Preprint, arXiv:1011.3487

SOME NEW SUPER-CONGRUENCES MODULO PRIME POWERS

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Abstract. Let $p>3$ be a prime. We show that

$$\sum_{k=0}^{p-1}\binom{-1/(p+1)}{k}^{p+1}\equiv 0\ (\mathrm{mod}\ p^5)\ \text{ and }\ \sum_{k=0}^{p-1}\binom{1/(p-1)}{k}^{p-1}\equiv 0\ (\mathrm{mod}\ p^4).$$

For any positive integer $m\not\equiv 0 \pmod p$, we prove that

$$\sum_{k=0}^{p-1}(-1)^{km}\binom{p/m-1}{k}^m\equiv 0 \pmod{p^4},$$

and

$$\sum_{k=1}^{p-1}\frac{(-1)^{km}}{k^2}\binom{p/m-1}{k}^m\equiv\frac{1}{p}\sum_{k=1}^{p-1}\frac{1}{k} \pmod{p^3}\quad \text{if } p>5.$$

The paper also contains some open conjectures.

1. Introduction

A p-adic congruence (with p a prime) is called a *super-congruence* if it happens to hold modulo higher powers of p. Here is a classical example due to J. Wolstenholme (cf. [W] or [HT]):

$$\sum_{k=1}^{p-1}\frac{1}{k}\equiv 0 \pmod{p^2}\ \text{ and }\ \binom{2p-1}{p-1}\equiv 1 \pmod{p^3}$$

2010 *Mathematics Subject Classification.* Primary 11B65; Secondary 05A10, 11A07, 11B68.

Keywords. Super-congruences, binomial coefficients, Bernoulli numbers.

Supported by the National Natural Science Foundation (grant 10871087) and the Overseas Cooperation Fund (grant 10928101) of China.

for any prime $p > 3$. The reader may consult [Su1] for some super-congruences modulo squares of primes.

In this paper we obtain some new super congruences modulo prime powers motivated by the well-known formula

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$$

Now we state our main results.

Theorem 1.1. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\binom{-1/(p+1)}{k}^{p+1} \equiv 0 \pmod{p^5}. \tag{1.1}$$

Theorem 1.2. *Let $p > 3$ be a prime and let m be a positive integer not divisible by p. Then we have*

$$\sum_{k=0}^{p-1}(-1)^{km}\binom{p/m-1}{k}^m \equiv 0 \pmod{p^4}. \tag{1.2}$$

In particular,

$$\sum_{k=0}^{p-1}\binom{1/(p-1)}{k}^{p-1} \equiv 0 \pmod{p^4}. \tag{1.3}$$

Remark 1.1. We conjecture that there are no composite numbers p satisfying (1.1) or (1.3). We also note that (1.1) and (1.3) can be refined as follows:

$$\sum_{k=0}^{p-1}\binom{-1/(p+1)}{k}^{p+1} \equiv \frac{p^5}{18}B_{p-3} \pmod{p^6} \quad \text{for any prime } p > 5, \tag{1.4}$$

and

$$\sum_{k=0}^{p-1}\binom{1/(p-1)}{k}^{p-1} \equiv \frac{2}{3}p^4 B_{p-3} \pmod{p^5} \quad \text{for any prime } p > 3, \tag{1.5}$$

where $B_0, B_1, B_2, \ldots$ are Bernoulli numbers (see [IR, pp. 228–241] for an introduction to Bernoulli numbers). However, the proofs of (1.4) and (1.5) are too complicated.

Theorem 1.3. *Let $p > 5$ be a prime and let m be a positive integer not divisible by p. Then*

$$\sum_{k=1}^{p-1} \frac{(-1)^{km}}{k^2} \binom{p/m-1}{k}^m \equiv \frac{1}{p} \sum_{k=1}^{p-1} \frac{1}{k} \pmod{p^3}, \tag{1.6}$$

and

$$\sum_{k=1}^{p-1} \frac{(-1)^{km}}{k^n} \binom{p/m-1}{k}^m \equiv -\frac{p}{n+1} B_{p-1-n} \pmod{p^2} \tag{1.7}$$

for all $n = 1, \ldots, p-3$.

Remark 1.2. We observe that if n is a positive integer and $p > 2n+1$ is a prime then

$$\sum_{k=1}^{p-1} \frac{1}{k^{2n-1}} \binom{1/(p-1)}{k}^{p-1} \equiv -\frac{2p^2 n^2}{2n+1} B_{p-1-2n} \pmod{p^3}$$

and

$$\sum_{k=1}^{p-1} \frac{1}{k^{2n-1}} \binom{-1/(p+1)}{k}^{p+1} \equiv \frac{p^2 n}{2n+1} B_{p-1-2n} \pmod{p^3}.$$

For a prime p and a p-adic number x, as usual we let $\nu_p(x)$ denote the p-adic valuation (i.e., p-adic order) of x.

Conjecture 1.1. *Let p be a prime and let n be a positive integer. Then*

$$\nu_p\left(\sum_{k=0}^{n-1} \binom{-1/(p+1)}{k}^{p+1} \right) \geqslant c_p \left\lfloor \frac{\nu_p(n)+1}{2} \right\rfloor,$$

where

$$c_p = \begin{cases} 1 & \text{if } p = 2, \\ 3 & \text{if } p = 3, \\ 5 & \text{if } p \geqslant 5. \end{cases}$$

If $p > 3$ then

$$\nu_p\left(\sum_{k=0}^{n-1} \binom{1/(p-1)}{k}^{p-1} \right) \geqslant 4 \left\lfloor \frac{\nu_p(n)+1}{2} \right\rfloor.$$

Now we raise one more conjecture.

Conjecture 1.2. *Let $m \geqslant 2$ and r be integers. And let $p > r$ be an odd prime not dividing m.*

(i) *If $m > 2$, $m \not\equiv r \pmod 2$, and $p \equiv r \pmod m$ with $r \geqslant -m/2$, then*

$$\sum_{k=0}^{p-1}(-1)^{km}\binom{r/m}{k}^m \equiv 0 \pmod{p^3}. \tag{1.8}$$

(ii) *If $p \equiv r \pmod{2m}$ with $r \geqslant -m$, then*

$$\sum_{k=0}^{p-1}(-1)^k\binom{r/m}{k}^{2n+1} \equiv 0 \pmod{p^2} \quad \text{for all } n = 1, \ldots, m-1. \tag{1.9}$$

Remark 1.3. The congruences in (1.8) and (1.9) modulo p are easy.

Theorems 1.1-1.2 and Theorem 1.3 will be proved in Sections 2 and 3 respectively.

2. Proofs of Theorems 1.1 and 1.2

For $m = 1, 2, 3, \ldots$ and $n = 0, 1, 2, \ldots$, we define

$$H_n^{(m)} := \sum_{0<k\leqslant n}\frac{1}{k^m}$$

and call it a harmonic number of order m. Those $H_n = H_n^{(1)}$ ($n = 0, 1, 2, \ldots$) are usually called *harmonic numbers*.

Lemma 2.1. *Let $p > 3$ be a prime. Then*

$$H_{p-1} \equiv -\frac{p^2}{3}B_{p-3} \pmod{p^3}, \quad H_{p-1}^{(2)} \equiv \frac{2}{3}pB_{p-3} \pmod{p^2}, \tag{2.1}$$

and

$$\sum_{1\leqslant i_1<i_2<\cdots<i_k\leqslant p-1}\frac{1}{i_1 i_2\cdots i_k} \equiv \frac{(-1)^{k-1}}{k+1}pB_{p-1-k} \tag{2.2}$$

for all $k = 1, 2, \ldots, p-1$.

Remark 2.1. The results in Lemma 2.1 are known, see Theorem 5.1, Corollary 5.1 and the proof of Theorem 6.1 of Z. H. Sun [S].

Lemma 2.2. *Let $p > 3$ be a prime. Then*

$$\sum_{k=1}^{p-1} H_k \equiv -\frac{p^3}{3}B_{p-3} - p + 1 \pmod{p^4}, \tag{2.3}$$

and

$$\sum_{k=1}^{p-1} H_k^{(2)} \equiv 0 \pmod{p^2} \quad \text{and} \quad \sum_{k=1}^{p-1} H_k^{(3)} \equiv 0 \pmod{p}. \tag{2.4}$$

Proof. For any positive integer m we have

$$\sum_{k=1}^{p-1} H_k^{(m)} = \sum_{k=1}^{p-1}\sum_{j=1}^{k} \frac{1}{j^m} = \sum_{j=1}^{p-1} \frac{\sum_{k=j}^{p-1} 1}{j^m} = \sum_{j=1}^{p-1} \frac{p-j}{j^m}.$$

Thus

$$\sum_{k=1}^{p-1} H_k = pH_{p-1} - p + 1, \ \sum_{k=1}^{p-1} H_k^{(2)} = pH_{p-1}^{(2)} - H_{p-1}, \ \sum_{k=1}^{p-1} H_k^{(3)} = pH_{p-1}^{(3)} - H_{p-1}^{(2)}.$$

Combining these with (2.1), we immediately get (2.3) and (2.4). □

Lemma 2.3. *Let $p > 3$ be a prime. Then*

$$\sum_{k=1}^{p-1} \sum_{1\leqslant i<j\leqslant k} \frac{1}{ij} \equiv -\frac{2}{3}p^2 B_{p-3} + p - 1 \pmod{p^3}, \tag{2.5}$$

$$\sum_{k=1}^{p-1} \sum_{1\leqslant i_1<i_2<i_3\leqslant k} \frac{1}{i_1 i_2 i_3} \equiv -\frac{p}{3}B_{p-3} - p + 1 \pmod{p^2}, \tag{2.6}$$

and

$$\sum_{k=1}^{p-1} \sum_{1\leqslant i_1<i_2<i_3<i_4\leqslant k} \frac{1}{i_1 i_2 i_3 i_4} \equiv -1 \pmod{p}. \tag{2.7}$$

We also have

$$\sum_{k=1}^{p-1} \sum_{1\leqslant i<j\leqslant k} \left(\frac{1}{ij^2} + \frac{1}{i^2 j}\right) \equiv 0 \pmod{p}. \tag{2.8}$$

Proof. For $s=2,\dots,p-1$ it is clear that

$$\begin{aligned}&\sum_{k=1}^{p-1}\sum_{1\leqslant i_1<i_2<\cdots<i_s\leqslant k}\frac{1}{i_1i_2\cdots i_s}\\
=&\sum_{1\leqslant i_1<i_2<\cdots<i_s\leqslant p-1}\frac{\sum_{k=i_s}^{p-1}1}{i_1i_2\cdots i_s}=\sum_{1\leqslant i_1<i_2<\cdots<i_s\leqslant p-1}\frac{p-i_s}{i_1i_2\cdots i_s}\\
=&p\sum_{1\leqslant i_1<i_2<\cdots<i_s\leqslant p-1}\frac{1}{i_1i_2\cdots i_s}-\sum_{1\leqslant i_1<i_2<\cdots<i_{s-1}\leqslant p-1}\frac{\sum_{i_{s-1}<i_s<p}1}{i_1i_2\cdots i_{s-1}}\\
=&p\sum_{1\leqslant i_1<i_2<\cdots<i_s\leqslant p-1}\frac{1}{i_1i_2\cdots i_s}-\sum_{1\leqslant i_1<i_2<\cdots<i_{s-1}\leqslant p-1}\frac{p-1-i_{s-1}}{i_1i_2\cdots i_{s-1}}.\end{aligned}$$

Thus, with the help of Lemmas 2.1, we have

$$\begin{aligned}\sum_{k=1}^{p-1}\sum_{1\leqslant i<j\leqslant k}\frac{1}{ij}=&p\sum_{1\leqslant i<j\leqslant p-1}\frac{1}{ij}-\sum_{i=1}^{p-1}\frac{p-1-i}{i}\\
\equiv&-\frac{p^2}{3}B_{p-3}-(p-1)H_{p-1}+p-1\\
\equiv&-\frac{2}{3}p^2B_{p-3}+p-1\pmod{p^3}.\end{aligned}$$

Also,

$$\begin{aligned}&\sum_{k=1}^{p-1}\sum_{1\leqslant i_1<i_2<i_3\leqslant k}\frac{1}{i_1i_2i_3}\\
=&p\sum_{1\leqslant i_1<i_2<i_3\leqslant p-1}\frac{1}{i_1i_2i_3}-\sum_{1\leqslant i_1<i_2\leqslant p-1}\frac{p-1-i_2}{i_1i_2}\\
\equiv&-(p-1)\sum_{1\leqslant i_1<i_2\leqslant p-1}\frac{1}{i_1i_2}+\sum_{i_1=1}^{p-1}\frac{\sum_{i_1<i_2<p}1}{i_1}\\
\equiv&(p-1)\frac{p}{3}B_{p-3}+\sum_{i=1}^{p-1}\frac{p-1-i}{i}\equiv-\frac{p}{3}B_{p-3}-p+1\pmod{p^2}\end{aligned}$$

and

$$\begin{aligned}
&\sum_{k=1}^{p-1}\sum_{1\leqslant i_1<i_2<i_3<i_4\leqslant k}\frac{1}{i_1i_2i_3i_4}\\
=&p\sum_{1\leqslant i_1<i_2<i_3<i_4\leqslant p-1}\frac{1}{i_1i_2i_3i_4}-\sum_{1\leqslant i_1<i_2<i_3\leqslant p-1}\frac{p-1-i_3}{i_1i_2i_3}\\
\equiv&\sum_{1\leqslant i_1<i_2<i_3\leqslant p-1}\frac{1+i_3}{i_1i_2i_3}\equiv\sum_{1\leqslant i_1<i_2\leqslant p-1}\frac{\sum_{i_2<i_3<p}1}{i_1i_2}\\
\equiv&\sum_{1\leqslant i<j\leqslant p-1}\frac{p-1-j}{ij}\equiv-\sum_{i=1}^{p-1}\frac{\sum_{i<j<p}1}{i}=-\sum_{i=1}^{p-1}\frac{p-1-i}{i}\equiv-1\ (\mathrm{mod}\ p).
\end{aligned}$$

Finally, we note that

$$\begin{aligned}
&\sum_{k=1}^{p-1}\sum_{1\leqslant i<j\leqslant k}\left(\frac{1}{ij^2}+\frac{1}{i^2j}\right)=\sum_{1\leqslant i<j\leqslant p-1}\left(\frac{p-j}{ij^2}+\frac{p-j}{i^2j}\right)\\
\equiv&-\sum_{1\leqslant i<j\leqslant p-1}\left(\frac{1}{ij}+\frac{1}{i^2}\right)\equiv-\sum_{i=1}^{p-1}\frac{\sum_{i<j<p}1}{i^2}\\
=&-\sum_{i=1}^{p-1}\frac{p-1-i}{i^2}=H_{p-1}-(p-1)H_{p-1}^{(2)}\equiv0\ (\mathrm{mod}\ p)
\end{aligned}$$

with the help of (2.1).

So far we have proved (2.5)-(2.8). □

Proof of Theorem 1.1. For each $k=1,\ldots,p-1$, clearly

$$\begin{aligned}
&\binom{-1/(p+1)}{k}^{p+1}=\binom{p/(p+1)-1}{k}^{p+1}=\prod_{j=1}^{k}\left(1-\frac{p}{(p+1)j}\right)^{p+1}\\
\equiv&\prod_{j=1}^{k}\left(1-\frac{(p+1)p}{(p+1)j}+\frac{(p+1)p}{2}\cdot\frac{p^2}{(p+1)^2j^2}-\frac{(p+1)p(p-1)}{3!}\cdot\frac{p^3}{(p+1)^3j^3}\right)\\
=&\prod_{j=1}^{k}\left(1-\frac{p}{j}+\frac{p^3}{2(p+1)j^2}-\frac{p^4(p-1)}{6(p+1)^2j^2}\right)\\
\equiv&\prod_{j=1}^{k}\left(1-\frac{p}{j}+\frac{p^3(1-p)}{2j^2}+\frac{p^4}{6j^3}\right)\ (\mathrm{mod}\ p^5)
\end{aligned}$$

and hence

$$\begin{aligned}\binom{-1/(p+1)}{k}^{p+1} \equiv & 1-pH_k+\frac{p^3(1-p)}{2}H_k^{(2)}+\frac{p^4}{6}H_k^{(3)}+p^2\sum_{1\leqslant i<j\leqslant k}\frac{1}{ij} \\ & -p^3\sum_{1\leqslant i_1<i_2<i_3\leqslant k}\frac{1}{i_1i_2i_3}+p^4\sum_{1\leqslant i_1<i_2<i_3<i_4\leqslant k}\frac{1}{i_1i_2i_3i_4} \\ & -\frac{p^4}{2}\sum_{1\leqslant i<j\leqslant k}\left(\frac{1}{ij^2}+\frac{1}{i^2j}\right) \pmod{p^5}.\end{aligned}$$

Thus, in view of Lemmas 2.1-2.3, we obtain

$$\begin{aligned}&\sum_{k=1}^{p-1}\binom{-1/(p+1)}{k}^{p+1} \\ \equiv & p-1-p\left(-\frac{p^3}{3}B_{p-3}-p+1\right)+p^2\left(-\frac{p}{3}B_{p-3}-p+1\right) \\ & -p^3\left(-\frac{p}{3}B_{p-3}-p+1\right)-p^4 \\ \equiv & -1 \pmod{p^5}\end{aligned}$$

and hence (1.1) follows.

The proof of Theorem 1.1 is now complete. □

Proof of Theorem 1.2. For each $k\in\{1,\ldots,p-1\}$, we have

$$\begin{aligned}(-1)^{km}\binom{p/m-1}{k}^m = & \prod_{j=1}^k\left(1-\frac{p}{jm}\right)^m \\ \equiv & \prod_{j=1}^k\left(1-\frac{pm}{jm}+\frac{m(m-1)}{2}\cdot\frac{p^2}{j^2m^2}-\binom{m}{3}\frac{p^3}{j^3m^3}\right) \\ \equiv & \prod_{j=1}^k\left(1-\frac{p}{j}+\frac{m-1}{2m}\cdot\frac{p^2}{j^2}-\frac{(m-1)(m-2)}{6m^2}\cdot\frac{p^3}{j^3}\right) \\ \equiv & 1-pH_k+\frac{m-1}{2m}p^2H_k^{(2)}-\frac{(m-1)(m-2)}{6m^2}p^3H_k^{(3)} \\ & +p^2\sum_{1\leqslant i<j\leqslant k}\frac{1}{ij}-p^3\sum_{1\leqslant i_1<i_2<i_3\leqslant k}\frac{1}{i_1i_2i_3} \\ & -\frac{m-1}{2m}p^3\sum_{1\leqslant i<j\leqslant k}\left(\frac{1}{ij^2}+\frac{1}{i^2j}\right) \pmod{p^4}.\end{aligned}$$

Therefore, applying Lemmas 2.2 and 2.3 we get

$$\sum_{k=1}^{p-1}(-1)^{km}\binom{p/m-1}{k}^m \equiv p-1-p(-p+1)+p^2(p-1)-p^3=-1 \pmod{p^4}.$$

This proves (1.2). Clearly (1.2) in the case $m = p-1$ yields (1.3). This concludes the proof. □

3. Proof of Theorem 1.3

Lemma 3.1. *Let $p > 3$ be a prime and let $n \in \{1, \ldots, p-3\}$.*

$$\sum_{k=1}^{p-1} \frac{H_k}{k^n} \equiv B_{p-1-n} \pmod{p}. \tag{3.1}$$

Proof. This is easy as mentioned in [Su2, Remark 1.1]. In fact, since $\sum_{k=1}^{p-1} k^m \equiv 0 \pmod{p}$ for any integer $m \not\equiv 0 \pmod{p-1}$ (see, e.g., [IR, p. 235]), we have

$$\begin{aligned}\sum_{k=1}^{p-1} \frac{1}{k^n} \sum_{j=0}^{k} j^{p-2} &= \sum_{k=1}^{p-1} \left(k^{p-2-n} + \frac{1}{k^n(p-1)} \sum_{i=0}^{p-2} \binom{p-1}{i} B_i k^{p-1-i} \right) \\ &= \sum_{k=1}^{p-1} k^{p-2-n} + \frac{1}{p-1} \sum_{i=0}^{p-2} \binom{p-1}{i} B_i \sum_{k=1}^{p-1} k^{p-1-n-i} \\ &\equiv \binom{p-1}{p-1-n} B_{p-1-n} \equiv (-1)^n B_{p-1-n} = B_{p-1-n} \pmod{p}\end{aligned}$$

and hence (3.1) follows. □

Lemma 3.2. *Let $p > 5$ be a prime. Then*

$$\sum_{k=1}^{p-1} \frac{1 - pH_k}{k^2} \equiv \frac{H_{p-1}}{p} \pmod{p^3}. \tag{3.2}$$

Proof. In view of Theorem 5.1(a) and Remark 5.1 of [S],

$$\frac{H_{p-1}^{(2)}}{2} \equiv p \left(\frac{B_{2p-4}}{2p-4} - 2\frac{B_{p-3}}{p-3} \right) \equiv -\frac{H_{p-1}}{p} \pmod{p^3}$$

and $H_{p-1}^{(3)} \equiv 0 \pmod{p^2}$. Also,

$$\sum_{k=1}^{p-1} \frac{H_{k-1}}{k^2} = \sum_{1 \leqslant j < k \leqslant p-1} \frac{1}{jk^2} \equiv -3\frac{H_{p-1}}{p^2} \pmod{p^2}$$

by [T, Theorem 2.3]. So we have

$$\sum_{k=1}^{p-1} \frac{1 - pH_k}{k^2} = H_{p-1}^{(2)} - pH_{p-1}^{(3)} - p\sum_{k=1}^{p-1} \frac{H_{k-1}}{k^2} \equiv \frac{H_{p-1}}{p} \pmod{p^3}.$$

This concludes the proof. □

Proof of Theorem 1.3. Let $k\in\{1,\ldots,p-1\}$. By the proof of Theorem 1.2,

$$(-1)^{km}\binom{p/m-1}{k}^m\equiv 1-pH_k+\frac{m-1}{2m}p^2H_k^{(2)}+p^2\sum_{1\leqslant i<j\leqslant k}\frac{1}{ij}\pmod{p^3}.$$

Thus, for any given $n\in\{1,\ldots,p-3\}$ we have

$$\begin{aligned}&\sum_{k=1}^{p-1}\frac{(-1)^{km}}{k^n}\binom{p/m-1}{k}^m\\ \equiv&\sum_{k=1}^{p-1}\frac{1-pH_k}{k^n}+\frac{m-1}{2m}p^2\sum_{k=1}^{p-1}\frac{H_k^{(2)}}{k^n}+\frac{p^2}{2}\sum_{k=1}^{p-1}\frac{H_k^2-H_k^{(2)}}{k^n}\pmod{p^3}.\end{aligned}$$

If n is even, then

$$\begin{aligned}\sum_{k=1}^{p-1}\frac{H_k^{(2)}}{k^n}=&\sum_{k=1}^{p-1}\frac{H_{p-k}^{(2)}}{(p-k)^n}\\ \equiv&\sum_{k=1}^{p-1}\frac{H_{p-1}^{(2)}-H_{k-1}^{(2)}}{k^n}\equiv-\sum_{k=1}^{p-1}\frac{H_k^{(2)}}{k^n}+\sum_{k=1}^{p-1}\frac{1}{k^{n+2}}\pmod{p}\end{aligned}$$

and hence $\sum_{k=1}^{p-1}H_k^{(2)}/k^n\equiv 0\pmod{p}$ since $n+2<p-1$. Thus,

$$\sum_{k=1}^{p-1}\frac{(-1)^{km}}{k^n}\binom{p/m-1}{k}^m\equiv\sum_{k=1}^{p-1}\frac{1-pH_k}{k^n}\pmod{p^2};\tag{3.3}$$

and

$$\sum_{k=1}^{p-1}\frac{(-1)^{km}}{k^n}\binom{p/m-1}{k}^m\equiv\sum_{k=1}^{p-1}\frac{1-pH_k}{k^n}+\frac{p^2}{2}\sum_{k=1}^{p-1}\frac{H_k^2}{k^n}\pmod{p^3}\tag{3.4}$$

if n is even.

Fix $n\in\{1,\ldots,p-3\}$. It is known that

$$\sum_{k=1}^{p-1}\frac{1}{k^n}\equiv\frac{pn}{n+1}B_{p-1-n}\pmod{p^2}$$

(see, e.g., [S, Corollary 5.1]). Combining this with (3.1) and (3.3) we get (1.7).

(3.4) in the case $n=2$, together with (3.2) and the congruence

$$\sum_{k=1}^{p-1}\frac{H_k^2}{k^2}\equiv 0\pmod{p}$$

(cf. [Su2, (1.5)]), yields (1.6).

The proof of Theorem 1.3 is now complete. □

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