Dynamics of a rational multi-parameter second order difference equation with cubic numerator and quadratic monomial denominator

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Abstract

The asymptotic behavior (such as convergence to an equilibrium, convergence to a 2-cycle, and divergence to ∞) of solutions of the following multi-parameter, rational, second order difference equation

$$x_{n+1} = \frac{ax_n^3 + bx_n^2 x_{n-1} + cx_n x_{n-1}^2 + dx_{n-1}^3}{x_n^2}, \quad x_{-1}, x_0 \in \mathbb{R},$$

is studied in this paper.
Keywords: Difference equation; equilibrium; 2-cycle; convergence; divergence

1 Introduction

Most of the work about rational difference equations treat the case where both numerator and denominator are linear polynomials. For second order rational difference equations with linear numerator and denominator we refer the reader to the monograph of Kulenovic and Ladass ([2]). In 2008, Sedaghat et al ([1]) extended the existing results about second order rational difference equations to second order rational difference equations with quadratic numerator and linear denominator.

In this paper we extend the existing results to the following difference equation

$$x_{n+1} = \frac{ax_n^3 + bx_n^2 x_{n-1} + cx_n x_{n-1}^2 + dx_{n-1}^3}{x_n^2}, \tag{1.1}$$

which is a second order rational difference equation with cubic numerator and quadratic monomial denominator. The parameters a, b, d are positive while the parameter c and initial conditions x_{-1}, x_0 could accept some negative values.

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In ([5]) we investigated the dynamics of the following difference equation

$$x_{n+1} = \frac{ax_n^3 + bx_n^2 + cx_n + d}{x_n^3}, \tag{1.2}$$

where it was shown that in most cases every positive solution of Eq.(1.2) converges to either an equilibrium or, a 2-cycle.

In this part we study the asymptotic behavior of solutions of Eq.(1.1) including convergence to an equilibrium, convergence to a 2-cycle, and divergence. Our analysis on the dynamics of Eq.(1.1) is essentially based on the dynamics of Eq.(1.2). The concepts of equilibrium point, 2-cycle, stability, asymptotic stability have been defined in the first part and will not be repeated here. Moreover, throughout the present paper we refer to some of the results in the first part.

Divide both sides of Eq.(1.1) by x_n to obtain

$$\frac{x_{n+1}}{x_n} = a + b\left(\frac{x_{n-1}}{x_n}\right) + c\left(\frac{x_{n-1}}{x_n}\right)^2 + d\left(\frac{x_{n-1}}{x_n}\right)^3,$$

In the preceding equation substitute

$$t_n = \frac{x_n}{x_{n-1}},$$

to obtain

$$t_{n+1} = \frac{at_n^3 + bt_n^2 + ct_n + d}{t_n^3},$$

which simply is the first order Eq.(1.2) (similar to the first part we use the function $\phi(t) = (at^3 + bt^2 + ct + d)/t^3$, which defines the right hand side of Eq.(1.2), in the present paper frequently). In fact the solutions of Eq.(1.2) are the successive ratios of the solutions of Eq.(1.1). So we call $\{t_n\}$ the sequence of ratios. Eq.(1.1) is a special semiconjugate factorization of Eq.(1.2) which is called semiconjugacy by ratios. For more about semiconjugacy and semiconjugacy by ratios see [3] and [4] respectively. We analyze the dynamics of Eq.(1.1) using the dynamics of Eq.(1.2) which was studied in the first part.

Now we discuss about the initial conditions of Eq.(1.1). Since in this paper we studied the dynamics of positive solutions of Eq.(1.2) then the initial conditions of Eq.(1.1) should be chosen in such a way that the sequence of ratios becomes positive eventually. If both x_{-1} and x_0 are positive or negative then the sequence of ratios is positive from the first step. On the other hand, if one of them be positive and the other one be negative then the ratio x_n/x_{n-1} may never becomes positive or even the iteration process may stop. For example if ϕ has a negative equilibrium which is attractive then it attracts some ratios in a neighborhood around itself and therefore such a ratio remains negative forever. Also, if at any step the ratio equals zero then the iteration process stops. We should avoid such cases. Although, the determination of these cases in general is not possible but we are able to determine some of them. Now, we mention one of them.

Consider the function ϕ on the interval $(-\infty, 0)$. Assume that $c_- < c \leq -c^* = \sqrt{3bd}$ or, $c > -c^*$ and $\phi(x_m) > 0$ (note that $x_m < 0$ when $c > -c^*$). Then, there exists a unique number $r < 0$ such that $\phi(r) = 0$. Suppose that $r < r' < 0$ is the unique number such that $\phi(r') = r$. Then, it is evident that any ratio in $(-\infty, r) \cup (r', 0)$ will eventually become positive after at most three steps. Note that if $\{x_n\}$ is a solution for Eq.(1.1) then so is $\{-x_n\}$. Also, first and third quadrants of $\mathbb{R}^2$, namely $(0, \infty)^2$ and $(-\infty, 0)^2$, are invariant under Eq.(1.1). By the discussions in the previous paragraph the ratio x_n/x_{n-1} should be positive eventually. Then there are two possibilities. Either $x_n > 0$ or, $x_n < 0$ for all $n \geq n_0$ for some $n_0 \in \mathbb{N}$. If the second case occurs then the change of variable $y_n = -x_n$ (or considering $\{-x_n\}$ as solution) reduces Eq.(1.1) to the first case. Therefore, without loss of generality we assume that both of initial conditions are positive, hereafter.

In the first part we discussed (in great detail) about the convergence of solutions of Eq.(1.2) (or the sequence of ratios) to both an equilibrium and a 2-cycle. Now, we want to study the dynamics of solutions of Eq.(1.1) in both of these cases.

2 Asymptotic stability when the sequence of ratios converges to an equilibrium

Theorem 2.1. *Assume that the sequence $\{x_n\}_{n=-1}^{\infty}$ is a positive solution for Eq.(1.1). Assume also that $\overline{t}$ is an equilibrium of Eq.(1.2) such that the sequence of ratios $\{x_n/x_{n-1}\}_{n=0}^{\infty}$ converges to it.*

(a) *If $\overline{t} > 1$ then $\{x_n\}$ diverges to ∞.*

(b) *If $\overline{t} < 1$ then $\{x_n\}$ converges to zero.*

(c) *Assume that $\overline{t} = 1$(or equivalently $a + b + c + d = 1$). Let $\mathcal{S} = \{\phi^{-n}(1)\}_{n=0}^{\infty}$. If $x_0/x_{-1} \in \mathcal{S}$ then $\{x_n\}$ is convergent to an equilibrium. Otherwise, $|b+2c+3d| \leq 1$ and also*

- *(c_1) If $|b + 2c + 3d| < 1$ then $\{x_n\}$ converges to an equilibrium. Moreover, if $0 < b + 2c + 3d < 1$ then one of subsequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ will be increasing and the other one will be decreasing eventually while $\{x_n\}$ will be increasing or decreasing eventually if $-1 < b + 2c + 3d \leq 0$.*

- *(c_2) If $b+2c+3d = -1$ then $\{x_n\}$ will be increasing or decreasing eventually. In the later case $\{x_n\}$ converges to an equilibrium obviously. In the former case if $c > -3d$ then $\{x_n\}$ diverges to ∞.*

- *(c_3) If $b + 2c + 3d = 1$ then both of subsequences of even and odd terms will be increasing eventually. In particular, if $c > \frac{-2d}{a+d} - b$ then $\{x_n\}$ diverges to ∞.*

Proof. (a) Since $\lim_{n\to\infty} x_n/x_{n-1} = \overline{t} > 1$ then there exist $L > 1$ and $N \in \mathbb{N}$ such that $x_n/x_{n-1} > L$ for all $n > N$. This simply shows that $x_n \to \infty$ as $n \to \infty$. The proof of (b) is similar and will be omitted.

(c) The equality $\overline{t} = 1$ is simply equivalent to the equality $a + b + c + d = 1$. If $x_0/x_{-1} \in \mathcal{S}$ then there exists $N \in \mathbb{N}$ such that $x_{n+1}/x_n = 1$ for all $n > N$. Thus x_n remains constant for all $n > N$. Hence, x_n converges to an equilibrium.

Next, assume that $x_0/x_{-1} \notin \mathcal{S}$. Then, since $\{x_n/x_{n-1}\}$ converges to the equilibrium $\overline{t} = 1$ we have $|\phi'(1)| \leq 1$ or equivalently

$$|b + 2c + 3d| \leq 1. \tag{2.1}$$

On the other hand, since $a + b + c + d = 1$ then we obtain by some computations that

$$x_{n+1} - x_n = r_n(x_n - x_{n-1}), \quad r_n = -\left(b + c + d + \frac{c+d}{t_n} + \frac{d}{t_n^2}\right), \tag{2.2}$$

notice that $r_n \to b+2c+3d$ as $n \to \infty$ since t_n converges to 1. Therefore, by (2.1) there are three cases to consider as follow:

Case I; $|b+2c+3d| < 1$: Thus there exist $0 < L < 1$ and $N \in \mathbb{N}$ such that $|r_n| < L$ for all $n > N$. So (2.2) implies for $n > N$ that

$$|x_{n+1} - x_n| < L|x_n - x_{n-1}|, \tag{2.3}$$

thus we have (by induction) for $n \geq N$ that

$$|x_{n+1} - x_n| < L^{n-N}|x_{N+1} - x_N|,$$

Therefore

$$\lim_{n\to\infty} x_{n+1} - x_n = 0. \tag{2.4}$$

On the other hand we obtain from (2.3) for $n > N$ that

$$\begin{aligned} |x_{n+1} - x_N| &\leq |x_{n+1} - x_n| + |x_n - x_{n-1}| + \ldots + |x_{N+1} - x_N| \\ &< \left(\sum_{i=0}^{n-N} L^i\right)|x_{N+1} - x_N| < \left(\sum_{i=0}^{\infty} L^i\right)|x_{N+1} - x_N| \\ &= \frac{|x_{N+1} - x_N|}{1 - L}, \end{aligned}$$

therefore, $\{x_n\}$ is bounded. This fact together with (2.4) imply that $\{x_n\}$ is convergent.

On the other hand , (2.2) implies that $t_n(t_{n+1} - 1) = r_n(t_n - 1)$. Therefore

$$\begin{aligned} t_{n+1}t_n - 1 &= t_{n+1}t_n \mp t_n - 1 \\ &= t_n(t_{n+1} - 1) + (t_n - 1) \\ &= (t_n - 1)(r_n + 1), \end{aligned}$$

or equivalently

$$\frac{x_{n+1}}{x_{n-1}} - 1 = (t_n - 1)(r_n + 1), \tag{2.5}$$

Note that since $|r_n| \to |b+2c+3d| < 1$ as $n \to \infty$ there exists $n_0 \in \mathbb{N}$ such that $r_n + 1 > 0$ for all $n > n_0$. Also, we know that when $\phi'(1) = -(b+2c+3d) \in (-1, 0)$, t_n oscillates alternately around 1 while when $-(b+2c+3d) \in [0, 1)$, t_n remains on one side of 1 forever. These facts together with (2.5) complete the proof of (c_1).

Case II; $b+2c+3d = -1$: In this case $\phi'(1) = \phi(1) = 1$ (recall that this case occurs when $c = c_m$ and 1 is the greater equilibrium of ϕ or, $c = c_M$ and 1 is the lower equilibrium of ϕ. This case also may occur when $x_m < 1 < x_M$) . Therefore, it's evident that there exists an $n_0 \in \mathbb{N}$ such that either $x_n/x_{n-1} < 1$ or, $x_n/x_{n-1} > 1$ for $n > n_0$. If the former case occurs then $\{x_n\}$ is decreasing for all $n > n_0$ and therefore it will converge to an equilibrium.

On the other hand if the later case occurs then $\{x_n\}$ is increasing for $n > n_0$. Define the following function

$$r(t) = -\left(b + c + d + \frac{c+d}{t} + \frac{d}{t^2}\right), \quad t > 0,$$

note that $r(1) = 1$, $r'(1) = c + 3d$. Thus, if $c > -3d$ then $r'(1) > 0$. As a result, there exists $\epsilon > 0$ such that $r(t) > 1$ for all $t \in (1, 1+\epsilon)$. Therefore, since $r_n = r(t_n)$, $t_n = x_n/x_{n-1} > 1$ for all $n > n_0$, and $t_n \to 1$ as $n \to \infty$ we conclude that there exists $n_1 > n_0$ such that $r_n > 1$ for all $n > n_1$. Thus by (2.2) the sequence of differences $\{x_{n+1} - x_n\}$ is increasing for $n > n_1$.

This fact together with the fact that $\{x_n\}$ is increasing eventually imply that $\{x_n\}$ diverges to ∞.

Case III; $b+2c+3d = 1$: In this case $\phi'(1) = -\phi(1) = -1$ (recall that this case may occur when $c \geq c^*$ or, $c < c^*$ and $1 < x_m$. Also note that by Lemma 2(c) in [5] this case never occurs when $x_M < 1$). Therefore, it's evident that there exists $n_0 \in \mathbb{N}$ such that the sequence of ratios oscillate alternately around 1 for all $n > n_0$. Some computations show that

$$x_{n+1} - x_{n-1} = \rho_n(x_n - x_{n-1}), \quad \rho_n = c + 2d - \frac{c+d}{t_n} - \frac{d}{t_n^2}, \tag{2.6}$$

since $\rho_n = (t_n - 1)[(c+2d)t_n + d]/t_n^2$, $c + 2d = a > 0$, and $t_n \neq 1$ for all $n \geq 0$ then

$$\rho_n(t_n - 1) > 0, \tag{2.7}$$

where the equality $c + 2d = a$ is gained by the subtraction of equalities $a + b + c + d = b+2c+3d = 1$. Now consider the consecutive ratios x_{2n}/x_{2n-1} and x_{2n+1}/x_{2n} for $n > n_0$. Since these ratios oscillate around 1 alternately then one of them is greater than 1 and the other one is less than 1. Without loss of generality assume that $x_{2n+1}/x_{2n} < 1 < x_{2n}/x_{2n-1}$. Thus by (2.7) $\rho_{2n+1} < 0 < \rho_{2n}$. Therefore, (2.6) implies that $x_{2n+1} > x_{2n-1}$ and $x_{2n+2} > x_{2n}$, i.e., both of subsequences of even and odd terms are increasing eventually.

Next, define the function

$$R(t) = r(t)r(\phi(t)), \quad t > 0,$$

where r is defined in the previous case. Some algebra shows that

$$R(1) = 1, \quad R'(1) = 0, \quad R''(1) = 2(b+c)(a+d) + 4d,$$

Therefore, if $R''(1) > 0$, i.e., $c > -2d/(a+d) - b$ then 1 is a local minimum point for R. As a result, there exists $\epsilon > 0$ such that $R(t) > 1$ for all $t \in (1-\epsilon, 1+\epsilon), t \neq 1$. Therefore, since $r_{n+1}r_n = R(t_n)$ and $t_n \to 1$ as $n \to \infty$ then there exists $n_1 > n_0$ such that $r_{n+1}r_n > 1$ for all $n > n_1$. Thus, we obtain from (2.2) that for $n > n_1$ $|x_{n+1} - x_n| = r_n r_{n-1}|x_{n-1} - x_{n-2}| > |x_{n-1} - x_{n-2}|$ or equivalently, $|d_{n+1}| > |d_{n-1}|$ where d_n is the sequence of differences. Therefore, both of sequences $\{|d_{2n+1}|\}$ and $\{|d_{2n}|\}$ are increasing. Hence, either they are convergent to a positive number or divergent to ∞.

Finally, we claim that both of subsequences of even and odd terms (and therefore $\{x_n\}$) diverge to ∞. Suppose for the sake of contradiction that one of them is convergent or both of them are convergent. If one of them is convergent and the other one is divergent then this simply is a contradiction since the ratio x_n/x_{n-1} converges to 1. On the other hand, if both of them are convergent then by the same reason both of them should be convergent to a same number. So $\{x_n\}$ is convergent. Thus, $|d_{n+1}| = |x_{n+1} - x_n| \to 0$ as $n \to \infty$ which is a contradiction. Therefore, $\{x_n\}$ diverges to ∞. The proof is complete.

3 Asymptotic stability when the sequence of ratios converges to a 2-cycle

Lemma 3.1. ***(a)*** *Eq.(1.2) has a unique 2-cycle (p, q) with $pq = 1$ if and only if*

$$\frac{a-c}{d} = \frac{d-b+1}{a} > 2. \tag{3.1}$$

(b) *Assume that (p, q) is an attractive 2-cycle of Eq.(1.2) with $pq = 1, p < 1 < q$. Then*

$$q + p\phi'(p) < 0 < p + q\phi'(q).$$

(c) *Assume that (3.1) holds. Then Eq.(1.1) has infinite number of 2-cycles. More precisely, the following set is the family of 2-cycles of Eq.(1.1)*

$$\mathcal{A} = \{(p', q') |\ p'/q' = p \ \text{ or } \ p'/q' = q\}.$$

Proof. (a) Assume that Eq.(1.2) has a 2-cycle (p, q) with $pq = 1$. So $q^2 = aq^3 + bq^2 + cq + d$ and $p^2 = ap^3 + bp^2 + cp + d$. Multiply the first equation by p and the second equation by q and apply some algebra to obtain

$$p + q = \frac{d-b+1}{a},$$

in a similar fashion multiply those two equations by p^2 and q^2 to obtain

$$p+q=\frac{a-c}{d}.$$

Therefore $(a-c)/d=(d-b+1)/a$. Since $pq=1$ and $p+q=(a-c)/d$ then both p and q satisfy the following quadratic polynomial

$$X^2-\frac{a-c}{d}X+1=0, \tag{3.2}$$

Therefore such a 2-cycle is unique. On the other hand, Eq.(3.2) should have positive determinant. So $(a-c)/d>2$ and therefore (18) holds.

Next, suppose that (18) holds. Then, it's easy to verify that the polynomial G in Lemma 1 in [5] is factored by Eq.(3.2). As a result, Eq.(1.2) has a 2-cycle (p,q) with $pq=1$.

(b) At first we show that both of quantities $q+p\phi'(p)$ and $p+q\phi'(q)$ have different signs. Since (p,q) is an attractive 2-cycle of Eq.(1.2) then

$$\phi'(p)\phi'(q)=(\phi^2)'(p)\leq 1. \tag{3.3}$$

On the other hand, since $pq=1$ then (a) implies that (3.1) holds. This fact together with the fact that $p+q=(a-c)/d$ imply that

$$\begin{aligned}
p^2\phi'(p)+q^2\phi'(q) &= -\left(\frac{bq^2+2cq+3d}{q^2}+\frac{bp^2+2cp+3d}{p^2}\right)\\
&= -\left(2b+2c(p+q)+3d((p+q)^2-2)\right)\\
&= -\left(2b+2c\left(\frac{a-c}{d}\right)+3d\left[\left(\frac{a-c}{d}\right)^2-2\right]\right)\\
&= -\left(2b+2a\frac{a-c}{d}+\frac{(a-c)^2}{d}-6d\right)\\
&= -\left(2b+2(d-b+1)+\frac{(a-c)^2}{d}-6d\right)\\
&= -\left(2+\frac{(a-c)^2-4d^2}{d}\right)\\
&< -2.
\end{aligned} \tag{3.4}$$

Thus (3.3), (3.4), and the equality $pq=1$ yield

$$(q+p\phi'(p))(p+q\phi'(q))=1+\phi'(p)\phi'(q)+p^2\phi'(p)+q^2\phi'(q)<2-2=0,$$

therefore, both of quantities $q+p\phi'(p)$ and $p+q\phi'(q)$ have different signs. On the other hand, the equality $pq=1$ together with (3.2) imply that

$$q+p\phi'(p)=q(1-b-2cq-3dq^2)=q(1-b+3d+(c-3a)q), \tag{3.5}$$

similarly

$$p + q\phi'(q) = p(1 - b + 3d + (c - 3a)p). \qquad (3.6)$$

If $p + q\phi'(q) < 0 < q + p\phi'(p)$ then (3.5) and (3.6) imply that

$$0 < (1 - b + 3d + (c - 3a)q) - (1 - b + 3d + (c - 3a)p) = (c - 3a)(q - p),$$

So since $p < q$ we obtain that $c > 3a$ which simply contradicts (3.1). Hence, $q+p\phi'(p) < 0 < p + q\phi'(q)$.

(d) The proof of (d) is clear and will be omitted.

The proof is complete.

The following theorem (whose proof somehow uses the ideas in the proof of Theorem 2.1) discusses about the dynamics of solutions of Eq.(1.1) when the sequence of ratios converges to a 2-cycle.

Theorem 3.1. *Assume that the sequence $\{x_n\}_{n=-1}^{\infty}$ is a positive solution for Eq.(1.1) and (p, q) is a 2-cycle of Eq.(1.2) such that the sequence of ratios $\{x_n/x_{n-1}\}_{n=0}^{\infty}$ converges to it.*

(a) *If $pq > 1$ then $\{x_n\}$ diverges to ∞.*

(b) *If $pq < 1$ then $\{x_n\}$ converges to zero.*

(c) *Assume that $pq = 1$(or equivalently (3.1) holds). Let $\mathcal{S} = \{\phi^{-n}(p), \phi^{-n}(q)\}_{n=0}^{\infty}$. If $x_0/x_{-1} \in \mathcal{S}$ then $\{x_n\}$ converges to a 2-cycle. Otherwise, we have $|\phi'(p)\phi'(q)| \le 1$ and we consider three cases as follow*

(c_1) $|\phi'(p)\phi'(q)| < 1$; *In this case $\{x_n\}$ converges to a 2-cycle. Moreover, if $-1 < \phi'(p)\phi'(q) < 0$ then the subsequences $\{x_{4n}\}$ and $\{x_{4n+3}\}$ will be increasing and the other two will be decreasing eventually or vice versa while both of subsequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ will be increasing or decreasing eventually if $0 \le \phi'(p)\phi'(q) < 1$.*

(c_2) $\phi'(p)\phi'(q) = 1$; *In this case both of subsequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ will be increasing or decreasing eventually. In the later case $\{x_n\}$ converges to a 2-cycle. In the former case $\{x_n\}$ diverges to ∞ if*

$$p + q\phi'(q) + (\phi'(q))^2\phi''(p)/2 + \phi'(p)\phi''(q)/2 > 0$$

(c_3) $\phi'(p)\phi'(q) = -1$; *Let $l = -(p^2 + (\phi'(q))^2q^2) + (q + p\phi'(p))\phi''(q)/2 + (p + q\phi'(q))(\phi'(q))^2\, \phi''(p)/2$. If $l < 0$ then all of subsequences $\{x_{4n}\}, \{x_{4n+1}\}, \{x_{4n+2}\}$, and $\{x_{4n+3}\}$ are decreasing eventually. In this case $\{x_n\}$ converges to a 2-cycle. If $l > 0$ then all of subsequences $\{x_{4n}\}, \{x_{4n+1}\}, \{x_{4n+2}\}$, and $\{x_{4n+3}\}$ are increasing eventually. In this case $\{x_n\}$ diverges to ∞ if*

$$-2s''(q) - 2(s'(q))^2 - s'(q)(\phi^2)''(q) > 0$$

where
$$s(t)=\frac{t\phi(t)\gamma(\phi(t))\theta(t)[\phi^2(t)\theta(\phi^2(t)+p]}{t\theta(t)+p},$$

$$\gamma(t)=-\left(\frac{b}{pt}+\frac{c(t+p)}{p^2t^2}+\frac{d(t^2+pt+p^2)}{p^3t^3}\right),$$

$$\theta(t)=-\left(\frac{b}{qt}+\frac{c(t+q)}{q^2t^2}+\frac{d(t^2+qt+q^2)}{q^3t^3}\right).$$

Proof. Throughout the proof we assume, without loss of generality, that

$$t_{2n}\to p,\quad t_{2n+1}\to q,\quad \text{as } n\to\infty, \tag{3.7}$$

therefore
$$\frac{x_{n+2}}{x_n}=t_{n+2}t_{n+1}\to pq\quad \text{as } n\to\infty,$$
Thus if $pq>1$ then there exist $N\in\mathbb{N}$ and $L>1$ such that $x_{n+2}/x_n>L$ for all $n>N$. This simply proves (a). In a similar fashion (b) is proved. Now we proceed to (c). Since $pq=1$ then we assume, without loss of generality, that $p<1<q$ hereafter. If $x_0/x_{-1}\in\mathcal{S}$ then there exists an integer N such that $x_{N+1}/x_N=p$ or $x_{N+1}/x_N=q$. Thus $x_{n+1}/x_n=p$ and $x_{n+2}/x_{n+1}=q$ for all $n\geq N$ or vice versa. Therefore for $n\geq N$

$$\frac{x_{n+2}}{x_n}=pq=1,$$

which means that $\{x_n\}$ converges to a 2-cycle. Now assume that $x_0/x_{-1}\notin\mathcal{S}$. Then since the 2-cycle (p,q) attracts the sequence of ratios $\{x_n/x_{n-1}\}$ we have $|\phi'(p)\phi'(q)|\leq 1$.

(c_1) Define $D_n=x_n-x_{n-2}$. Therefore, using (3.7) and hopital law in calculous one can write

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{D_{2n+2}}{D_{2n}}\right| &= \lim_{n\to\infty}\left|\frac{t_{2n+2}t_{2n+1}t_{2n}t_{2n-1}-t_{2n}t_{2n-1}}{t_{2n}t_{2n-1}-1}\right|\\ &= \lim_{t\to q}\left|\frac{t\phi(t)\phi^2(t)\phi^3(t)-t\phi(t)}{t\phi(t)-1}\right|\\ &= |\phi'(p)\phi'(q)|\\ &< 1.\end{aligned}$$

In a similar fashion
$$\lim_{n\to\infty}\left|\frac{D_{2n+1}}{D_{2n-1}}\right|=|\phi'(p)\phi'(q)|<1,$$
Consequently, there exist $n_0\in\mathbb{N}$ and $0<L<1$ such that for $n>n_0$

$$|D_{2n+2}|<L|D_{2n}|,\quad |D_{2n+1}|<L|D_{2n-1}|.$$

Therefore, by an analysis precisely similar to what was applied in Theorem 2.1(c) it could be shown that both of subsequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are convergent and hence $\{x_n\}$ converges to a 2-cycle.

Some calculations show that

$$t_{n+1} - q = \gamma_n(t_n - p), \quad \gamma_n = -\left(\frac{b}{pt_n} + \frac{c(t_n + p)}{p^2 t_n^2} + \frac{d(t_n^2 + pt_n + p^2)}{p^3 t_n^3}\right), \tag{3.8}$$

and

$$t_{n+1} - p = \theta_n(t_n - q), \quad \theta_n = -\left(\frac{b}{qt_n} + \frac{c(t_n + q)}{q^2 t_n^2} + \frac{d(t_n^2 + qt_n + q^2)}{q^3 t_n^3}\right). \tag{3.9}$$

by (3.7) we obtain

$$\gamma_{2n} \to \phi'(p), \quad \theta_{2n+1} \to \phi'(q), \quad \text{as } n \to \infty. \tag{3.10}$$

Now, suppose that $-1 < \phi'(p)\phi'(q) < 0$. By Lemma 3.1(b), $\phi'(p) < -q/p < 0$. So $\phi'(q) > 0$. Therefore, in a neighborhood around p and another neighborhood around q ϕ is decreasing and increasing respectively. This fact together with (3.7) imply that there exists $n_0 \in BBbN$ such that for $n \geq n_0$

(i) either $t_{4n} < p, t_{4n+1} > q, t_{4n+2} > p, t_{4n+3} < q$ or,

(ii) $t_{4n} > p, t_{4n+1} < q, t_{4n+2} < p, t_{4n+3} > q$.

On the other hand, (3.8) and (3.9) imply that

$$\begin{aligned}
t_{n+4}t_{n+3}t_{n+2}t_{n+1} - 1 &= t_{n+4}t_{n+3}t_{n+2}t_{n+1} \mp p^2 t_{n+3}t_{n+1} - p^2 q^2 \\
&= t_{n+3}t_{n+1}(t_{n+4}t_{n+2} - p^2) + p^2(t_{n+3}t_{n+1} - q^2) \\
&= t_{n+3}t_{n+1}(t_{n+4}t_{n+2} \mp pt_{n+2} - p^2) + p^2(t_{n+3}t_{n+1} \mp qt_{n+1} - q^2) \\
&= t_{n+3}t_{n+1}[t_{n+2}(t_{n+4} - p) + p(t_{n+2} - p)] + \\
&\quad p^2[t_{n+1}(t_{n+3} - q) + q(t_{n+1} - q)] \\
&= t_{n+3}t_{n+1}(t_{n+2}\theta_{n+3}\gamma_{n+2} + p)(t_{n+2} - p) + \\
&\quad p^2(t_{n+1}\gamma_{n+2}\theta_{n+1} + q)(t_{n+1} - q) \\
&= [t_{n+3}t_{n+1}\theta_{n+1}(t_{n+2}\theta_{n+3}\gamma_{n+2} + p) + p^2(t_{n+1}\gamma_{n+2}\theta_{n+1} + q)] \times \\
&\quad (t_{n+1} - q),
\end{aligned}$$

Therefore

$$\frac{x_{n+4}}{x_n} - 1 = \lambda_n(t_{n+1} - q), \quad \lambda_n = \&t_{n+3}t_{n+1}\theta_{n+1}(t_{n+2}\theta_{n+3}\gamma_{n+2} + p) + p^2(t_{n+1}\gamma_{n+2}\theta_{n+1} + q), \tag{3.11}$$

In a similar fashion one can write

$$\frac{x_{n+4}}{x_n} - 1 = \xi_n(t_{n+1} - p), \quad \xi_n = \&t_{n+3}t_{n+1}\gamma_{n+1}(t_{n+2}\gamma_{n+3}\theta_{n+2} + q) + q^2(t_{n+1}\theta_{n+2}\gamma_{n+1} + p), \tag{3.12}$$

notice that (3.7) and (3.10) imply that

$$\lambda_{2n} \to (\phi'(p)\phi'(q)+1)(p+q\phi'(q)), \quad \xi_{2n+1} \to (\phi'(p)\phi'(q)+1)(q+p\phi'(p)), \quad \text{as } n \to \infty. \tag{3.13}$$

Consequently, by the fact that $\phi'(p)\phi'(q) > -1$, Lemma 3.1(b), (3.11), (3.12), and (3.13) the subsequences $\{x_{4n}\}$ and $\{x_{4n+3}\}$ will be increasing while the other two will be decreasing eventually if (i) holds. Otherwise, the subsequences $\{x_{4n}\}$ and $\{x_{4n+3}\}$ will be decreasing while the other two will be increasing eventually.

Next, assume that $0 \le \phi'(p)\phi'(q) < 1$. By Lemma 3.1(b) $\phi'(p) < -q/p < 0$. Thus $\phi'(q) \le 0$. If $\phi'(q) < 0$ then in a neighborhood around p and another neighborhood around q ϕ is decreasing. As a result by (3.7) we conclude that there exists an integer $n_0 \in \mathbb{N}$ such that for $n \ge n_0$

(i) either $t_{2n} > p, t_{2n+1} < q$ or,

(ii) $t_{2n} < p, t_{2n+1} > q$.

If, on the other hand $\phi'(q) = 0$ then $q = x_m$ or $q = x_M$. It's easy to show that if $q = x_m$ then case (i) occurs while case (ii) occurs if $q = x_M$. By an analysis somehow similar to that of applied for the expression $x_{n+4}/x_n - 1$ we obtain

$$\frac{x_{n+2}}{x_n} - 1 = \lambda'_n(t_{n+1} - q) = \xi'_n(t_{n+1} - p), \quad \lambda'_n = t_{n+1}\theta_{n+1} + p, \quad \xi'_n = t_{n+1}\gamma_{n+1} + q, \tag{3.14}$$

with

$$\lambda'_{2n} \to p + q\phi'(q), \quad \xi'_{2n+1} \to q + p\phi'(p), \quad \text{as } n \to \infty. \tag{3.15}$$

Therefore, Lemma 3.1(b), (3.14), and (3.15) imply that both of subsequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are decreasing eventually if (i) holds and vice versa if (ii) holds.

(c_2) By an analysis precisely similar to what was applied for the case $0 \le \phi'(p)\phi'(q) < 1$ in (c_1) one can prove that both of subsequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are increasing or decreasing eventually. If the later case occurs (note that this case occurs when case (i) in (c_1) occurs, i.e., $t_{2n} > p, t_{2n+1} < q$ for $n > n_0$) then $\{x_n\}$ converges to a 2-cycle obviously. Now assume that the former case occurs (note that in this case $t_{2n} < p, t_{2n+1} > q$ for $n > n_0$). Then

$$\begin{aligned} \frac{D_{n+2}}{D_n} &= \frac{x_{n+2} - x_n}{x_n - x_{n-2}} \\ &= \frac{t_n t_{n-1}(t_{n+2}t_{n+1} - 1)}{t_n t_{n-1} - 1}, \end{aligned}$$

Therefore (3.14), (3.8), (3.9), and some algebra imply that

$$D_{n+2} = s_n D_n, \quad s_n = \frac{t_n t_{n-1}\gamma_n\theta_{n-1}\lambda'_n}{\lambda'_{n-2}}. \tag{3.16}$$

Some computations show that

$$\gamma(p) = \phi'(p), \quad \theta(q) = \phi'(q), \quad \gamma'(p) = \frac{\phi''(p)}{2}, \quad \theta'(q) = \frac{\phi''(q)}{2}, \tag{3.17}$$

Notice that $s_n = s(t_{n-1})$ and by (3.17) $s(q) = 1$. Also using (3.17) and some algebra we obtain that

$$s'(q) = p + q\phi'(q) + (\phi'(q))^2\frac{\phi''(p)}{2} + \phi'(p)\frac{\phi''(q)}{2} > 0,$$

As a result there exists $\epsilon > 0$ such that $s(t) > 1$ for all $t \in (q, q + \epsilon)$. Therefore, since $s_{2n} = s(t_{2n-1})$, $t_{2n-1} > q$ for all $n > n_0$, and $t_{2n-1} \to q$ as $n \to \infty$ then there exists $n_1 > n_0$ such that $s_{2n} > 1$ for all $n > n_1$. Thus by (3.16) the sequence $\{D_{2n}\}$ is increasing eventually.

Consequently, since $\{x_{2n}\}$ is increasing eventually then $\{x_{2n}\}$ should be divergent to ∞ and hence by (3.7) $\{x_{2n+1}\}$ should be divergent to ∞, too. This means that $\{x_n\}$ is divergent to ∞.

(c_3) Note that similar to the case $-1 \leq \phi'(p)\phi'(q) < 0$ in (c_1) there exists $n_0 \in \mathbb{N}$ such that either $t_{4n} < p, t_{4n+1} > q, t_{4n+2} > p, t_{4n+3} < q$ or, $t_{4n} > p, t_{4n+1} < q, t_{4n+2} < p, t_{4n+3} > q$ for $n > n_0$. Consider the quantities λ_n and ξ_n in (c_1) and define the following functions for $t > 0$

$$\lambda(t) = t\phi^2(t)\theta(t)[\phi(t)\theta(\phi^2(t))\gamma(\phi(t)) + p] + p^2[t\gamma(\phi(t))\theta(t) + q],$$
$$\xi(t) = t\phi^2(t)\gamma(t)[\phi(t)\gamma(\phi^2(t))\theta(\phi(t)) + p] + p^2[t\theta(\phi(t))\gamma(t) + q],$$

Notice that $\lambda_n = \lambda(t_{n+1}), \xi_n = \xi(t_{n+1})$, and by (3.17) $\lambda(q) = \xi(p) = 0$. Also by (3.17) and some algebra we have

$$(\phi'(q))^2\xi'(p) = \lambda'(q) = l,$$

Therefore both of quantities $\xi'(p)$ and $\lambda'(q)$ have the same signum. Now assume that both of them are negative, i.e., $l < 0$. Then there are neighborhoods around p and q that ξ and λ are decreasing on respectively. Assume that $t_{4n} < p, t_{4n+1} > q, t_{4n+2} > p, t_{4n+3} < q$ for $n > n_0$. Thus since $\lambda_{4n} = \lambda(t_{4n+1}), \lambda_{4n+2} = \lambda(t_{4n+3}), \xi_{4n+1} = \xi(t_{4n+2})$, and $\xi_{4n+3} = \xi(t_{4n+4})$ then by (3.7) we obtain that there exists $n_1 > n_0$ such that for $n > n_1$

$$\lambda_{4n} < 0 < \lambda_{4n+2}, \quad \xi_{4n+1} < 0 < \xi_{4n+3},$$

Consequently by (3.11) and (3.12) we conclude that all of subsequences $\{x_{4n}\}$,$\{x_{4n+1}\}$, $\{x_{4n+2}\}$, and $\{x_{4n+3}\}$ are decreasing eventually (note that similar result obtains if $t_{4n} > p, t_{4n+1} < q, t_{4n+2} < p, t_{4n+3} > q$ for $n > n_0$). As a result, all of these four subsequences are convergent and by the fact that $x_{n+2}/x_n \to 1$ as $n \to \infty$ we obtain that both of subsequences $\{x_{4n}\}, \{x_{4n+2}\}$ of even terms should be convergent to a same number. The same result holds for the subsequences $\{x_{4n+1}\}, \{x_{4n+3}\}$ of odd terms. Hence, $\{x_n\}$ converges to a 2-cycle.

Next, suppose that $l > 0$. Then similar arguments show that all of subsequences $\{x_{4n}\}$,$\{x_{4n+1}\}$,$\{x_{4n+2}\}$, and $\{x_{4n+3}\}$ are increasing eventually. Define the function

$$S(t) = s(t)s(\phi^2(t)), \quad t > 0,$$

using the fact that $s(q) = \phi'(p)\phi'(q) = -1$ and by some algebra we obtain that

$$S(q) = 1, \quad S'(q) = 0, \quad S''(q) = -2s''(q) - 2(s'(q))^2 - s'(q)(\phi^2)''(q) > 0,$$

Thus q is a local minimum point for S. So there exists $\epsilon > 0$ such that $S(t) > 1$ for $t \in (q - \epsilon, q + \epsilon), t \neq q$. Therefore, since $s_{4n+2}s_{4n} = s(t_{4n+1})s(t_{4n-1}) = S(t_{4n-1})$ and $t_{4n-1} \to q$ as $n \to \infty$ then there exists $n_0 \in \mathbb{N}$ such that $s_{4n+2}s_{4n} > 1$ for all $n > n_0$. As a result (3.16) implies that $|D_{4n+4}| = s_{4n+2}s_{4n}|D_{4n}| > |D_{4n}|$, i.e., the sequence $\{|D_{4n}|\}$ is increasing eventually. Thus, either it converges to a positive number or, diverges to ∞.

We claim that both of subsequences of even terms, i.e., $\{x_{4n}\}$ and $\{x_{4n+2}\}$ are divergent to ∞ (and therefore by (3.7) the other two subsequences are divergent, too. Hence, $\{x_n\}$ diverges to ∞). Otherwise, at least one of them should be convergent and therefore since $x_{4n+2}/x_{4n} \to 1$ as $n \to \infty$ we conclude that both of them are convergent. As a result $D_{4n} \to 0$ as $n \to \infty$ which simply is a contradiction. The proof is complete.

Remark 3.1. *In Theorem 2.1 and Theorem 3.1 dynamical behavior of solutions of Eq.(1.1) was studied where the sequence of ratios converges to an equilibrium and a 2-cycle respectively. By Theorem 4 and Theorem 5 in [5] we know that one of these two cases occur definitely when* $c \geq c^*$ *or,* $c < c^*, x_M \leq \bar{t}$ *or,* $c < c^*, x_m \leq \bar{t} \leq x_M$*. But if* $c < c^*$ *and* $\bar{t} < x_m$ *the sequence of ratios may fail to be convergent to an equilibrium or a 2-cycle. In this case according to Theorem* 6(*a*) *in [5] the interval* $I = [\phi(x_m), \phi^2(x_m)]$ *is invariant under hypothesis* (H) *or even ratios eventually end up in* I *if* $c \leq c_1^*$*. Therefore, if* $x_0/x_{-1} \in I$ *and* (H) *holds, or* $x_0/x_{-1} \notin I$ *but* $c \leq c_1^*$ *then* $\{x_n\}$ *diverges to* ∞ *when* $\phi(x_m) \geq 1$ *while* $\{x_n\}$ *converges to zero when* $\phi^2(x_m) \leq 1$ *obviously.*

4 Some examples

Example 1. *Consider the first example in Remark 4 in [5]. Note that* $c > c_- \approx -4.1305$ *where* c_- *is the unique negative root of the cubic polynomial* Q *in Theorem 1 in [5]. So By Theorem* 1(*b*) *in [5] nonpositive iterations of Eq.(1.2) do not occur. In this example Eq.(1.2) has two equilibria and no 2-cycle. Some computations show that*

$$x_m \approx 0.7133, \bar{t}_1 \approx 0.7845, \bar{t}_2 = 1, \delta \approx 0.5833,$$

where δ *has been defined in Theorem* 7(*a*) *in [5]. Notice that here* $a + b + c + d = 1$ *and hence one of two equilibria is* 1*. By Theorem* 7(a_1) *if* $t_0 \in (\delta, \bar{t}_2)$ *then* $\{t_n\}$ *converges to* $\bar{t}_1$ *otherwise, it converges to* $\bar{t}_2$*. Moreover, if* $t_0 \notin (\delta, \bar{t}_2)$ *then* $\{t_n\}$ *converges to* t_2 *from the right.*

Therefore, since $\bar{t}_1 < 1$*,* $a + b + c + d = 1$*,* $b + 2c + 3d = -1$*, and* $c > -3d$ *then Theorem 2.1(*c_2*) and Theorem 2.1(b) imply that*

(i) *If* $x_0/x_{-1} \in (\delta, \bar{t}_2)$ *then* $\{x_n\}$ *converges to zero.*

(ii) *If* $t_0 \notin (\delta, \bar{t}_2)$ *then* $\{x_n\}$ *diverges to* ∞*.*

Example 2. *In Eq.(1.1) set* $a = 0.2, b = 1.7, c = -2, d = 1.1$*. So* $c > c_- \approx -2.8540$ *and therefore similar to the arguments in the previous example nonpositive iterations of Eq.(1.2) do not occur. In this example Eq.(1.2) has a unique equilibrium* $\bar{t} = 1$ *(notice that* $a + b + c + d = 1$*) and two 2-cycles* $(p_1, q_1) \approx (0.2262, 63.6517)$ *and* $(p_2, q_2) \approx (0.5110, 4.1111)$*. Since* $c > c^* = -\sqrt{3bd} \approx -2.3685$ *then by Theorem 4(c) in [5]* $\{t_n\}$ *converges to* 1 *if* $t_0 \in (p_2, q_2)$ *and converges to the 2-cycle* (p_1, q_1) *if* $t_0 \in (0, p_2) \cup (q_2, \infty)$*.*

Therefore, since $p_1q_1 > 1$*,* $a + b + c + d = b + 2c + 3d = 1$*, and* $c > \frac{-2d}{a+d} - b$ *then Theorem 2.1(*c_3*), Theorem 3.1(a), and Theorem 3.1(c) imply that*

(i) *If* $x_0/x_{-1} \in (0, \infty) \setminus \{p_1, p_2, \bar{t}, q_2, q_1\}$ *then* $\{x_n\}$ *diverges to* ∞*.*

(ii) *if* $x_0/x_{-1} = \bar{t}$ *then* $\{x_n\}$ *converges to an equilibrium.*

(iii) *If* $x_0/x_{-1} \in \{p_1, p_2, q_2, q_1\}$ *then* $\{x_n\}$ *converges to a 2-cycle.*

Example 3. *In Eq.(1.1) set* $a = 0.1, b = 1.79, c = -2, d = 1$*. Thus again* $c > c_- \approx -2.7295$ *which similar to the previous examples this guarantees that all iterations of Eq.(1.2) remain positive forever. In this example Eq.(1.2) has a unique equilibrium* $\bar{t} \approx 0.9423$ *and three 2-cycles* $(p_1, q_1) \approx (0.1024, 759.2585)$*,* $(p_2, q_2) = (0.6021, 2.1370)$ *and* $(p_3, q_3) \approx (0.7298, 1.3702)$*. Since* $c > c^* \approx -5.37$ *then by Theorem 4(d) in [5]* $\{t_n\}$ *converges to the 2-cycle* (p_1, q_1) *if* $t_0 \in (0, p_2) \cup (q_2, \infty)$ *and converges to the 2-cycle* (p_3, q_3) *if* $t_0 \in (p_2, q_2) \setminus \{\bar{t}\}$*. Notice that* $p_3q_3 = 1$*. This is evident since in this example (3.1) holds easily.*

Consequently, since $p_3q_3 = 1$*,* $0 < \phi'(p_3)\phi'(q_3) < 1$*, and* $p_1q_1 > 1$ *then by Theorem 2.1, Theorem 3.1(a), and Theorem 3.1(*c_1*) we conclude that*

(i) *If* $x_0/x_{-1} \in (0, p_2) \cup (q_2, \infty)$ *then* $\{x_n\}$ *diverges to* ∞*.*

(ii) *If* $x_0/x_{-1} \in [p_2, q_2] \setminus \{\bar{t}\}$ *then* $\{x_n\}$ *converges to a 2-cycle.*

(iii) *If* $x_0/x_{-1} = \bar{t}$ *then* $\{x_n\}$ *simply converges to an equilibrium.*

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