The Generalized Schur Decomposition and the rank-R set of real $I \times J \times 2$ arrays

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Abstract

It is known that a best low-rank approximation to multi-way arrays or higher-order tensors may not exist. This is due to the fact that the set of multi-way arrays with rank at most R is not closed. Nonexistence of the best low-rank approximation results in diverging rank-1 components when an attempt is made to compute the approximation. Recently, a solution to this problem has been proposed for real $I \times J \times 2$ arrays. Instead of a best rank-R approximation the best fitting Generalized Schur Decomposition (GSD) is computed. Under the restriction of nonsingular upper triangular matrices in the GSD, the set of GSD solutions equals the interior and boundary of the rank-R set. Here, we show that this holds even without the restriction. We provide a complete classification of interior, boundary, and exterior points of the rank-R set of real $I \times J \times 2$ arrays, and show that the set of GSD solutions equals the interior and boundary of this set.

Keywords: tensor decomposition, low-rank approximation, Candecomp, Parafac, Schur decomposition, matrix pencils, diverging components.

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1 Introduction

This paper is an addendum to Stegeman and De Lathauwer [11] who study the following subject. Let $\circ$ denote the outer-product, and define the outer-product rank of $\underline{\mathbf{Y}} \in \mathbb{R}^{I\times J\times K}$ as

$$\mathrm{rank}_\circ(\underline{\mathbf{Y}}) = \min\{R \mid \underline{\mathbf{Y}} = \sum_{r=1}^{R} \mathbf{x}_r \circ \mathbf{y}_r \circ \mathbf{z}_r\}\,. \tag{1.1}$$

Let

$$\mathcal{S}_R(I,J,K) = \{\underline{\mathbf{Y}} \in \mathbb{R}^{I\times J\times K} \mid \mathrm{rank}_\circ(\underline{\mathbf{Y}}) \leq R\}\,, \tag{1.2}$$

and let $\overline{\mathcal{S}}_R(I,J,K)$ denote the closure of $\mathcal{S}_R(I,J,K)$, i.e. the union of the set itself and its boundary points in $\mathbb{R}^{I\times J\times K}$.

Let $\underline{\mathbf{Z}} \in \mathbb{R}^{I\times J\times K}$ and $||\cdot||$ denote the Frobenius norm on $\mathbb{R}^{I\times J\times K}$. Consider the following low-rank approximation problem.

$$\min\{||\underline{\mathbf{Z}} - \underline{\mathbf{Y}}|| \mid \underline{\mathbf{Y}} \in \mathcal{S}_R(I,J,K)\}\,. \tag{1.3}$$

Assuming $\mathrm{rank}_\circ(\underline{\mathbf{Z}}) > R$, an optimal solution of (1.3) will be a boundary point of the set $\mathcal{S}_R(I,J,K)$. However, the set $S_R(I,J,K)$ is not closed for $R \geq 2$, and problem (1.3) may not have an optimal solution due to this fact; see De Silva and Lim [2]. Nonexistence of an optimal solution results in diverging rank-1 components when an attempt is made to compute a best rank-R approximation, see Krijnen, Dijkstra and Stegeman [5]. In order to overcome this fallacy, [2] proposed to consider instead

$$\min\{||\underline{\mathbf{Z}} - \underline{\mathbf{Y}}|| \mid \underline{\mathbf{Y}} \in \overline{\mathcal{S}}_R(I,J,K)\}\,. \tag{1.4}$$

Note that if (1.3) has an optimal solution, then it is also an optimal solution of (1.4). To solve problem (1.4), we need to characterize the boundary points of $\mathcal{S}_R(I,J,K)$ and we need an algorithm to find an optimal boundary point. For $R = 2$, the boundary points are determined in [2], and an algorithm to solve (1.4) is proposed in Rocci and Giordani [7]. For a general approach to obtain an optimal solution to (1.4) from an attempt to solve (1.3), see Stegeman [10].

In Stegeman and De Lathauwer [11] the case $K = 2$ is considered. Let

$$\mathcal{P}_R(I,J,2) = \{\underline{\mathbf{Y}} \in \mathbb{R}^{I\times J\times 2} \mid \mathbf{Y}_k = \mathbf{Q}_a\,\mathbf{R}_k\,\mathbf{Q}_b^T,\ k=1,2,\ \text{with}\ \mathbf{Q}_a^T\mathbf{Q}_a = \mathbf{Q}_b^T\mathbf{Q}_b = \mathbf{I}_R$$

$$\text{and}\ \mathbf{R}_k\ (R\times R)\ \text{upper triangular}\}\,, \tag{1.5}$$

denote the set of arrays with a full Generalized Schur Decomposition (GSD). Here, $\mathbf{Y}_k$ $(I \times J)$ denotes the kth frontal slice of $\underline{\mathbf{Y}}$. Note that a GSD exists only for $R \leq \min(I,J)$. In [11] it is shown that the problem

$$\min\{||\underline{\mathbf{Z}} - \underline{\mathbf{Y}}|| \mid \underline{\mathbf{Y}} \in \mathcal{P}_R(I,J,2)\}\,, \tag{1.6}$$

is guaranteed to have an optimal solution. Moreover, it holds that $\mathcal{P}_R(I,J,2) = \overline{\mathcal{S}}_R(I,J,2)$ under the restriction that only arrays are considered that have a GSD with $\mathbf{R}_1$ and $\mathbf{R}_2$ nonsingular. Also, a Jacobi algorithm (based on De Lathauwer, De Moor and Vandewalle [1]) is presented for solving (1.6). Hence, under the above restriction, for $K=2$ problem (1.4) can be solved by solving problem (1.6).

In this note we show that the restriction used in [11] is not necessary. That is, we prove that $\mathcal{P}_R(I,J,2) = \overline{\mathcal{S}}_R(I,J,2)$ holds for $R \leq \min(I,J)$.

We use the notation $(\mathbf{S},\mathbf{T},\mathbf{U}) \cdot \underline{\mathbf{Y}}$ to denote the multilinear matrix multiplication of an array $\underline{\mathbf{Y}} \in \mathbb{R}^{I\times J\times K}$ with matrices $\mathbf{S}$ $(I_2 \times I)$, $\mathbf{T}$ $(J_2 \times J)$, and $\mathbf{U}$ $(K_2 \times K)$. The result of the multiplication is an $I_2 \times J_2 \times K_2$ array. We refer to $(\mathbf{I}_I, \mathbf{I}_J, \mathbf{U}) \cdot \underline{\mathbf{Y}}$ with $\mathbf{U}$ $(K \times K)$ nonsingular as a slicemix.

For later use we mention that, for nonsingular $\mathbf{S}$, $\mathbf{T}$, $\mathbf{U}$, and $\underline{\mathbf{X}} = (\mathbf{S},\mathbf{T},\mathbf{U}) \cdot \underline{\mathbf{Y}}$, we have $\text{rank}_\circ(\underline{\mathbf{X}}) = \text{rank}_\circ(\underline{\mathbf{Y}})$ and $\underline{\mathbf{X}}$ is an interior (boundary, exterior) point of $\mathcal{S}_R(I,J,K)$ if and only if $\underline{\mathbf{Y}}$ is an interior (boundary, exterior) point of $\mathcal{S}_R(I,J,K)$.

2 The case $I = J = R$

Here, we consider the case where the arrays have two $I \times I$ slices and the number of components equals I. In [11] only arrays are considered that have two nonsingular slices. In Proposition 2.2 below we present a complete classification of $I \times I \times 2$ arrays into interior, boundary, and exterior points of the set $\mathcal{S}_R(I,J,2)$. This classification is used to show that $\mathcal{P}_I(I,I,2) = \overline{\mathcal{S}}_I(I,I,2)$ in

Theorem 2.3. In the proofs of Proposition 2.2 and Theorem 2.3 we use the following lemma which extends the theory on real matrix pencils and may be of interest in itself.

Lemma 2.1 *Let* $\underline{\mathbf{Y}} \in \mathbb{R}^{I \times I \times 2}$ *with* $I \times I$ *slices* $\mathbf{Y}_1$ *and* $\mathbf{Y}_2$ *such that* $\det(\mu\,\mathbf{Y}_1 + \lambda\,\mathbf{Y}_2) = 0$ *for all* $\mu, \lambda \in \mathbb{R}$. *Then* $\underline{\mathbf{Y}} \in \mathcal{P}_I(I, I, 2)$.

Proof. As shown in Moler and Stewart [6] (see also Golub and Van Loan [3, Section 7.7.2]) there exist orthonormal $\mathbf{Q}$ and $\mathbf{Z}$ such that $\mathbf{G} = \mathbf{Q}\,\mathbf{Y}_2\,\mathbf{Z}$ is upper triangular and $\mathbf{F} = \mathbf{Q}\,\mathbf{Y}_1\,\mathbf{Z}$ is quasi-upper triangular. That is, $\mathbf{F}$ is block-upper triangular where its diagonal blocks are 2×2 or 1×1 in size. The proof is complete if we show that there exist orthonormal $\tilde{\mathbf{Q}}$ and $\tilde{\mathbf{Z}}$ such that $\tilde{\mathbf{Q}}\,\mathbf{F}\,\tilde{\mathbf{Z}}$ and $\tilde{\mathbf{Q}}\,\mathbf{G}\,\tilde{\mathbf{Z}}$ are upper triangular.

Let $\mathbf{F}$ have diagonal blocks $\mathbf{F}_1, \ldots, \mathbf{F}_m$, where $\mathbf{F}_i$ is 2×2 or 1×1. Denote the corresponding diagonal blocks of $\mathbf{G}$ by $\mathbf{G}_i$, $i = 1, \ldots, m$. We have

$$\det(\mu\,\mathbf{Y}_1 + \lambda\,\mathbf{Y}_2) = \prod_{i=1}^{m} \det(\mu\,\mathbf{F}_i + \lambda\,\mathbf{G}_i) = 0\,, \qquad \text{for all } \mu, \lambda \in \mathbb{R}\,. \tag{2.1}$$

This can only hold if

$$\text{for some } l\,, \qquad \det(\mu\,\mathbf{F}_l + \lambda\,\mathbf{G}_l) = 0\,, \qquad \text{for all } \mu, \lambda \in \mathbb{R}\,. \tag{2.2}$$

It is shown in Moler and Stewart [6, Section 5] that if $\mathbf{F}_i$ is a 2×2 block and $\det(\mathbf{F}_i + \lambda\,\mathbf{G}_i) = 0$ for some $\lambda \in \mathbb{R}$, then 2×2 orthonormal $\tilde{\mathbf{Q}}$ and $\tilde{\mathbf{Z}}$ can be found such that $\tilde{\mathbf{Q}}\,\mathbf{F}_i\,\tilde{\mathbf{Z}}$ and $\tilde{\mathbf{Q}}\,\mathbf{G}_i\,\tilde{\mathbf{Z}}$ are upper triangular. Hence, we may assume without loss of generality that if $\mathbf{F}_i$ is a 2×2 block, then $\det(\mathbf{F}_i + \lambda\,\mathbf{G}_i) \neq 0$ for all $\lambda \in \mathbb{R}$.

Let index l be as in (2.2). From the discussion above it follows that we may assume that $\mathbf{F}_l$ is 1×1. Hence, $\mathbf{F}$ and $\mathbf{G}$ have a zero on their diagonals in the same position. Suppose the common zero appears right after a 2×2 block $\mathbf{F}_i$, i.e.

$$\begin{bmatrix} \mathbf{F}_i & \mathbf{f} \\ \mathbf{0}^T & 0 \end{bmatrix} = \left[\begin{array}{cc|c} * & * & * \\ * & * & * \\ \hline 0 & 0 & 0 \end{array}\right], \qquad \begin{bmatrix} \mathbf{G}_i & \mathbf{g} \\ \mathbf{0}^T & 0 \end{bmatrix} = \left[\begin{array}{cc|c} * & * & * \\ 0 & * & * \\ \hline 0 & 0 & 0 \end{array}\right]. \tag{2.3}$$

Let $\mathbf{z}_1 \in \mathbb{R}^3$ be orthogonal to the second rows of the matrices in (2.3). Then postmultiplying (2.3) by any orthonormal $\tilde{\mathbf{Z}} = [\mathbf{z}_1|\mathbf{z}_2|\mathbf{z}_3]$ brings both matrices into upper triangular form and leaves the common zero in the same position. Analogously, if the common zero appears right before a 2×2 block $\mathbf{F}_i$, then premultiplying by a suitable orthonormal $\tilde{\mathbf{Q}}$ does the trick.

If the common zero on the diagonals of $\mathbf{F}$ and $\mathbf{G}$ is not adjacent to a 2×2 block $\mathbf{F}_i$, then we resort to simultaneously reordering the diagonal blocks of $\mathbf{F}$ and $\mathbf{G}$ (except the common zero) such that it is. It suffices to show that swapping adjacent 2×2 and 1×1 blocks is possible by orthonormal transformations. Let $\mathbf{F}_i$ be 2×2 and consider the 3×3 matrices

$$\begin{bmatrix} \mathbf{F}_i & \mathbf{f} \\ \mathbf{0}^T & f_{i+1} \end{bmatrix}, \qquad \begin{bmatrix} \mathbf{G}_i & \mathbf{g} \\ \mathbf{0}^T & g_{i+1} \end{bmatrix}. \tag{2.4}$$

Swapping the diagonal blocks i and $i+1$ by orthonormal transformations is possible if $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ exist such that they satisfy the so-called generalized Sylvester equation (see e.g. Kressner [4, Section 5]):

$$\mathbf{F}_i \, \mathbf{x} - f_{i+1} \, \mathbf{y} = \mathbf{f}\,, \qquad \mathbf{G}_i \, \mathbf{x} - g_{i+1} \, \mathbf{y} = \mathbf{g}\,. \tag{2.5}$$

We may assume that f_{i+1} and g_{i+1} are not both zero. Let $g_{i+1} \neq 0$ (the proof for $f_{i+1} \neq 0$ is analogous). Then (2.5) is satisfied for

$$\mathbf{y} = (\mathbf{G}_i \, \mathbf{x} - \mathbf{g})/g_{i+1}\,, \qquad \mathbf{x} = (\mathbf{F}_i - (f_{i+1}/g_{i+1}) \, \mathbf{G}_i)^{-1} \, (\mathbf{f} - (f_{i+1}/g_{i+1}) \, \mathbf{g})\,. \tag{2.6}$$

Note that $\det(\mathbf{F}_i - (f_{i+1}/g_{i+1}) \, \mathbf{G}_i) \neq 0$ by assumption, and the solution (2.6) is unique. Hence, the diagonal blocks can be swapped. Analogously, it can be shown that a 1×1 block i and a 2×2 block $i+1$ can be swapped. This completes the proof. $\square$

Proposition 2.2 *Let* $\underline{\mathbf{Y}} \in \mathbb{R}^{I \times I \times 2}$.

(a) *If there exists a* $\mathbf{U}$ *nonsingular such that* $\underline{\mathbf{X}} = (\mathbf{I}_I, \mathbf{I}_I, \mathbf{U}) \cdot \underline{\mathbf{Y}}$ *has nonsingular slice* $\mathbf{X}_1$*, then*

- (a1) $\underline{\mathbf{Y}}$ *is an interior point of* $\mathcal{S}_I(I, I, 2)$ *if* $\mathbf{X}_2 \mathbf{X}_1^{-1}$ *has* I *distinct real eigenvalues.*
- (a2) $\underline{\mathbf{Y}}$ *is a boundary point of* $\mathcal{S}_I(I, I, 2)$ *if* $\mathbf{X}_2 \mathbf{X}_1^{-1}$ *has* I *real eigenvalues but not all distinct.*
- (a3) $\underline{\mathbf{Y}}$ *is an exterior point of* $\mathcal{S}_I(I, I, 2)$ *if* $\mathbf{X}_2 \mathbf{X}_1^{-1}$ *has at least one pair of complex eigenvalues.*

(b) *If there does not exist a* $\mathbf{U}$ *nonsingular such that* $\underline{\mathbf{X}} = (\mathbf{I}_I, \mathbf{I}_I, \mathbf{U}) \cdot \underline{\mathbf{Y}}$ *has nonsingular slice* $\mathbf{X}_1$*, then* $\underline{\mathbf{Y}}$ *is a boundary point of* $\mathcal{S}_I(I, I, 2)$.

Proof. The proofs of (a) follow from the fact that multilinear matrix multiplication leaves the property interior (boundary, exterior) point invariant, and application of Stegeman and De Lathauwer [11, Lemma 3.1] which is due to Stegeman [8].

Next we prove (b). From Lemma 2.1 it follows that $\underline{\mathbf{Y}} = (\mathbf{Q}_a, \mathbf{Q}_b, \mathbf{I}_2) \cdot \underline{\mathbf{R}}$, where $\underline{\mathbf{R}}$ has two upper diagonal slices. Below, we show that $\underline{\mathbf{R}}$ is a boundary point of $\mathcal{S}_I(I, I, 2)$. Since $\mathbf{Q}_a$ and $\mathbf{Q}_b$ are nonsingular, it follows that also $\underline{\mathbf{Y}}$ is a boundary point of $\mathcal{S}_I(I, I, 2)$.

It holds that $\det(\mu\,\mathbf{R}_1 + \lambda\,\mathbf{R}_2) = 0$ for all $\mu, \lambda \in \mathbb{R}$, which implies that $\mathbf{R}_1$ and $\mathbf{R}_2$ have a zero on their diagonals in the same position. A small perturbation of the diagonals of $\mathbf{R}_1$ and $\mathbf{R}_2$ yields slices $\mathbf{H}_1$ (nonsingular) and $\mathbf{H}_2$, with $\mathbf{H}_2\mathbf{H}_1^{-1}$ (upper triangular) having I real eigenvalues, and $||\underline{\mathbf{R}} - \underline{\mathbf{H}}|| < \epsilon$ for any $\epsilon > 0$. Next, we show that it is possible to choose the perturbation such that $\mathbf{H}_2\mathbf{H}_1^{-1}$ has a pair of identical eigenvalues. For simplicity, we assume that the diagonals of $\mathbf{R}_1$ and $\mathbf{R}_2$ contain one common zero. A proof for the general case is analogous.

Let $(\mathbf{R}_1)_{ii} = (\mathbf{R}_2)_{ii} = 0$ and set $(\mathbf{H}_1)_{ii} = \delta_1$ and $(\mathbf{H}_2)_{ii} = \delta_2$. This yields a nonzero eigenvalue δ_2/δ_1 for $\mathbf{H}_2\mathbf{H}_1^{-1}$ (assuming small perturbations of the other zeros on the diagonal of $\mathbf{R}_1$, such that $\mathbf{H}_1$ is nonsingular). Unless stated otherwise, we only perturb the zero diagonal elements of $\mathbf{R}_1$ and $\mathbf{R}_2$. If, for some $j \neq i$, $(\mathbf{R}_1)_{jj} \neq 0$ and $(\mathbf{R}_2)_{jj} \neq 0$, then let $\lambda = (\mathbf{R}_2)_{jj}/(\mathbf{R}_1)_{jj}$, and choose $\delta_2 = \lambda\,\delta_1$. This yields $\mathbf{H}_2\mathbf{H}_1^{-1}$ with two identical real eigenvalues $\delta_2/\delta_1 = \lambda$ for any $\delta_1 > 0$. If no common nonzero diagonal elements of $\mathbf{R}_1$ and $\mathbf{R}_2$ exist, then we proceed as follows. If, for some $j \neq i$, $(\mathbf{R}_1)_{jj} \neq 0$ and $(\mathbf{R}_2)_{jj} = 0$, then let $(\mathbf{H}_2)_{jj} = \eta$, and choose $\delta_1 = \sqrt{\delta_2}\,(\mathbf{R}_1)_{jj}$ and $\eta = \sqrt{\delta_2}$. This yields $\mathbf{H}_2\mathbf{H}_1^{-1}$ with two identical real eigenvalues $\delta_2/\delta_1 = \eta/(\mathbf{R}_1)_{jj}$ for any $\delta_2 > 0$. If, for some $j \neq i$, $(\mathbf{R}_1)_{jj} = 0$ and $(\mathbf{R}_2)_{jj} \neq 0$, then let $(\mathbf{H}_1)_{jj} = \eta$, and choose $\delta_2 = \sqrt{\delta_1}\,(\mathbf{R}_2)_{jj}$ and $\eta = \sqrt{\delta_1}$. This yields $\mathbf{H}_2\mathbf{H}_1^{-1}$ with two identical real eigenvalues $\delta_2/\delta_1 = (\mathbf{R}_2)_{jj}/\eta$ for any $\delta_1 > 0$. Hence, it is possible to get $\mathbf{H}_2\mathbf{H}_1^{-1}$ with a pair of identical eigenvalues.

By Proposition 2.2 (a2), the array $\underline{\mathbf{H}}$ is a boundary point of $\mathcal{S}_I(I, I, 2)$. Since $||\underline{\mathbf{R}} - \underline{\mathbf{H}}|| < \epsilon$ for any $\epsilon > 0$, it follows that $\underline{\mathbf{R}}$ can be approximated arbitrarily closely from $\overline{\mathcal{S}}_I(I, I, 2)$. Hence, we obtain $\underline{\mathbf{R}} \in \overline{\mathcal{S}}_I(I, I, 2)$. Moreover, since for any $\epsilon > 0$ the array $\underline{\mathbf{H}}$ is a boundary point of $\mathcal{S}_I(I, I, 2)$, it follows that $\underline{\mathbf{R}}$ itself must be a boundary point of $\mathcal{S}_I(I, I, 2)$. $\square$

We are now ready to present our result for $I = J = R$.

Theorem 2.3 *It holds that $\mathcal{P}_I(I,I,2) = \overline{\mathcal{S}}_I(I,I,2)$.*

Proof. First, observe that $\underline{\mathbf{X}} = (\mathbf{I}_I, \mathbf{I}_I, \mathbf{U}) \cdot \underline{\mathbf{Y}}$ with nonsingular $\mathbf{U}$ has a full GSD if and only if $\underline{\mathbf{Y}}$ has a full GSD. Indeed, a slicemix of upper triangular slices results in upper triangular slices.

This observation, together with Stegeman and De Lathauwer [11, Lemma 5.1], yields the following results for the arrays $\underline{\mathbf{Y}}$ in Proposition 2.2. If $\underline{\mathbf{Y}}$ satisfies (a1) or (a2), then $\underline{\mathbf{Y}} \in \mathcal{P}_I(I,I,2)$. If $\underline{\mathbf{Y}}$ satisfies (a3), then $\underline{\mathbf{Y}} \notin \mathcal{P}_I(I,I,2)$.

Lemma 2.1 shows that an array $\underline{\mathbf{Y}}$ satisfying (b) lies in $\mathcal{P}_I(I,I,2)$. Since (a)-(b) defines a partition of $\mathbb{R}^{I\times I\times 2}$, we have shown that $\underline{\mathbf{Y}} \in \mathcal{P}_I(I,I,2)$ if and only if $\underline{\mathbf{Y}} \in \overline{\mathcal{S}}_I(I,I,2)$. This completes the proof. □

3 Extension to general I, J, R

A GSD exists only for $R \leq \min(I,J)$. However, nonexistence of an optimal solution to problem (1.3) for $I \times J \times 2$ arrays does not seem to occur for $R > I$ or $R > J$; see Stegeman [9]. In Theorem 3.2 below we show that $\mathcal{P}_R(I,J,2) = \overline{\mathcal{S}}_R(I,J,2)$ for $R \leq \min(I,J)$. This extends Theorem 2.3. In the proof of Theorem 3.2, we make use of Theorem 2.3 and the following lemma, which concerns an orthogonal equivalence between interior and boundary points of $\mathcal{S}_R(I,J,2)$ and those of $\mathcal{S}_R(R,R,2)$.

Lemma 3.1 *Let $\underline{\mathbf{Y}} \in \mathbb{R}^{I\times J\times 2}$ with $R \leq \min(I,J)$. Then $\underline{\mathbf{Y}} \in \overline{\mathcal{S}}_R(I,J,2)$ if and only if there exist $\mathbf{S}$ $(I \times R)$ and $\mathbf{T}$ $(J \times R)$ with $\mathbf{S}^T\mathbf{S} = \mathbf{T}^T\mathbf{T} = \mathbf{I}_R$ such that $\underline{\mathbf{Y}} = (\mathbf{S}, \mathbf{T}, \mathbf{I}_2) \cdot \underline{\mathbf{X}}$ with $\underline{\mathbf{X}} \in \overline{\mathcal{S}}_R(R,R,2)$. Moreover, $\underline{\mathbf{Y}} \in \mathcal{S}_R(I,J,2)$ if and only if $\underline{\mathbf{X}} \in \mathcal{S}_R(R,R,2)$.*

Proof. See [2, Theorem 5.2]. □

Theorem 3.2 *Let $R \leq \min(I,J)$. It holds that $\mathcal{P}_R(I,J,2) = \overline{\mathcal{S}}_R(I,J,2)$.*

Proof. Let $\underline{\mathbf{Y}} \in \overline{\mathcal{S}}_R(I,J,2)$. By Lemma 3.1 and Theorem 2.3 we have $\underline{\mathbf{Y}} = (\mathbf{S}, \mathbf{T}, \mathbf{I}_2) \cdot \underline{\mathbf{X}}$ with $\underline{\mathbf{X}} \in \overline{\mathcal{S}}_R(R,R,2) = \mathcal{P}_R(R,R,2)$. This implies

$$\mathbf{Y}_k = \mathbf{S}\,\mathbf{X}_k\,\mathbf{T}^T = (\mathbf{S}\mathbf{Q}_a)\,\mathbf{R}_k\,(\mathbf{T}\mathbf{Q}_b)^T\,, \qquad k = 1, 2\,. \tag{3.1}$$

Since the matrices $\mathbf{SQ}_a$ and $\mathbf{TQ}_b$ are column-wise orthonormal and $\mathbf{R}_k$ is $R \times R$ upper triangular, (3.1) implies that $\underline{\mathbf{Y}} \in \mathcal{P}_R(I, J, 2)$.

Next, let $\underline{\mathbf{Y}} \in \mathcal{P}_R(I, J, 2)$. Then $\mathbf{Y}_k = \mathbf{Q}_a \mathbf{R}_k \mathbf{Q}_b^T$ for $k = 1, 2$, which is equivalent to $\underline{\mathbf{Y}} = (\mathbf{Q}_a, \mathbf{Q}_b, \mathbf{I}_2) \cdot \underline{\mathbf{R}}$, where $\underline{\mathbf{R}} \in \mathbb{R}^{R \times R \times 2}$ has two upper triangular slices. Hence, $\underline{\mathbf{R}} \in \mathcal{P}_R(R, R, 2) = \overline{\mathcal{S}}_R(R, R, 2)$ by Theorem 2.3. An application of Lemma 3.1 yields $\underline{\mathbf{Y}} \in \overline{\mathcal{S}}_R(I, J, 2)$. This completes the proof. $\square$

4 Conclusion

We have shown that the set of $I \times J \times 2$ arrays with a full GSD of size R equals the closure of the set of $I \times J \times 2$ arrays with at most rank R. Also, we have provided a complete classification of interior, boundary, and exterior points of the latter set. This extends the theoretical results in [11], which were limited to the case of nonsingular upper triangular matrices in the GSD.

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