On Archimedean Decompositions of Linearly Ordered Fields PREPRINT

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1 Functor Mechanics

We recall that an *ordered field* is a field F together with a linear order $\leq$ that respects addition and multiplication. Similarly an *ordered group* is a group G together with a linear order $\leq$ that respects the group operation.

Definition 1.1 (Category of Ordered Fields). Define **OrdFld** to be the category with its objects ordered fields, and its morphisms order-preserving field homomorphisms. (Note that, since field homomorphisms are always injective, the morphisms of **OrdFld** are *strictly* order-preserving.)

Definition 1.2 (Category of Ordered Abelian Groups). Define **OrdAbGrp** to be the category with its objects ordered abelian groups, and its morphisms strictly order-preserving group homomorphisms.

Remark 1.3. Since in this paper we are interested exclusively in ordered fields and ordered abelian groups, when we refer to homomorphisms of ordered fields or ordered groups, it will be implicit that we refer to morphisms in the appropriate category – that is, that we refer to strictly order-preserving homomorphisms.

Definition 1.4 (Simple Map). Let G be an ordered group. A function $p : \mathbb{N} \to G$ is called *simple* if it is injective and its image is right-well-ordered – that is, if every non-empty subset of $\mathrm{im}(p)$ has a largest element.

Definition 1.5 (Index). Let G be an ordered group, F be an ordered field, $p : \mathbb{N} \to G$ simple, and $a : \mathbb{N} \to F$ a sequence. Then define the *index of p over a* to be

$$\mathrm{ind}(p, a) \quad = \quad p^{-1}\left(\max_{n\in\mathbb{N}} \{p(n) \ : \ a(n) \neq 0\}\right)$$

if the maximum exists, and $\mathrm{ind}(p, a) = 0$ otherwise. (We note that since $\mathrm{im}(f)$ is right-well-ordered, the maximum always exists unless $a(n) = 0$ for all n.)

Definition 1.6 (Long Category of Ordered Groups). Define $\mathbf{OrdAbGrp}^\infty$ to be the category with its objects of the form

$$(I, G)$$

where I is an ordered set, and $G : I \to \mathrm{Obj}(\mathbf{OrdAbGrp})$. The morphisms are of the form

$$(I, f) : (I, G) \to (I, H)$$

where $f : I \to \mathrm{Mor}(\mathbf{OrdAbGrp})$ with $f(i) : G(i) \to H(i)$. (This notation implicitly conveys that if I and J are different index sets, the set $\mathrm{Hom}((I, G), (J, H))$ is empty.) Morphism composition is the obvious:

$$(I, f) \circ (I, g) \quad = \quad (I, f \circ g)$$

where $(f \circ g)(i) = f(i) \circ g(i)$.

Definition 1.7 (Box Product Functor). Let

$$\boxtimes : \mathbf{OrdFld} \times \mathbf{OrdAbGrp} \to \mathbf{OrdFld}$$

be defined as follows. For $F \in \mathrm{Obj}(\mathbf{OrdFld})$ and $G \in \mathrm{Obj}(\mathbf{OrdAbGrp})$, define

$$F \boxtimes G \quad = \quad \left\{ \sum_{i=0}^{\infty} a_i x^{p(i)} \; : \; a_i \in F, p \text{ simple} \right\}$$

Here x is a formal variable, and addition, multiplication, and division of the formal power series work as usual. We take the linear order to be induced by

$$\sum_{i=0}^{\infty} a_i x^{p(i)} > 0 \quad \Longleftrightarrow \quad a_{\mathrm{ind}(p,a)} > 0$$

Then if $f : F \to F'$ and $g : G \to G'$ are morphisms in $\mathbf{OrdFld}$ and $\mathbf{OrdAbGrp}$ respectively, then define $f \boxtimes g : F \boxtimes G \to F' \boxtimes G'$ by

$$(f \boxtimes g) \left(\sum_{i=0}^{\infty} a_i x^{p(i)} \right) \quad = \quad \sum_{i=0}^{\infty} f(a_i) x^{g(p(i))}$$

From the above definition, we can see the motivation for our definition of $\mathrm{ind}(p, a)$. Effectively, we wish to order our formal sum by the (nonzero) coefficient of the term with the largest exponent, and $\mathrm{ind}(p, a)$ gives us the index where this occurs.

Proposition 1.8. The category map $\boxtimes$ is a functor.

Definition 1.9 (Box Sum Functor). Let

$$\boxplus : \mathbf{OrdAbGrp}^\infty \to \mathbf{OrdAbGrp}$$

be defined as follows. For $(I, G) \in \mathrm{Obj}(\mathbf{OrdAbGrp}^{\infty})$, define

$$\boxplus(I, G) \quad = \quad \left(\bigoplus_{i \in I} G(i), \leq \right)$$

where the ordering $\leq$ is given by

$$(g_i)_{i \in I} > 0 \quad \Longleftrightarrow \quad g_k > 0$$

where $k \in I$ is the largest element such that $g_k \neq 0$. Then if $(I, f) : (I, G) \to (I, H)$ is a morphism in $\mathrm{Mor}(\mathbf{OrdAbGrp}^{\infty})$, we define $\boxplus(I, f) : \boxplus(I, G) \to \boxplus(I, H)$ by

$$\boxplus(I, f)\left((g_i)_{i \in I}\right) \quad = \quad \left(f_i(g_i)\right)_{i \in I}$$

as expected.

Proposition 1.10. The category map $\boxplus$ is a functor.

Notation 1.11. For G, H ordered groups, we will sometimes write $G \boxplus H$ to denote $\boxplus(I, F)$ where $I = \{0, 1\}$ with the usual ordering, where $F(0) = G$ and $F(1) = H$. We may also write $f \boxplus g$ to have the equivalent meaning on morphisms.

Theorem 1.12. Let F be an ordered field and let G, H be ordered groups. Then there is a natural isomorphism

$$F \boxtimes (G \boxplus H) \simeq (F \boxtimes G) \boxtimes H$$

Proof:

This follows by the formal variable identification $x^{(a,b)} \mapsto y^a z^b$, together with a standard index argument. ■

2 Generalized Metric Mechanics

2.1 Completeness

In (CITATION HERE) it was shown that ordered fields have a natural β-structure. We recall that β-spaces are generalizations of metric space given by a triple (X, R, β) where $\beta : X \times R \to \wp(X)$ satisfies

1. $x \in \beta(x, r)$
2. For all $z \in \beta(x, r) \cap (y, s)$, there is a t such that $\beta(z, t) \subseteq \beta(x, r) \cap \beta(y, s)$
3. For all $r \in R$ there is an $s \in R$, called a swing value for r, such that if $y \in \beta(x, s)$, then $\beta(x, s) \subseteq \beta(y, r)$.

For an ordered field F, this structure is given by $X = F$, $R = F^{>0}$, and $y \in \beta(x, r)$ precisely when $|x - y| < r$. We recall the following definitions:

Definition 2.1 (Net).

Definition 2.2 (Cauchy).

Definition 2.3 (Converge).

Definition 2.4 (Completion). The completion of a field F will be denoted by $\overline{F}$.

Definition 2.5 (r-Cauchy).

Definition 2.6 (r-Converge).

Definition 2.7 (Radially Complete). The radial completion of a field F will be denoted by $\overline{\overline{F}}$.

Theorem 2.8. For any $F \in \mathrm{Obj}(\mathbf{OrdFld})$ and any nontrivial $G \in \mathrm{Obj}(\mathbf{OrdAbGrp})$, the field $F \boxtimes G$ is complete.

Example 2.9. The Levi-Civita field LC is a well-known non-Archimedean ordered field. It is constructed as the set of all formal power series of the form

$$\sum_{q \in \mathbb{Q}} a_q \varepsilon^q$$

where the collection $(a_q)_{q \in \mathbb{Q}} \subseteq \mathbb{R}$ is left-finite – that is, for each k, the collection (a_q) only contains finitely many points smaller than a_k. The ordering on this field is given by the usual dictionary ordering; that is,

$$\sum_{q \in \mathbb{Q}} a_q \varepsilon^q > 0 \quad \Longleftrightarrow \quad a_k > 0$$

where k is the smallest rational such that $a_k \neq 0$. Of course we can re-index by replacing q with $-q$. Then we equivalently require that the collection (a_q) be right-finite, and specify that our element $\sum_{q \in \mathbb{Q}} a_q \left(\varepsilon^{-1}\right)^q$ is considered positive when $a_k > 0$ for k the largest index such that $a_k \neq 0$. Clearly right-finite implies right-well-ordered. Therefore, enumerating the rationals by $\alpha : \mathbb{N} \to \mathbb{Q}$, the function $p(n) = a_{\alpha(n)}$ is simple, and so, making the identification $\varepsilon^{-1} \mapsto x$, we have the anticipated (canonical) isomorphism

$$LC \quad = \quad \mathbb{R} \boxtimes \mathbb{Q}$$

The completeness of LC (which is well-known) then follows immediately from Theorem 2.8.

Remark 2.10. Theorem 2.8 has the interesting (and unexpected) consequence that a field that is not complete can form one that is complete under the $\boxtimes$ functor. We can easily see that $\mathbb{Q} \boxtimes \mathbb{Z}$ is complete by the above theorem. Of course this is somewhat unappealing, and we would like there to be some version of completeness that is preserved under the $\boxtimes$ operation. Our next theorem shows that the concept of radial completeness is precisely the definition we need.

Theorem 2.11. *$F \boxtimes G$ is radially complete if and only if F is radially complete.*

Example 2.12. The Levi-Civita field LC is radially complete.

2.2 Level Operations

Definition 2.13 (Swing-Sequence). A *swing-sequence* for a value $a \in R$ is a sequence $(a_i)_{i=1}^{\infty}$ such that $a_1 = a$ and a_{i+1} is a swing value for a_i for all i. When $(a_i)_{i=1}^{\infty}$ is a swing sequence for a, we write $(a_i) \prec a$.

Definition 2.14 (Level Set). Given $x \in X$ and $r \in R$, the *level set about x of radius r* is given by

$$L(x,r) \quad = \quad \bigcup_{(r_i) \prec r} \left(\bigcap_{i=1}^{\infty} \beta(x, r_i) \right)$$

Definition 2.15 (Level-Equivalence). Two values $a, b \in R$ are said to be *level-equivalent*, written $a \sim b$, if $L(x,a) = L(x,b)$ for all x.

Proposition 2.16. Given an ordered field F and $a, b \in F^{>0}$, $a \sim b$ in the induced β-structure if and only if there are positive integers m, n such that $ma > b$ and $a < nb$.

Definition 2.17 (Level Group Functor). Let $\mathcal{L} : \mathbf{OrdFld} \to \mathbf{OrdAbGrp}$ be defined as follows. For $F \in \mathrm{Obj}(\mathbf{OrdFld})$, let

$$\mathcal{L}(F) \quad = \quad F^{>0} / \sim$$

where for $x, y \in F^{>0}$, $[x] + [y] = [x \cdot y]$, and where the ordering is given by

$$[x] > [y] \quad \iff \quad x > y \cdot n \text{ for all } n \in \mathbb{N}$$

If $f : F \to F'$ is a morphism in $\mathbf{OrdFld}$, we define $\mathcal{L}(f) : \mathcal{L}(F) \to \mathcal{L}(F')$ by

$$\mathcal{L}f\,([x]) \quad = \quad [f(x)]$$

The level group of an ordered field will be an essential component of our toolbox, and is in itself an interesting invariant. Before we proceed, we need to verify that the above operations are actually well-defined and functorial.

Proposition 2.18. The category map $\mathcal{L}$ is a well-defined functor.

Remark 2.19. It is natural to ask the following questions:

1. What are the weakest possible relationships between fields F and F' such that $\mathcal{L}(F) \simeq \mathcal{L}(F')$?
2. When (if ever) does $\mathcal{L}(F) \simeq \mathcal{L}(F')$ guarantee that $F \simeq F'$?

These questions are answered by Corollaries 3.3 and 3.13.

The equivalence relation $\sim$ clearly does not require the full structure of a field; in fact, its definition can be applied equally well to an ordered abelian group. Noting that $\mathcal{L}(F)$ is abelian for any F, we proceed with a lemma.

Lemma 2.20. For any ordered abelian group G, the set $G^{>0}/\sim$ has a natural ordering.

Definition 2.21 (Non-Archimedean Generator Set). The *non-Archimedean generator set*, or the *generator set*, of an ordered field F, is the ordered set defined by

$$\mathrm{Gen}(F) \quad = \quad \mathcal{L}(F)^{>0}/\sim$$

It should be noted that, as with our previous constructions, the construction of the generator set is indeed functorial; however, this fact does not seem to showcase any interesting or useful structure, and so it is omitted from formal treatment. For many of the theorems that follow, a categorical structure is obvious but not stated for the same reason.

Definition 2.22 (Non-Archimedean Degree). The *non-Archimedean degree* of an ordered field F is defined as

$$\deg_\infty(F) \quad = \quad |\mathrm{Gen}(F)|$$

As the names suggest, the non-Archimedean degree of an ordered field can intuitively be thought of as the number of non-Archimedean generators of the field. While this intuition will be made precise in Corollary 3.12, we can immediately see a compelling, if obvious, justification of this.

Proposition 2.23. An ordered field F is Archimedean if and only if $\deg_\infty(F) = 0$.

3 Field Components

Definition 3.1 (Maximal Archimedean Subfield). The *maximal Archimedean subfield* of an ordered field F is

$$\mathrm{Arch}(F) \quad = \quad \bigcup_{\substack{K \text{ Archimedean} \\ K \leq F}} K$$

Since an ordered field necessarily has characteristic 0, any ordered field F contains $\mathbb{Q}$ as a subfield. This guarantees that $\mathrm{Arch}(F)$ is always nonempty.

Theorem 3.2. If F is an ordered non-Archimedean field, then $\overline{F} \simeq \mathrm{Arch}(F) \boxtimes \mathcal{L}(F)$.

Of course, the above theorem doesn't hold for F Archimedean, since the closure of any Archimedean field is always the reals. In particular, if $F = \mathbb{Q}$, clearly $\mathrm{Arch}(F) = \mathbb{Q}$, and $\mathcal{L}(F) = \{0\}$. But

$$\mathbb{R} \not\simeq \mathbb{Q} \boxtimes \{0\} \simeq \mathbb{Q}$$

This is another instance of the effect illustrated in Theorem 2.8, where completeness on non-Archimedean fields does not rely on the structure of the underlying Archimedean subfield.

Corollary 3.3. Let F_1 and F_2 be ordered non-Archimedean fields. Then $\overline{F_1} \simeq \overline{F_2}$ if and only if $\mathcal{L}(F_1) \simeq \mathcal{L}(F_2)$ and $\mathrm{Arch}(F_1) \simeq \mathrm{Arch}(F_2)$.

Proof:

The $\Longrightarrow$ direction is trivial. The $\Longleftarrow$ direction follows from the fact that $\boxtimes$ is a functor, so that if $f : \mathrm{Arch}(F_1) \to \mathrm{Arch}(F_2)$ and $g : \mathcal{L}(F_1) \to \mathcal{L}(F_2)$, then

$$f \boxtimes g : \mathrm{Arch}(F_1) \boxtimes \mathcal{L}(F_1) \to \mathrm{Arch}(F_2) \boxtimes \mathcal{L}(F_2)$$

In particular, since f and g are isomorphisms, $f \boxtimes g$ is an isomorphism. ■

Definition 3.4 (Upper and Lower Generator Groups). Let F be an ordered field, and let $[x] \in \mathrm{Gen}(F)$. We define the *upper generator group for x* to be

$$G^{[x]} = \left\langle \bigcup_{[y] \leq [x]} [y] \right\rangle$$

where $\langle A \rangle$ indicates the subgroup of $\mathcal{L}(F)$ generated by all the elements of the set $A \subseteq \mathcal{L}(F)$. Similarly, we define the *lower generator group for x* to be

$$G_{[x]} = \left\langle \bigcup_{[y] < [x]} [y] \right\rangle$$

Definition 3.5 (Class Group). Given an ordered field F and some $[x] \in \mathrm{Gen}(F)$, we define the *class group of x* to be the group

$$\mathcal{G}\,[x] = G^{[x]}/G_{[x]}$$

Proposition 3.6. For any $[x] \in \mathrm{Gen}(F)$, $\mathcal{G}[x]$ is Archimedean, with a natural order induced by the order on $\mathcal{L}(F)$.

Theorem 3.7. For any ordered field F,

$$\mathcal{L}(F) \quad \simeq \quad \boxplus(\mathrm{Gen}(F), \mathcal{G})$$

Corollary 3.8. For any ordered field F and any ordered abelian group G,

$$\mathcal{L}(F \boxtimes G) \quad \simeq \quad \mathcal{L}(F) \boxplus G$$

Corollary 3.9. For any Archimedean field F and any ordered abelian group G,

$$\mathcal{L}(F \boxtimes G) \quad \simeq \quad G$$

Lemma 3.10. An Archimedean field F is complete if and only if it is radially complete.

Theorem 3.11. Let F be an ordered field. Then there exists an Archimedean field F_0, an ordered index set I and a collection of ordered Archimedean abelian groups $(G_i)_{i \in I}$ – all unique up to isomorphism – such that

$$\overline{F} \quad \simeq \quad F_0 \boxtimes (\boxplus(I, G))$$

Additionally, $\overline{F}$ is radially complete if and only if $F_0 = \mathbb{R}$.

Proof:

Of course we take $F_0 = \mathrm{Arch}(F)$, $I = \mathrm{Gen}(F)$, and $G = \mathcal{G}$. The isomorphism follows from Theorem 3.2 together with Theorem 3.7. Uniqueness follows from the functoriality of $\mathcal{L}$, $\boxtimes$, and $\boxplus$, together with Corollary 3.9. Radial completeness follows from Theorem 2.11 together with Lemma 3.10. ■

Corollary 3.12.

If $\deg_\infty(F) = n < \infty$, then there exist F_0 and groups $G_1, G_2, \ldots, G_n$ as in Theorem 3.11 such that

$$\overline{F} \quad \simeq \quad F_0 \boxtimes G_1 \boxtimes \ldots \boxtimes G_n$$

Proof:

This follows from Theorem 1.12. ■

Corollary 3.13. For any ordered fields F, F', $\overline{\overline{F}} \simeq \overline{\overline{F'}}$ if and only if $\mathcal{L}(F) \simeq \mathcal{L}(F')$.

Corollary 3.14. Any countable ordered field can be embedded in $\mathbb{R}$.