MORI'S PROGRAM FOR THE MODULI SPACE OF POINTED STABLE RATIONAL CURVES

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ABSTRACT. We prove that, assuming the F-conjecture, the log canonical model of the pair $(\overline{M}_{0,n}, \sum a_i\psi_i)$ is the Hassett's moduli space $\overline{M}_{0,\mathcal{A}}$ without any modification of weight coefficients. For the boundary weight cases, we prove that the birational model is the GIT quotient of the product of the projective lines. This is a generalization of Simpson's theorem for symmetric weight cases.

1. INTRODUCTION

During the last several decades, the *log canonical model* of a pair (X, D) which is defined by

$$X(K_X + D) := \text{Proj}\ \left(\oplus_{l\ge 0} H^0(X, \mathcal{O}(l(K_X + D)))\right) \tag{1}$$

where the sum is taken over l sufficiently divisible, plays an important role in birational geometry especially in Mori's program. In this paper, we prove the following theorem.

Theorem 1.1. *Let $\mathcal{A} = (a_1, a_2, \cdots, a_n)$ be a weight datum, that is, a sequence of rational numbers such that $0 < a_i \le 1$. Let $\overline{M}_{0,n}$ be the moduli space of pointed stable rational curves. Let ψ_i be the i-th psi class* ([2, Section 2]).

(1) Assume the F-conjecture. Suppose that $\sum_{i=1}^n a_i > 2$. Then $\overline{M}_{0,n}(K_{\overline{M}_{0,n}} + \sum_{i=1}^n a_i\psi_i)$ is isomorphic to $\overline{M}_{0,\mathcal{A}}$, the Hassett's moduli space of weighted pointed stable rational curves.
(2) Suppose that $\sum_{i=1}^n a_i = 2$. Then $\overline{M}_{0,n}(K_{\overline{M}_{0,n}} + \sum_{i=1}^n a_i\psi_i)$ is isomorphic to the geometric invariant theory (GIT) quotient $(\mathbb{P}^1)^n /\!/_L SL(2)$ with respect to the linearization $L = \mathcal{O}(a_1, \cdots, a_n)$.

For the definition and properties of $\overline{M}_{0,\mathcal{A}}$, see [5]. The precise statement of the F-conjecture is in [8, Question 1.1]. Note that item (2) does not rely on the F-conjecture.

Theorem 1.1 is an outcome of an attempt to generalize the following theorem of M. Simpson. Set $m = \lfloor \frac{n}{2} \rfloor$. Let ϵ_k be a rational number in the range $\frac{1}{m+1-k} < \epsilon_k \le \frac{1}{m-k}$ for $k = 1, 2, \cdots, m-2$. Let $n \cdot \epsilon = (\epsilon, \cdots, \epsilon)$ be a *symmetric* weight datum.

Theorem 1.2. [1, 4, 9, 10] *Let β be a rational number satisfying $\frac{2}{n-1} < \beta \le 1$ and let $D = \overline{M}_{0,n} - M_{0,n}$ denote the total boundary divisor.*

(1) If $\frac{2}{m-k+2} < \beta \le \frac{2}{m-k+1}$ for $k = 1, 2, \cdots, m-2$, then $\overline{M}_{0,n}(K_{\overline{M}_{0,n}} + \beta D) \cong \overline{M}_{0,n\cdot\epsilon_k}$.
(2) If $\frac{2}{n-1} < \beta \le \frac{2}{m+1}$, then $\overline{M}_{0,n}(K_{\overline{M}_{0,n}} + \beta D) \cong (\mathbb{P}^1)^n /\!/_L SL(2)$ where $L = \mathcal{O}(1, \cdots, 1)$.

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Indeed, Theorem 1.2 is a special case of Theorem 1.1 under the F-conjecture. See Remark 2.1.

In [3], Fedorchuk proved that for every weight datum $\mathcal{A}$ and genus g, there exists a divisor $D_{g,\mathcal{A}}$ on $\overline{M}_{g,n}$ such that $(\overline{M}_{g,n}, D_{g,\mathcal{A}})$ is a *log canonical pair* and $\overline{M}_{g,n}(K_{\overline{M}_{0,n}} + D_{g,\mathcal{A}}) \cong \overline{M}_{g,\mathcal{A}}$. His divisor $K_{\overline{M}_{0,n}} + D_{0,\mathcal{A}}$ is not proportional to $K_{\overline{M}_{0,n}} + \sum_{i=1}^n a_i\psi_i$.

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2. Proof of the Main Theorem

Throughout this section, we will assume $n \geq 4$. Fix a weight datum $\mathcal{A} = (a_1, a_2, \cdots, a_n)$ such that $\sum a_i > 2$. Let T' be the set of nonempty proper subsets of $[n] := \{1, 2, \cdots, n\}$. For each $I \in T'$ we can define the *weight* of I as $w_I := \sum_{i \in I} a_i$. Let $T \subset T'$ be the subset of $I \subset [n]$ such that $w_I < w_{I^c}$ or $w_I = w_{I^c}$, $|I| < |I^c|$ or $w_I = w_{I^c}$, $|I| = |I^c|$, $1 \in I$. So there is a bijection between the set $\{I \in T \mid 2 \leq |I| \leq n-2\}$ and the set of irreducible components D_I of the boundary divisor of $\overline{M}_{0,n}$. Set $D_I = 0$ in the Neron-Severi vector space $N^1(\overline{M}_{0,n})$ for $I \in T$ with $|I| = 1$.

Let $\Delta_{\mathcal{A}} := K_{\overline{M}_{0,n}} + \sum_{i=1}^n a_i\psi_i$. By [2, Lemma 1] and [8, Lemma 3.5], it is straightforward to show that

$$(2) \quad \Delta_{\mathcal{A}} = \sum_{\substack{I \in T \\ 2 \leq |I| \leq n-2}} \left(\frac{|I|(n-|I|)}{n-1} - 2 + \frac{(n-|I|)(n-|I|-1)}{(n-1)(n-2)} w_I + \frac{|I|(|I|-1)}{(n-1)(n-2)} w_{I^c} \right) D_I.$$

By the geometry of reduction morphism $\varphi_{\mathcal{A}} : \overline{M}_{0,n} \to \overline{M}_{0,\mathcal{A}}$ ([5, §4]), the following formulas are results of routine calculations. Let $C \subset T$ (chosen for *contracted*) be a subset such that $w_I \leq 1$. Let $D_I \in N^1(\overline{M}_{0,\mathcal{A}})$ be $\varphi_{\mathcal{A}*}(D_I)$ with abuse of notation.

$$(3) \quad \varphi_{\mathcal{A}*}(\Delta_{\mathcal{A}}) = \sum_{I \in C^c} \left(\frac{|I|(n-|I|)}{n-1} - 2 + \frac{(n-|I|)(n-|I|-1)}{(n-1)(n-2)} w_I + \frac{|I|(|I|-1)}{(n-1)(n-2)} w_{I^c} \right) D_I,$$

(4)

$$\begin{aligned}
\varphi_{\mathcal{A}}^*\varphi_{\mathcal{A}*}(\Delta_{\mathcal{A}}) &= \sum_{I \in C^c} \left(\frac{|I|(n-|I|)}{n-1} - 2 + \frac{(n-|I|)(n-|I|-1)}{(n-1)(n-2)} w_I + \frac{|I|(|I|-1)}{(n-1)(n-2)} w_{I^c} \right) D_I \\
&\quad + \sum_{I \in C} \sum_{\substack{J \subset I \\ |J|=2}} \left(\frac{2(n-2)}{n-1} - 2 + \frac{(n-2)(n-3)}{(n-1)(n-2)} w_J + \frac{2 \cdot 1}{(n-1)(n-2)} w_{J^c} \right) D_I \\
&= \sum_{I \in C^c} \left(\frac{|I|(n-|I|)}{n-1} - 2 + \frac{(n-|I|)(n-|I|-1)}{(n-1)(n-2)} w_I + \frac{|I|(|I|-1)}{(n-1)(n-2)} w_{I^c} \right) D_I \\
&\quad + \sum_{I \in C} \left(\binom{|I|}{2} \left(\frac{2(n-2)}{n-1} - 2 \right) + \frac{n-3}{n-1}(|I|-1) w_I + \frac{(|I|-1)(|I|-2)}{(n-1)(n-2)} w_I + \frac{|I|(|I|-1)}{(n-1)(n-2)} w_{I^c} \right) D_I.
\end{aligned}$$

Observe that if $|I| = 1$ (so $I \in C$), then the coefficient of D_I is zero.

From a direct calculation, it is immediate to check that

$$\Delta_{\mathcal{A}} - \varphi_{\mathcal{A}}^* \varphi_{\mathcal{A}*}(\Delta_{\mathcal{A}}) = \sum_{\substack{I \in C \\ 2 \le |I| \le n-2}} \Big((|I| - 2)(1 - w_I) \Big) D_I. \tag{5}$$

Note that for every $I \in C$ with nonzero D_I, $w_I \le 1$ by the definition of C. So the difference $\Delta_{\mathcal{A}} - \varphi_{\mathcal{A}}^* \varphi_{\mathcal{A}*}(\Delta_{\mathcal{A}})$ is effective and supported on the exceptional locus of $\varphi_{\mathcal{A}}$. This implies that $H^0(\overline{M}_{0,n}, \Delta_{\mathcal{A}}) \cong H^0(\overline{M}_{0,n}, \varphi_{\mathcal{A}}^* \varphi_{\mathcal{A}*}(\Delta_{\mathcal{A}})) \cong H^0(\overline{M}_{0,\mathcal{A}}, \varphi_{\mathcal{A}*}(\Delta_{\mathcal{A}}))$. The same statement holds for a positive multiple of the same divisors. Thus we get

$$\overline{M}_{0,n}(\Delta_{\mathcal{A}}) = \mathrm{Proj} \left(\bigoplus_{l \ge 0} H^0(\overline{M}_{0,n}, \mathcal{O}(l\Delta_{\mathcal{A}})) \right) = \mathrm{Proj} \left(\bigoplus_{l \ge 0} H^0(\overline{M}_{0,\mathcal{A}}, \mathcal{O}(l\varphi_{\mathcal{A}*}(\Delta_{\mathcal{A}}))) \right). \tag{6}$$

If we prove $\varphi_{\mathcal{A}*}(\Delta_{\mathcal{A}})$ is ample, then the last birational model is exactly $\overline{M}_{0,\mathcal{A}}$.

Due to the F-conjecture ([8, Question 1.1]), we may assume that the Mori cone $\overline{\mathrm{NE}}_1(\overline{M}_{0,n})$ is generated by vital curve classes. This implies that $\overline{\mathrm{NE}}_1(\overline{M}_{0,\mathcal{A}})$ is *finitely* generated by the images of non-contracted vital curve classes. Thus by Kleiman's criterion, a Cartier divisor $\varphi_{\mathcal{A}*}(\Delta_{\mathcal{A}})$ is ample on $\overline{M}_{0,\mathcal{A}}$ if and only if for every non-contracted vital curve class $C \in \overline{\mathrm{NE}}_1(\overline{M}_{0,\mathcal{A}})$, $C \cdot \varphi_{\mathcal{A}*}(\Delta_{\mathcal{A}}) > 0$. By projection formula, this is equivalent to $\varphi_{\mathcal{A}}^* \varphi_{\mathcal{A}*}(\Delta_{\mathcal{A}})$ is nef and it contracts exceptional curves only. Therefore to check the ampleness of $\varphi_{\mathcal{A}*}(\Delta_{\mathcal{A}})$, it suffices to show that $\varphi_{\mathcal{A}}^* \varphi_{\mathcal{A}*}(\Delta_{\mathcal{A}})$ intersects positively with non-contracted vital curves.

Although the number of vital curve classes increases exponentially, the intersection numbers of vital curves and $\varphi_{\mathcal{A}}^* \varphi_{\mathcal{A}*}(\Delta_{\mathcal{A}})$ have surprisingly simple patterns. For a partition $S_1 \sqcup S_2 \sqcup S_3 \sqcup S_4 = [n]$, let $C(S_1, S_2, S_3, S_4)$ be the corresponding vital curve class. Let $w_j := \sum_{i \in S_j} a_i$, $T_j := |S_j|$ and $w := w_1 + w_2 + w_3 + w_4$. We may assume that $w_1 \le w_2 \le w_3 \le w_4$.

We encode the data of weights and partitions into a sequence of symbols of length 7. Let $(s_1, s_2, s_3, s_4, s_{12}, s_{13}, s_{14})$ be a sequence of symbols such that each s_i or s_{ij} is one of $-, +, *$. A vital curve class $C(S_1, S_2, S_3, S_4)$ is called of *type* $(s_1, s_2, s_3, s_4, s_{12}, s_{13}, s_{14})$ if the following conditions are satisfied:

- If s_i (resp. s_{ij}) is $-$, then $w_i \le 1$ (resp. $w_i + w_j \le 1$);
- If s_i (resp. s_{ij}) is $+$, then $1 < w_i < w - 1$ (resp. $1 < w_i + w_j < w - 1$);
- If s_i (resp. s_{ij}) is $*$, then $w_i \ge w - 1$ (resp. $w_i + w_j \ge w - 1$).

It is straightforward to see that a vital curve class $C(S_1, S_2, S_3, S_4)$ is contracted by $\varphi_{\mathcal{A}}$ if and only if there exists j such that $\sum_{i \notin S_j} a_i \le 1$. So we may assume that $w_i + w_j + w_k > 1$ for all $\{i, j, k\} \subset \{1, 2, 3, 4\}$, hence no s_i is $*$. Then we can divide all non-contracted vital curve classes into 13 types. By using [8, Lemma 4.3], we conclude that the formulas of intersection numbers $C(S_1, S_2, S_3, S_4) \cdot \varphi_{\mathcal{A}}^* \varphi_{\mathcal{A}*}(\Delta_{\mathcal{A}})$ are depend only on the type of vital curves. It is easy to calculate the intersection numbers by using a computer algebra system. The list of all intersection numbers are in Table 1 in page 4. Note that all terms are nonnegative and for each intersection number the last term is positive. This completes the proof of item (1) of Theorem 1.1.

Next, we prove item (2) of Theorem 1.1. Set T', T and C as before. Define

$$\Delta'_{\mathcal{A}} := (n-4) \sum_{I \in T} \left(-\binom{|I|}{2} \frac{2}{(n-1)(n-2)} + \frac{|I| - 1}{n-2} w_I \right) D_I. \tag{7}$$

Types	Intersection numbers
$(-,-,-,-,-,-,-)$	$(n-4)w_1 + T_1(w-2)$
$(-,-,-,-,-,-,+)$	$(T_1-1)(w_1+w_2+w_3-1)+(T_2+T_3-2)w_1+(T_4-1)(1-w_4)+(w-2)$
$(-,-,-,-,-,+,+)$	$(T_2-1)w_1+(T_1-1)w_2+(T_3-1)(1-w_3)+(T_4-1)(1-w_4)+(w-2)$
$(-,-,-,-,+,+,+)$	$(T_1-1)(1-w_1)+(T_2-1)(1-w_2)+(T_3-1)(1-w_3)+(T_4-1)(1-w_4)+(w-2)$
$(-,-,-,+,-,-,+)$	$(T_2+T_3-2)w_1+T_1(w_1+w_2+w_3-1)$
$(-,-,-,+,-,+,+)$	$(T_2-1)w_1+(T_1-1)w_2+(T_3-1)(1-w_3)+(w_1+w_2+w_3-1)$
$(-,-,-,+,+,+,+)$	$(T_1-1)(1-w_1)+(T_2-1)(1-w_2)+(T_3-1)(1-w_3)+(w_1+w_2+w_3-1)$
$(-,-,+,+,-,+,+)$	$T_2w_1+T_1w_2$
$(-,-,+,+,+,+,+)$	$(T_1-1)(1-w_1)+(T_2-1)(1-w_2)+w_1+w_2$
$(-,+,+,+,+,+,+)$	$(T_1-1)(1-w_1)+w_1+1$
$(+,+,+,+,+,+,+)$	2
$(-,-,-,-,-,-,*)$	$(T_1+T_2+T_3-3)(w_1+w_2+w_3-1)+(T_4-1)(1-w_4)+(w-2)$
$(-,-,-,+,-,-,*)$	$(T_1+T_2+T_3-2)(w_1+w_2+w_3-1)$

TABLE 1. Intersection numbers for each vital curve types

Then it is straightforward to check that $\Delta_{\mathcal{A}} - \Delta'_{\mathcal{A}}$ is equal to the right side of (5).

By [6], there exists a birational morphism $\pi_{\mathcal{A}} : \overline{M}_{0,n} \to (\mathbb{P}^1)^n //_L SL(2)$ for any ample linearization $L = \mathcal{O}(a_1, \cdots, a_n)$. Every boundary divisors except D_I for $|I| = 2$ are contracted by $\pi_{\mathcal{A}}$. The coefficients of D_I in $\Delta_{\mathcal{A}} - \Delta'_{\mathcal{A}}$ is nonnegative since $w_I \le 1$ and is zero when $|I| = 2$. Thus $\Delta_{\mathcal{A}} - \Delta'_{\mathcal{A}}$ is also effective and supported on the exceptional locus of $\pi_{\mathcal{A}}$. Therefore, by the same argument of the proof of item (1), $\overline{M}_{0,n}(\Delta_{\mathcal{A}}) \cong \overline{M}_{0,n}(\Delta'_{\mathcal{A}})$.

For a vital curve class $C(S_1, S_2, S_3, S_4)$, by [8, Lemma 4.3],

$$\Delta'_{\mathcal{A}} \cdot C(S_1,S_2,S_3,S_4) = \begin{cases} 0, & w_4 \ge 1 \\ (n-4)(1-w_4), & w_4 \le 1 \text{ and } w_1 + w_4 \ge 1 \\ (n-4)w_1, & w_4 \le 1 \text{ and } w_1 + w_4 \le 1. \end{cases} \tag{8}$$

These intersection numbers are propotional to that of $\pi^*_{\mathcal{A}}(\mathcal{O}(a_1, \cdots, a_n)//SL(2))$ in [1, Lemma 2.2]. Since $\overline{NE}_1(\overline{M}_{0,n})$ is generated by vital curves, $\Delta'_{\mathcal{A}}$ is proportional to the pull-back of the *ample* divisor $\mathcal{O}(a_1, \cdots, a_n)//SL(2)$ on $(\mathbb{P}^1)^n //_L SL(2)$. Therefore $\overline{M}_{0,n}(\Delta_{\mathcal{A}}) \cong (\mathbb{P}^1)^n //_L SL(2)$.

Remark 2.1. The total psi-class $\psi := \sum_i \psi_i$ is $\psi = \sum_{j=2}^{\lfloor n/2 \rfloor} \sum_{|I|=j} \frac{j(n-j)}{n-1} D_I = K_{\overline{M}_{0,n}} + 2D$ by [2, Lemma 1] and [8, Lemma 3.5]. So for $\alpha > 0$, $K_{\overline{M}_{0,n}} + \alpha\psi = (1+\alpha)(K_{\overline{M}_{0,n}} + \frac{2\alpha}{1+\alpha}D)$. Therefore if $a_1 = \cdots = a_n = \alpha$ for some $2/n < \alpha \le 1$, then $\overline{M}_{0,n}(K_{\overline{M}_{0,n}} + \sum a_i\psi_i) = \overline{M}_{0,n}(K_{\overline{M}_{0,n}} + \alpha\psi)$ is equal to $\overline{M}_{0,n}(K_{\overline{M}_{0,n}} + \frac{2\alpha}{1+\alpha}D)$. If we substitute $\beta = \frac{2\alpha}{1+\alpha}$, then we get item (1) of Theorem 1.2. Similarly, we can prove that $\overline{M}_{0,n}(K_{\overline{M}_{0,n}} + \frac{2}{n/2+1}D) \cong (\mathbb{P}^1)^n //_L SL(2)$ with $L = \mathcal{O}(2/n, 2/n, \cdots, 2/n)$ which is proportial to $\mathcal{O}(1, 1, \cdots, 1)$. So we can recover item (2) of Theorem 1.2 except the range of bigness.

Remark 2.2. Theorem 1.1 shows a mysterious duality. For $(C, x_1, x_2, \cdots, x_n) \in \overline{M}_{0,n}$, by the definition, the log canonical model $C(\omega_C + \sum a_i x_i)$ is an $\mathcal{A}$-stable curve and it is $\varphi_{\mathcal{A}}(C, x_1, x_2, \cdots, x_n)$. The *same weight datum* determines $\overline{M}_{0,n}(K_{\overline{M}_{0,n}} + \sum a_i\psi_i)$ of the moduli space $\overline{M}_{0,n}$ itself.

Remark 2.3. In [7], Keel proved that $\dim N^1(\overline{M}_{0,n}) = 2^{n-1} - \binom{n}{2} - 1$. By Theorem 1.1, if the F-conjecture is true, then the family $\{\overline{M}_{0,\mathcal{A}}\}$ of birational models of $\overline{M}_{0,n}$ are detected by only an n-dimensional subcone of the effective cone of $\overline{M}_{0,n}$. So we can expect that still there is a huge wild world of unknown birational models of $\overline{M}_{0,n}$.

Remark 2.4. Although there is a strong belief on the F-conjecture, it seems that the proof of the F-conjecture is far from our hands. So it is necessary finding a proof of Theorem 1.1 without relying on the F-conjecture. As in the proof, proving the ampleness of $\varphi_{\mathcal{A}*}(\Delta_{\mathcal{A}})$ is a crucial step. We can express $\varphi_{\mathcal{A}*}(\Delta_{\mathcal{A}})$ in terms of tautological divisors on the universal curve of $\overline{M}_{0,\mathcal{A}}$. The author is working on proving the ampleness by using the expression and the technique of Fedorchuk in [3].

REFERENCES

[1] V. Alexeev and D. Swinarski. *Nef divisors on $\overline{M}_{0,n}$ from GIT.* arXiv:0812.0778. 1, 4

[2] G. Farkas and A. Gibney. *The Mori cones of moduli spaces of pointed curves of small genus.* Trans. Amer. Math. Soc. 355 (2003), no. 3, 1183–1199. 1, 2, 4

[3] M. Fedorchuk. *Moduli spaces of weighted stable curves and log canonical models of $\overline{M}_{g,n}$.* arXiv:1004.4938. 2, 5

[4] M. Fedorchuk and D. Smyth. *Ample divisors on moduli spaces of pointed rational curves.* to appear in Journal of Algebraic Geometry, arXiv:0810.1677. 1

[5] B. Hassett. *Moduli spaces of weighted pointed stable curves.* Adv. Math. 173 (2003), no. 2, 316–352. 1, 2

[6] M. Kapranov. *Chow quotients of Grassmannians. I.* I. M. Gelfand Seminar, 29–110, Adv. Soviet Math., 16, Part 2, Amer. Math. Soc., Providence, RI, 1993. 4

[7] S. Keel. *Intersection theory of moduli space of stable n-pointed curves of genus zero.* Trans. Amer. Math. Soc. 330 (1992), no. 2, 545–574. 5

[8] S. Keel and J. McKernan. *Contractible extreamal rays on $\overline{M}_{0,n}$.* arXiv:9607009. 1, 2, 3, 4

[9] Y.-H. Kiem and H.-B. Moon. *Moduli spaces of weighted pointed stable rational curves via GIT.* arXiv:1002.2461. 1

[10] M. Simpson. *On Log canonical models of the moduli space of stable pointed curves.* arXiv:0709.4037. 1

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