SMALL RANDOM PERTURBATIONS OF A DYNAMICAL SYSTEM WITH BLOW-UP

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ABSTRACT. We study small random perturbations by additive white-noise of a spatial discretization of a reaction-diffusion equation with a stable equilibrium and solutions that blow up in finite time. We prove that the perturbed system blows up with total probability and establish its order of magnitude and asymptotic distribution. For initial data in the domain of explosion we prove that the explosion time converges to the deterministic one while for initial data in the domain of attraction of the stable equilibrium we show that the system exhibits metastable behavior.

1. INTRODUCTION

We consider small random perturbations of the following ODE

$$\left\{\begin{array}{lcll} U_1' &=& \frac{2}{h^2}(-U_1+U_2), & \\ U_i' &=& \frac{1}{h^2}(U_{i+1}-2U_i+U_{i-1}) & 2\le i\le d-1, \\ U_d' &=& \frac{2}{h^2}(-U_d+U_{d-1}+hg(U_d)). & \end{array}\right. \tag{1.1}$$

Here $g\colon \mathbb{R}\to\mathbb{R}$ is a reaction term given by $g(x)=(x^+)^p-x$ with $p>1$, and $h>0$ is a parameter. We also impose an initial condition $U_0\in\mathbb{R}^d$. This kind of systems arise as spatial discretizations of diffusion equations with nonlinear boundary conditions of Neumann type. In fact, it is known that as $h\to 0$ solutions to this system converge to solutions of the PDE

$$\left\{\begin{array}{lcll} u_t(t,x) &=& u_{xx}(t,x) & 0<x<1, 0\le t<T, \\ u_x(0,t) &=& 0 & 0\le t<T, \\ u_x(1,t) &=& g(u(1,t)) & 0\le t<T, \\ u(x,0) &=& u_0(x) & 0\le x\le 1. \end{array}\right.$$

This and more general reaction-diffusion problems including for instance the possibility of a nonlinear source term like g and other type of boundary conditions appear in several branches of pure and applied mathematics. They have been used to model heat transfer, exothermic chemical reactions, population growth models, geometric flows, etc.

An important feature of this type of problems is that they admit solutions which are local in time, with the possibility of blow-up in finite time. The asymptotic behavior of solutions to (1.1) can be briefly summarized as follows (we give a detailed description afterwards): the system has two equilibriums $U_0\equiv 0$ and $U_0\equiv 1$. The first one is stable while the second is unstable. Hence, there exists a domain of attraction D_0 for the zero solution such that if $U_0\in D_0$ then the solution $U(t)=(U_1(t),\dots,U_d(t))$ with initial condition U_0 is globally defined and $U(t)\to 0$ as $t\to\infty$. There exists

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also a stable manifold for the unstable equilibrium which is of co-dimension one and coincides with the boundary of D_0. For $U_0 \in \overline{D}_0{}^c$ the solution U blows up in finite time $T = T(U_0)$.

Since mathematical models are not exact, it is important to understand what changes arise in the behavior of the system when it is subject to perturbations. We study random perturbations given by additive white-noise. More precisely, we consider Stochastic Differential Equations (SDE) of the form

$$\left\{\begin{array}{lcll} \mathrm{d}U_1^{u,\varepsilon} & = & \frac{2}{h^2}(-U_1^{u,\varepsilon}+U_2^{u,\varepsilon})\,\mathrm{d}t+\varepsilon \mathrm{d}W_1, & \\ \mathrm{d}U_i^{u,\varepsilon} & = & \frac{1}{h^2}(U_{i+1}^{u,\varepsilon}-2U_i^{u,\varepsilon}+U_{i-1}^{u,\varepsilon})\,\mathrm{d}t+\varepsilon \mathrm{d}W_i & 2\le i\le d-1,\\ \mathrm{d}U_d^{u,\varepsilon} & = & \frac{2}{h^2}(-U_d^{u,\varepsilon}+U_{d-1}^{u,\varepsilon}+hg(U_d^{u,\varepsilon}))\,\mathrm{d}t+\varepsilon \mathrm{d}W_d, & \end{array}\right. \tag{1.2}$$

which can be written in matrix form as

$$dU^{u,\varepsilon} = (-AU^{u,\varepsilon}+\frac{2}{h}g(U_d^{u,\varepsilon})e_d)\,dt+\varepsilon dW. \tag{1.3}$$

Here $W=(W_1,\dots W_d)$ is a $d-$dimensional standard Brownian motion, $\varepsilon>0$ is a small parameter and $e_d=(0,\dots,1)$ is the d-th canonical vector on $\mathbb{R}^d$. In the sequel we use $U^{u,\varepsilon}$ for a solution to (1.2) with initial condition $U^{u,\varepsilon}(0)=u\in\mathbb{R}^d$. In the case $\varepsilon=0$ we are left with the deterministic equation and so we use the notation $U^u := U^{u,0}$ to denote a solution to (1.1).

The field $b(U) := -AU+\frac{2}{h}g(U_d)e_d$ is a gradient ($b=-\nabla\phi$) with potential given by

$$\phi(U)=\frac{1}{2}\langle AU,U\rangle-\frac{2}{h}\Big(\frac{|U_d^+|^{p+1}}{p+1}-\frac{{U_d}^2}{2}\Big).$$

The SDE associated to this energy functional can be compared with the classic double-well potential model, which we now briefly summarize. We refer to [10, p. 294] for a more detailed description.

In the double-well potential model one considers a stochastic differential equation of the form

$$dX^\varepsilon = r(X^\varepsilon)\,dt+\varepsilon\,dW \tag{1.4}$$

where W is a standard d-dimensional Brownian motion and r is a globally Lipschitz gradient field over $\mathbb{R}^d$ given by the double-well potential $\tilde{\phi}$. More precisely, this potential $\tilde{\phi}$ possesses exactly three critical points: two local minima p and q of different depth and a saddle point z with higher energy, that is $\tilde{\phi}(z)>\tilde{\phi}(p)>\tilde{\phi}(q)$. Each minimum corresponds to a stable equilibrium and hence for initial data lying outside the stable manifold of z, the deterministic system ($\varepsilon=0$) converges to one of them depending on the initial condition. When considering random perturbations, for compact time intervals the stochastic system converges as $\varepsilon\to 0$ to the deterministic one uniformly but the qualitative behavior of the perturbed system is quite different from that of the deterministic solution for large times. If the potential grows fast enough at infinity the resulting stochastic system admits a stationary probability measure which converges to a Dirac delta concentrated at the bottom of the deepest well q. Hence, for initial data in the domain of attraction of the shortest well p we observe that

(i) Due to the action of the field r, the process is attracted towards the bottom of the shortest well p; once near p, the field becomes negligible and the process is then pushed away from the bottom of the well by noise. Being apart from p, noise becomes overpowered by the field r and this allows for the previous pattern to repeat itself: a large number of attempts to escape from the given well, followed by a strong attraction towards its bottom. This phase is known as *thermalization*.

(ii) Eventually, after many frustrated attempts, the process succeeds in overcoming the barrier of potential and reaches the deepest well. Since the probability of such an event is small, we

expect this *tunneling time* to be exponentially large. Moreover, due to the large number of attempts that are necessary, we expect this time to show little memory.

(iii) Once in the deepest well, the process behaves as in (i). Since the new barrier of potential is higher, the next tunneling time is expected to happen on a larger time scale.

This description was proved rigorously in [3, 5, 7, 4, 8] using different techniques. The phenomenon is known as *metastability*. For a detailed description of it we refer to [10].

Coming back to our potential ϕ, the situation is slightly more complex. Instead of having a deepest well, we have a direction along which the potential goes to $-\infty$ and, hence, the size of the "deepest well" is now infinity and there is no return from there. Moreover, since the potential behaves like $-s^{p+1}$ in this direction, if the system falls in this "well", it reaches infinity in finite time (explosion).

The purpose of this paper is to study the metastability phenomenon for this kind of potentials where there is a shortest (finite) well and a deepest well which yields to infinity in finite time. The ideas developed here can be extended to other systems with the same structure. The typical situation with this kind of geometry is the case of reaction-diffusion equations where the reaction comes from a nonlinear source with superlinear behavior at infinity such as

$$u_t = u_{xx} + u_+^p,$$

with $p > 1$, in a bounded domain of $\mathbb{R}$ and homogeneous Dirichlet boundary conditions. In this case the diffusive term pushes the solution towards zero (a stable equilibrium) while the source u_+^p pushes it to infinity. In this situation we expect the same behavior as the one of solutions to (1.2).

Since the drift in (1.2) is not globally Lipschitz, we are only able to prove the existence of local solutions and in fact, explosions occur for solutions of (1.2). In particular, classical large deviation principles as well as other Freidlin-Wentzell estimates do not apply directly. All of these results deal with globally Lipschitz coefficients. Also, the loss of memory for the tunneling time was proved only in the globally Lipschitz case where explosions do not occur. The only exception is the work of Azencott [2] where locally Lipschitz coefficients are considered and explosions are allowed, but the large deviations estimates developed there apply only to neighborhoods of solutions which do not explode in a fixed time interval (and hence the perturbed system is automatically defined in the whole interval for ε small enough). In that work the author also considers the exit from a domain problem, but explosions are not allowed in his analysis.

As opposed to this last case, we specifically focus on trajectories that blow up in finite time. The asymptotic behavior (as $\varepsilon \to 0$) of the explosion time for (1.2) is not understood yet, and this is the goal of this article.

In order to study this kind of systems, localization techniques may be applied but this has to be done carefully. The main difficulties lie in (i) the geometry of the potential (and its respective truncations) which is far from being as simple as in the double-well potential and (ii) the explosion phenomena itself. Localization techniques apply reasonably well to deal with the process until it escapes any bounded domain, but dealing with process from there up to the explosion time requires different tools, which include a careful study of the blow-up phenomenon. Clearly, localization arguments are useless for this last part.

The paper is organized as follows. In Section 2 we give the necessary definitions, review some Freidlin-Wentzell estimates and detail the results of this article. Section 3 is devoted to giving a detailed description of the deterministic system (1.1). In Section 4 we begin our analysis of the stochastic system. We prove that explosions occur with probability one for every initial datum. In

Section 5 we prove that for initial data in the domain of explosion, the explosion time converges to the deterministic one as $\varepsilon \to 0$. Finally, throughout Section 6 we study the characteristics associated to metastability for initial datum in the domain of attraction of the origin: exponential magnitude of the explosion time and asymptotic loss of memory.

2. Definitions and results

2.1. **Solutions up to an explosion time.** Throughout the paper we study stochastic differential equations of the form

$$dX = \tilde{b}(X)\,dt + \varepsilon\,dW \tag{2.1}$$

where $\varepsilon > 0$ and $\tilde{b} : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is locally Lipschitz. It is possible that such equations do not admit strong solutions in the usual sense as these may not be globally defined but defined *up to an explosion time* instead. We now formalize the idea of explosion and properly define the concept of solutions for this kind of equations. We follow [9].

Definition 2.1. A *solution up to an explosion time* of the stochastic differential equation (2.1) on the probability space $(\Omega, \mathcal{F}, P)$, with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions and a fixed Brownian motion $(W_t, \mathcal{F}_t)_{t\geq 0}$ with (a.s. finite) initial condition ξ is an adapted process X with continuous paths taking values in $\mathbb{R}^d \cup \{\infty\}$ which satisfies the following properties:

- If we define $\tau^n = \inf\{t > 0 : |X(t)| = n\}$ then for every $n \geq 1$ we have

$$P\left(\int_0^{t\wedge\tau^n} |\tilde{b}(X(s))|\,ds < +\infty \right) = 1 \qquad \forall\ 0 \leq t < +\infty$$

  and

$$P\left(X(t\wedge\tau^n) = \xi + \int_0^t \tilde{b}(X(s))\mathbb{1}_{\{s\leq\tau^n\}}\,ds + \varepsilon W(t\wedge\tau^n);\ \forall\ 0 \leq t < +\infty \right) = 1.$$

- X has the strong Markov property, i.e. if we note $\tau := \lim_{n\to+\infty}\tau^n$ and $\tilde{\tau}$ is a stopping time of X then, conditional on $\tilde{\tau} < \tau$ and $X(\tilde{\tau}) = x$, the future $\{X^+(t) = X(t+\tilde{\tau}) : t < \tau - \tilde{\tau}\}$ is independent of the past $\{X(s) : s \leq \tilde{\tau}\}$ and identical in law to the process started at x.

We call τ the *explosion time* for X. Notice that the assumption of continuity of X in $\mathbb{R}^d \cup \{\infty\}$ implies that

$$\tau = \inf\{t > 0 : X(t) \notin \mathbb{R}^d\} \qquad \text{and} \qquad X(\tau-) = X(\tau) = \infty \ \text{ on } \ \{\tau < +\infty\}.$$

We stipulate that $X(t) = \infty$ provided that $\tau \leq t < +\infty$ but we do not assume that $\lim_{t\to+\infty} X(t)$ exists when $\tau = +\infty$.

Notice that the assumption of finiteness of ξ grants us $P(\tau > 0) = 1$. Also, if $P(\tau = +\infty) = 1$ then we are left with the usual definition of strong solution to the equation.

Remark 2.1. It can be proved that if $\tilde{b} \in C^1(\mathbb{R}^d)$ then there exists a unique solution of (2.1) up to an explosion time (see [6, 9]).

2.2. **Freidlin-Wentzell estimates.** One of the most valuable tools in the study of perturbations by additive white noise of an ODE is the Freidlin-Wentzell theory, whose main results we briefly describe here.

Let $X^{x,\varepsilon}$ be a solution to the SDE

$$dX^{x,\varepsilon} = \tilde{b}(X^{x,\varepsilon})\,dt + \varepsilon\,dW$$

with initial condition $x \in \mathbb{R}^d$, where $\tilde{b}$ is globally Lipschitz with Lipschitz constant K. Fix $T > 0$ and let $P_x^{\varepsilon,T}$ denote the law of $X^{x,\varepsilon}$ on $C([0,T],\mathbb{R}^d)$. Let us also consider X^x the unique solution to the deterministic equation

$$\dot{X}(t) = \tilde{b}(X(t))$$

with initial condition $x \in \mathbb{R}^d$.

Theorem 2.2 (Freidlin and Wentzell, [4])**.** For each $x \in \mathbb{R}^d$ and $T > 0$ the family $(P_x^{\varepsilon,T})_{\varepsilon>0}$ satisfies a large deviations principle on $C([0,T],\mathbb{R}^d)$ with scaling ε^{-2} and (good) rate function I_T^x given by

$$I_T^x(\varphi) = \begin{cases} \frac{1}{2}\int_0^T |\dot{\varphi}(s) - \tilde{b}(\varphi(s))|^2\,ds & \text{if } \varphi \text{ is absolutely continuous and } \varphi(0) = x \\ \\ +\infty & \text{otherwise} \end{cases}$$

As a matter of fact, we need only the following weaker statement for our analysis: for every fixed $T > 0$ and $\delta > 0$ there exist positive constants C_1 and C_2 depending on T, δ and K such that for all $0 < \varepsilon \leq 1$

$$\sup_{x\in\mathbb{R}^d} P\left(\sup_{t\in[0,T]} |X^{x,\varepsilon}(t) - X^x(t)| > \delta\right) \leq C_1 e^{-\frac{C_2}{\varepsilon^2}}. \tag{2.2}$$

2.3. **Main results.** We now state the main results of the article. The first of them concerns the explosion time of solutions to (1.2). In the following P_u denotes the law of the solution to (1.2) up to the explosion time τ_ε^u with initial condition u. When the initial condition is clear we often write τ_ε instead of τ_ε^u to simplify the notation.

Theorem 2.3. Let $U^{u,\varepsilon}$ be a solution to (1.2). Then $P_u(\tau_\varepsilon < \infty) = 1$.

Let us notice that this result establishes a first difference in behavior with respect to the deterministic system. While global solutions exist in the deterministic equation, they do not for the stochastic one.

We then focus on establishing the order of magnitude and asymptotic distribution of the explosion time for the different initial conditions $u \in \mathbb{R}^d$. We deal first with initial conditions in the domain of explosion D_e and show the following result.

Theorem 2.4. Given $\delta > 0$ and $u \in D_e$ we have

$$\lim_{\varepsilon\to 0} P_u(|\tau_\varepsilon - \tau_0| > \delta) = 0. \tag{2.3}$$

Moreover, the convergence is exponentially fast.

This last theorem shows that for small $\varepsilon > 0$ the behavior of the stochastic system does not differ significantly from the deterministic one for initial conditions in D_e. However, this is not the case for initial data in the domain of attraction of the origin. Here is where important differences appear and where characteristics associated with metaestability are observed. In order to properly state the results achieved in this matter, we need to introduce some notation.

For each $\varepsilon > 0$ we define

$$\beta_\varepsilon = \inf\{t \geq 0 : P_0(\tau_\varepsilon > \beta_\varepsilon) \leq e^{-1}\}$$

which is well defined since $P_0(\tau_\varepsilon < +\infty) = 1$ for every $\varepsilon > 0$. We first show that the family $(\beta_\varepsilon)_{\varepsilon>0}$ verifies

$$\lim_{\varepsilon\to 0} \varepsilon^2 \log \beta_\varepsilon = \Delta$$

with $\Delta := 2(\phi(\mathbf{1}) - \phi(0))$. In fact, we prove the stronger statement featured in the following theorem.

Theorem 2.5. For each $u \in D_0$ and $\delta > 0$

$$\lim_{\varepsilon\to+\infty} P_u\Big(e^{\frac{\Delta-\delta}{\varepsilon^2}} < \tau_\varepsilon < e^{\frac{\Delta+\delta}{\varepsilon^2}}\Big) = 1,$$

where the convergence is uniform over compact subsets of D_0.

This theorem characterizes the asymptotic order of magnitude of the explosion time for any initial condition $u \in D_0$. Regarding its distribution, we show the asymptotic loss of memory in our last result.

Theorem 2.6. For each $u \in D_0$ and $t > 0$

$$\lim_{\varepsilon\to 0} P_u(\tau_\varepsilon > t\beta_\varepsilon) = e^{-t}$$

where the convergence is uniform over compact subsets of D_0.

3. The deterministic system

Throughout this section we state some properties and study the behavior of solutions to (1.1). This is carried out in [1] for solutions with nonnegative initial conditions. The purpose of this section is to extend the analysis in [1] to any arbitrary initial data $u \in \mathbb{R}^d$.

Let us start by noticing that equation (1.1) can be written as

$$\dot{U}(t) = b(U(t))$$

for $b = -\nabla\phi$ where ϕ is defined as

$$\phi(U) = \frac{1}{2}\langle AU, U\rangle - \frac{2}{h}\Big(\frac{|U_d^+|^{p+1}}{p+1} - \frac{{U_d}^2}{2}\Big). \tag{3.1}$$

Here A is as in (1.2)-(1.3). Notice that the potential ϕ has exactly two critical points: $\mathbf{1} := (1,\dots,1)$ and the origin. Both of them are hyperbolic. The origin is the only local minimum of ϕ while $\mathbf{1}$ is a saddle point. Our goal is to decompose $\mathbb{R}^d$ into distinct regions, each of them having different asymptotic characteristics under our system. To be able to accomplish such decomposition we need a few results concerning solutions to (1.1). We begin with the following proposition.

Proposition 3.1. Let $U = (U_1,\dots,U_d)$ be a solution to (1.1). Then the application $t \mapsto \phi(U(t))$ is monotone decreasing.

Proof. Since A is symmetric and $\dot{U} = -AU + \frac{2}{h}g(U_d)e_d$, a direct calculation shows that

$$\frac{d\phi\big(U(t)\big)}{dt} = \langle \dot{U}(t), AU(t)\rangle - \frac{2}{h}g\big(U_d(t)\big)\dot{U}_d(t) = -|\dot{U}(t)|^2 \leq 0.$$

□

Next we show that solutions to (1.1) satisfy a Maximum Principle.

Lemma 3.2 (Maximum Principle). Let $U=(U_1,\dots,U_d)$ be a solution to (1.1). Then U satisfies

$$\max_{k=1,\dots,d}|U_k(t)|\le \max\{\max_{k=1,\dots,d}|U_k(0)|,\max_{0\le s\le t}U_d(s)\} \tag{3.2}$$

Proof. We prove first that

$$\max_{k=1,\dots,d}|U_k(t)|\le \max\{\max_{k=1,\dots,d}|U_k(0)|,\max_{0\le s\le t}|U_d(s)|\} \tag{3.3}$$

and then we check that if (3.3) holds then

$$\max\{\max_{k=1,\dots,d}|U_k(0)|,\max_{0\le s\le t}|U_d(s)|\}=\max\{\max_{k=1,\dots,d}|U_k(0)|,\max_{0\le s\le t}U_d(s)\}$$

which allows us to conclude (3.2). Let j be the node that maximizes $\max_{0\le s\le t}|U_j(s)|$. Let us observe that if $j=d$ then (3.3) is immediately verified. Hence, we can assume that $1\le j<d$. Consider $t_0=\min\{t'\in[0,t]:\max_{0\le s\le t}|U_j(s)|=|U_j(t')|\}$, the first time in which the maximum is attained. Note that $|U_j(t_0)|=\max_{k=1,\dots,d}(\max_{0\le s\le t}|U_k(s)|)$. If $t_0=0$ then

$$\max_{k=1,\dots,d}|U_k(0)|\ge|U_j(t_0)|=\max_{k=1,\dots,d}\Big(\max_{0\le s\le t}|U_k(s)|\Big)\ge\max_{k=1,\dots,d}|U_k(t)|$$

and we get (3.3). If $t_0>0$ we must consider two cases: $U_j(t_0)\ge 0$ and $U_j(t_0)<0$. If $U_j(t_0)\ge 0$ then by definition of t_0 we get that $U_j(t_0)\ge U_j(s)$ for all $0\le s\le t$. From this it follows that $U_j'(t_0)\ge 0$. On the other hand, the choice of j guarantees that $U_j(t_0)\ge U_k(t_0)$ for all $k=1,\dots,d$. This implies that

$$U_j'(t_0)=\frac{1}{h^2}\big((U_{j+1}(t_0)-U_j(t_0))+(U_{j-1}(t_0)-U_j(t_0))\big)\le 0\qquad\text{if } 1<j<d$$

and

$$U_1'(t_0)=\frac{2}{h^2}(U_2(t_0)-U_1(t_0))\le 0\qquad\text{if } j=1.$$

In any case we conclude that $U_j'(t_0)=0$ and, in particular, that $U_{j+1}(t_0)=U_j(t_0)$. We conclude that $|U_{j+1}(t_0)|=\max_{k=1,\dots,d}(\max_{0\le s\le t}|U_k(s)|)$ which allows us to repeat the same argument, now for $j+1$ instead of j. Thus, an inductive procedure eventually yields that $U_d(t_0)=U_j(t_0)$. From here we obtain (3.3) if $U_j(t_0)\ge 0$. The case $U_j(t_0)<0$ is analogous. To conclude (3.2) we notice that if $t_1=\min\{t'\in[0,t]:\max_{0\le s\le t}|U_d(s)|=|U_d(t')|\}>0$ then $U_d(t_1)\ge 0$ because, otherwise, from (1.1) and (3.3) we get that $U_d'(t_1)>0$ which contradicts the definition of t_1. □

As a consequence of the Maximum Principle we have the following characterization of globally defined solutions to (1.1).

Lemma 3.3. Let U be a globally defined solution to (1.1). Then U is bounded.

Proof. Let us suppose that U is not bounded. Then by the Maximum Principle we obtain that $\max_{0\le s\le t}|U_d(s)|\to+\infty$ as $t\to+\infty$.

1. Given $M>0$ we define $t_M:=\inf\{t\ge 0:|U_d(t)|>M\}$. From this definition it follows that $|U_d(t_M)|\ge M$ and that $|U_d(t_M)|=\max_{0\le s\le t_M}|U_d(s)|$. If $M>\max_{k=1,\dots,d}|U_k(0)|$ then $t_M>0$ and by the Maximum Principle we have $|U_{d-1}(t_M)|\le U_d(t_M)$. This gives us the inequality

$$U_d'(t_M)\ge\frac{2}{h}U_d^p(t_M)-\Big(\frac{4}{h^2}+\frac{2}{h}\Big)U_d(t_M).$$

2. From here it is easy to see that if M is large enough we have that $U_d : [t_M, +\infty) \longrightarrow \mathbb{R}$ is monotone increasing. This implies that for $t \geq t_M$ we have $U_d(t) = \max_{0\leq s\leq t} |U_d(s)| \geq M$ and, as a consequence, that $U_d'(t) \geq \frac{2}{h}U_d^p(t) - \left(\frac{4}{h^2} + \frac{2}{h}\right)U_d(t)$. If M is taken large enough then U verifies $U_d'(t) \geq \frac{1}{h}U_d^p(t)$ for $t \geq t_M$ and, therefore, cannot be globally defined. This is a contradiction which implies that U must be bounded. □

From the previous lemma and the fact that (1.1) admits the Lyapunov functional (3.1) we obtain the following corollary.

Corollary 3.4. Let U be a solution to (1.1). Then either U explodes in finite time or is globally defined and converges to a stationary solution as $t \to +\infty$.

With this result at our disposal we can obtain the following theorem, whose proof is in [1].

Theorem 3.5.

(1) Equation (1.1) has exactly two equilibriums $U \equiv 0$ and $U \equiv 1$. The first one is stable and the second one is unstable.
(2) Let u be a nonnegative initial datum such that U^u is globally defined and $\lim_{t\to+\infty} U^u(t) = \mathbf{1}$. Then
   - $0 \leq v \lneq u \Longrightarrow U^v$ is globally defined and $\lim\limits_{t\to+\infty} U^v(t) = 0$.
   - $u \lneq v \Longrightarrow U^v$ explodes in finite time.
(3) Consider $\lambda > 0$ and a nonnegative initial condition u. Then there exists $\lambda_c > 0$ such that
   (a) $\lambda < \lambda_c \Longrightarrow U^{\lambda u}$ is globally defined and $\lim_{t\to+\infty} U^{\lambda u}(t) = 0$
   (b) $\lambda_c < \lambda \Longrightarrow U^{\lambda u}$ explodes in finite time
   (c) $\lambda = \lambda_c \Longrightarrow U^{\lambda u}$ is globally defined and $\lim_{t\to+\infty} U^{\lambda u}(t) = \mathbf{1}$.

This results allow us to give a good description of the behavior of the deterministic system U for the different initial conditions $u \in \mathbb{R}^d$. Indeed, we have a decomposition

$$\mathbb{R}^d = D_0 \cup \mathcal{W}_1^s \cup D_e$$

where D_0 denotes the stable manifold of the origin, $\mathcal{W}_1^s$ is the stable manifold of $\mathbf{1} := (1, \ldots, 1)$ and D_e is the domain of explosion, i.e., if $u \in D_e$ then U^u explodes in finite time. The sets D_0 and D_e are open in $\mathbb{R}^d$. The origin is an asymptotically stable equilibrium of the system. $\mathcal{W}_1^s$ is a manifold of codimension one. Also $\mathbf{1}$ admits an unstable manifold of dimension one which we shall note by $\mathcal{W}_{\mathbf{1}}^u$. This unstable manifold is contained in $\mathbb{R}_+^d$, has nonempty intersection with both D_0 and D_e and joins $\mathbf{1}$ with the origin. An illustration of this decomposition is given in Figure 1 for the 2-dimensional case.

4. Explosions in the stochastic model

In this section we focus on proving that solutions to (1.2) blow up in finite time with probability one for any initial condition $u \in \mathbb{R}^d$ and every $\varepsilon > 0$. The idea is to show that, conditioned on non-explosion, the system is guaranteed to enter a specific region of space in which we can prove that explosion occurs with total probability. From this we can conclude that non-explosion must happen with zero probability. We do this by comparison with an adequate Ornstein-Uhlenbeck process.

Proof of Theorem 2.3. Let $Y^{y,\varepsilon}$ be the solution to

$$dY^{y,\varepsilon} = -\left(AY^{y,\varepsilon} + \frac{2}{h}Y_d^{y,\varepsilon}e_d\right)dt + \varepsilon dW$$

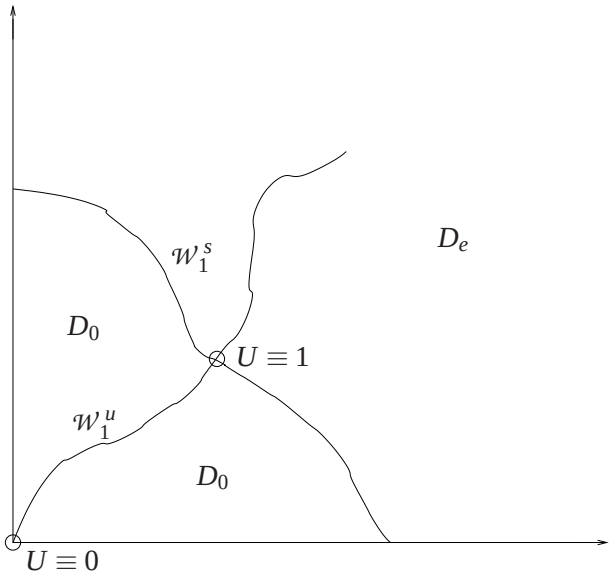

FIGURE 1. The phase diagram of equation (1.1).

with initial condition $Y^{y,\varepsilon}(0) = y$. Notice that the drift term is linear, and given by a negative definite matrix. Hence, $Y^{y,\varepsilon}$ is in fact a d-dimensional Ornstein-Uhlenbeck process which admits an invariant distribution supported in $\mathbb{R}^d$. We also have convergence to this equilibrium measure for any initial distribution and therefore the hitting time of $Y^{y,\varepsilon}$ of any open set is finite almost surely.

On the other hand, since the drift term of (4) is smaller or equal than b we can apply the stochastic comparison principle to obtain that $U^{u,\varepsilon}(t) \geq Y^{y,\varepsilon}$ holds a.s. as long as $U^{u,\varepsilon}$ is finite, if $u \geq y$. From here, the result follows applying the following lemma and the strong Markov property. □

Lemma 4.1. Consider the set

$$\Theta^M := \{y \in \mathbb{R}^d : y_k \geq 0 \text{ for all } 0 \leq k \leq d-1\,, y_d \geq M\},$$

then we have

$$\lim_{M\to\infty} \inf_{y\in\Theta^M} P_y(\tau_\varepsilon < \infty) = 1.$$

Proof. Consider the auxiliary process $Z^{y,\varepsilon} := U^{y,\varepsilon} - \varepsilon W$. Notice that this process verifies the random differential equation

$$dZ^{y,\varepsilon} = b(Z^{y,\varepsilon} + \varepsilon W)dt, \quad Z^{y,\varepsilon}(0) = y.$$

Let us also observe that $Z^{y,\varepsilon}$ has the same explosion time as $U^{y,\varepsilon}$. For each $k \in \mathbb{N}$ let us define the set $A_k := \{\sup_{0\leq t\leq 1} |W_d(t)| \leq k\}$. On A_k we have that $Z^{y,\varepsilon}$ verifies the inequality

$$\frac{dZ^{y,\varepsilon}}{dt} \geq -AZ^{y,\varepsilon} - \frac{4}{h^2}\varepsilon k \sum e_i + \frac{2}{h}((Z_d^{y,\varepsilon} - \varepsilon k)_+^p - Z_d^{y,\varepsilon} - \varepsilon k)e_d. \tag{4.1}$$

Observe that (4.1) can be written as

$$\frac{dZ^{y,\varepsilon}}{dt} \geq QZ^{y,\varepsilon} + q + (Z_d^{y,\varepsilon} - \varepsilon k)_+^p \geq QZ^{y,\varepsilon} + q,$$

where $Q \in \mathbb{R}^{d\times d}$ and $q \in \mathbb{R}^d$ both depend on ε, h and k, but not on M. This allows us to conclude the inequality $|Z^{y,\varepsilon}| \leq (M+|q|)\exp(|Q|)$ for all $0 \leq t \leq 1$. In particular, for the last coordinate we get

$$\begin{cases} \frac{dZ_d^{y,\varepsilon}}{dt} \geq -\alpha_1 M + \alpha_2 (Z_d^{y,\varepsilon})^p & \text{if } 0 \leq t \leq 1 \\ \\ Z_d^{y,\varepsilon}(0) \geq M \end{cases}$$

for constants α_1, α_2 which do not depend on M. It is a straightforward calculation to check that solutions to this one-dimensional inequality blow-up in a finite time that converges to zero as $M \to +\infty$. Therefore, for each $k \in \mathbb{N}$ there exists M_k such that $P(A_k) \leq \inf_{y\in\Theta^M} P_y(\tau_\varepsilon < \infty)$ for all $M \geq M_k$. Since $\lim_{k\to+\infty} P(A_k) = 1$, this concludes the proof. □

5. Convergence of τ_ε^u for initial conditions in D_e

This section is devoted to prove that for initial data in the domain of explosion of the deterministic system, the explosion time is of order one and, moreover, as $\varepsilon \to 0$ converges to the explosion time of the deterministic system. Observe that do to the lack of boundedness this result do not follow from standard perturbation arguments for dynamical system (deterministic or stochastic). We first introduce the truncations of the drift that we use here to prove one of the bounds and we are going to make more profit of them in Section 6 when we deal with initial data in the domain of attraction of the origin.

5.1. **Truncations of the potential and localization.**

The large deviations principle originally formulated by Freidlin and Wentzell for solutions of stochastic differential equations like (2.1) require a global Lipschitz condition on the drift term $\tilde{b}$. While this condition is met on the classic double-well potential model, it is not in our case. As a consequence, we cannot apply such estimates to our system directly. Nonetheless, the use of localization techniques helps us to solve this problem and allows us to take advantage of the theory developed by Freidlin and Wentzell despite the fact that our drift term is not globally Lipschitz. In the following lines we give details about the localization procedure to be employed in the study of our system.

For every $n \in \mathbb{N}$ let $G_n : \mathbb{R} \longrightarrow \mathbb{R}$ be of class C^2 such that

$$G_n(u) = \begin{cases} \frac{|u^+|^{p+1}}{p+1} - \frac{u^2}{2} & \text{if } |u| \leq n \\ 0 & \text{if } |u| \geq 2n. \end{cases}$$

We consider then the family $\left(\phi^n\right)_{n\in\mathbb{N}}$ of potentials over $\mathbb{R}^d$ given by

$$\phi^n(u) = \frac{1}{2}\langle Au, u\rangle - \frac{2}{h} G_n(u_d).$$

This family satisfies the following properties:

(i) For every $n \in \mathbb{N}$ the potential ϕ^n is of class C^2 and $b^n = -\nabla\phi^n$ is globally Lipschitz.
(ii) For $n \leq m \in \mathbb{N}$ we have $b^n \equiv b^m$ over the region $\Pi^n = \{u \in \mathbb{R}^d : |u_d| < n\}$.
(iii) For every $n \in \mathbb{N}$ we have $\liminf_{|u|\to+\infty} \frac{\phi^n}{|u|} > 0$.

Since b^n is globally Lipschitz, for each $u \in \mathbb{R}^d$ there exists a unique solution to the ordinary differential equation

$$\dot{U}^{n,u} = b^n(U^{n,u})$$

with initial condition u. Such solution is globally defined and describes the same trajectory as the solution to (1.1) starting at u until the escape from Π^n. In the same way, for each $x \in \mathbb{R}^d$ and $\varepsilon > 0$ there exists a unique global solution to the stochastic differential equation

$$dU^{n,u,\varepsilon} = b^n\big(U^{n,u,\varepsilon}\big)\,dt + \varepsilon\,dW \tag{5.1}$$

with initial condition u.

As before we use $U^{n,u}$ for $U^{n,u,0}$. Since b^n coincides with b over the ball $B_n(0)$ of radius n centered at the origin, if we write

$$\tau_\varepsilon^{n,u} = \inf\{t \geq 0 : |U^{n,u,\varepsilon}(t)| \geq n\}, \qquad \tau_\varepsilon^u := \lim_{n\to+\infty} \tau_\varepsilon^{n,u},$$

then for $t < \tau_\varepsilon^u$ we have that $U^{u,\varepsilon}(t) := \lim_{n\to+\infty} U^{n,u,\varepsilon}(t)$ is a solution to

$$dU^{u,\varepsilon} = b(U^{u,\varepsilon})\,dt + \varepsilon\,dW \tag{5.2}$$

until the explosion time τ_ε^u with initial condition u. Moreover, if we define the stopping times

$$\pi_\varepsilon^{n,u} = \inf\{t \geq 0 : U^{n,u,\varepsilon}(t) \notin \Pi^n\},$$

it can be seen that (ii) implies that

$$\tau_\varepsilon^u = \lim_{n\to+\infty} \pi_\varepsilon^{n,u}$$

and that $U^{u,\varepsilon}$ coincides with the process $U^{n,u,\varepsilon}$ until the escape from Π^n. On the other hand, (i) guarantees that for each $n \in \mathbb{N}$ and $u \in \mathbb{R}^d$ the family $\big(U^{n,u,\varepsilon}\big)_{\varepsilon>0}$ satisfies a large deviations principle. Finally, from (iii) we get that there is an unique invariant probability measure for the process $U^{n,\varepsilon}$ for each $\varepsilon > 0$ given by the formula

$$\mu_\varepsilon^n(A) := \frac{1}{Z_\varepsilon^n} \int_A e^{-\frac{2}{\varepsilon^2}\phi^n(u)}\,du, \qquad A \in \mathcal{B}(\mathbb{R}^d)$$

where $Z_\varepsilon^n = \int_{\mathbb{R}^d} e^{-\frac{2\phi^n(u)}{\varepsilon^2}}\,du$. Hereafter, when we refer to the solution of (5.2) we mean the solution constructed in this particular way.

5.2. **Proof of Theorem 2.4.** We split the proof of Theorem 2.4 in two parts, the first one is immediate from the continuity of the solutions of (1.2) with respect to ε in intervals where the deterministic solution is bounded.

Proposition 5.1. For any fixed $\delta > 0$ and $u \in D_e$ we have

$$\lim_{\varepsilon\to 0} P_u(\tau_\varepsilon < \tau_0 - \delta) = 0.$$

Proof. We may assume that $\tau_0^u > \delta$ since the proof is trivial otherwise. Now, as the deterministic system U^u is defined up until τ_0^u, if we take $M := \sup_{0\leq t\leq \tau_0^u-\delta} |U_t^u| < +\infty$ then $\tau_\varepsilon^u < \tau_u - \delta$ implies that

$$\sup_{0\leq t\leq \tau_u-\delta} \big|U^{2M,u,\varepsilon}(t) - U^{2M,u}(t)\big| > 1.$$

By (2.2) we get (5.1). □

Proposition 5.2. For any $\delta > 0$ and $u \in D_e$ we have

$$\lim_{\varepsilon\to 0} P_u(\tau_\varepsilon > \tau_0 + \delta) = 0.$$

Moreover, the convergence is uniform over compact subsets of D_e.

Proof. Fix $\delta > 0$, $\mathcal{K}$ a compact set contained in D_e and let Y^u be the solution to the ordinary differential equation

$$\dot{Y}^u = -\Big(AY^u + \frac{2}{h}Y_d^{u,\varepsilon}e_d\Big)$$

with initial condition $u \in \mathcal{K}$. By the Comparison principle we have that $U^u \geq Y^u$ for as long as U^u is defined. Since Y^u is the solution to a linear system of ordinary differential equations whose associated matrix is symmetric and negative definite, we get that there exists $\rho_{\mathcal{K}} \in \mathbb{R}$ such that for all $u \in \mathcal{K}$ every coordinate of U^u remains bounded from below by $\rho_{\mathcal{K}} + 1$ up until τ_0^u. If for $\rho \in \mathbb{R}$ and $M > 0$ we write

$$\Theta_\rho^M := \{y \in \mathbb{R}^d : y_k \geq \rho \text{ for all } 0 \leq k \leq d-1\,, y_d \geq M\}$$

then by the Maximum Principle and the previous statement we have that $T_u := \inf\{t \geq 0 : U_t^u \in \Theta_{\rho_{\mathcal{K}}+1}^{M+1}\}$ is finite. Moreover, as $U^{M+2,u}$ agrees with U^u until the escape from Π_{M+2}, we obtain the expression $T_u = \inf\{t \geq 0 : U_t^{M+2,u} \in \Theta_{\rho_{\mathcal{K}}+1}^{M+1}\}$. Taking $T_{\mathcal{K}} := \sup_{u\in\mathcal{K}} T_u < +\infty$ we may compute

$$P_u\big(\tau_\varepsilon(\Theta_{\rho_{\mathcal{K}}}^M) > T_u\big) \leq P_u\big(\pi_\varepsilon^{M+2} \wedge \tau_\varepsilon(\Theta_{\rho_{\mathcal{K}}}^M) > T_u\big) + P_u\big(\pi_\varepsilon^{M+2} \leq T_u\,, \tau_\varepsilon(\Theta_{\rho_{\mathcal{K}}}^M) > T_u\big)$$

$$\leq 2P_u\Big(\sup_{0\leq t\leq T_u} |U^{M+2,\varepsilon}(t) - U^{M+2}(t)| > 1\Big)$$

$$\leq 2P_u\Big(\sup_{0\leq t\leq T_{\mathcal{K}}} |U^{M+2,\varepsilon}(t) - U^{M+2}(t)| > 1\Big),$$

from which by (2.2) we obtain

$$\limsup_{\varepsilon\to 0}\sup_{u\in\mathcal{K}} P_u\big(\tau_\varepsilon(\Theta_{\rho_{\mathcal{K}}}^M) > T_u\big) = 0. \tag{5.3}$$

On the other hand, by the strong Markov property for $U^{u,\varepsilon}$ we get

$$P_u\big(\tau_\varepsilon > \tau_0 + \delta\big) \leq P_u\big(\tau_\varepsilon > T_u + \delta\big) \leq \sup_{y\in\Theta_{\rho_{\mathcal{K}}}^M} P_y(\tau_\varepsilon > \delta) + \sup_{u\in\mathcal{K}} P_u\big(\tau_\varepsilon(\Theta_{\rho_{\mathcal{K}}}^M) > T_u\big).$$

Taking into consideration (5.3), in order to finish the proof we only need to show that the first term on the right hand side tends to zero as $\varepsilon \to 0$ for an adequate choice of M. To see this we consider for each $\varepsilon > 0$ and $y \in \Theta_{\rho_{\mathcal{K}}}^M$ the processes $Y^{y,\varepsilon}$ and $Z^{y,\varepsilon}$ defined by

$$dY^{y,\varepsilon} = -\Big(AY^{y,\varepsilon} + \frac{2}{h}Y_d^{y,\varepsilon}e_d\Big)\,dt + \varepsilon dW,$$

and $Z^{y,\varepsilon} := U^{y,\varepsilon} - Y^{y,\varepsilon}$, respectively. Notice that since $Y^{y,\varepsilon}$ is globally defined and both $U^{y,\varepsilon}$ and $Z^{y,\varepsilon}$ have the same explosion time. Also note that $Z^{y,\varepsilon}$ satisfies the random differential equation

$$dZ^{y,\varepsilon} = -\Big(AZ^{y,\varepsilon} + \frac{2}{h}\Big(\Big[\big(U_d^{y,\varepsilon}\big)^+\Big]^p - Z_d^{y,\varepsilon}\Big)e_d\Big)\,dt.$$

The continuity of trajectories allows us to use the Fundamental Theorem of Calculus to show that almost surely $Z^{y,\varepsilon}(\omega)$ is a solution to the ordinary differential equation

$$\dot{Z}^{y,\varepsilon}(t)(\omega) = -AZ^{y,\varepsilon}(\omega) + \frac{2}{h}\Big(\Big[\big(U_d^{y,\varepsilon}\big)^+\Big]^p(\omega) - Z_d^{y,\varepsilon}(\omega)\Big)e_d. \tag{5.4}$$

For each $y \in \Theta^M_{\rho_{\mathcal{K}}}$ and $\varepsilon > 0$ let Ω^y_ε be a set of probability one in which (5.4) holds. Notice that for every $\omega \in \Omega^y_\varepsilon$ we have the inequality

$$\dot{Z}^{y,\varepsilon}(\omega) \geq -AZ^{y,\varepsilon}(\omega) - \frac{2}{h} Z^{y,\varepsilon}_d(\omega) e_d.$$

Using the Comparison Principle we conclude that $Z^{y,\varepsilon}(\omega) \geq 0$ for every $\omega \in \Omega^y_\varepsilon$ and, therefore, that the inequality $U^{y,\varepsilon}(\omega) \geq Y^{y,\varepsilon}(\omega)$ holds for as long as $U^{y,\varepsilon}(\omega)$ is defined.

For each $y \in \Theta^M_{\rho_{\mathcal{K}}}$ and $\varepsilon > 0$ let us also consider the set

$$\tilde{\Omega}^y_\varepsilon = \Big\{ \omega \in \Omega : \sup_{0 \leq t \leq \delta} |Y^{y,\varepsilon}(\omega,t) - Y^y(\omega,t)| \leq 1, \ \sup_{0 \leq t \leq \delta} |\varepsilon W(\omega,t)| \leq 1 \Big\}.$$

Note that $\lim_{\varepsilon \to 0} \inf_{y \in \Theta^M_{\rho_{\mathcal{K}}}} P(\tilde{\Omega}^y_\varepsilon) = 1$. Our goal is to show that if M is chosen adequately then for fixed $y \in \Theta^M_{\rho_{\mathcal{K}}}$ the trajectory $U^{y,\varepsilon}(\omega)$ explodes before time δ for all $\omega \in \Omega^y_\varepsilon \cap \tilde{\Omega}^y_\varepsilon$. From this we get that

$$\inf_{y \in \Theta^M_{\rho_{\mathcal{K}}}} P(\tilde{\Omega}^y_\varepsilon) = \inf_{y \in \Theta^M_{\rho_{\mathcal{K}}}} P(\Omega^y_\varepsilon \cap \tilde{\Omega}^y_\varepsilon) \leq \inf_{y \in \Theta^M_{\rho_{\mathcal{K}}}} P_y(\tau_\varepsilon \leq \delta).$$

and by letting $\varepsilon \to 0$ we conclude the result.

So let us take $y \in \Theta^M_{\rho_{\mathcal{K}}}$, $\omega \in \Omega^y_\varepsilon \cap \tilde{\Omega}_\varepsilon$ and suppose that $U^{y,\varepsilon}(\omega)$ is defined in the interval $[0,\delta]$. Notice that since $\omega \in \Omega^y_\varepsilon \cap \tilde{\Omega}_\varepsilon$ then the $(d-1)$-th coordinate of $Y^{y,\varepsilon}(\omega,t)$ is bounded from below by $\rho_{\mathcal{K}} - 1$ for $t \in [0,\delta]$. By comparison we know that the $(d-1)$-th coordinate of $U^{y,\varepsilon}_t(\omega,t)$ is bounded from below by $\rho_{\mathcal{K}} - 1$ as well.

From here we deduce that the last coordinate of $U^{y,\varepsilon}(\omega)$ verifies the integral equation

$$U^{y,\varepsilon}_d(\omega,t) \geq U^{y,\varepsilon}_d(\omega,s) + \int_s^t \frac{2}{h^2} \Big(-U^{y,\varepsilon}_d(\omega,r) + \rho_{\mathcal{K}} - 1 + hg\big(U^{y,\varepsilon}_d(\omega,r)\big) \Big) dr - 1$$

for $s < t \in [0,\delta]$. We can take $M \in \mathbb{N}$ large enough to guarantee that there exists a constant $\alpha > 0$ such that for all $m \geq M$ we have

$$\frac{2}{h^2} \big(-m + \rho_{\mathcal{K}} - 1 + hg(m) \big) \geq \alpha m^p.$$

If we recall that $U^{y,\varepsilon}_d(\omega,0) \geq M$ then our selection of M implies that

$$U^{y,\varepsilon}_d(\omega,t) \geq M - 1 + \alpha \int_0^t \big(U^{y,\varepsilon}_d(\omega,u) \big)^p du$$

for all $t \in [0,\delta]$. But if this inequality holds and M is large enough, one can check that $U^{y,\varepsilon}(\omega)$ explodes before time δ, which contradicts our assumptions. Therefore, if $y \in \Theta^M_{\rho_{\mathcal{K}}}$ and $\omega \in \Omega^y_\varepsilon \cap \tilde{\Omega}_\varepsilon$ then $U^{y,\varepsilon}(\omega)$ explodes before time δ and this fact concludes our proof. □

Combining these two propositions we get Theorem 2.4. Observe that the bounds obtained decay to zero exponentially fast due to Proposition (2.2).

6. Metastable behavior for initial conditions in D_0

Finally we focus on initial data in D_0, where the metastability phenomenon can be appreciated. We start with the construction of an auxiliary domain that contains the origin and such that the exit time from this domain is asymptotically equivalent to the explosion time.

6.1. **Construction of an auxiliary domain.** In order to proceed with our analysis of the explosion time we must first construct an auxiliary bounded domain. The purpose behind this construction is to reduce our problem to a simpler one, the escape from this domain. This is easier because we may assume that the drift coefficient b is globally Lipschitz, as the escape only depends on the behavior of the system while it remains inside a bounded region. In this case, large deviations estimates as the ones proved by Freidlin and Wentzell apply. We need a bounded domain G which verifies the following properties:

(1) G is bounded, contains $\mathbf{1}$ and the origin.
(2) There exists $c>0$ such that $B_c(0)\subseteq G$ and for all $y\in B_c(0)$ the system U^y is globally defined and tends to zero without escaping G.
(3) The border of G can be decomposed in two parts: ∂^1 and $\partial G\setminus\partial^1$. The region of the border ∂^1 is closed and satisfies $\min_{u\in\partial G}\phi(u)=\min_{u\in\partial^1}\phi(u)$ and

$$\inf_{u\in\partial G\setminus\partial^1}\phi(u)>\min_{u\in\partial G}\phi(u).$$

(4) For all $y\in\partial^1$ the deterministic system U^y explodes in finite time.

The domain G can be constructed as follows. Let us consider the value of ϕ at the saddle point $\mathbf{1}$, $\phi(\mathbf{1})=-1/(p+1)+1/2>0=\phi(0)$ and $c>0$ such that $\phi(u)<\phi(\mathbf{1})$ for $u\in B_c(0)$.

For each point $u\in\partial B_c(0)$ consider the ray $R_u:=\{\lambda u:\lambda>0\}$. Since the vector $\mathbf{1}$ is not tangent to $\mathcal{W}^s_{\mathbf{1}}$ at $\mathbf{1}$, we may take a sufficiently small neighborhood V of $c\mathbf{1}$ such that for all $u\in V\cap\partial B_c(0)$ the ray R_u intersects $\mathcal{W}^s_{\mathbf{1}}\cap(\mathbb{R}_{>0})^d$. For such V we may then define $\bar\lambda_u=\inf\{\lambda>0:\lambda u\in\mathcal{W}^s_{\mathbf{1}}\}$ for $u\in V\cap\partial B_c(0)$. If we consider[1]

$$\eta:=\inf_{u\in\partial[V\cap\partial B_c(0)]}\phi(\bar\lambda_u u)>\phi(\mathbf{1})$$

then the fact that $\phi(U(t))$ is strictly decreasing (see Proposition 3.1) allows us to shrink V into a smaller neighborhood V^* of $c\mathbf{1}$ such that $\phi(v)=\eta$ for all $v\in\partial[V^*\cap\partial B_c(0)]$. Let us also observe that since $\mathbf{1}$ is the only saddle point we can take V sufficiently small so as to guarantee that $\max\{\phi(\lambda u):\lambda>0\}\geq\eta$ for all $u\in\partial B_c(0)\setminus V^*$. Then if we take the level curve $C_\eta=\{x\in\mathbb{R}^d:\phi(x)=\eta\}$ every ray R_u with $u\in\partial B_c(0)\setminus V^*$ intersects C_η. With this we may define for each $u\in\partial B_c(0)$

$$\lambda^*_u=\begin{cases}\bar\lambda_u & \text{if } u\in V^*\\ \\ \inf\{\lambda>0:\lambda u\in C_\eta\} & \text{if } u\in B_c(0)\setminus V^*\end{cases}.$$

Notice that the application $u\mapsto\lambda^*_u$ is continuous. Due to this fact, if $\tilde G:=\{\lambda u:0\leq\lambda<\lambda^*_u,\,u\in\partial B_c(0)\}$ then $\partial\tilde G=\{\lambda^*_u u:u\in\partial B_c(0)\}$. To finish the construction of our domain we must make a slight radial expansion of $\tilde G$, i.e., for $\alpha>0$ consider G defined by the formula

$$G:=\{\lambda u:0\leq\lambda<(1+\alpha)\lambda^*_u,\,u\in\partial B_c(0)\}.$$

Let us observe that Theorem 3.5 insures that G verifies condition (1). Since $\lambda^*_u>1$ for all $u\in\partial B_c(0)$ then it must also verify (2). Also, if we define $\partial^1:=\{(1+\alpha)\lambda^*(u):\lambda^*(u)u\in\overline{V^*}\}$ then ∂^1 is closed and if $\alpha>0$ is taken small enough then (3) holds. Finally, due to Theorem 3.5 we have $\partial^1\subset D_e$ and so (4) is verified. See Figure 6.1.

[1]By $\partial[V\cap\partial B_c(0)]$ we mean the border of the $(d-1)$-dimensional manifold $V\cap\partial B_c(0)$.

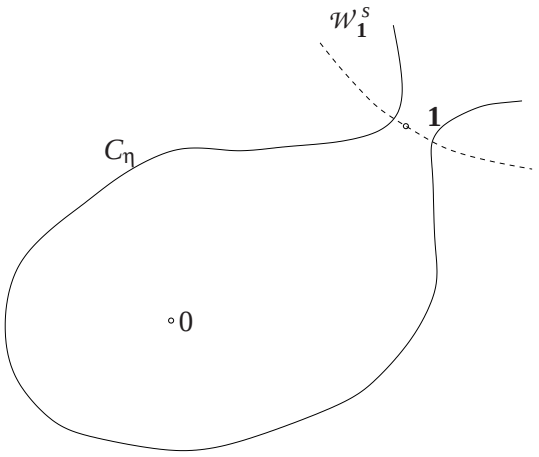

FIGURE 2. The level curve C_η and the stable manifold of **1**.

6.2. **The escape from** G**.** The behavior of the explosion time for initial data $u \in D_0$ is proved by showing that, with overwhelming probability as $\varepsilon \to 0$, the stochastic system describes the following path:

(i) It enters a neighborhood of the origin $B_c(0)$ in before a finite time T that does not depend on ε.
(ii) Once in $B_c(0)$ the system remains in G for a time of order e^{Δ/ε^2} and then escapes from G through ∂^1 since the barrier imposed by the potential is the lowest there.
(iii) After escaping G through ∂^1 the system explodes before a finite time τ which does not depend on ε.

The fact that the domain G is bounded allows us to assume that b is globally Lipschitz if we wish to study the behavior of our system while it remains inside G. Indeed, we may take $n_0 \in \mathbb{N}$ such that $G \subset B_{n_0}(0)$ and study the behavior of the solution to (5.1) since it coincides with our process until the escape from G. Then we can proceed as in the double-well potential case to obtain the following results (see [10, pp 295–300] for their proofs). Hereafter, $B_c(0)$ denotes the neighborhood of the origin highlighted in the construction of G in the previous section.

Theorem 6.1. Given $\delta > 0$ we have

$$\lim_{\varepsilon\to 0} \sup_{u\in\overline{B}_c(0)} P_u\Big(e^{\frac{\Delta-\delta}{\varepsilon^2}} < \tau_\varepsilon(\partial G) < e^{\frac{\Delta+\delta}{\varepsilon^2}}\Big) = 1.$$

Theorem 6.2. The stochastic system verifies

$$\lim_{\varepsilon\to 0} \sup_{u\in\overline{B}_c(0)} P_u\big(U^\varepsilon(\tau_\varepsilon(\partial G)) \notin \partial^1\big) = 0.$$

From these two theorems we can obtain the following useful corollary.

Corollary 6.3. For any $\delta > 0$ we have

$$\lim_{\varepsilon\to 0} \sup_{u\in\overline{B}_c(0)} P_u\Big(\tau_\varepsilon(\partial^1) > e^{\frac{\Delta+\delta}{\varepsilon^2}}\Big) = 0.$$

Proof. One can easily check that

$$\sup_{u\in\overline{B}_c(0)} P_u\Big(\tau_\varepsilon(\partial^1) > e^{\frac{\Delta+\delta}{\varepsilon^2}}\Big) \le \sup_{u\in\overline{B}_c(0)} P_u\Big(\tau_\varepsilon(\partial G) \ge e^{\frac{\Delta+\delta}{\varepsilon^2}}\Big) + \sup_{u\in\overline{B}_c(0)} P_u\Big(U^\varepsilon_{\tau_\varepsilon(\partial G)} \notin \partial^1\Big).$$

□

Concerning the asymptotic distribution of $\tau_\varepsilon(\partial G)$ we can obtain the following result.

Theorem 6.4. Let $\gamma_\varepsilon > 0$ be defined by the relation

$$P_0(\tau_\varepsilon(\partial G) > \gamma_\varepsilon) = e^{-1}.$$

Then there exists $\rho > 0$ such that for all $t \ge 0$ we have

$$\lim_{\varepsilon\to 0} \sup_{u\in\overline{B}_\rho(0)} |P_u(\tau_\varepsilon(\partial G) > t\gamma_\varepsilon) - e^{-t}| = 0.$$

6.3. **Bounds for the explosion time.** This section is devoted to the lower and upper bounds for the explosion time. More precisely, in this section we show that given $\delta > 0$, for all $u \in D_0$ one has

$$\lim_{\varepsilon\to 0} P_u\Big(\tau_\varepsilon < e^{\frac{\Delta-\delta}{\varepsilon^2}}\Big) = 0$$

and

$$\lim_{\varepsilon\to 0} P_u\Big(\tau_\varepsilon > e^{\frac{\Delta+\delta}{\varepsilon^2}}\Big) = 0,$$

where the convergence can be taken uniform over compact subsets of D_0. The proofs of these bounds essentially follow [10], where analogous bounds are given for the tunneling time. However, unlike the double-well potential model, the use of localization techniques becomes necessary at some points throughout our work. We begin first with the lower bound.

Proposition 6.5. Given $\delta > 0$ and $u \in D_0$ we have

$$\lim_{\varepsilon\to 0} P_u\Big(\tau_\varepsilon < e^{\frac{\Delta-\delta}{\varepsilon^2}}\Big) = 0. \tag{6.1}$$

Moreover, the convergence is uniform over compact subsets of D_0.

Proof. First observe that since for $u \in G$ we have $P_u(\tau_\varepsilon \ge \tau_\varepsilon(\partial G)) = 1$ then (6.1) holds uniformly over any small neighborhood of the origin by Lemma 6.1. Next, we generalize the result for any $u \in D_0$. For each $u \in D_0$ there exist $T_u > 0$, $\delta_u > 0$ and $n_u \in \mathbb{N}$ such that the deterministic system beginning at u reaches $B_{\frac{\rho}{2}}(0)$ before T_u, remaining in $B_{n_u}(0)$ and at a distance δ_u from $\partial B_{n_u}(0)$ on $[0, T_u]$. It follows that $U^{n_u,u}$ does so as well. From this we obtain

$$P_u\big(\tau_\varepsilon(\overline{B}_\rho(0)) > T_u\big) \le P_u\big(\min\{\tau_\varepsilon^{n_u}, \tau_\varepsilon(\overline{B}_\rho(0))\} > T_u\big) + P_u\big(\tau_\varepsilon^{n_u} \le T_u\big)$$

$$\le P_u\Big(\sup_{0\le t\le T_u} |U^{n_u,\varepsilon}(t) - U^{n_u}(t)| > \frac{\rho}{2}\Big) + P_u\Big(\sup_{0\le t\le T_u} |U^{n_u,\varepsilon}(t) - U^{n_u}(t)| > \frac{\delta_u}{2}\Big).$$

Using estimation (2.2) for the family $\big(U^{n_u,u,\varepsilon}\big)_{\varepsilon>0}$ we conclude

$$\lim_{\varepsilon\to 0} P_u\big(\tau_\varepsilon(\overline{B}_\rho(0)) > T_u\big) = 0. \tag{6.2}$$

Therefore, if we write

$$P_u\Big(\tau_\varepsilon < e^{\frac{\Delta-\delta}{\varepsilon^2}}\Big) \le P_u\Big(\tau_\varepsilon(\overline{B}_\rho(0)) < \tau_\varepsilon < e^{\frac{\Delta-\delta}{\varepsilon^2}}\Big) + P_u(\tau_\varepsilon \le T_u) + P_u(\tau_\varepsilon(\overline{B}_\rho(0)) > T_u),$$

then the last two terms on the right tend to zero when $\varepsilon \to 0$ as a consequence of what we stated above. By the strong Markov property for $U^{u,\varepsilon}$ we have

$$P_u\Big(\tau_\varepsilon(\overline{B}_\rho(0)) < \tau_\varepsilon < e^{\frac{\Delta-\delta}{\varepsilon^2}}\Big) \leq \sup_{y\in\overline{B}_\rho(0)} P_y\Big(\tau_\varepsilon < e^{\frac{\Delta-\delta}{\varepsilon^2}}\Big) \leq \sup_{y\in\overline{B}_\rho(0)} P_y\Big(\tau^{n_0}(\partial G) < e^{\frac{\Delta-\delta}{\varepsilon^2}}\Big)$$

where n_0 is taken as in the first step. Since the rightmost term tends to zero by Lemma 6.1 we conclude the result for arbitrary $u \in D_0$. The uniform convergence over compact subsets $\mathcal{K}$ of D_0 is proved in a similar fashion by taking δ_u and T_u uniformly over $\mathcal{K}$ as in Proposition 5.2. □

Now we turn to the proof of the upper bound. As we stated before, when studying the behavior of the stochastic system under initial conditions $u \in G$ and for small $\varepsilon > 0$ we typically observe that the process $U^{u,\varepsilon}$ escapes from G through ∂^1 since the cost imposed by the potential is the lowest there. Once in ∂^1 the influence of noise becomes negligible and the process then describes a path similar to the deterministic trajectory until exploding in a finite time. We formalize this statement in the following proposition.

Proposition 6.6. There exists $T_0 > 0$ such that

$$\limsup_{\varepsilon\to 0} \sup_{u\in\partial^1} P_u(\tau_\varepsilon > T_0) = 0.$$

Proof. Since ∂^1 is a compact set contained in D_e, the proof follows from Proposition 5.2 and the fact that $\sup_{u\in\partial^1} \tau_u^0 < +\infty$. □

With this proposition we are able to conclude the upper bound.

Proposition 6.7. For each $\delta > 0$ and $u \in D_0$ we have

$$\lim_{\varepsilon\to 0} P_u\Big(\tau_\varepsilon > e^{\frac{\Delta+\delta}{\varepsilon^2}}\Big) = 0.$$

Moreover, the convergence is uniform over compact subsets of D_0.

Proof. We proceed in two steps.

1. We check that given $\delta > 0$ we get

$$\lim_{\varepsilon\to 0} \sup_{x\in\overline{B}_c(0)} P_x\Big(\tau_\varepsilon > e^{\frac{\Delta+\delta}{\varepsilon^2}}\Big) = 0. \tag{6.3}$$

It is not hard to show that for $\varepsilon > 0$ small enough the strong Markov property yields

$$\sup_{u\in\overline{B}_c(0)} P_u\Big(\tau_\varepsilon > e^{\frac{\Delta+\delta}{\varepsilon^2}}\Big) \leq \sup_{u\in\overline{B}_c(0)} P_u\left(\tau_\varepsilon(\partial^1) > e^{\frac{\Delta+\frac{\delta}{2}}{\varepsilon^2}}\right) + \sup_{u\in\partial^1} P_u(\tau_\varepsilon > T_0) + \sup_{u\in\overline{B}_c(0)} P_u(U^\varepsilon_{\tau_\varepsilon(\partial G)} \notin \partial^1)$$

where $T_0 > 0$ is taken as in Proposition 6.6. We finish this first step by observing that the right hand side converges to zero. Indeed, the first term does so by Corollary 6.3, the second by Proposition 6.6 and the third by Lemma 6.2.

2. We now generalize the result for $u \in D_0$. This follows from the fact that

$$P_u\Big(\tau_\varepsilon > e^{\frac{\Delta+\delta}{\varepsilon^2}}\Big) \leq \sup_{u\in\overline{B}_c(0)} P_u\left(\tau_\varepsilon > \frac{e^{\frac{\Delta+\delta}{\varepsilon^2}}}{2}\right) + P_u\big(\tau_\varepsilon(\overline{B}_c(0)) > T_u\big)$$

by the strong Markov property. Observing that the first term on the right hand side of the equation tends to zero by (6.3) and that the second term does by (6.2), we obtain our result. The convergence over compact subsets of D_0 can be seen as in Proposition 5.2. □

6.4. **Asymptotic distribution of the explosion time.** Our main objective in this section is to prove the asymptotic memory loss of the normalized explosion time $\frac{\tau_\varepsilon}{\beta_\varepsilon}$. The proof focuses on studying the escape from G. The asymptotic memory loss for τ_ε can be deduced once we show that the time in which the process exits from G and the explosion time are asymptotically similar. We formalize this last statement in the following proposition.

Proposition 6.8. There exists a positive constant T_0 such that for all $u \in D_0 \cap G$

$$\lim_{\varepsilon\to 0} P_x(\tau_\varepsilon > \tau_\varepsilon(\partial G) + T_0) = 0.$$

Proof. Let us observe that by the strong Markov property

$$P_u(\tau_\varepsilon > \tau_\varepsilon(\partial G) + T_0) \le \sup_{y\in\partial^1} P_y(\tau_\varepsilon > T_0) + P_u\big(\tau_\varepsilon(\partial G) < \tau_\varepsilon(\overline{B}_c(0))\big) + \sup_{u\in\overline{B}_c(0)} P_u\big(U^\varepsilon_{\tau_\varepsilon(\partial G)} \notin \partial^1\big).$$

We can now conclude our desired result by the use of Proposition 6.6 and Lemma 6.2. □

We are now ready to establish the asymptotic memory loss of the explosion time. Having the former proposition at our disposal, the rest of the proof is very similar to the one offered in the double-well potential model. We emphasize that the main difference with this case lies in how to show this last proposition. In the double-well potential the corresponding statement to Proposition 6.8 holds due to the fact that the tunneling time for initial conditions in the deepest well is of order one. This can be easily deduced from the Freidlin-Wentzell estimates. Analogously, in our model Proposition 6.8 holds since now the explosion time for initial data in D_e is of order one. However, the lack of a global Lipschitz condition forces us to proceed differently in order to show this last fact. We recall that a proof of this is contained essentially in Proposition 5.2. We now give a brief sketch of the rest of the proof of Theorem 2.6 in the following lines and refer to [5] for further details.

Sketch of proof of Theorem 2.6.

(1) We first check that, for $\rho > 0$ small enough, $\lim_{\varepsilon\to 0} \sup_{u\in\overline{B}_\rho(0)} |P_u(\tau_\varepsilon(\partial G) > t\beta_\varepsilon) - e^{-t}| = 0$. This is due to the fact that $\lim_{\varepsilon\to 0} \frac{\beta_\varepsilon}{\gamma_\varepsilon} = 1$.
(2) Next, we prove that $P_0(\tau_\varepsilon > t\beta_\varepsilon) = e^{-t}$ for $t > 0$. This is done with the help of Proposition 6.8 and the previous step.
(3) With the help of appropriate coupling techniques we establish the uniform convergence over any small enough neighborhood of the origin.
(4) Finally, by using the strong Markov property, we conclude the result for arbitrary initial data $u \in D_0$.

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