TWO-SYMMETRIC LORENTZIAN MANIFOLDS

D.V.ALEKSEEVSKY AND A.S.GALAEV

ABSTRACT. The local form of all two-symmetric Lorentzian manifolds is found. To do this, the methods of the theory of the holonomy groups is used.

CONTENTS

1. INTRODUCTION

Symmetric pseudo-Riemannian manifolds is an important class of spaces. The direct generalization of these manifolds form the so called k-symmetric pseudo-Riemannian spaces (M, g) satisfying the condition

$$\nabla^k R = 0, \quad \nabla^{k-1} R \neq 0,$$

Date: November 16, 2010.

where $k \geq 1$ and R is the curvature tensor of (M, g). For Riemannian manifolds the condition $\nabla^k R = 0$ implies $\nabla R = 0$ [13].

The paper [12] deals with the two-symmetric Lorentzian manifolds. It contains a historical review of the problem and a long list of literature. In this paper it is shown that such space must admit a parallel null vector field.

In [1] the local structure of four-dimensional two-symmetric Lorentzian manifolds is found. It is shown that these spaces are special pp-waves. For the proof the Petrov classification and the computations in local coordinates are used.

In the present paper we generalize the result of [1] for the arbitrary dimension. We prove the following theorem.

Theorem 1. *Let (M, g) be a Lorentzian manifold of dimension $n+2$. Then (M, g) is two-symmetric if and only if locally there exist coordinates $v, x^1, ..., x^n, u$ such that*

$$g = 2dvdu + \sum_{i=1}^{n}(dx^i)^2 + (H_{ij}u + F_{ij})x^i x^j (du)^2,$$

where H_{ij} is a diagonal matrix with the diagonal elements $\lambda_1 \leq \cdots \leq \lambda_n$ that are simultaneously non-zero real numbers, F_{ij} is a symmetric real matrix.

Any other metric of this form isometric to g is given by the same H_{ij} and by $\tilde{F}_{ij} = cH_{ij} + F_{kl}a_i^k a_j^l$, where $c \in \mathbb{R}$ and a_i^j is an orthogonal matrix such that $H_{kl}a_i^k a_j^l = H_{ij}$.

For the proof we used the methods of the theory of holonomy groups. We may assume that the manifold is locally indecomposable. The condition that a Lorentzian manifold (M, g) is two-symmetric implies that the holonomy algebra $\mathfrak{hol}_m$ of (M, g) at a point $m \in M$ annihilates the value ∇R_m that can be assumed to be non-zero. This can not happen if the holonomy algebra is the whole Lorentzian Lie algebra $\mathfrak{so}(1, n+1)$. Hence the holonomy algebra must preserve a null line and it is contained in the maximal Lie algebra with this property [6],

$$\mathfrak{hol}_m \subset \mathfrak{sim}_n = (\mathbb{R} \oplus \mathfrak{so}(n)) + \mathbb{R}^n.$$

We show that in fact $\mathfrak{hol}_m \subset \mathfrak{so}(n) + \mathbb{R}^n$ and it is enough to consider the following two case: $\mathfrak{hol}_m = \mathbb{R}^n$ and $\mathfrak{hol}_m = \mathfrak{h} + \mathbb{R}^n$, where $\mathfrak{h} \subset \mathfrak{so}(n)$ is an irreducible subalgebra. The first case corresponds to pp-waves. In the second case we find the form of ∇R. Using the result of [9], we show that the Weyl conformal curvature tensor W is parallel. This and the results of [4, 5, 10] give a contradiction. Thus $\mathfrak{hol}_m = \mathbb{R}^n$, i.e. we deal with a pp-wave. The condition $\nabla^2 R = 0$ and simple computations allow to find its coordinate form.

2. Holonomy groups of Lorentzian manifolds

We recall some basic facts about holonomy groups of Lorentzian manifold. Let (M, g) be a Lorentzian d-dimensional manifold and $\mathrm{Hol}^0(M) = \mathrm{Hol}^0(M)_m$ its connected holonomy group

at a point $m \in M$. It is a subgroup of the (connected) Lorentz group $SO(V)^0$ where $V = T_m M$ is the tangent space and it is determined by its Lie algebra $\mathfrak{hol}(M) \subset \mathfrak{so}(V)$ which is called the holonomy algebra of M.
The manifold M is indecomposable (i.e. locally is not decomposable into a direct product of two pseudo-Riemannian manifolds) if and only if the holonomy group Hol $^0(M)$ (or the holonomy algebra $\mathfrak{hol}(M)$) is weakly irreducible, i.e. it does not preserve any proper nondegenerate subspace of V. Any weakly irreducible holonomy group Hol (M) different from the Lorentz group $SO(V)^0$ is a subgroup of the horospheric group $SO(V)_{[p]}$, the subgroup of $SO^0(V)$ which preserves a null line $[p] = \mathbb{R}p$.
This group is identified with the group $\mathrm{Sim_n} = \mathbb{R}^* \cdot \mathrm{SO_n} \cdot \mathbb{R}^n$, $\mathrm{n = d - 2}$ of the Euclidean space E^n as follows.
The Lorentzian group $SO(V)^0$ acts transitively on the celestial sphere $S^n = PV^0$ (the space of null lines) which is the projectivization of the null cone $V^0 \subset V$ with the stabilizer $SO(V)_{[p]}$. The stabilizer has an open orbit $S^n \setminus [p]$ which is identified via the stereographic projection with the Euclidean space E^n. Having in mind this isomorphism, we will call the group $SO(V)_{[p]}$ the similarity group and denote it by $\mathrm{Sim_n}$.

Using the metric $< .,. >= g_m$, we will identify the Lorentz Lie algebra $\mathfrak{so}(V) \simeq \mathfrak{so}(1, n+1)$ with the space $\Lambda^2 V$ of bivectors.
Then the Lie algebra $\mathfrak{sim}_n$ of the similarity group can be written as

$$\mathfrak{sim}_n = \mathfrak{so}(V)_{[p]} = \mathbb{R}p \wedge q + p \wedge E + \mathfrak{so}(E)$$

where p, q are isotropic vectors with $< p, q >= 1$ which span 2-dimensional Minkowski subspace U and $E = U^\perp$ is its orthogonal complement. The commutative ideal $p \wedge E$ generates the commutative normal subgroup $T_E \subset \mathrm{Sim_n}$ which acts on E^n by parallel translations. This group is called the vector group. The one-dimensional subalgebra $\mathbb{R}p \wedge q = \mathfrak{so}(U)$ generates the maximal diagonal subgroup A of $\mathrm{Sim_n}$ which is the Lorentz group $SO(U)^0$ and the maximal compact subalgebra $\mathfrak{so}(E)$ generates the group $SO(E)$ of orthogonal transformations of E.
The above decomposition of the Lie algebra $\mathfrak{sim}_n$ defines the Iwasawa decomposition

$$\mathrm{Sim_n = K \cdot A \cdot N = SO(E) \cdot SO(U)^0 \cdot T_E}$$

of the group $\mathrm{Sim_n}$.
The list of connected weakly irreducible connected holonomy groups Hol $^0(M)$ of Lorentzian manifolds is known, see [11, 6]. Assume for simplicity that $Hol^0(M)$ is an algebraic group. Then it contains the vector group T_E and has one of the following forms:
(type I) $Hol^0(M) = K \cdot SO(U)^0 \cdot T_E$
(type II) $Hol^0(M) = K \cdot T_E$ where $K \subset SO(E)$ is a connected holonomy group of a Riemannian $n-2$-dimensional manifold , i.e. a product of the Lie groups from the Berger list :
$SO_m, U_m, SU_m, Sp_1 \cdot Sp_m, Sp_m, G_2, Spin_7$ and the isotropy groups of irreducible symmetric Riemannian manifolds.

If the holonomy group is not algebraic, it is obtained from one of the holonomy groups of type I or II by some twisting (holonomy groups of type III and IV).
Note that all these holonomy groups act transitively on the Euclidean space $E^n = PV^0 \setminus [p]$ [8].

The Lorentzian holonomy algebras $\mathfrak{g} \subset \mathfrak{sim}_n$ are the following (in all cases $\mathfrak{h} \subset \mathfrak{so}(E)$ is a Riemannian holonomy algebra):
(type I) $\mathbb{R}p \wedge q + \mathfrak{h} + p \wedge E$;
(type II) $\mathfrak{h} + p \wedge E$;
(type III) $\{\varphi(A)p \wedge q + A | A \in \mathfrak{h}\} + p \wedge E$, where $\varphi : \mathfrak{h} \to \mathbb{R}$ is a linear map that is zero on the commutant $[\mathfrak{h}, \mathfrak{h}]$;
(type IV) $\{A + p \wedge \psi(A) | A \in \mathfrak{h}\} + p \wedge E_1$, where $E = E_1 \oplus E_2$ is an orthogonal decomposition, $\mathfrak{h}$ annihilates E_2, i.e. $\mathfrak{h} \subset \mathfrak{so}(E_1)$, and $\psi : \mathfrak{h} \to E_2$ is a surjective linear map that is zero on the commutant $[\mathfrak{h}, \mathfrak{h}]$.

A simply connected Lorentzian manifold admits a parallel null vector field if and only if its holonomy group is of type II or IV.

3. The holonomy group of a 2-symmetric Lorentzian manifold

Definition 1. *A pseudo-Riemannian manifold (M, g) with the curvature tensor R is called k-symmetric if*

$$\nabla^k R = 0, \quad \nabla^{k-1} R \neq 0.$$

So 1-symmetric spaces is the same as locally symmetric spaces ($\nabla R = 0$). Recall that a complete simply connected locally symmetric space is a symmetric space, that is it admits a central symmetry S_m with center at any point m, i.e. an involutive isometry S_m which has m as an isolated fixed point.

Remark that any k-symmetric Riemannian manifold is in fact locally symmetric [13].

All irreducible simply connected Lorentzian symmetric spaces are exhausted by the De Sitter and the anti De Sitter spaces and the Cahen-Wallach spaces, which have the vector holonomy group T_E.
Below we prove that any indecomposable Lorentzian 2-symmetric space has vector holonomy group T_E.

Theorem 2. *The holonomy group* $\mathrm{Hol}^0(M)$ *of an $(n+2)$-dimensional locally indecomposable two-symmetric Lorentz manifold (M, g) is the vector group T_E with the Lie algebra $p \wedge E \subset \mathfrak{so}(V)$.*

It is known that any Lorentzian manifold with the holonomy algebra $p \wedge E$ is a pp-wave (see e.g. [6]), i.e. locally there exist coordinates such that the metric g can be written in the form

$$g = 2dvdu + \delta_{ij} dx^i dy^j + H(du)^2, \quad \partial_v H = 0.$$

We will need only to decide which functions H corresponds to two-symmetric spaces.

3.1. **Algebraic curvature tensors and their derivatives.** Let (W, g) be a pseudo-Euclidean space and $\mathfrak{f} \subset \mathfrak{so}(W)$ be a subalgebra. The vector space

$$\mathcal{R}(\mathfrak{f}) = \{R \in \Lambda^2 W^* \otimes \mathfrak{f} | R(u,v)w + R(v,w)u + R(w,u)v = 0 \text{ for all } u, v, w \in W\}$$

is called *the space of algebraic curvature tensors of type* $\mathfrak{f}$. It is known that if $\mathfrak{f} \subset \mathfrak{so}(W)$ is the holonomy algebra of a pseudo-Riemannian manifold (M, g), then the values of the curvature tensor of (M, g) belong to $\mathcal{R}(\mathfrak{f})$ and

$$\mathfrak{f} = \operatorname{span}\{R(u,v) | R \in \mathcal{R}(\mathfrak{f}),\ u, v \in W\},$$

i.e. $\mathfrak{f}$ is spanned by the images of the elements $R \in \mathcal{R}(\mathfrak{f})$.

The spaces $\mathcal{R}(\mathfrak{g})$ for holonomy algebras of Lorentzian manifolds are found in [7]. Let e.g. $\mathfrak{g} = \mathbb{R}p \wedge q + \mathfrak{h} + p \wedge E$. For the subalgebra $\mathfrak{h} \subset \mathfrak{so}(n)$ define the space

$$\mathcal{P}(\mathfrak{h}) = \{P \in E^* \otimes \mathfrak{h} | g(P(x)y, z) + g(P(y)z, x) + g(P(z)x, y) = 0 \text{ for all } x, y, z \in E\}.$$

Any $R \in \mathcal{R}(\mathfrak{g})$ is uniquely given by

$$\lambda \in \mathbb{R},\ v \in E,\ P \in \mathcal{P}(\mathfrak{h}),\ R_0 \in \mathcal{R}(\mathfrak{h}), \text{ and } T \in \operatorname{End}(E) \text{ with } T^* = T$$

in the following way:

$$\begin{aligned} R(p,q) &= -\lambda p \wedge q - p \wedge v, \qquad R(x,y) = R_0(x,y) - p \wedge (P(y)x - P(x)y),\\ R(x,q) &= -g(v,x)p \wedge q + P(x) - p \wedge T(x), \qquad R(p,x) = 0 \end{aligned}$$

for all $x, y \in E$. For the algebras $\mathfrak{g}$ of the other types, any $R \in \mathcal{R}(\mathfrak{g})$ can be given in the same way and by the condition that R takes values in $\mathfrak{g}$. For example, $R \in \mathcal{R}(\mathfrak{h} + p \wedge E)$ if and only if $\lambda = 0$ and $v = 0$.

Let again $\mathfrak{f} \subset \mathfrak{so}(W)$. Consider the vector space

$$\mathcal{R}^\nabla(\mathfrak{f}) = \{S \in W^* \otimes \mathcal{R}(\mathfrak{f}) | S_u(v,w) + S_v(w,u) + S_w(u,v) = 0 \text{ for all } u, v, w \in W\}.$$

If $\mathfrak{f} \subset \mathfrak{so}(W)$ is the holonomy algebra of a pseudo-Riemannian manifold (M, g), then the values of the covariant derivative of the curvature tensor of (M, g) belong to $\mathcal{R}(\mathfrak{f})$. The spaces $\mathcal{R}^\nabla(\mathfrak{so}(r,s))$ and $\mathcal{R}^\nabla(\mathfrak{u}(r,s))$ are found in [9].

To find the spaces $\mathcal{R}^\nabla(\mathfrak{g})$ for the Lorentzian holonomy algebras $\mathfrak{g} \subset \mathfrak{sim}_n$ it is enough to consider an element $S \in V^* \otimes \mathcal{R}(\mathfrak{g})$, then for any $u \in V$ its value $S_u \in \mathcal{R}(\mathfrak{g})$ can be expressed in terms of some elements λ_u, v_u, P_u, R_{0u}, T_u as above, and it is enough to write down the second Bianchi identity.

3.2. **Adapted coordinates and a reduction lemma.** Let (M,g) be an $(n+2)$-dimensional locally indecomposable two-symmetric Lorentz manifold, i.e. the tensor ∇R is non-zero and parallel. Suppose that the holonomy algebra of (M,g) is $\mathfrak{so}(1,n+1)$. Then for any point $m\in M$, the holonomy algebra $\mathfrak{so}(T_mM)\simeq\mathfrak{so}(1,n+1)$ must annihilate the value $\nabla R_m\in\mathcal{R}^\nabla(\mathfrak{so}(T_mM))$. From [9] it follows that the space $\mathcal{R}^\nabla(\mathfrak{so}(1,n+1))$ does not contain non-zero elements annihilated by $\mathfrak{so}(1,n+1)$. We get a contradiction. The Lie algebra $\mathfrak{so}(1,n+1)$ is the only irreducible holonomy algebra [6]. Hence the holonomy algebra of (M,g) preserves a null line, i.e. it is contained in $\mathfrak{sim}_n$. Consequently (M,g) admits (locally) a parallel distribution of null lines.

Let (M,g) be Lorentzian manifold (of dimension $d=n+2$) that admits a parallel distribution of null lines. Then locally there exist the so called Walker coordinates $v,x^1,...,x^n,u$ and the metric g has the form

$$g=2dvdu+h+2Adu+H(du)^2, \tag{3.1}$$

where $h=h_{ij}(x^1,...,x^n,u)dx^idx^j$ is an u-dependent family of Riemannian metrics, $A=A_i(x^1,\dots,x^n,u)dx^i$ is an u-dependent family of one-forms, and H is a local function on M [14]. The vector field ∂_v defines the parallel distribution of null lines.

Let $\mathfrak{g}\subset\mathfrak{sim}_n$ be the holonomy algebra of the Lorentzian manifold (M,g) and $\mathfrak{h}\subset\mathfrak{so}(E)$ be the associated Riemannian holonomy algebra. Then there exists an orthogonal decomposition

$$E=E_0\oplus E_1\oplus\cdots\oplus E_r \tag{3.2}$$

and the corresponding decomposition into the direct sum of ideals

$$\mathfrak{h}=\{0\}\oplus\mathfrak{h}_1\oplus\cdots\oplus\mathfrak{h}_r \tag{3.3}$$

such that $\mathfrak{h}$ annihilates E_0, $\mathfrak{h}_i(E_j)=0$ for $i\neq j$, and $\mathfrak{h}_i\subset\mathfrak{so}(E_i)$ is an irreducible subalgebra for $1\leq i\leq s$. In [2] it is proved that there exist Walker coordinates

$$v,x_0^1,\dots,x_0^{n_0},\dots,x_r^1,...,x_r^{n_r},u$$

that are adapted to the decompositions (3.2) and (3.3). This means that

$$h=h_0+h_1+\cdots+h_r,\quad h_0=\sum_{i=1}^{n_0}(dx_0^i)^2$$

and

$$A=\sum_{\alpha=1}^{r}\sum_{k=1}^{n_\alpha}A_k^\alpha dx_\alpha^k,\quad (A_0=0)$$

and for each $1\leq\alpha\leq r$ it holds

$$h_\alpha=\sum_{i,j=1}^{n_\alpha}h_{\alpha ij}dx_\alpha^idx_\alpha^j$$

with

$$\frac{\partial}{\partial x_\beta^k}h_{\alpha ij}=\frac{\partial}{\partial x_\beta^k}A_i^\alpha=0$$

for all $1\leq i,j\leq n_\alpha$ if $\beta\neq\alpha$.

For $i = 0, ..., r$ consider the metric

$$g_i = 2dvdu + h_i + 2A_i du + H_i(du)^2,$$

where H_i equals to H assuming that all coordinates except $v, x_i^1, ..., x_i^{n_i}, u$ are parameters.

Lemma 1. *If the metric g is two-symmetric, then each metric g_i satisfies $\nabla^2 R = 0$.*

Proof. It is easy to see that the Christoffel symbols of any metric g_i equal to the corresponding Christoffel symbols of the metric g. Consequently, the components of the curvature tensor of g_i and its derivatives equal to the corresponding components of the corresponding tensors for the metric g. □

It is clear that the projection on $\mathfrak{so}(E_i)$ of the holonomy algebra of the metric g_i equals to $\mathfrak{h}_i$ $(i = 1, ..., r)$.

3.3. **Proof of Theorem 2.** First we prove the following two propositions.

Proposition 1. *Any two-symmetric Lorentzian manifold (M, g) admits a parallel null vector field.*

Proof. We may assume that (M, g) is locally indecomposable. The metric g is locally given by (3.1). The above arguments allow us to assume that the projection $\mathfrak{h} \subset \mathfrak{so}(E)$ of the holonomy algebra $\mathfrak{g}$ on $\mathfrak{so}(E)$ is irreducible. It is enough to prove that $\mathfrak{g}$ is of type 2 or 4, i.e. it is not of type 1 or 3.

The condition $\nabla^2 R = 0$ means that ∇R is parallel. The holonomy principle shows that $\mathfrak{g}$ must annihilate a tensor in the space $\mathcal{R}^\nabla(\mathfrak{g})$. If $\mathfrak{g}$ is of type 1, then it contains $p \wedge q$. Using this element and the second Bianchi identity it can be proven that there are no non-zero elements in $\mathcal{R}^\nabla(\mathfrak{g})$ that are annihilated by $\mathfrak{g}$. If $\mathfrak{g}$ is of type 3, then $\mathfrak{h} \subset \mathfrak{u}(E)$ and for some $a \in \mathbb{R}$, the element $p \wedge q + aJ$ belongs to $\mathfrak{g}$. Simple computations show that there are no non-zero elements in $\mathcal{R}^\nabla(\mathfrak{g})$ that are annihilated by $\mathfrak{g}$.

Thus $\mathfrak{g}$ is of type 2 or 4, in this case (M, g) admits a parallel null vector field. □

Proposition 2. *A Lorentzian manifold with the holonomy algebra $\mathfrak{h} + p \wedge E$ with $\mathfrak{h} \neq 0$ can not be two-symmetric.*

Proof. Suppose that (M, g) is two-symmetric and its holonomy algebra equals to $\mathfrak{h} + p \wedge E$ with $\mathfrak{h} \neq 0$. We may assume that $\mathfrak{h} \subset \mathfrak{so}(E)$ is irreducible.

Lemma 2. *The subspace of $\mathcal{R}^\nabla(\mathfrak{g})$ annihilated by $\mathfrak{g}$ is one-dimensional and it is spanned by the tensor S with the only non-zero value*

$$S_q(x, q) = p \wedge x, \quad x \in E.$$

Proof. Let $S\in\mathcal{R}^\nabla(\mathfrak{g})$ and assume that $\mathfrak{g}$ annihilates S. For any $u\in V$ the element $S_u\in\mathcal{R}(\mathfrak{g})$ can be expressed in terms of some R_{0u}, P_u and T_u as above. Since $S_\cdot(p,\cdot)=0$, it holds $S_p=0$. The fact that $\mathfrak{g}$ annihilates S can be expressed as

$$[\xi,S_u(u_1,u_2)]-S_{\xi u}(u_1,u_2)-S_u(\xi u_1,u_2)-S_u(u_1,\xi u_2)=0$$

for all $\xi\in\mathfrak{g}$ and $u,u_1,u_2\in V$. Let $U,X,Y,Z\in E$. We have

$$[p\wedge X,S_U(Y,Z)]=0.$$

Hence, $R_{0U}(Y,Z)X=0$, i.e. $R_{0U}=0$. Next,

$$[p\wedge X,S_U(Y,q)]-S_U(Y,X)=0.$$

Consequently,

$$-p\wedge P_U(Y)X-p\wedge(P_U(Y)X-P_U(X)Y)=0,$$

i.e. $2P_U(Y)X=P_U(X)Y$. Since this equality holds for any $X,Y\in E$, we conclude $P_U=0$. We have got $S_U(X,Y)=0$. Similarly,

$$[p\wedge X,S_q(Y,Z)]=0,$$

i.e. $R_{0q}=0$. The equality

$$[p\wedge X,S_q(Y,q)]-S_X(Y,q)-S_q(Y,X)=0$$

implies

$$T_X(Y)=2P_q(Y)X-P_q(X)Y.$$

From the second Bianchi identity

$$S_q(X,Y)+S_X(Y,q)+S_Y(q,X)=0$$

it follows that

$$T_X(Y)-T(Y)X=P_q(X)Y-P_q(Y)X.$$

We conclude $P_q(Y)X-P_q(X)Y=0$. This and the definition of the space $\mathcal{P}(\mathfrak{h})$ imply $P_q=0$. Consequently, $T_X=0$. Finally, let $A\in\mathfrak{h}$, then

$$[A,S_q(X,q)]-S_q(AX,q)=0.$$

This implies $AT(X)=T(AX)$, i.e. T commutes with $\mathfrak{h}$. Since T is symmetric, by the Schur Lemma, T is proportional to the identity. This proves the lemma. □

We may write the metric g in the form (3.1). In this case ∂_v is parallel and $\partial_v H=0$. Consider the local frame basis

$$p=\partial_v,\quad X_i=\partial_i-A_i\partial_v,\quad q=\partial_u-\frac{1}{2}H\partial_v.$$

Let $E=\mathrm{span}\{X_1,...,X_n\}$. We obtain that the only non-zero value of ∇R is of the form

$$\nabla_q R(X,q)=fp\wedge X,\quad X\in\Gamma(E),\tag{3.4}$$

for some function f.

From [9] it follows that the tensor ∇R can be decomposed into four components

$$\nabla R = S_0' + S_0'' + S' + S_1,$$

where S_0' can be expressed through the covariant derivative ∇W of the Weyl conformal tensor W and the Cotton tensor C; S_0'' is defined by the symmetrization of the tensor $\mathrm{Ric} - \frac{2s}{d+2}g$; S' is defined by the Cotton tensor C; S_1 is defined by the gradient $\mathrm{grad}s$ of the scalar curvature s. From (3.4) it follows that $\mathrm{tr}\,_{1,5}\nabla R = 0$. This implies $C = 0$ and $\mathrm{grad}s = 0$. Consequently,

$$\nabla R = \nabla W + S_0''.$$

The tensor S_0'' is defined as

$$(S_0'')_X(Y, Z) = T_X Y \wedge Z + Y \wedge T_X Z,$$

where T is defined by the equality

$$\mathrm{tr}\,_{2,4}\nabla R = (2 - d)T.$$

Hence the only non-zero value of T is

$$T_q q = -fp.$$

Consequently, the only non-zero value of S_0'' is

$$(S_0'')_q(X, q) = fX \wedge p.$$

Thus, $\nabla R = S_0''$ and

$$\nabla W = 0.$$

The local form of Lorentzian manifolds with $\nabla W = 0$ are found in [5, 4], where it is shown that this condition implies one of the following: $W = 0$ (i.e. (M, g) is locally conformally flat), $\nabla R = 0$, (M, g) is a pp-wave. In [10] it is shown that if the metric (3.1) is conformally flat, than this is a metric of a pp-wave. Thus the holonomy algebra of (M, g) is contained in $p \wedge E$ and we get a contradiction. □

This proposition and Lemma 1 prove Theorem 2. □

4. Lorentzian manifolds with vector holonomy group T_E (pp-waves)

In this section we derive formulas for the curvature tensor and its covariant derivatives for an $(n+2)$-dimensional Lorentzian manifold with the vector holonomy group $\mathrm{Hol}\,(M) = T_E$ (or, equivalently, the holonomy algebra $\mathfrak{hol}(M) = p \wedge E$).

4.1. **Adapted local coordinates and associated pseudo-group of transformations.** It is well known that a Lorentzian manifold (M,g) (of dimension $n+2$) has the connected holonomy group T_E if and only if in a neighborhood of any point $x \in M$ with respect to some local coordinates $v, x^1, \cdots, x^n, u$ (called adapted coordinates) the metric is given by

$$g = 2dudv + \delta_{ij}dx^i dx^j + Hdu^2, \tag{4.1}$$

where H is a function depending on x^i and u. Such Lorentzian manifolds are called pp-waves.

It is not hard to prove the following.

Lemma 3. *Any two adapted coordinate systems with the same ∂_v are related by*

$$\tilde{u} = u + c, \quad \tilde{x}^i = a^i_j x^j + b^i(u), \quad \tilde{v} = v - \sum_j a^j_i \frac{db^j(u)}{du} x^i + d(u), \tag{4.2}$$

where $c \in \mathbb{R}$, a^j_i is an orthogonal matrix, and $b^i(u)$, $d(u)$ are arbitrary functions of u.

4.2. **Levi-Civita connection.** We associate with an adapted coordinates (u, x^i, v) of a pp-wave space (M,g) with a potential $H = H(x^i, u)$ a standard field of frames

$$p = \partial_v, \quad e_i = \partial_i = \frac{\partial}{\partial x^i}, \quad q = \partial_u - \frac{1}{2}H\partial_v$$

and the dual field of coframes

$$p' = dv + \frac{1}{2}Hdu, \quad e^i = dx^i, \quad q' = du.$$

The Gram matrix of these bases is given by

$$G = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1_n & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

We will consider coordinates of all tensor fields with respect to these non-holonomic frame and coframe. Then the covariant derivative of a vector $Y = Y^p p + Y^i e_i + Y^q q$ and a covector $\omega = \omega_p p' + \omega_i e^i + \omega_q q'$ in direction of a vector field X can be written as

$$\nabla_X Y = \partial_X Y + A_X Y, \ \nabla_X \omega = \partial_X \omega - A_X^T \omega$$

where ∂_X is the derivative of coordinates in direction of X and A_X is a matrix and A_X^T is the transposed matrix.

Lemma 4. *The matrix A_u, A_i, A_v of the connection which correspond to the coordinate vector fields $\partial_u, \partial_i, \partial_v$ and their transposed are given by*

$$A_u = \begin{pmatrix} 0 & \frac{1}{2}H_i & 0 \\ 0 & 0 & -\frac{1}{2}H_i \\ 0 & 0 & 0 \end{pmatrix}, \ A_u^T = \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{2}H_i & 0 & 0 \\ 0 & -\frac{1}{2}H_i & 0 \end{pmatrix}, \ A_i = A_i^T = A_v = A_v^T = 0.$$

In particular, $\nabla p = \nabla p' = 0$.

Proof: The only non zero Christoffel symbols are

$$\Gamma^v_{uu} = \frac{1}{2}H_{,u}, \quad \Gamma^i_{uu} = -\frac{1}{2}H_{,i}, \quad \Gamma^v_{iu} = \frac{1}{2}H_{,i}$$

where the comma means the partial derivative. Then we calculate

$$\nabla\partial_v = \nabla p = 0,\ \nabla_u\partial_i = \frac{1}{2}H_{,i}p,$$

$$\nabla_u q = \nabla_u(\partial_u - \frac{1}{2}H\partial_v) = \frac{1}{2}H_{,u}p - \frac{1}{2}H_{,i}e_i - \frac{1}{2}H_{,u}p = -\frac{1}{2}H_{,i}e_i.$$

$$\nabla_i\partial_j = 0,\ \nabla_i\partial_u = \frac{1}{2}H_{,i}p,\ \nabla_i q = \nabla_i(\partial_u - \frac{1}{2}Hp) = 0,\ \nabla_v\partial_u = \nabla_v\partial_i = \nabla_v\partial_v = 0.$$

□

Corollary 1. *A Lorentzian manifold* M *with vector holonomy group* $\mathrm{Hol}\,(M) = T_E$ *has the (globally defined) parallel vector field* $p = \partial_v$ *and parallel 1-form* $q' = du$.

4.3. **The curvature tensor of a pp-wave space.**

Lemma 5. *With respect to the standard frame* $p = \partial_v$, $e_i = \partial_i$, $q = \partial_u - \frac{1}{2}H\partial_v$ *and the dual coframe* p', e^i, q', *the curvature tensor of a pp-wave with potential* $H(u, x^i)$ *is given by*

$$R = \sum_{i,j}\frac{1}{2}H_{,ij}(p\wedge e_i \vee p\wedge e_j) \text{ (the contravariant curvature tensor)}$$

$$\bar{R} = \frac{1}{2}H_{,ij}(q'\wedge e^i \vee q'\wedge e^j) \text{ (the covariant curvature tensor).}$$

Proof: It follows from the formula $R(X,Y) = \partial_X A_Y - \partial_Y A_X - A_{[X,Y]}$ for vector fields X, Y on M. □

Corollary 2. *The Ricci tensor of* M *is given by*

$$\mathrm{ric} = \frac{1}{2}\Delta H q'\otimes q' = \frac{1}{2}\Delta H du^2$$

where Δ *is the Euclidean Laplacian.*

4.4. **The covariant derivatives of the curvature tensor.**

Note that for any i, j, the covariant tensor $q'\wedge e^i \vee q'\wedge e^j$ and the contravariant tensor $p\wedge e_i \vee p\wedge e_j$ are parallel. Hence the first covariant derivative of the curvature tensor is the following:

$$\nabla\bar{R} = \frac{1}{2}H_{,ijk}e^k\otimes(q'\wedge e^i \vee q'\wedge e^j) + \frac{1}{2}H_{,iju}q'\otimes(q'\wedge e^i \vee q'\wedge e^j). \tag{4.3}$$

We get

Corollary 3. *The manifold* (M, g) *is a locally symmetric space if and only if the Hessian* $H_{,ij}$ *of the potential* H *is a constant, that is* $H = H_{ij}x^ix^j + G_i(u)x^i + K(u)$.

It can be shown that in the last case the coordinates can be chosen in such a way that $H=\lambda_1(x^1)^2+\dots+\lambda_n(x^n)^2$ for some non-zero real numbers λ_i such that $\lambda_1\leq\dots\leq\lambda_n$ [3].

The second covariant derivative of the curvature tensor has the following form:

$$\nabla^2\bar{R}=\left(\frac{1}{2}H_{,ijk}-\frac{1}{4}\sum_k H_{,k}H_{,ijk}\right)q'^2\otimes(q'\wedge e^i\vee q'\wedge e^j) \\ +\frac{1}{2}H_{,ijku}(q'\vee e^k)\otimes(q'\wedge e^i\vee q'\wedge e^j)+\frac{1}{2}H_{,ijk\ell}(e^k\otimes e^\ell)\otimes(q'\wedge e^i\vee q'\wedge e^j). \tag{4.4}$$

This implies the following.

Theorem 3. *A pp-wave with the metric* (4.1) *is two-symmetric if and only if*

$$H=(uH_{ij}+F_{ij})x^ix^j+G_i(u)x^i+K(u),$$

where H_{ij} and F_{ij} are symmetric real matrices, the matrix H_{ij} is non-zero, $G_i(u)$ and $K(u)$ are functions depending on u.

5. Proof of Theorem 1

To prove the theorem we start with the metric (4.1) and H as in Theorem 3 and use transformation (4.2) in order to write the metric as in Theorem 1. Let $\tilde{v},\tilde{x}^1,...,\tilde{x}^n,\tilde{u}$ be a new coordinate system. We may assume that the inverse transformation is given by

$$u=\tilde{u}+c,\quad x^i=a^i_j\tilde{x}^j+b^i(\tilde{u}),\quad v=\tilde{v}-\sum_j a^j_i\frac{db^j(\tilde{u})}{d\tilde{u}}\tilde{x}^i+d(\tilde{u}). \tag{5.1}$$

For the new function $\tilde{H}$ written as in Theorem 3 we get

$$\tilde{H}_{kl}=H_{ij}a^i_ka^j_l, \tag{5.2}$$

$$\tilde{F}_{kl}=(cH_{ij}+F_{ij})a^i_ka^j_l, \tag{5.3}$$

$$\tilde{G}_k(\tilde{u})=-2\sum_j a^j_k\frac{d^2b^j}{(d\tilde{u})^2}+2((\tilde{u}+c)H_{ij}+F_{ij})b^ia^j_k+G_ia^i_k, \tag{5.4}$$

$$\tilde{K}(\tilde{u})=2\frac{dd(\tilde{u})}{d\tilde{u}}+\sum_j\left(\frac{db^j}{du}\right)^2+((\tilde{u}+c)H_{ij}+F_{ij})b^ib^j+G_ib^i+K. \tag{5.5}$$

Equation (5.4) shows that there exist functions $b^j(\tilde{u})$ such that $\tilde{G}_k=0$. Then using the last equation it is possible to find $d(\tilde{u})$ such that $\tilde{K}=0$. From equation (5.2) it follows that there exists an orthogonal matrix a^j_i such that $\tilde{H}_{kl}$ is a diagonal matrix with the diagonal elements $\lambda_1,...,\lambda_n$ such that $\lambda_1\leq\dots\leq\lambda_n$.

Since $\nabla R\neq 0$, Corollary 3 shows that H_{ij} must be non-zero.

Clearly, the transformations that do not change the form of the metric from Theorem 1 are defined by the transformation (5.1) such that $H_{kl}a_i^k a_j^l = H_{ij}$ and with certain $b^i(\tilde{u})$ and $d(\tilde{u})$. This and (5.3) prove the theorem. □

References

[1] O. F. Blanco, M. Sánchez, J. M. Senovilla, *Complete classifcation of second-order symmetric spacetimes.* Journal of Physics: Conference Series 229 (2010), 012021, 5pp.

[2] C. Boubel, *On the holonomy of Lorentzian metrics.* Ann. Fac. Sci. Toulouse Math. (6) 16 (2007), no. 3, 427–475.

[3] M. Cahen, N. Wallach, *Lorentzian symmetric spaces,* Bull. Amer. Math. Soc. 76 (1970), 585-591.

[4] A. Derdzinski, W. Roter, *On conformally symmetric manifolds with metrics of indices* 0 *and* 1*,* Tensor (N.S.) 31 (1977), no. 3, 255–259.

[5] A. Derdzinski, W. Roter, *The local structure of conformally symmetric manifolds,* Bull. Belg. Math. Soc. Simon Stevin 16 (2009), no. 1, 117–128.

[6] A. S. Galaev, T. Leistner, *Holonomy groups of Lorentzian manifolds: classification, examples, and applications,* Recent developments in pseudo-Riemannian geometry, 53–96, ESI Lect. Math. Phys., Eur. Math. Soc., Zürich, 2008.

[7] A. S. Galaev, *The spaces of curvature tensors for holonomy algebras of Lorentzian manifolds*, Diff. Geom. and its Applications 22 (2005), 1–18.

[8] A. S. Galaev, *Isometry groups of Lobachevskian spaces, similarity transformation groups of Euclidean spaces and Lorentzian holonomy groups*, Rend. Circ. Mat. Palermo (2) Suppl. No. 79 (2006), 87–97.

[9] A. S. Galaev, *Decomposition of the covariant derivative of the curvature tensor of a pseudo-Kählerian manifold,* arXiv:1010.0861.

[10] A. S. Galaev, *Holonomy groups of conformally flat Lorentzian manifolds,* in preparation.

[11] T. Leistner, *On the classification of Lorentzian holonomy groups*, J. Differential Geom. 76 (2007), no. 3, 423–484.

[12] J. M. Senovilla, *Second-order symmetric Lorentzian manifolds. I. Characterization and general results,* Classical Quantum Gravity 25 (2008), no. 24, 245011, 25 pp.

[13] S. Tanno, *Curvature tensors and covariant derivatives.* Ann. Mat. Pura Appl. (4) 96 (1972), 233–241.

[14] A. G. Walker, *On parallel fields of partially null vector spaces,* Quart. J. Math., Oxford Ser., 20 (1949), 135–145.