TRIANGULINE REPRESENTATIONS

by

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Abstract. — Trianguline representations are a certain kind of p-adic representations of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ like the crystalline, semistable and de Rham representations of Fontaine. Their definition involves the theory of (φ,Γ)-modules. In this survey, we explain the theory of (φ,Γ)-modules and the definition and properties of trianguline representations. After that, we give some examples of their occurrence in arithmetic geometry.

Contents

2000 *Mathematics Subject Classification.* — 11-02; 11F11; 11F33; 11F80; 11F85; 11S20; 14G22.
Key words and phrases. — Galois representations; p-adic Hodge theory; (φ,Γ)-modules; trianguline representations; p-adic local Langlands correspondence; overconvergent modular forms.

1. Introduction

1.1. Representations of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. — The starting point for this survey is that one can attach representations of the group $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ to some objects which occur in arithmetic geometry, for example elliptic curves and modular forms. Suppose for instance that A is an elliptic curve defined over $\mathbf{Q}$ and choose a prime number p. The group $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ acts on the p^n-th torsion points $A[p^n](\overline{\mathbf{Q}})$ of A and this gives rise to the Tate module of A, a 2-dimensional $\mathbf{Q}_p$-vector space V_pA which is the p-adic representation of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ attached to A.

Let ℓ be a prime number and choose an embedding $\iota_\ell : \overline{\mathbf{Q}} \to \overline{\mathbf{Q}}_\ell$. This gives rise to a map $\mathrm{Gal}(\overline{\mathbf{Q}}_\ell/\mathbf{Q}_\ell) \to \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ which is injective and whose image is the decomposition group D_ℓ of a place above ℓ (a different choice of ι_ℓ gives rise to another subgroup of $\mathrm{Gal}(\overline{\mathbf{Q}}_\ell/\mathbf{Q}_\ell)$ which is conjugate to D_ℓ). The group D_ℓ contains the inertia subgroup I_ℓ and the quotient D_ℓ/I_ℓ is isomorphic to $\mathrm{Gal}(\overline{\mathbf{F}}_\ell/\mathbf{F}_\ell) = \widehat{\mathbf{Z}}$ which is topologically generated by the Frobenius map $\mathrm{Frob}_\ell = [z \mapsto z^\ell]$. We then have the following theorem which says that the representation V_pA is also "attached to A" in a deeper way.

Theorem 1.1.1. — *If $\ell \nmid p \cdot \mathrm{Disc}(A)$, then the restriction of V_pA to I_ℓ is trivial and* $\det(X - \mathrm{Frob}_\ell \mid V_pA) = X^2 - a_\ell X + \ell$ *where* $a_\ell = \ell + 1 - \mathrm{Card}(A(\mathbf{F}_\ell))$.

As ℓ runs through a set of primes of density 1, the groups D_ℓ and their conjugates form a dense subset of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ by Chebotarev's theorem and therefore theorem 1.1.1 determines the semisimplification of V_pA. If $\ell \neq p$ but $\ell \mid \mathrm{Disc}(A)$, then we also have a description of $V_pA \mid_{D_\ell}$ which now depends on the geometry of $A \bmod \ell$. A much deeper problem is the description of the restriction of V_pA to D_p and this is the goal of Fontaine's theory, which we'll discuss in the next §.

Before we do that, let us recall that one can also attach p-adic representations of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ to modular forms as follows. Let $f = \sum_{n \geqslant 1} a_n q^n$ be a normalized cuspidal eigenform of weight k and level N and character ε, and let E be the field generated over $\mathbf{Q}_p$ by the images of the a_n's in $\overline{\mathbf{Q}}_p$ under the chosen embedding. The field E is a finite extension of $\mathbf{Q}_p$ and we have the following result of Deligne [**Del71**] (see theorem 6.1 of Deligne-Serre [**DS74**]).

Theorem 1.1.2. — *There exists an irreducible 2-dimensional E-linear representation V_pf of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ such that for every prime number $\ell \nmid pN$, the restriction of V_pf to I_ℓ is trivial and* $\det(X - \mathrm{Frob}_\ell \mid V_pf) = X^2 - a_\ell X + \varepsilon(\ell)\ell^{k-1}$.

1.2. Trianguline representations and (φ, Γ)-modules. — Let E be a finite extension of $\mathbf{Q}_p$ which is the field of coefficients of the representations we consider. The goal of Fontaine's theory is to study the E-linear representations of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$. These may arise as the restriction to D_p of representations of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ as above but they are also interesting considered on their own. A p-adic representation of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ is then a finite dimensional E-vector space V along with a continuous E-linear action of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$.

Fontaine's approach has been to construct some rings of periods, for example $\mathbf{B}_{\mathrm{cris}}$, $\mathbf{B}_{\mathrm{st}}$ and $\mathbf{B}_{\mathrm{dR}}$ and to use these rings to define and study crystalline, semistable and de Rham representations (see §3.1 for reminders about this). These constructions allow one to give a complete description of the restriction to D_p of the representations V_pA and V_pf of §1.1 (see §3.2). Another construction of Fontaine's which is crucial in this survey is the theory of (φ,Γ)-modules which we now describe (and will describe again in more detail in §§2.1–2.4).

Let $\mathcal{R}$ be the ring of power series $f(X) = \sum_{n\in\mathbf{Z}} a_n X^n$ where $a_n \in E$ and for which there exists $\rho(f)$ such that $f(X)$ converges on the p-adic annulus $\rho(f) \leqslant |X|_p < 1$. This ring is endowed with a Frobenius φ given by $(\varphi f)(X) = f((1+X)^p - 1)$ and with an action of $\mathbf{Z}_p^\times$ (now called Γ) given by $([a]f)(X) = f((1+X)^a - 1)$ if $a \in \mathbf{Z}_p^\times$.

A (φ,Γ)-module is a free $\mathcal{R}$-module of finite rank d endowed with a semilinear Frobenius φ such that $\mathrm{Mat}(\varphi) \in \mathrm{GL}_d(\mathcal{R})$ and with a commuting semilinear continuous action of Γ. The main result relating (φ,Γ)-modules and p-adic Galois representations is the following (it combines theorems of Fontaine, Fontaine-Wintenberger, Cherbonnier-Colmez and Kedlaya). The ring $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$ below denotes one of Fontaine's rings of periods. We say that a (φ,Γ)-module is étale if there exists a basis in which $\mathrm{Mat}(\varphi) \in \mathrm{GL}_d(\mathcal{O}^\dagger_{\mathcal{E}})$ where $\mathcal{O}^\dagger_{\mathcal{E}}$ is the set of $f(X) \in \mathcal{R}$ with $|a_n|_p \leqslant 1$ for all $n \in \mathbf{Z}$.

Theorem 1.2.1. — *If* D *is an étale* (φ,Γ)*-module, then* $V(\mathrm{D}) = (\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \otimes_{\mathcal{R}} \mathrm{D})^{\varphi=1}$ *is a* p*-adic representation of* $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ *and the resulting functor* $\mathrm{D} \mapsto V(\mathrm{D})$ *gives rise to an equivalence of categories:* $\{$*étale* (φ,Γ)*-modules*$\} \to \{$$p$*-adic representations*$\}$.

We denote by $V \mapsto \mathrm{D}(V)$ the inverse functor. The category of étale (φ,Γ)-modules is a full subcategory of the larger category of all (φ,Γ)-modules. In particular, if V is an irreducible p-adic representation, then $\mathrm{D}(V)$ is irreducible in the category of étale (φ,Γ)-modules but it can be reducible in the larger category of all (φ,Γ)-modules.

Definition 1.2.2. — If V is a p-adic representation of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$, then we say that V is trianguline if $\mathrm{D}(V)$ is a successive extension of (φ,Γ)-modules of rank 1 (after possibly enlarging E).

This definition can be seen as some far reaching generalization of the notion of ordinary representation. It was first given by Colmez in his construction of the "unitary principal series of $\mathrm{GL}_2(\mathbf{Q}_p)$" which is an important building block of the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbf{Q}_p)$ (see §4.1). Some important examples of trianguline representations are (1) the semistable representations of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ and (2) the restriction to $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ of the representations of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ attached to finite slope overconvergent modular forms.

This survey has three chapters. In the first one, we give a more detailed description of the definition and properties of (φ, Γ)-modules, including Kedlaya's theory of Frobenius slopes. In the second one, we give some examples of trianguline representations by relating the theory of (φ, Γ)-modules to p-adic Hodge theory, and then we give Colmez' construction of a parameter space for all 2-dimensional trianguline representations. In the last chapter, we explain how trianguline representations occur in the p-adic local Langlands correspondence, in the theory of overconvergent modular forms and in the study of Selmer groups.

1.3. Notations and conventions. — The field E is a finite extension of $\mathbf{Q}_p$ with ring of integers $\mathcal{O}_E$ whose maximal ideal is $\mathfrak{m}_E$ and residue field k_E. All the representations and characters in this survey are assumed to be continuous (note that a character $\delta : \mathbf{Q}_p^\times \to E^\times$ is necessarily continuous by exercise 6 of §4.2 of [**Ser94**]). When we say that an E-linear object is irreducible, we mean that it is absolutely irreducible, that is it remains irreducible when we extend scalars from E to a finite extension.

The cyclotomic character χ_{cycl} gives an isomorphism $\chi_{\mathrm{cycl}} : \mathrm{Gal}(\mathbf{Q}_p(\mu_{p^\infty})/\mathbf{Q}_p) \to \mathbf{Z}_p^\times$. The maximal abelian extension of $\mathbf{Q}_p$ is $\mathbf{Q}_p^{\mathrm{ab}} = \mathbf{Q}_p^{\mathrm{nr}} \cdot \mathbf{Q}_p(\mu_{p^\infty})$ and every element of $\mathrm{Gal}(\mathbf{Q}_p^{\mathrm{ab}}/\mathbf{Q}_p)$ can be written as $\mathrm{Frob}_p^n \cdot g$ where Frob_p is the lift of $[z \mapsto z^p]$ and $n \in \widehat{\mathbf{Z}}$ and $g \in \mathrm{Gal}(\mathbf{Q}_p(\mu_{p^\infty})/\mathbf{Q}_p)$. If $\delta : \mathbf{Q}_p^\times \to \mathcal{O}_E^\times$ is a unitary character, then by local class field theory δ gives rise to a character (still denoted by δ) of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ which is determined by the formula $\delta(\mathrm{Frob}_p^n \cdot g) = \delta(p)^{-n} \cdot \delta(\chi(g))$ if $n \in \mathbf{Z}$. In other words, we normalize class field theory so that p corresponds to the geometric Frobenius Frob_p^{-1}.

2. Galois representations and (φ, Γ)-modules

In this chapter, we explain the theory of (φ, Γ)-modules and its relation to p-adic representations. This allows us to define trianguline representations.

2.1. The Robba ring and (φ, Γ)-modules. — The *Robba ring* $\mathcal{R}$ is the ring of power series $f(X) = \sum_{n \in \mathbf{Z}} a_n X^n$ where $a_n \in E$ such that $f(X)$ converges on an annulus of the form $\rho(f) \leqslant |X|_p < 1$. For example, the power series $t = \log(1 + X)$ belongs the Robba ring (and here $\rho(t) = 0$).

The Robba ring is endowed with a Frobenius map φ given by $(\varphi f)(X) = f((1+X)^p - 1)$. Let Γ be another notation for $\mathbf{Z}_p^\times$ with the isomorphism $\mathbf{Z}_p^\times \to \Gamma$ denoted by $a \mapsto [a]$.

The Robba ring is endowed with an action of Γ given by $([a]f)(X) = f((1+X)^a - 1)$ and this action commutes with φ. For example, we have $\varphi(t) = pt$ and $[a](t) = at$.

Definition 2.1.1. — A *(φ, Γ)-module over $\mathcal{R}$* is a free $\mathcal{R}$-module of finite rank d, endowed with a semilinear Frobenius φ such that $\mathrm{Mat}(\varphi) \in \mathrm{GL}_d(\mathcal{R})$ and a semilinear action of Γ which commutes with φ.

For example, if $\delta : \mathbf{Q}_p^\times \to E^\times$ is a character, then we define $\mathcal{R}(\delta)$ as the (φ, Γ)-module of rank 1 having e_δ as a basis where $\varphi(e_\delta) = \delta(p)e_\delta$ and $[a](e_\delta) = \delta(a)e_\delta$.

Theorem 2.1.2. — *Every (φ, Γ)-module of rank 1 over $\mathcal{R}$ is isomorphic to $\mathcal{R}(\delta)$ for a well-defined character $\delta : \mathbf{Q}_p^\times \to E^\times$.*

The proof of this theorem (proposition 3.1 of [**Col08**]) uses the results of §2.2 (the equivalence between étale (φ, Γ)-modules and p-adic representations) and it would be nice to have a more direct proof which uses only computations in (φ, Γ)-modules.

2.2. Etale (φ, Γ)-modules and Galois representations. — The ring $\mathcal{E}^\dagger$ is the subring of $\mathcal{R}$ consisting of those $f(X) = \sum_{n \in \mathbf{Z}} a_n X^n$ for which the sequence $\{a_n\}_{n \in \mathbf{Z}}$ is bounded. The subring of $\mathcal{E}^\dagger$ consisting of those $f(X) = \sum_{n \in \mathbf{Z}} a_n X^n$ for which $|a_n|_p \leqslant 1$ is denoted by $\mathcal{O}_{\mathcal{E}}^\dagger$. This is a henselian local ring with residue field $k_E(\!(X)\!)$.

Definition 2.2.1. — We say that a (φ, Γ)-module over $\mathcal{R}$ is *étale* if it has a basis in which $\mathrm{Mat}(\varphi) \in \mathrm{GL}_d(\mathcal{O}_{\mathcal{E}}^\dagger)$.

In §2.3 of [**Ber02**], a ring $\widetilde{\mathbf{B}}_{\mathrm{rig}}^\dagger$ is constructed which has the following properties: it is endowed with a Frobenius φ and a commuting action of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ and it contains the Robba ring $\mathcal{R}$. This inclusion is compatible with φ and with the action of Γ on $\mathcal{R}$ in the sense that if $y \in \mathcal{R}$, then $g(y) = [\chi_{\mathrm{cycl}}(g)](y)$. One can think of $\widetilde{\mathbf{B}}_{\mathrm{rig}}^\dagger$ as some sort of "algebraic closure" of $\mathcal{R}$.

If D is a (φ, Γ)-module over $\mathcal{R}$, then $V(\mathrm{D}) = (\widetilde{\mathbf{B}}_{\mathrm{rig}}^\dagger \otimes_{\mathcal{R}} \mathrm{D})^{\varphi=1}$ is an E-vector space, endowed with the action of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ given by $g(x \otimes e) = g(x) \otimes [\chi_{\mathrm{cycl}}(g)](e)$. This E-vector space can be finite or infinite-dimensional in general, but we have the following theorem which combines results of Fontaine (theorem 3.4.3 of [**Fon90**]), Cherbonnier-Colmez (corollary III.5.2 of [**CC98**]) and Kedlaya (theorem 6.3.3 of [**Ked05**]).

Theorem 2.2.2. — *If D is an étale (φ, Γ)-module of rank d over $\mathcal{R}$, then $V(\mathrm{D})$ is an E-linear representation of dimension d of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ and the resulting functor, from the category of étale (φ, Γ)-modules over $\mathcal{R}$ to the category of E-linear representations of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$, is an equivalence of categories.*

We denote by $V \mapsto \mathrm{D}(V)$ the inverse functor which to a p-adic representation attaches the corresponding étale (φ, Γ)-module over $\mathcal{R}$.

For example, the (φ, Γ)-module $\mathcal{R}(\delta)$ is étale if and only if $\mathrm{val}_p(\delta(p)) = 0$. In this case, the representation $V(\mathcal{R}(\delta))$ is the character of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ corresponding to δ by local class field theory as recalled in §1.3.

2.3. Trianguline representations. — We can now give the definition of trianguline representations (see §0.4 of [**Col08**]).

Definition 2.3.1. — If V is a p-adic representation of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$, then

1. we say that V is *split trianguline* if the (φ, Γ)-module $\mathrm{D}(V)$ is a successive extension of (φ, Γ)-modules of rank 1;
2. we say that V is *trianguline* if there exists a finite extension F of E such that $F \otimes_E V$ is split trianguline.

The possible extension of scalars from E to F is harmless and on the level of (φ, Γ)-modules consists in extending scalars from the Robba ring with coefficients in E to the Robba ring with coefficients in F. For example, we'll see later on that semistable representations are always trianguline, and they are split trianguline if and only if E contains the eigenvalues of φ on $\mathrm{D}_{\mathrm{st}}(V)$.

It is important to understand that a representation V may well be trianguline without V itself being an extension of representations of dimension 1. Indeed, the definition is that $\mathrm{D}(V)$ is a successive extension of (φ, Γ)-modules of rank 1, but these (φ, Γ)-modules are generally not étale and therefore do not correspond to subquotients of V.

Note also that a (φ, Γ)-module may be written as a successive extension of (φ, Γ)-modules of rank 1 in several different ways, but that the actual triangulation is not part of the data. This additional data of a triangulation amounts to what Mazur calls a *refinement* in [**Maz00**].

In the rest of this survey, we'll see several examples of trianguline representations, but we give here the two main classes:

1. the representations of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ which become semistable when restricted to $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p(\zeta_{p^n}))$ for some $n \geqslant 0$;
2. the representations of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ which arise from overconvergent modular eigenforms of finite slope.

In [**Ber10b**], some explicit families of 2-dimensional (φ, Γ)-modules are constructed and the trianguline ones are determined.

2.4. Slopes of (φ,Γ)-modules. — We now recall Kedlaya's theory of slopes for φ-modules over the ring $\mathcal{R}$ (free $\mathcal{R}$-modules of finite rank d with a semilinear φ such that $\mathrm{Mat}(\varphi)\in \mathrm{GL}_d(\mathcal{R})$). If $a/h\in\mathbf{Q}$, then we say that a φ-module over $\mathcal{R}$ is *pure of slope a/h* if it has a basis in which $\mathrm{Mat}(p^{-a}\varphi^h)\in\mathrm{GL}_d(\mathcal{O}_{\mathcal{E}}^{\dagger})$ (being étale is therefore equivalent to being pure of slope zero). A φ-module over $\mathcal{R}$ which is pure of a certain slope is said to be *isoclinic*. For example, the (φ,Γ)-module $\mathcal{R}(\delta)$ is pure of slope $\mathrm{val}_p(\delta(p))$. The main result of the theory of slopes is theorem 6.10 of [**Ked04**].

Theorem 2.4.1. — *If* D *is a φ-module over $\mathcal{R}$, then there exists a unique filtration* $\{0\}=\mathrm{D}_0\subset\mathrm{D}_1\subset\cdots\subset\mathrm{D}_\ell=\mathrm{D}$ *of* D *by sub-φ-modules such that:*

1. *for all $i\geqslant 1$, the φ-module* $\mathrm{D}_i/\mathrm{D}_{i-1}$ *is isoclinic;*
2. *if s_i is the slope of* $\mathrm{D}_i/\mathrm{D}_{i-1}$*, then $s_1<s_2<\cdots<s_\ell$.*

If D is a (φ,Γ)-module, then each of the D_i is stable under the action of Γ since the filtration is unique and hence each D_i is itself a (φ,Γ)-module.

A delicate but crucial point of the theory of slopes is that a φ-module over $\mathcal{R}$ which is pure of slope s has no subobject of slope $<s$ by theorem 2.4.1, but it may well have subobjects of slope $>s$. This helps to explain the definition of trianguline representations: an étale (φ,Γ)-module over $\mathcal{R}$ may be irreducible in the category of étale (φ,Γ)-modules but it can still admit some nontrivial subobjects in the larger category of all (φ,Γ)-modules.

Theorem 2.4.1 also helps to understand theorem 2.2.2. If D is a (φ,Γ)-module, then $V(\mathrm{D})=(\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}\otimes_{\mathcal{R}}\mathrm{D})^{\varphi=1}$ is constructed by solving φ-equations determined by the matrix of φ on D. If the slopes of D are >0 then these equations have no nonzero solutions while if the slopes of D are <0 then the space of solutions if infinite dimensional (see theorem A of [**Ber09**] for more precise results). The condition that D is étale is exactly the right one for $V(\mathrm{D})$ to be a finite dimensional E-vector space of the correct dimension.

3. Examples of trianguline representations

In this chapter, we explain how to relate (φ,Γ)-modules and p-adic Hodge theory, which allows us to give important examples of trianguline representations. After that, we explain how to compute extensions of (φ,Γ)-modules and Colmez' resulting construction of all 2-dimensional trianguline representations.

3.1. Fontaine's rings of periods. — The purpose of Fontaine's theory is to sort through p-adic representations and to classify the interesting ones by using objects from semilinear algebra. Recall that Fontaine has constructed in [**Fon94a**] a number of rings among which $\mathbf{B}_{\mathrm{cris}}$, $\mathbf{B}_{\mathrm{st}}$ and $\mathbf{B}_{\mathrm{dR}}$. The construction of these rings is quite complicated but they have a number of properties some of which we now recall and which suffice for this survey. All of them are $\mathbf{Q}_p$-algebras endowed with an action of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ and some extra structures which are all compatible with the action of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$. The ring $\mathbf{B}_{\mathrm{st}}$ has a Frobenius φ and a monodromy operator N which satisfy the relation $N\varphi = p\varphi N$ and the ring $\mathbf{B}_{\mathrm{cris}}$ is then $\mathbf{B}_{\mathrm{st}}^{N=0}$. The ring $\mathbf{B}_{\mathrm{dR}}$ is actually a field and is endowed with a filtration. The ring $\mathbf{B}_{\mathrm{cris}}$ contains $\widehat{\mathbf{Q}}_p^{\mathrm{nr}}$ and the choice of $\log_p(p)$ gives rise to an injective map $\overline{\mathbf{Q}}_p \otimes_{\mathbf{Q}_p^{\mathrm{nr}}} \mathbf{B}_{\mathrm{st}} \to \mathbf{B}_{\mathrm{dR}}$.

If V is a p-adic representation of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ and $* \in \{\mathrm{cris, st, dR}\}$, then we set $\mathrm{D}_*(V) = (\mathbf{B}_* \otimes_{\mathbf{Q}_p} V)^{\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)}$. The space $\mathrm{D}_*(V)$ is then an E-vector space of dimension $\leqslant \dim_E(V)$ and we say that V is *crystalline* or *semistable* or *de Rham* if we have equality of dimensions with $*$ being cris, st or dR.

The E-vector space $\mathrm{D}_{\mathrm{dR}}(V)$ is then endowed with an E-linear filtration, the space $\mathrm{D}_{\mathrm{st}}(V) \subset \mathrm{D}_{\mathrm{dR}}(V)$ is a filtered (φ, N)-module and $\mathrm{D}_{\mathrm{cris}}(V) = \mathrm{D}_{\mathrm{st}}(V)^{N=0}$ is a filtered φ-module. If D is a filtered (φ, N)-module, then we define $t_N(D)$ as the p-adic valuation of φ on $\det(D)$ and $t_H(D)$ as the unique integer h such that $\mathrm{Fil}^h(\det(D)) = \det(D)$ and $\mathrm{Fil}^{h+1}(\det(D)) = \{0\}$. We say that D is *admissible* if $t_H(D) = t_N(D)$ and if $t_H(D') \leqslant t_N(D')$ for every subobject D' of D. The following theorem combines results of Fontaine (§5.4 of [**Fon94c**]) and the Colmez-Fontaine theorem (theorem A of [**CF00**]).

Theorem 3.1.1. — *If V is a semistable representation of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$, then $\mathrm{D}_{\mathrm{st}}(V)$ is an admissible filtered (φ, N)-module and the functor $V \mapsto \mathrm{D}_{\mathrm{st}}(V)$ gives an equivalence of categories: $\{$semistable representations$\} \to \{$admissible filtered (φ, N)-modules$\}$.*

All of these constructions also work for representations of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/K)$; in particular, we say that a p-adic representation of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ is *potentially semistable* if its restriction to $\mathrm{Gal}(\overline{\mathbf{Q}}_p/K)$ is semistable for some finite extension K of $\mathbf{Q}_p$. Following §2.3 of [**Fon94b**], we can attach to a potentially semistable representation V of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ a Weil-Deligne representation $\mathrm{WD}(V)$. Let $W_{\mathbf{Q}_p} = \{g \in \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ such that $n(g) \in \mathbf{Z}\}$ be the Weil group of $\mathbf{Q}_p$ and suppose that the restriction of V to $\mathrm{Gal}(\overline{\mathbf{Q}}_p/K)$ is semistable. The space of the representation $\mathrm{WD}(V)$ is $D = \mathrm{D}_{\mathrm{st}}(V|_{\mathrm{Gal}(\overline{\mathbf{Q}}_p/K)})$ with $N_{\mathrm{WD}} = N$ and $\rho_{\mathrm{WD}}(w) = w\varphi^{-n(w)}$ if $w \in W_{\mathbf{Q}_p}$ where $W_{\mathbf{Q}_p}$ acts on D through $\mathrm{Gal}(K/\mathbf{Q}_p)$.

3.2. p-adic Hodge theory. — If X is a proper and smooth scheme over $\mathbf{Q}_p$, then the étale cohomology groups $\mathrm{H}^i_{\mathrm{\acute{e}t}}(X_{\overline{\mathbf{Q}}_p}, \mathbf{Q}_p)$ are p-adic representations of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ and we have the following theorem of Tsuji (theorem 0.2 of [**Tsu99**]), which is the former conjecture C_{st} of Fontaine-Jannsen (see §6.2 of [**Fon94c**]).

Theorem 3.2.1. — *If* X *is a proper scheme over* $\mathbf{Z}_p$ *with semistable reduction, then* $\mathrm{H}^i_{\text{ét}}(X_{\overline{\mathbf{Q}}_p}, \mathbf{Q}_p)$ *is a semistable representation of* $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ *and there is a natural isomorphism of filtered* (φ, N)*-modules:* $\mathrm{D}_{\mathrm{st}}(\mathrm{H}^i_{\text{ét}}(X_{\overline{\mathbf{Q}}_p}, \mathbf{Q}_p)) = \mathrm{H}^i_{\text{log-cris}}(X)$.

If f is a modular eigenform, then one can attach to it a p-adic representation $V_p f$ as recalled in theorem 1.1.2 as well as a smooth admissible representation $\Pi_p f$ of $\mathrm{GL}_2(\mathbf{Q}_p)$ (see [**Del73**]), and we then have the following result of Saito (the main theorem of [**Sai97**]), which is the "missing part" of theorem 1.1.2.

Theorem 3.2.2. — *If* f *is a cuspidal eigenform, then* $V_p f$ *is potentially semistable at* p *and* $\mathrm{WD}(V_p f)$ *is the Weil-Deligne representation attached to* $\Pi_p f$ *by the local Langlands correspondence.*

If in addition $p \nmid N$, then $V_p f$ is crystalline and the above theorem completely determines $\mathrm{D}_{\mathrm{cris}}(V_p f)$ because there is only one possible choice for the filtration (in this case, theorem 3.2.2 was previously proved by Scholl, see theorem 1.2.4 of [**Sch90**]). We get $\mathrm{D}_{\mathrm{cris}}((V_p f)^*) = D_{k,a_p}$ where $k = k(f)$ and $a_p = a_p(f)$ and $D_{k,a_p} = Ee_1 \oplus Ee_2$ with

$$\mathrm{Mat}(\varphi) = \begin{pmatrix} 0 & -1 \\ \varepsilon(p)p^{k-1} & a_p \end{pmatrix} \text{ and } \mathrm{Fil}^i D_{k,a_p} = \begin{cases} D_{k,a_p} & \text{if } i \leqslant 0, \\ Ee_1 & \text{if } 1 \leqslant i \leqslant k-1, \\ \{0\} & \text{if } i \geqslant k. \end{cases}$$

The converse of theorem 3.2.1 is known as the Fontaine-Mazur conjecture (conjecture 1 of [**FM95**]).

Conjecture 3.2.3. — *If* V *is an irreducible* p*-adic representation of* $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ *which is unramified except at a finite number of primes and which is de Rham at* p*, then* V *is a subquotient of the étale cohomology of some algebraic variety over* $\mathbf{Q}$.

If in addition $\dim(V) = 2$, then we actually expect V to be the representation attached to a modular eigenform, and we have the following precise conjecture (conjecture 3c of [**FM95**]).

Conjecture 3.2.4. — *If* V *is an irreducible* 2*-dimensional* p*-adic representation of* $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ *which is unramified except at a finite number of primes and which is de Rham at* p *with distinct Hodge-Tate weights, then* V *is a twist of the Galois representation attached to a cuspidal eigenform with weight* $k \geqslant 2$.

Let us write $\overline{V}$ for the reduction modulo $\mathfrak{m}_E$ of V.

Theorem 3.2.5. — *The Fontaine-Mazur conjecture is true, if we suppose that* $\overline{V}$ *satisfies some technical hypotheses.*

This theorem has been proved independently by Kisin (this is the main theorem of [**Kis09**]) and by Emerton (theorem 1.2.4 of [**Eme10**]). The "technical hypotheses" of Kisin are the following (χ_{cycl} is now the reduction mod p of the cyclotomic character).

1. $p \neq 2$ and $\overline{V}$ is odd,
2. $\overline{V} \mid_{\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}(\zeta_p))}$ is irreducible,
3. $\overline{V} \mid_{\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)}$ is not of the form $\left(\begin{smallmatrix} \eta\chi_{\mathrm{cycl}} & * \\ 0 & \eta \end{smallmatrix}\right)$ for any character η.

The "technical hypotheses" of Emerton are (1) and (2) and

3'. $\overline{V} \mid_{\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)}$ is not of the form $\left(\begin{smallmatrix} \eta & * \\ 0 & \eta\chi_{\mathrm{cycl}} \end{smallmatrix}\right)$ nor of the form $\left(\begin{smallmatrix} \eta & * \\ 0 & \eta \end{smallmatrix}\right)$ for any character η.

3.3. Crystalline and semistable (φ, Γ)-modules. — In §3.1, we recalled the definition of $\mathrm{D}_{\mathrm{cris}}(V)$ and $\mathrm{D}_{\mathrm{st}}(V)$ for a p-adic representation V. We now explain how to extend this definition to (φ, Γ)-modules. Recall that we denote by t the element $\log(1+X) \in \mathcal{R}$.

Definition 3.3.1. — If D is a (φ, Γ)-module, let $\mathrm{D}_{\mathrm{cris}}(\mathrm{D}) = (\mathcal{R}[1/t] \otimes_{\mathcal{R}} \mathrm{D})^{\Gamma}$.

In order to define $\mathrm{D}_{\mathrm{st}}(\mathrm{D})$, we add a variable to $\mathcal{R}$. The power series $\log(\varphi(X)/X^p)$ and $\log(\gamma(X)/X)$ (for $\gamma \in \Gamma$) both converge in $\mathcal{R}$. Let $\log(X)$ be a variable which we adjoin to $\mathcal{R}$, with the Frobenius and the action of Γ extending to $\mathcal{R}[\log(X)]$ by $\varphi(\log(X)) = p\log(X) + \log(\varphi(X)/X^p)$ and $\gamma(\log(X)) = \log(X) + \log(\gamma(X)/X)$. We also define a monodromy map N on $\mathcal{R}[\log(X)]$ by $N = -p/(p-1) \cdot d/d\log(X)$.

Definition 3.3.2. — If D is a (φ, Γ)-module, let $\mathrm{D}_{\mathrm{st}}(\mathrm{D}) = (\mathcal{R}[\log(X), 1/t] \otimes_{\mathcal{R}} \mathrm{D})^{\Gamma}$.

Definitions 3.3.1 and 3.3.2 make sense for any (φ, Γ)-module. We say that D is *crystalline* or *semistable* if $\mathrm{D}_{\mathrm{cris}}(\mathrm{D})$ or $\mathrm{D}_{\mathrm{st}}(\mathrm{D})$ is an E-vector space of dimension $\mathrm{rk}(\mathrm{D})$. The space $\mathrm{D}_{\mathrm{st}}(\mathrm{D})$ is then a (φ, N)-module and $\mathrm{D}_{\mathrm{cris}}(\mathrm{D}) = \mathrm{D}_{\mathrm{st}}(\mathrm{D})^{N=0}$. One can also define a filtration on these two spaces by using the filtration of $\mathcal{R}$ given by "the order of vanishing at $\zeta_{p^n} - 1$ for $n \gg 0$" so that $\mathrm{D}_{\mathrm{st}}(\mathrm{D})$ becomes a filtered (φ, N)-module (which in general will not be admissible). The following result is theorem 0.2 of [**Ber02**].

Theorem 3.3.3. — *If V is a p-adic representation of* $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ *and if* $\mathrm{D}(V)$ *is the attached* (φ, Γ)*-module, then* $\mathrm{D}_{\mathrm{cris}}(V) = \mathrm{D}_{\mathrm{cris}}(\mathrm{D}(V))$ *and* $\mathrm{D}_{\mathrm{st}}(V) = \mathrm{D}_{\mathrm{st}}(\mathrm{D}(V))$.

The proof of this requires a number of delicate computations in several of Fontaine's rings of periods. Recall that $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$ is the ring used in §2.2 in order to attach p-adic representations to (φ, Γ)-modules. One can show that the ring $\mathbf{B}_{\mathrm{cris}}$ of Fontaine admits a subring $\widetilde{\mathbf{B}}^{+}_{\mathrm{rig}}$ such that

1. for any p-adic representation V the inclusion $(\widetilde{\mathbf{B}}^{+}_{\mathrm{rig}}[1/t] \otimes_{\mathbf{Q}_p} V)^{\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)} \subset \mathrm{D}_{\mathrm{cris}}(V)$ is an isomorphism;
2. there is a natural inclusion $\widetilde{\mathbf{B}}^{+}_{\mathrm{rig}} \subset \widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$.

These facts allow one to go from the usual p-adic periods to the theory of (φ, Γ)-modules and then to prove theorem 3.3.3. The spaces $\mathrm{D}_{\mathrm{st}}(V)$ and $\mathrm{D}_{\mathrm{st}}(\mathrm{D}(V))$ are then equal as subspaces of $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}[1/t] \otimes_{\mathbf{Q}_p} V$. It is also possible to define $\mathrm{D}_{\mathrm{dR}}(\mathrm{D})$ as well as de Rham (φ, Γ)-modules in the same way and to prove an analogue of theorem 3.3.3 but this is slightly more complicated and we do not give the recipe here.

If V is a semistable representation and if M is a sub-(φ, N)-module of $\mathrm{D}_{\mathrm{st}}(V)$, then it is easy to see that $(\mathcal{R}[\log(X), 1/t] \otimes_E M)^{N=0} \cap \mathrm{D}(V)$ is a sub (φ, Γ)-module of $\mathrm{D}(V)$ of rank $\dim(M)$. Using this observation and theorem 3.3.3, we get the following result.

Theorem 3.3.4. — *Semistable representations of* $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ *are trianguline.*

We see that the (φ, Γ)-module of a semistable representation may then admit several different triangulations, corresponding to flags of $\mathrm{D}_{\mathrm{st}}(V)$ stable under φ and N. Another consequence of theorem 3.3.3 which is proved in the same way is the following useful result (proposition 4.3 of [**Col08**]).

Theorem 3.3.5. — *If V is a p-adic representation of dimension 2, then V is trianguline if and only if there exists a character η of* $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ *such that* $\mathrm{D}_{\mathrm{cris}}(V(\eta)) \neq 0$.

3.4. Weights of trianguline representations. — Recall that p-adic representations of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ have weights: Sen's theory (§2.2 of [**Sen80**]) allows us to attach to V a polynomial $P(X) \in E[X]$ of degree $\dim(V)$ whose roots are the *generalized Hodge-Tate weights of V* (warning: in [**BC09**] as in other places, the opposite sign is chosen for the weights). For example if V is de Rham, then these weights are the opposites of the jumps of the filtration on $\mathrm{D}_{\mathrm{dR}}(V)$ and are then the classical Hodge-Tate weights of V.

If V is a trianguline representation and if $\{0\} = \mathrm{D}_0 \subset \mathrm{D}_1 \subset \cdots \subset \mathrm{D}_d = \mathrm{D}(V)$ is a triangulation of V, then each $\mathrm{D}_i/\mathrm{D}_{i-1}$ is of rank 1 and hence of the form $\mathcal{R}(\delta_i)$ by theorem 2.1.2. We say that the ordered set of characters $\delta_1, \ldots, \delta_d$ is *attached* to V. If $\delta : \mathbf{Q}_p^\times \to E^\times$ is a character, then $w(\delta) = \log_p \delta(u)/\log_p u$ does not depend on the choice of $u \in 1 + p\mathbf{Z}_p$ and is called the *weight* of δ.

Theorem 3.4.1. — *If V is a trianguline representation and $\delta_1, \ldots, \delta_d$ are the characters attached to V, then $w(\delta_1), \ldots, w(\delta_d)$ are the generalized Hodge-Tate weights of V.*

The following theorem (proposition 2.3.4 of [**BC09**]) can be seen as a generalization of Perrin-Riou's theorem 1.5 of [**PR94**] that "ordinary representations are semistable".

Theorem 3.4.2. — *Let V be a trianguline representation and let $\delta_1, \ldots, \delta_d$ be the characters attached to V. If $w(\delta_1), \ldots, w(\delta_d)$ are integers and if $w(\delta_1) > \cdots > w(\delta_d)$, then V is de Rham.*

3.5. Cohomology of (φ,Γ)-modules. — Since trianguline representations are successive extensions of (φ,Γ)-modules of rank 1, an important part of the study of these representations is the determination of the extension groups of (φ,Γ)-modules.

Let D be a (φ,Γ)-module and let γ be a topological generator of Γ (the group $\mathbf{Z}_p^\times$ is topologically cyclic if $p \neq 2$; if $p = 2$, then the definitions have to be slightly modified). Let $C(\varphi,\gamma)$ be the complex

$$0 \to \mathrm{D} \xrightarrow{z\mapsto((\gamma-1)z,(\varphi-1)z)} \mathrm{D}\oplus\mathrm{D} \xrightarrow{(x,y)\mapsto(\varphi-1)x-(\gamma-1)y} \mathrm{D} \to 0.$$

The E-vector spaces $\mathrm{H}^i(C(\varphi,\gamma))$ do not depend on the choice of γ and we define the cohomology groups of D to be $\mathrm{H}^i(\mathrm{D}) = \mathrm{H}^i(C(\varphi,\gamma))$. Note that by construction $\mathrm{H}^i(\mathrm{D}) = 0$ if $i \geqslant 3$.

The following result (theorems 1.1 and 1.2 and §3.1 of [**Liu08**]) summarizes several properties of the groups $\mathrm{H}^i(\mathrm{D})$.

Theorem 3.5.1. — *If* D *is a* (φ,Γ)*-module, then:*

1. *the* $\mathrm{H}^i(\mathrm{D})$ *are finite dimensional* E*-vector spaces and* $h^1(\mathrm{D})-h^0(\mathrm{D})-h^2(\mathrm{D}) = \mathrm{rk}(\mathrm{D})$*;*
2. $\mathrm{H}^0(\mathrm{D}) = \mathrm{D}^{\Gamma=1,\varphi=1}$ *and* $\mathrm{H}^1(\mathrm{D}) = \mathrm{Ext}^1(\mathcal{R},\mathrm{D})$*;*
3. *if* V *is a* p*-adic representation, then* $\mathrm{H}^i(\mathrm{D}(V)) = \mathrm{H}^i(\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p),V)$*;*

In the special case when D is of rank 1, Colmez has computed explicitely $\mathrm{H}^1(\mathrm{D})$. This way we have the following result (theorem 0.2 of [**Col08**]) which we use in §3.6. Let $x : \mathbf{Q}_p^\times \to E^\times$ be the map $z \mapsto z$ and let $|\cdot|_p : \mathbf{Q}_p^\times \to E^\times$ be the map $z \mapsto p^{-\mathrm{val}_p(z)}$.

Theorem 3.5.2. — *If* δ_1 *and* $\delta_2 : \mathbf{Q}_p^\times \to E^\times$ *are two characters, then* $\mathrm{Ext}^1(\mathcal{R}(\delta_2),\mathcal{R}(\delta_1))$ *is a* 1*-dimensional* E*-vector space, unless* $\delta_1\delta_2^{-1}$ *is either of the form* x^{-i} *with* $i \geqslant 0$ *or* $|x|_p x^i$ *with* $i \geqslant 1$*, in which case* $\mathrm{Ext}^1(\mathcal{R}(\delta_2),\mathcal{R}(\delta_1))$ *is of dimension* 2.

In the first case, there is therefore one nonsplit extension $0 \to \mathcal{R}(\delta_1) \to \mathrm{D} \to \mathcal{R}(\delta_2) \to 0$ while in the second case, the set of such extensions is parameterized by $\mathbf{P}^1(E)$. The parameter for such an extension is called the $\mathcal{L}$*-invariant* and turns out to be a generalization of the usual $\mathcal{L}$-invariant (see [**Col10a**]).

3.6. Trianguline representations of dimension 2. — If $\delta : \mathbf{Q}_p^\times \to E^\times$ is a character, then we set $u(\delta) = \mathrm{val}_p(\delta(p))$ so that $u(\delta)$ is the slope of $\mathcal{R}(\delta)$. Recall that $w(\delta)$ is the weight of δ defined in §3.4.

If V is a trianguline representation of dimension 2, then $\mathrm{D}(V)$ is an extension of two (φ,Γ)-modules of rank 1 so that we have an exact sequence $0 \to \mathcal{R}(\delta_1) \to \mathrm{D}(V) \to \mathcal{R}(\delta_2) \to 0$. The fact that $\mathrm{D}(V)$ is étale implies that $u(\delta_1)+u(\delta_2) = 0$ and (because of theorem 2.4.1) $u(\delta_1) \geqslant 0$. If $u(\delta_1) = u(\delta_2) = 0$, then $\mathcal{R}(\delta_1)$ and $\mathcal{R}(\delta_2)$ are étale and V itself is an extension of two representations.

We denote by $\mathcal{S}$ the space $\mathcal{S} = \{(\delta_1, \delta_2, \mathcal{L})\}$ where $\mathcal{L} = \infty$ if $\delta_1\delta_2^{-1}$ is neither of the form x^{-i} with $i \geqslant 0$, nor of the form $|x|_p x^i$ with $i \geqslant 1$, and $\mathcal{L} \in \mathbf{P}^1(E)$ otherwise. Theorem 3.5.2 above allows us to construct for every $s \in \mathcal{S}$ a nontrivial extension $\mathrm{D}(s)$ of $\mathcal{R}(\delta_2)$ by $\mathcal{R}(\delta_1)$ and vice versa.

If $s \in \mathcal{S}$, then we set $w(s) = w(\delta_1) - w(\delta_2)$. We define $\mathcal{S}_*$ as the set of $s \in \mathcal{S}$ such that $u(\delta_1) + u(\delta_2) = 0$ and $u(\delta_1) > 0$ and we then set $u(s) = u(\delta_1)$ if $s \in \mathcal{S}_*$. We define the "crystalline", "semistable" and "nongeometric" parameter spaces as follows.

1. $\mathcal{S}_*^{\mathrm{cris}} = \{s \in \mathcal{S}_*$ such that $w(s) \geqslant 1$ and $u(s) < w(s)$ and $\mathcal{L} = \infty\}$;
2. $\mathcal{S}_*^{\mathrm{st}} = \{s \in \mathcal{S}_*$ such that $w(s) \geqslant 1$ and $u(s) < w(s)$ and $\mathcal{L} \neq \infty\}$;
3. $\mathcal{S}_*^{\mathrm{ng}} = \{s \in \mathcal{S}_*$ such that $w(s)$ is not an integer $\geqslant 1\}$.

Let $\mathcal{S}_{\mathrm{irr}} = \mathcal{S}_*^{\mathrm{cris}} \sqcup \mathcal{S}_*^{\mathrm{st}} \sqcup \mathcal{S}_*^{\mathrm{ng}}$.

Theorem 3.6.1. — *If $s \in \mathcal{S}_{\mathrm{irr}}$, then $\mathrm{D}(s)$ is étale and the attached representation $V(s)$ is trianguline and irreducible. Every 2-dimensional irreducible trianguline representation is of the form $V(s)$ (after possibly extending scalars) and we have $V(s) = V(s')$ if and only if $s \in \mathcal{S}_*^{\mathrm{cris}}$ and $s' = (x^{w(s)}\delta_2, x^{-w(s)}\delta_1, \infty)$.*

In particular, if $s \in \mathcal{S} \setminus \mathcal{S}_{\mathrm{irr}}$ then either $\mathrm{D}(s)$ is étale but $V(s)$ is reducible or $\mathrm{D}(s)$ is not even étale (this happens for example if $w(s) \geqslant 1$ and $u(s) > w(s)$).

The representation $V(s)$ becomes crystalline (or semistable) on an abelian extension of $\mathbf{Q}_p$ after possibly twisting by a character if $s \in \mathcal{S}_*^{\mathrm{cris}}$ (or if $s \in \mathcal{S}_*^{\mathrm{st}}$), while $V(s)$ is not a twist of a de Rham representation if $s \in \mathcal{S}_*^{\mathrm{ng}}$. In the cases where $V(s)$ is crystalline or semistable, Colmez has explicitly determined in §4.5 and 4.6 of [**Col08**] the filtered φ- and (φ, N)-modules $\mathrm{D}_{\mathrm{cris}}(V(s))$ and $\mathrm{D}_{\mathrm{st}}(V(s))$ in terms of s.

Let us give as an example the description of the parameter s corresponding to the representation $V_p f$ arising from a modular eigenform of level N prime to p, weight k, character ε and coefficient $a_p \in \mathfrak{m}_E$. If $y \in E^\times$, let $\mu_y : \mathbf{Q}_p^\times \to E^\times$ be the character defined by $\mu_y(z) = y^{\mathrm{val}_p(z)}$. Let $x_0 : \mathbf{Q}_p^\times \to E^\times$ be the character defined by $x_0(z) = z|z|_p$ so that $x_0(p) = 1$ and $x_0(z) = z$ if $z \in \mathbf{Z}_p^\times$. The result below then follows from the computations of §4.5 of [**Col08**].

Theorem 3.6.2. — *We have $(V_p f)^* = V(\mu_y, \mu_{\varepsilon(p)/y} x_0^{1-k}, \infty)$ where $y \in \mathfrak{m}_E$ is such that $a_p = y + \varepsilon(p)p^{k-1}/y$.*

The results of this § have been generalized to 2-dimensional trianguline representations of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/K)$ by Nakamura in [**Nak09**].

4. Arithmetic applications

In this chapter, we explain the role that trianguline representations play in the p-adic local Langlands correspondence and then in the theory of overconvergent modular forms.

4.1. The p-adic local Langlands correspondence. — The p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbf{Q}_p)$ is a bijection between certain 2-dimensional p-adic representations of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ and certain representations of $\mathrm{GL}_2(\mathbf{Q}_p)$. The first examples of this correspondence were constructed by Breuil, for semistable and crystalline representations of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$. These examples inspired Colmez to use (φ,Γ)-modules in order to give a "functorial" construction of these examples, and he realized that the natural condition to impose on the p-adic representations which he was considering was that the attached (φ,Γ)-module be an extension of two (φ,Γ)-modules of rank 1. This is what led him to define trianguline representations. In the notations of §3.6, if $s \in \mathcal{S}_{\mathrm{irr}}$ then the representation of $\mathrm{GL}_2(\mathbf{Q}_p)$ corresponding to $V(s)$ by the p-adic local Langlands correspondence is a p-adic unitary Banach space representation $\Pi(s)$ of $\mathrm{GL}_2(\mathbf{Q}_p)$ constructed as follows.

Let $\log_{\mathcal{L}}$ be the logarithm normalised by $\log_{\mathcal{L}}(p) = \mathcal{L}$ (if $\mathcal{L} = \infty$, we set $\log_\infty = \mathrm{val}_p$) and if $s \in \mathcal{S}$, let δ_s be the character $(x|x|_p)^{-1}\delta_1\delta_2^{-1}$. Note that if $s \in \mathcal{S}_{\mathrm{irr}}$ then we can have $\mathcal{L} \neq \infty$ only if δ_s is of the form x^i with $i \geqslant 0$. We can define the notion of a class $\mathcal{C}^u$ function for $u \in \mathbf{R}_{\geqslant 0}$ generalizing the usual case $u \in \mathbf{Z}_{\geqslant 0}$. We denote by $\mathrm{B}(s)$ the space of functions $f : \mathbf{Q}_p \to E$ which are of class $\mathcal{C}^{u(s)}$ and such that $x \mapsto \delta_s(x) f(1/x)$ extends at 0 to a function of class $\mathcal{C}^{u(s)}$. The space $\mathrm{B}(s)$ is then endowed with an action of $\mathrm{GL}_2(\mathbf{Q}_p)$ given by the formula:

$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot f\right](y) = (x|x|_p\delta_1^{-1})(ad-bc) \cdot \delta_s(cy+d) \cdot f\left(\frac{ay+b}{cy+d}\right).$$

The space $M(s)$ is defined by

1. if δ_s is not of the form x^i with $i \geqslant 0$, then $M(s)$ is the space generated by 1 and by the functions $y \mapsto \delta_s(y-a)$ with $a \in \mathbf{Q}_p$;
2. if δ_s is of the form x^i with $i \geqslant 0$, then $M(s)$ is the intersection of $\mathrm{B}(s)$ with the space generated by the functions $y \mapsto \delta_s(y-a)$ and $y \mapsto \delta_s(y-a)\log_{\mathcal{L}}(y-a)$ with $a \in \mathbf{Q}_p$.

We finally set $\Pi(s) = \mathrm{B}(s)/\widehat{M}(s)$ where $\widehat{M}(s)$ is the closure of $M(s)$ inside $\mathrm{B}(s)$.

Theorem 4.1.1. — *The unitary Banach space representation $\Pi(s)$ of $\mathrm{GL}_2(\mathbf{Q}_p)$ is nonzero, topologically irreducible and admissible in the sense of Schneider-Teitelbaum.*

These representations $\Pi(s)$ are called the "unitary principal series" and the above theorem is theorem 0.4 of [**Col10b**]. Colmez then proceeds in [**Col10c**] to attach to any 2-dimensional p-adic representation of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ a representation of $\mathrm{GL}_2(\mathbf{Q}_p)$ and he proves that they have the required properties by using the fact that this is true for trianguline representations, that his construction is suitably continuous, and that trianguline representations are Zariski dense in the space of all 2-dimensional p-adic representations. See [**Ber10a**] for a detailed survey.

4.2. Families of Galois representations. — A character $1 + p\mathbf{Z}_p \to 1 + \mathfrak{m}_E$ is determined by its value at $1 + p$ so that if $\eta : \mathbf{Z}_p^\times \to \mathcal{O}_E^\times$ is a character, then the natural parameter space for characters $\eta' : \mathbf{Z}_p^\times \to \mathcal{O}_E^\times$ which have the same reduction modulo p as η is the rigid analytic space attached to $\mathbf{Q}_p \otimes_{\mathbf{Z}_p} \mathbf{Z}_p[\![X]\!]$ (the rigid analytic space attached to such a ring is the set of its maximal ideals). We call this space the *universal deformation space* of $\overline{\eta}$ and denote it by $\mathscr{X}_{\overline{\eta}}$.

There is also a parameter space $\mathscr{X}_{\overline{\delta}}^u$ for characters $\delta : \mathbf{Q}_p^\times \to E^\times$ which have a fixed slope u and such that $\overline{\delta(p)/p^u}$ and $\overline{\delta} \mid_{\mathbf{Z}_p^\times}$ are fixed, and this parameter space is the rigid analytic space attached to $\mathbf{Q}_p \otimes_{\mathbf{Z}_p} \mathbf{Z}_p[\![X_1, X_2]\!]$. Denote by $\delta(x)$ the character corresponding to a point $x \in \mathscr{X}_{\overline{\delta}}^u$. Colmez proves in §5.1 of [**Col08**] that the representations $V(s)$ live in analytic families of trianguline representations.

Theorem 4.2.1. — *If* $(\delta_1, \delta_2, \infty) \in \mathcal{S}_{\mathrm{irr}}$ *and if* $\delta_1\delta_2^{-1}(p) \notin p^{\mathbf{Z}}$*, then there exists a neighborhood* $\mathscr{U}$ *of* $(\delta_1, \delta_2) \in \mathscr{X}_{\overline{\delta}_1}^{u_1} \times \mathscr{X}_{\overline{\delta}_2}^{u_2}$ *and a free* $\mathcal{O}_{\mathscr{U}}$*-module* V *of rank* 2 *with an action of* $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ *such that* $V(u) = V(\delta_1(u), \delta_2(u), \infty)$ *if* $u \in \mathscr{U}$.

Recall that Mazur generalized the construction of $\mathscr{X}_{\overline{\eta}}$ in [**Maz89**] and proved that for certain groups G and representations $\overline{\rho} : G \to \mathrm{GL}_d(\overline{\mathbf{F}}_p)$, there exists a universal deformation space $\mathscr{X}_{\overline{\rho}}$ which is a parameter space for all representations $\rho : G \to \mathrm{GL}_d(\overline{\mathbf{Z}}_p)$ having reduction modulo $\mathfrak{m}_{\overline{\mathbf{Z}}_p}$ isomorphic to $\overline{\rho}$. This applies for example if $\mathrm{End}(\overline{\rho}) = \overline{\mathbf{F}}_p$ and if either $G = \mathrm{Gal}(\mathbf{Q}_S/\mathbf{Q})$ is the Galois group of the maximal extension of $\mathbf{Q}$ which is unramified outside of a finite set of places S or if $G = \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$.

In the case when $G = \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ and $d = 2$, the corresponding space $\mathscr{X}_{\overline{\rho}}$ is usually the rigid analytic space attached to $\mathbf{Q}_p \otimes_{\mathbf{Z}_p} \mathbf{Z}_p[\![X_1, X_2, X_3, X_4, X_5]\!]$. Theorem 4.2.1 then shows that inside the 5-dimensional space $\mathscr{X}_{\overline{\rho}}$ there is a countable number (one for each slope) of 4-dimensional subspaces corresponding to trianguline representations. In particular, the "trianguline locus" of $\mathscr{X}_{\overline{\rho}}$ is Zariski dense (it is however a "thin subset" of $\mathscr{X}_{\overline{\rho}}$ in the terminology of §4 of [**BC10**]). This can be compared with the following result (theorems B and C of [**BC08**]).

Theorem 4.2.2. — *If $b \geqslant a$, then the locus of $\mathscr{X}_{\overline{\rho}}$ corresponding to crystalline (or semistable or de Rham or Hodge-Tate) representations with Hodge-Tate weights in the range $[a, b]$ is a closed subspace of $\mathscr{X}_{\overline{\rho}}$.*

4.3. Overconvergent modular forms. — Overconvergent modular forms are objects defined by Coleman in [**Col96**] which are p-adic generalizations of classical modular forms. We do not define them in this survey because we don't really need to (for a survey about overconvergent modular forms, see [**Eme09**]). Suffice to say that an overconvergent modular form has a q-expansion which is a p-adic limit of q-expansions of classical modular forms and that one can attach Galois representations to them. In fact in this § we directly define some p-adic representations of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ by a p-adic interpolation process and merely recall that these representations are the ones which are attached to "overconvergent modular eigenforms of finite slope".

Let $N \geqslant 1$ be an integer prime to p and let S be the set of primes dividing pN and ∞. Fix some 2-dimensional $\overline{\mathbf{F}}_p$-representation $\overline{\rho}$ of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. Let $\mathscr{X}_{\overline{\rho}}^S$ be the universal deformation space for representations ρ of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ whose reduction is $\overline{\rho}$ and which are unramified outside of S so that every $x \in \mathscr{X}_{\overline{\rho}}^S(E)$ corresponds to an E-linear representation V_x of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ which is unramified outside of S. Note that $\mathscr{X}_{\overline{\rho}}^S$ is usually a 3-dimensional rigid analytic ball by results of Weston (see theorem 1 of [**Wes04**]).

Let $\mathcal{C}_{\mathrm{cl}}$ be the set of points $(x, \lambda) \in \mathscr{X}_{\overline{\rho}}^S \times \mathbf{G}_{\mathrm{m}}$ such that V_x is the representation attached to a modular eigenform f on $\Gamma_1(Np^r)$ for some $r \geqslant 1$ with $U_p(f) = \lambda f$, and let $\mathcal{C}$ be the Zariski closure of $\mathcal{C}_{\mathrm{cl}}$ inside $\mathscr{X}_{\overline{\rho}}^S \times \mathbf{G}_{\mathrm{m}}$. By §1.5 of [**CM98**], we have the following result.

Theorem 4.3.1. — *The variety $\mathcal{C}$ is a rigid analytic curve.*

Coleman and Mazur then show in [**CM98**] that the Galois representations V_x corresponding to points $(x, \lambda) \in \mathcal{C}(E)$ are the ones which are attached to the "overconvergent modular eigenforms of finite slope" defined by Coleman. The curve $\mathcal{C}$ is called the *eigencurve*. The projection of $\mathcal{C}$ on $\mathscr{X}_{\overline{\rho}}^S$ is then a complicated space (for instance, it has infinitely many double points) which is the "infinite fern" of [**Maz97**] and [**GM98**], see §2.5 of [**Eme09**]. The following result (a consequence of theorem 6.3 of [**Kis03**] combined with theorem 3.3.5) describes the restriction to $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ of the representations of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ which are constructed in this way.

Theorem 4.3.2. — *If f is an overconvergent modular eigenform of finite slope of level N (i.e. if $(V_p f, \lambda) \in \mathcal{C}(E)$ by the above remark), then $V_p f$ is a trianguline representation.*

The idea is that this theorem is true if f is a classical modular eigenform by using theorem 3.2.2, and Kisin deduces theorem 4.3.2 from the classical case by a p-adic interpolation argument using theorem 3.3.5. We then have the following converse to theorem 4.3.2, Emerton's generalization of the Fontaine-Mazur conjecture.

Theorem 4.3.3. — *If V is an irreducible 2-dimensional p-adic representation of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ such that*

1. *V is unramified at almost every prime,*
2. *V is trianguline at p,*
3. *$\overline{V}$ satisfies hypotheses (1), (2) and (3') of §3.2,*

then V is a twist of the Galois representation attached to an overconvergent cuspidal eigenform of finite slope.

We now describe the parameter $s \in \mathcal{S}$ such that $(V_p f)^* = V(s)$ just as we did for classical modular forms at the end of §3.6. Let f be a finite slope overconvergent modular eigenform of level N and character ε. Let $k = w(\det(V_p f)) + 1$ (so that if f is classical, then k is the weight of f) and let $\lambda \in E$ be such that $U_p(f) = \lambda f$. The following result is proposition 5.2 of [**Che08**], where $\mu_\lambda : \mathbf{Q}_p^\times \to E^\times$ is the character $z \mapsto \lambda^{\mathrm{val}_p(z)}$.

Theorem 4.3.4. — *If $k \geqslant 1$ and either $\mathrm{val}_p(\lambda) = 0$ or $\mathrm{val}_p(\lambda) = k-1$, then $V_p f$ is reducible and otherwise $V_p f$ is irreducible and $(V_p f)^* = V(\delta_1, \det(V_p f)^{-1} \cdot \delta_1^{-1}, \mathcal{L})$ where*

1. *if $k \geqslant 1$ and $0 < \mathrm{val}_p(\lambda) < k-1$, then $\delta_1 = \mu_\lambda$;*
2. *if $k \geqslant 1$ and $\mathrm{val}_p(\lambda) > k-1$, then $\delta_1 = x^{1-k}\mu_\lambda$;*
3. *if k is not an integer $\geqslant 1$, then $\delta_1 = \mu_\lambda$.*

Note that case (1) corresponds to $\mathcal{S}_*^{\mathrm{cris}} \sqcup \mathcal{S}_*^{\mathrm{st}}$ while cases (2) and (3) correspond to $\mathcal{S}_*^{\mathrm{ng}}$. Coleman's "small slope criterion" for the classicality of overconvergent modular eigenforms (§6 of [**Col96**]) can then be interpreted as follows in terms of Galois representations: if $k \geqslant 1$ and $0 < \mathrm{val}_p(\lambda) < k-1$, then the representation $V_p f$ is potentially semistable at p and therefore the overconvergent modular form f is classical by the Fontaine-Mazur conjecture (theorem 3.2.5).

We finish this § by discussing the weight of overconvergent cuspidal eigenforms of finite slope. Let $\mathscr{W}$ be the *weight space*, that is the parameter space for characters of $\mathbf{Z}_p^\times$. The space $\mathscr{W}$ is the union of the $p-1$ balls $\mathscr{X}_{\omega^i}$ where $0 \leqslant i \leqslant p-2$. If V is a p-adic representation of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$, then by class field theory $\det(V)$ gives a character of $\mathbf{Q}_p^\times$ whose restriction to $\mathbf{Z}_p^\times$ is the *weight* κ_V of V (this definition is more precise than the one given in §3.4). This gives rise to a map $\kappa : \mathscr{X}_{\overline{\rho}}^S \to \mathscr{W}$ and by composition to a map $\mathcal{C} \to \mathscr{W}$ which satisfies the following property by §1.5 of [**CM98**].

Theorem 4.3.5. — *The map $\mathcal{C} \to \mathscr{W}$ is locally finite and flat.*

We now explain that if $N = 1$ and $p \in \{2, 3, 5, 7\}$, then one can give a "local" realization of the eigencurve. A point $(\kappa, \lambda) \in \mathscr{W} \times \mathbf{G}_{\mathrm{m}}$ is said to be *special* if $\kappa = x^k$ for some $k \geqslant 2$ and $\lambda^2 = p^{k-2}$. Let $\mathscr{W} \widetilde{\times} \mathbf{G}_{\mathrm{m}}$ be the blow-up of $\mathscr{W} \times \mathbf{G}_{\mathrm{m}}$ at the special points. Consider the map $\mathcal{C} \to \mathscr{W} \widetilde{\times} \mathbf{G}_{\mathrm{m}}$ given by $(V_x, \lambda) \mapsto (\kappa_x, \lambda, \mathcal{L}_x)$ at the special points and by $(V_x, \lambda) \mapsto (\kappa_x, \lambda)$ elsewhere. The following theorem is the main result of [**Che08**].

Theorem 4.3.6. — *The map $\mathcal{C} \to \mathscr{W} \widetilde{\times} \mathbf{G}_{\mathrm{m}}$ is a rigid analytic map and if $N = 1$ and $p \in \{2, 3, 5, 7\}$, then it is a closed immersion.*

The main ideas underlying this theorem are Colmez' theorem 0.5 of [**Col10a**] expressing the $\mathcal{L}$-invariant as the derivative of the U_p-eigenvalue and the fact that if $p \in \{2, 3, 5, 7\}$ and $S = \{p, \infty\}$, then an odd 2-dimensional p-adic representation of $\mathrm{Gal}(\overline{\mathbf{Q}}_S/\mathbf{Q})$ is determined by its restriction to $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ (proposition 1.8 of [**Che08**]).

4.4. Trianguline representations and Selmer groups. — Since the (φ, Γ)-module attached to a trianguline representation V has a particularly easy structure, one can use this structure to study the cohomology groups attached to V, in particular the Selmer group and its variants. Some of the techniques which are available in the ordinary case for that study (such as [**Gre89**]) can be extended to the case of trianguline representations.

For example, it is possible to give a generalized definition of the usual $\mathcal{L}$-invariant (see Benois' [**Ben09**]), and to study the Selmer groups corresponding to families of trianguline representations such as those carried by the eigencurve or more general eigenvarieties (as in the book [**BC09**] by Bellaïche and Chenevier and in Pottharst's [**Pot08**] and [**Pot10**]). In this way, it is possible to prove some new cases of the Bloch-Kato conjectures by establishing some "lower semicontinuity" results about the rank of the Selmer groups (see [**BC09**] and Bellaïche's [**Bel10**]).

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