Discrete Math.* **78** (1989) 135–142.
[25] B. P. Mull, Enumerating the orientable 2-cell imbeddings of complete bipartite graphs, *J. Graph Theory* **30** (1999), 77–90.
[26] M. I. Poshni, I. F. Khan, and J. L. Gross, Genus distribution of edge- amalgamations, *Ars Math. Contemporanea* **3** (2010), 69–86.
[27] Th. Rivlin, Chebyshev polynomials. From approximation theory to algebra and number theory, John Wiley, New York, 1990.
[28] R. G. Rieper, The enumeration of graph imbeddings, PhD thesis, Western Michigan University, 1990.
[29] G. Ringel, Map Color Theory, Springer, Berlin, 1974.
[30] S. Stahl, Generalized embedding schemes, *J. Graph Theory* **2** (1978) 41–52.
[31] S. Stahl, Region distributions of graph embeddings and Stirling numbers, *Discrete Math.* **82** (1990), 57–78.
[32] S. Stahl, Permutation-partition pairs III: Embedding distributions of linear fam- ilies of graphs, *J. Combin. Theory* (B) **52** (1991), 191–218.
[33] S. Stahl, Region distributions of some small diameter graphs, *Discrete Math.* **89** (1991), 281–299.
[34] S. Stahl, On the zeros of some polynomial, *Canad. J. Math.* **49** (1996) 617–640
[35] E. H. Tesar, Genus distribution of Ringel ladders, *Discrete Math.* **216** (2000) 235–252.
[36] T. I. Visentin and S. W. Wieler, On the genus distribution of (p, q, n)-dipoles, *Electronic J. of Combin.* **14** (2007), Art. No. R12.
[37] L. X. Wan and Y. P. Liu, Orientable embedding distributions by genus for certain types of graphs, *Ars Combin.* **79** (2006), 97–105.
[38] L. X. Wan and Y. P. Liu, Orientable embedding genus distribution for certain types of graphs, *J. Combin. Theory* (B) **47** (2008), 19–32.
[39] A. T. White, Graphs of Groups on Surfaces, North-Holland, 2001.