Green Correspondence for Virtually Pro-p Groups

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1 Introduction

Let p be a prime number, G a finite group, Q a p-subgroup of G and L any subgroup of G containing the normalizer $\mathrm{N}_G(Q)$ of Q in G. Let k be a field of positive characteristic p. In [2] J.A. Green demonstrates a fundamental correspondence between finitely generated kG-modules with vertex Q and finitely generated kL-modules with vertex Q. When $L = \mathrm{N}_G(Q)$ the Green correspondence allows for the reduction of many questions about general modules to questions about modules with a normal vertex.

Now let G be a profinite group and k a finite field of characteristic p. In [3] we took some first steps towards a modular representation theory of profinite groups. In particular we demonstrated a classification theorem for relatively projective finitely generated $k[[G]]$-modules, introduced vertices and sources, and showed that the expected uniqueness properties hold for these objects (under additional hypotheses in the case of sources). Here we generalize the Green correspondence (properly interpreted) to the class of virtually pro-p groups. We will reference [3] frequently, since many necessary foundational results are discussed therein.

Our main result is the following:

Theorem 1.1. *Let G be a virtually pro-p group, let Q be a closed pro-p subgroup of G and let L be a closed subgroup of G containing $\mathrm{N}_G(Q)$. Let S be a finitely generated indecomposable profinite $k[[Q]]$-module with vertex Q. Then there is a canonical bijection between the set of isomorphism classes of indecomposable profinite $k[[L]]$-modules with vertex Q and source S, and the set of isomorphism classes of indecomposable profinite $k[[G]]$-modules with vertex Q and source S.*

More explicitly, if V is an indecomposable $k[[L]]$-module with vertex Q and source S, then the correspondent of V under the above bijection is the unique indecomposable summand of $V \uparrow^G$ having vertex Q.

We approach the proof in two main steps. We first demonstrate a correspondence which is word-for-word analogous to the finite case under the additional assumption that L is open in G. Using this special case we then demonstrate the truth of the above theorem. First let us establish some notation to be assumed throughout our discussion.

The main concepts mentioned in this paragraph are introduced and discussed in [3]. Let G be a virtually pro-p group and k a finite field of characteristic p. All modules are assumed to be profinite left modules. If U is a $k[[G]]$-module and N is a closed normal subgroup of G, then we denote by U_N the coinvariant module $k\widehat{\otimes}_{k[[N]]}U$ - note that this is not a restriction. If U is finitely generated and N is open in G, then U_N is finite. If U is non-zero, finitely generated and indecomposable, then by [3, 2.8, 2.9] we can choose a cofinal inverse system of open normal pro-p subgroups of G for which each U_N is non-zero and indecomposable. As usual, if H is a closed subgroup of G and V is a $k[[H]]$-module, then we denote by $V\uparrow^G$ the $k[[G]]$-module induced from V. If U is a $k[[G]]$-module, then we denote by $U\downarrow_H$ the $k[[H]]$-module obtained by restricting the coefficients of U.

Let Q be a closed pro-p subgroup of G and let L be any closed subgroup of G containing $\mathrm{N}_G(Q)$. We define the following two sets of subgroups of G:

$$\mathfrak{X} = \{X \leq_C G \,|\, X \leq xQx^{-1} \cap Q, x \notin L\}$$

$$\mathfrak{Y} = \{Y \leq_C G \,|\, Y \leq xQx^{-1} \cap L, x \notin L\}.$$

If $\mathfrak{H}$ is a collection of subgroups of G, then we say a finitely generated $k[[G]]$-module U is relatively $\mathfrak{H}$-projective if each indecomposable summand of U is projective relative to an element of $\mathfrak{H}$. As in the finite case we note that $\mathfrak{X}$ consists of proper subgroups of Q, while $\mathfrak{Y}$ may contain a conjugate of Q.

2 The case where L is open

Essentially following the treatment in [1, 3.12] we prove three lemmas which constitute the bulk of the work for the case of open L.

Lemma 2.1. *Let V be a finitely generated indecomposable Q-projective $k[[L]]$-module. Then $V\uparrow^G\downarrow_L \cong V \oplus V_1$, where V_1 is $\mathfrak{Y}$-projective.*

Proof. By the Mackey decomposition formula [5, 2.2] we have

$$V\uparrow^G\downarrow_L \cong \bigoplus_{x\in L\backslash G/L} x(V)\downarrow_{xLx^{-1}\cap L}\uparrow^L = V \oplus \bigoplus_{x\in L\backslash G/L, x\notin L} x(V)\downarrow_{xLx^{-1}\cap L}\uparrow^L$$

so we need only show that a summand of the form $x(V)\downarrow_{xLx^{-1}\cap L}\uparrow^L$ with $x \notin L$ is $\mathfrak{Y}$-projective. By [3, 5.2] the module $x(V)$ is projective relative to xQx^{-1} so by [3, 3.7] we can choose a $k[[xQx^{-1}]]$-module S such that $x(V)\,|\,S\uparrow^{xLx^{-1}}$. Then

$$x(V)\downarrow_{xLx^{-1}\cap L}\uparrow^L \,\Big|\, S\uparrow^{xLx^{-1}}\downarrow_{xLx^{-1}\cap L}\uparrow^L \cong \bigoplus_y y(S)\downarrow_{(yx)Q(yx)^{-1}\cap xLx^{-1}\cap L}\uparrow^L$$

where y runs through a set of double coset representatives of

$$(xLx^{-1} \cap L)\backslash xLx^{-1}/xQx^{-1}.$$

Note that $yx = xlx^{-1}x = xl$ for some $l \in L$ and $x \notin L$ implies $xl \notin L$ so that each $((yx)Q(yx)^{-1} \cap xLx^{-1} \cap L) \in \mathfrak{Y}$. Hence the module $x(V)\downarrow_{xLx^{-1}\cap L}\uparrow^L$ is relatively $\mathfrak{Y}$-projective as required. $\square$

Lemma 2.2. *Let V be a finitely generated indecomposable Q-projective $k[[L]]$-module. Then $V\uparrow^G \cong U \oplus U_1$, where U is indecomposable, $V \mid U\downarrow_L$, and U_1 is $\mathfrak{X}$-projective.*

Proof. Since $V \mid V\uparrow^G\downarrow_L$, by the Krull–Schmidt theorem [4, 2.1] there is an indecomposable summand U of $V\uparrow^G$ with $V \mid U\downarrow_L$. Write $V\uparrow^G \cong U \oplus U_1$ and take an indecomposable summand U' of U_1. We wish to show that U' is relatively $\mathfrak{X}$-projective. Note that U' is relatively Q-projective.

Since $N_G(Q) \leq L$ and L is open, a standard compactness argument allows us to consider a cofinal inverse system of open normal subgroups N of G such that $N_G(QN) \leq L$. Fix some N in our system. The module U' is projective relative to QN, so $U' \mid U'\downarrow_{QN}\uparrow^G$ by [3, 3.7]. Since $U'\downarrow_{QN}$ is finitely generated, we can find some indecomposable $k[[QN]]$-module S such that $S \mid U'\downarrow_{QN}$ and $U' \mid S\uparrow^G$. Now $U'\downarrow_{QN} \cong U'\downarrow_L\downarrow_{QN}$ so there is an indecomposable finitely generated $k[[L]]$-module V' such that $V' \mid U'\downarrow_L$ and $S \mid V'\downarrow_{QN}$. Note that V' is a direct summand of $V\uparrow^G\downarrow_L$ distinct from V, so by Lemma 2.1 it is projective relative to a subgroup of the form $tQt^{-1} \cap L$ with $t \in G, t \notin L$. Let T be a $k[[tQt^{-1} \cap L]]$-module such that $V' \mid T\uparrow^L$. From the Mackey decomposition theorem [5, 2.2] we have

$$S \mid V'\downarrow_{QN} \mid T\uparrow^L\downarrow_{QN} \cong \bigoplus_{l \in QN\backslash L/tQt^{-1}\cap L} l(T)\downarrow_{(lt)Q(lt)^{-1}\cap QN}\uparrow^{QN} .$$

Since $t \notin L$ it follows that S is projective relative to a subgroup of the form $xQx^{-1} \cap QN$ for some $x \notin L$. Since $U' \mid S\uparrow^G$ we have shown that for each N in our system the module U' is projective relative to a subgroup of the form $xQx^{-1} \cap QN$ for some $x \notin L$ that depends on N.

We would like to find some $x \in G, x \notin L$ for which U' is projective relative to $xQx^{-1} \cap QN$ for every N in our system. Denote by C_N the non-empty set of $x \in G, x \notin L$ for which U' is relatively $[xQx^{-1} \cap QN]$-projective. If ever $N \leq M$ and $x \in C_N$ then certainly U' is projective relative to $xQx^{-1}\cap QM$, so if $x \in C_N$ then $x \in C_M$. Since each C_N is closed in G the standard compactness argument now shows that $\bigcap_N C_N \neq \emptyset$.

Choose some $x \in G, x \notin L$ for which U' is projective relative to $xQx^{-1}\cap QN$ for each N in our system. By [3, 4.2] it follows that U' is projective relative to

$$\bigcap_N (xQx^{-1} \cap QN) = xQx^{-1} \cap (\bigcap_N QN) = xQx^{-1} \cap Q$$

as required. $\square$

In the finite case the following lemma is an easy corollary of Lemma 2.1. In our more general context it requires a little more care.

Lemma 2.3. *Let U be a finitely generated indecomposable $k[[G]]$-module with vertex Q. There is a finitely generated indecomposable $k[[L]]$-module V with vertex Q such that $U \mid V{\uparrow}^G$ and $V \mid U{\downarrow}_L$.*

Proof. We work in a cofinal inverse system of $N \lhd_O G$ with $\mathrm{N}_G(QN) \leq L$. We first show that $U\downarrow_L$ has an indecomposable summand with vertex Q. Since $U \mid U\downarrow_L{\uparrow}^G$ we have $U\downarrow_L$ has at least one summand with vertex conjugate to Q in G. Let $\mathcal{X}$ denote the non-empty set of isomorphism classes of $V \mid U\downarrow_L$ having vertex conjugate to Q. For each N, the fact that $U \mid U\downarrow_{QN}{\uparrow}^G$ implies that $U \mid V\downarrow_{QN}^L{\uparrow}^G$ for some $V \in \mathcal{X}$, and so $U \mid W{\uparrow}^G$ for some $W \mid V\downarrow_{QN}$. Clearly W has vertex $yQy^{-1} \subseteq QN$.

Suppose V has vertex xQx^{-1} and let S be a $k[[xQx^{-1}]]$-module with $V \mid S{\uparrow}^L$. Applying Mackey's formula to $W \mid S{\uparrow}^L\downarrow_{QN}$ it follows that V has vertex L-conjugate to a subgroup of QN, and so V has a vertex contained in QN. Note that $\mathcal{X}$ is a finite set, so some element of $\mathcal{X}$ must have vertex contained in QN for a cofinal subset of $N \lhd_O G$ and hence some element of $\mathcal{X}$ has vertex Q.

Let Z be an indecomposable summand of $U\downarrow_L$ and suppose that for all N in our system there is some $x \notin L$ such that Z is $xQNx^{-1}$-projective. Denote by C_N the non-empty set of all such $x \notin L$. If $x \in C_N$ then $xq \in C_N$ for all $q \in QN$ and so each set C_N is closed in G.

If $N \leq M$ and $x \in C_N$ then certainly $x \in C_M$. By compactness we now have that $\bigcap_N C_N \neq \emptyset$. Fix $x \in \bigcap_N C_N$. It follows that Z is $xQx^{-1}N$-projective for each N, and so Z is xQx^{-1}-projective. Note that for any $l \in L$ we have $(lx)Q(lx)^{-1} \neq Q$ since $lx \notin L$. From the conjugacy of vertices [3, 4.6] it follows that Z does not have vertex Q.

Since there is an indecomposable summand of $U\downarrow_L$ with vertex Q the contrapositive of the previous argument shows there is some $N_0 \lhd_O G$ such that this summand is not projective relative to xQN_0x^{-1} for any $x \notin L$. From now on we work within the cofinal system of $N \lhd_O G$ with $N \leq N_0$.

Let $\mathcal{T}$ denote the (finite, non-empty) set of isomorphism classes of indecomposable $V \mid U\downarrow_L$ such that $U \mid V{\uparrow}^G$. We wish to find an element of $\mathcal{T}$ with vertex Q. Choose N in our system. Since $U \mid U\downarrow_{QN}{\uparrow}^G$ we take some indecomposable summand $V \mid U\downarrow_{QN}{\uparrow}^L$ such that $U \mid V{\uparrow}^G$. Since $V \mid V{\uparrow}^G\downarrow_L$, by Lemma 2.1 we have two possibilities:

- $V \mid U\downarrow_L$ or

- Each summand of $U\downarrow_L$ is projective relative to $xQNx^{-1} \cap L$ for some $x \notin L$.

By our choice of N the latter cannot happen, so that $V \mid U\downarrow_L$ and so $V \in \mathcal{T}$. Thus for all N there is an element of $\mathcal{T}$ which is QN-projective, and so there is an element of $\mathcal{T}$ which has vertex Q, as required. $\square$

Proposition 2.4. *Let G be a virtually pro-p group, Q a closed pro-p subgroup of G and let L be an open subgroup of G containing $\mathrm{N}_G(Q)$. Then we have*

the following correspondence between finitely generated indecomposable $k[[G]]$-modules with vertex Q, and finitely generated indecomposable $k[[L]]$-modules with vertex Q:

1. *If U is a finitely generated indecomposable $k[[G]]$-module with vertex Q, then there is a unique indecomposable summand $f(U)$ of $U\downarrow_L$ with vertex Q, and the rest have vertex in $\mathfrak{Y}$.*

2. *If V is a finitely generated indecomposable $k[[L]]$-module with vertex Q, then there is a unique indecomposable summand $g(V)$ of $V\uparrow^G$ with vertex Q, and the rest have vertex in $\mathfrak{X}$.*

3. *The given correspondence is one-one in the sense that $f(g(V))\cong V$ and $g(f(U))\cong U$.*

Proof.   1. By Lemma 2.3 we have that $U\,\big|\,V\uparrow^G$ for some finitely generated indecomposable $k[[L]]$-module V with vertex Q. Thus $U\downarrow_L\,\big|\,V\uparrow^G\downarrow_L$. By Lemma 2.1, V is the only summand of $V\uparrow^G\downarrow_L$ with vertex Q and the rest have vertex in $\mathfrak{Y}$, so that $U\downarrow_L$ has at most one summand with vertex Q. On the other hand, again by Lemma 2.3 we have that $U\downarrow_L$ has at least one summand with vertex Q. Hence we set $f(U)=V$ and the claim holds.

2. We have $V\,\big|\,V\uparrow^G\downarrow_L$ so we choose an indecomposable summand $U\,\big|\,V\uparrow^G$ such that $V\,\big|\,U\downarrow_L$. By Lemma 2.2, we have $V\uparrow^G\cong U\oplus U_1$ where U_1 is $\mathfrak{X}$-projective. The module U has vertex Q since if it had smaller vertex then the Mackey decomposition theorem shows that V would as well. Thus, we take $g(V)=U$ and we are done.

3. This is clear.

$\square$

3   A more general case

We retain the notation from above but drop the assumption that L is open in G. When L was open and U,V were Green correspondents as above, we see in particular that $V\,\big|\,U\downarrow_L$. This need not be the case when L has infinite index in G - an example of this phenomenon can be found in the last section of [5]. For this reason we now focus on the map g.

Let V be an indecomposable finitely generated $k[[L]]$-module with vertex Q. By [3, 5.1] we can choose a cofinal inverse system of $N\lhd_O G$ for which $V\uparrow^{LN}$ is indecomposable. We work in this system as we prove the following key lemma:

Lemma 3.1. *For any given $M\lhd_O G$ in our inverse system the module $V\uparrow^{LM}$ has vertex Q.*

Proof. Certainly $V\uparrow^{LM}$ is relatively Q-projective, so we choose some vertex R of $V\uparrow^{LM}$ contained in Q. We will show that V is R-projective. Consider the

cofinal inverse system of those $N \triangleleft_O G$ contained inside M, noting that for each such N the module $V\uparrow^{LN}$ is indecomposable.

Let S be a $k[[R]]$-module such that $V\uparrow^{LM} \big| S\uparrow^{LM}$. Then for each $N \leq M$ we have

$$V\uparrow^{LN} \big| V\uparrow^{LM}\downarrow_{LN} \big| S\uparrow^{LM}_R\downarrow_{LN} \cong \bigoplus_{x \in LN \backslash LM / R} x(S)\downarrow_{xRx^{-1} \cap LN}\uparrow^{LN}$$

so that $V\uparrow^{LN}$ is xRx^{-1}-projective for some $x \in LM$ and hence has vertex xRx^{-1}. Denote by C_N the non-empty set of all $y \in LM$ with the property that yRy^{-1} is a vertex of $V\uparrow^{LN}$. Then C_N is a finite union of right cosets of LN so is a closed subset of LM. We would like to show that $\bigcap_N C_N \neq \emptyset$.

Given $N_1, \ldots, N_n$, let $N' = N_1 \cap \ldots \cap N_n$. Then by the argument above $C_{N'} \neq \emptyset$. But $C_{N'} \subseteq C_{N_i}$ for each i, since if $V\uparrow^{LN'}$ is induced from a yRy^{-1}-module, then so is each $V\uparrow^{LN_i}$. Thus, $\emptyset \neq C_{N'} \subseteq C_{N_1} \cap \ldots \cap C_{N_n}$ and so by compactness $\bigcap_N C_N \neq \emptyset$. It follows that we can find some $y \in LM$ so that $V\uparrow^{LN}$ is yRy^{-1}-projective for each $N \leq M$.

We move now from induced modules to coinvariant modules. Note that if $V\uparrow^{LN}$ is yRy^{-1}-projective then it is certainly $yRNy^{-1}$-projective, so for some $yRNy^{-1}$-module T we have $V\uparrow^{LN} \big| T\uparrow^{LN}$. Now

$$V_{L \cap N} \cong (V\uparrow^{LN})_N \big| (T\uparrow^{LN})_N \cong T_N\uparrow^{LN}$$

by [3, 2.6] so that $V_{L \cap N}$ is $yRNy^{-1}$-projective for each N in our system. Now by [3, 3.5] the module V is yRy^{-1}-projective and so some conjugate of yRy^{-1} contains Q. Thus $R \leq Q \leq zRz^{-1}$ for some $z \in LM$, so $R = Q$ and we are done. $\square$

Recall that L contains the normalizer of Q in G.

Corollary 3.2. *Let V be an indecomposable finitely generated $k[[L]]$-module with vertex Q. Then $V\uparrow^G$ has a unique summand $g(V)$ with vertex Q, and the rest have vertex in $\mathfrak{X}$.*

Proof. We choose some $M \triangleleft_O G$ for which $V\uparrow^{LM}$ is indecomposable. By Lemma 3.1, $V\uparrow^{LM}$ has vertex Q. But now by Proposition 2.4, $V\uparrow^G \cong V\uparrow^{LM}\uparrow^G$ has a unique summand $g(V)$ with vertex Q and the rest have vertex in

$$\{X \leq_C G \,|\, X \leq xQx^{-1} \cap Q, x \notin LM\}$$

but this is a subset of $\mathfrak{X}$ and so we are done. $\square$

We can now prove Theorem 1.1:

Proof. The map g from Corollary 3.2 restricted to those modules with source S has the appropriate image and domain. We need only check that g is bijective.

First we show that if U is an indecomposable $k[[G]]$-module with vertex Q and source S, then there is some indecomposable $k[[L]]$-module V with vertex Q

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and source S such that $U \cong g(V)$. But this is clear since if $S{\uparrow}^L \cong V_1 \oplus \ldots \oplus V_n$ is a decomposition into indecomposable summands then

$$U \mid S{\uparrow}^G \cong S{\uparrow}^L{\uparrow}^G \cong V_1{\uparrow}^G \oplus \ldots \oplus V_n{\uparrow}^G$$

and so $U \mid V_i{\uparrow}^G$ for some i since U has local endomorphism ring by [3, 4.4]. Clearly V_i has vertex Q. This shows that g is surjective.

It remains to show that if V, W are finitely generated indecomposable $k[[L]]$-modules having vertex Q and source S, and $g(V) \cong g(W)$ as $k[[G]]$-modules, then $V \cong W$ as $k[[L]]$-modules. Choose a cofinal inverse system of $N \lhd_O G$ for which both $V{\uparrow}^{LN}$ and $W{\uparrow}^{LN}$ are indecomposable. Let $g(V) \cong U \cong g(W)$. The modules $V{\uparrow}^{LN}$ and $W{\uparrow}^{LN}$ are both Green correspondents of U in the sense of Proposition 2.4 and so $V{\uparrow}^{LN} \cong W{\uparrow}^{LN}$ for each N in our inverse system. But

$$V{\uparrow}^{LN} \cong W{\uparrow}^{LN}$$
$$\implies (V{\uparrow}^{LN})_N \cong (W{\uparrow}^{LN})_N$$
$$\implies V_{L \cap N} \cong W_{L \cap N}$$

for each N, and so $V \cong W$ by [3, 3.4]. $\qquad\square$

4 Acknowledgements

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