# On the Erdös distinct distance problem in the plane 

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November 19, 2010


#### Abstract

In this paper, we prove that a set of $N$ points in $\mathbf{R}^{2}$ has at least $c \frac{N}{\log N}$ distinct distances, thus obtaining the sharp exponent in a problem of Erdös. We follow the set-up of Elekes and Sharir which, in the spirit of the Erlangen program, allows us to study the problem in the group of rigid motions of the plane. This converts the problem to one of point-line incidences in space. We introduce two new ideas in our proof. In order to control points where many lines are incident, we create a cell decompostion using the polynomial ham sandwich theorem. This creates a dichotomy: either most of the points are in the interiors of the cells, in which case we immediately get sharp results, or alternatively the points lie on the walls of the cells, in which case they are in the zero set of a polynomial of suprisingly low degree, and we may apply the algebraic method. In order to control points where only two lines are incident, we use the flecnode polynomial of the Rev. George Salmon to conclude that most of the lines lie on a ruled surface. Then we use the geometry of ruled surfaces to complete the proof.


## 1 Introduction

In 1946, Paul Erdös posed the question: how few distinct distances are determined by $N$ points in the plane. [E] By choosing the points to lie in a lattice, one can achieve as few as $O\left(\frac{N}{\sqrt{\log N}}\right)$. But the only lower bound he was then able to show was that the number of distances was $\gtrsim N^{\frac{1}{2}}$. (Throughout this paper, we use the notation $A \gtrsim B$ to mean that there is a universal constant $C>0$ with $A \geq C B$.)

Various authors have improved this exponent $\frac{1}{2}$. These include but are not limited to [M], [CSzT, SoT0, T]. Most recently, it was shown in [KT], that the number of distances is $\gtrsim N^{\frac{48-14 e}{55-16 e}-\epsilon}$ for any $\epsilon>0$. Note that the exponent

$$
\frac{48-14 e}{55-16 e} \approx 0.8641
$$

falls far short of the lattice upper bound.

Erdös' problem inspired the Szemerédi-Trotter incidence theorem of [SzT] which is the most basic and important result in extremal incidence geometry.

For a more thorough presentation of the history of the subject see the forthcoming book GIS].

Incidence theory took a rather dramatic turn more recently with the advent of the algebraic method introduced by Dvir in his paper [D] which settled the finite field Kakeya problem. Suddenly it became possible to solve completely problems where previously only incremental gains in the exponent were possible. The authors in [GK] solved the Joints problem, a basic question in three dimensional incidence theory using Dvir's idea. Previously only incremental progress had been possible because techniques like cell decompositions, projections, and cuttings were being used. These kind of ideas were introduced in the seminal paper CEGSW, where it was shown among other things that they could be used to prove the Szemerédi Trotter theorem.

But while cell methods were extremely powerful in the plane, they seemed inadequate in space because they did not well account for the possibility that the lines all lie in a hypersurface and that the problem is somehow lower dimensional. On the other hand, the algebraic method seemed to work magically. It guaranteed that the lines all lay in a hypersurface, an algebraic surface of high degree, but for the joints problem not too high.

An exciting development occurred when Elekes and Sharir [ES] pointed out advances in three-dimensional point line incidence theory could shed light on the Erdös problem. They reduced Erdós' problem to a sequence of incidence questions with a parameter $k$ which counted the number of lines incident at a point. (In their set-up, they had curves, not lines, but this can be repaired with a coordinate change in the group of rigid motions.)

Unfortunately, Elekes and Sharir were unable to complete their program for various technical reasons. For $k=2$, they had only one and not two polynomials that vanished on the lines and we are able to rectify this problem by invoking Salmon's flecnode polynomial and using the nineteenth century theory of ruled surfaces.
Conversely for $k$ large, they were unable to apply the algebraic method because their lines did not need to lie in a sufficiently low degree surface. This is a serious problem and helps explain why their incidence conjecture does not hold unrestrictedly in finite fields. They realized that the problem was fundamentally topological and resorted to cell-type techniques which seemed to condemn them to incremental exponents.

In the present paper, we prove
Theorem 1.1. A set of $N$ points in the plane determines $\gtrsim \frac{N}{\log N}$ distinct distances.
Our most dramatic contribution is to introduce a new kind of cell decomposition produced by the polynomial ham sandwich theorem. This procedure has what turns out to be the
advantage of not always resulting in a decomposition. Instead there is a dichotomy. One possibility is that we get an extremely efficient decomposition providing us with exactly the incidence theorems we would like. The alternative is that the procedure fails in which case most of our lines lie in the zero set of a polynomial of fairly low degree. This is an acceptable alternative because it allows us to apply the algebraic method.

We have now largely described the plan of the paper. There are some other technicalities which the reader should not take too seriously. These involve pigeonholing. When dealing with dichotomies, there are two types of objects and we often must prune our set of lines. One danger is that one will prune too many. Usually one can overcome that danger by proceeding inductively, but it is a little difficult in this paper because the sets of lines have certain restrictions which one must reestablish in the inductive step. Most of the technical estimates in the paper deal with these issues.

Acknowledgements: The first author is partially supported by NSF grant DMS-0635607 and by the Monell Foundation. The second author is partially supported by NSF grant DMS-1001607. He would like to thank Michael Larsen for some very helpful discussions about algebraic geometry. He would also like to thank the Institute of Advanced Study for the use of its magnificent duck pond during a visit which resulted in this paper.

## 2 Elekes-Sharir framework

Our goal in this paper will be to obtain a lower bound on the set of pairwise distances $d(P)$ between the points of a set

$$
P \subset \mathbf{R}^{\mathbf{2}}
$$

with

$$
|P|=N .
$$

Elekes and Sharir [ES] created a framework that connects $d(P)$ with problems of incidence geometry in 3 -dimensional space. They first observed that to obtain a lower bound on the size of $d(P)$, it suffices to obtain an upper bound on a set of quadruples. We let $Q(P)$ be the set of quadruples, $\left(p_{1}, p_{2}, p_{3}, p_{4}\right) \in P^{4}$ satisfying

$$
\begin{equation*}
d\left(p_{1}, p_{2}\right)=d\left(p_{3}, p_{4}\right) . \tag{2.1}
\end{equation*}
$$

Henceforth, we refer to the elements of $Q(P)$ as distance quadruples.
By applying the Cauchy-Schwarz inequality, we easily obtain

$$
\begin{equation*}
|d(P)| \geq \frac{N^{4}}{|Q(P)|} \tag{2.2}
\end{equation*}
$$

To prove Theorem [1.1, it suffices to bound $|Q(P)| \lesssim N^{3} \log N$.
Now we consider what it means for a quadruple $\left(p_{1}, p_{2}, p_{3}, p_{4}\right)$ to be a distance quadruple in terms of the group $G$ of positively oriented rigid motions of the plane.
Proposition 2.1. The quadruple ( $p_{1}, p_{2}, p_{3}, p_{4}$ ) is a distance quadruple if and only if there is $g \in G$ so that $g\left(p_{1}\right)=p_{3}$ and $g\left(p_{2}\right)=p_{4}$.

Note that the $g$ given by the proposition is unique. All positively oriented rigid motions taking $p_{1}$ to $p_{3}$ can be obtained from the translation from $p_{1}$ to $p_{3}$ by applying a rotation $R$ about the point $p_{3}$. There is a unique such rotation sending $p_{2}+p_{3}-p_{1}$ into $p_{4}$.

Elekes and Sharir introduce the sets $L_{p q} \subset G$ given by

$$
L_{p q}=\left\{g \in G: g\left(p_{1}\right)=p_{2}\right\} .
$$

Then by the proposition we have that $\left(p_{1}, p_{2}, p_{3}, p_{4}\right)$ is a distance quadruple if and only if $L_{p_{1} p_{3}} \cap L_{p_{2} p_{4}} \neq \emptyset$. So the number of quadruples $|Q(P)|$ is the number of intersecting pairs of $L_{p q}$ 's. We can count the number of intersecting pairs by considering elements of $G$ where the intersections occur. We let $G_{k} \subset G$ be the rigid motion that lie in at least $k$ and at most $2 k$ of the sets $\left\{L_{p q}\right\}_{p, q \in P}$. Each rigid motion in $G_{k}$ contributes roughly $k^{2}$ quadruples.

$$
|Q(P)| \lesssim \sum_{j=1}^{\infty}\left(2^{j}\right)^{2} G_{2^{j}}
$$

There are only $N^{2}$ sets $L_{p q}$, and so this sum only has $\lesssim \log N$ terms. Elekes and Sharir conjectured that $\left|G_{k}\right| \lesssim N^{3} k^{-2}$, which implies that each term in the sum is bounded by $N^{3}$.

Elekes and Sharir refer to the sets $L_{p q}$ as helices. We shall now see that by a good choice of the parametrization of $G$, we may actually view them as lines in $\mathbf{R}^{3}$.

Let $G^{\prime}$ denote the open subset of the orientable rigid motion group $G$ given by rigid motions which are not translations. Each element of $G^{\prime}$ has a unique fixed point $(x, y)$ and an angle $\theta$ of rotation about the fixed point with $0<\theta<2 \pi$. We define the map

$$
\rho: G \longrightarrow \mathbf{R}^{3}
$$

by

$$
\rho(x, y, \theta)=\left(x, y, \cot \frac{\theta}{2}\right) .
$$

Proposition 2.2. Let $p=\left(p_{x}, p_{y}\right)$ and $q=\left(q_{x}, q_{y}\right)$ be points in $\mathbf{R}^{2}$. Then with $\rho$ as above, the set $\rho\left(L_{p q} \cap G^{\prime}\right)$ is a line in $\mathbf{R}^{3}$.

Proof. Noting that the fixed point of any transformation taking $p$ to $q$ must lie on the perpindicular bisector of $p$ and $q$, the reader will easily verify that the set $\rho\left(L_{p q} \cap G^{\prime}\right)$ can be parametrized as

$$
\begin{equation*}
\left(\frac{p_{x}+q_{x}}{2}, \frac{p_{y}+q_{y}}{2}, 0\right)+t\left(\frac{q_{y}-p_{y}}{2}, \frac{p_{x}-q_{x}}{2}, 1\right) . \tag{2.3}
\end{equation*}
$$

We will for the remainder of the paper be concerned with $\mathfrak{L}$, the set of lines in $\mathbf{R}^{3}$ given by $\left\{\rho\left(L_{p q} \cap G^{\prime}\right)\right\}_{p, q \in P}$. A rigid motion in $G^{\prime} \cap G_{k}$ corresponds to a point in $\mathbb{R}^{3}$ which lies in at least $k$ and at most $2 k$ lines of $\mathfrak{L}$.

We record some genericity properties of the set of lines $\mathfrak{L}$. (These are all observed in one form or another in the Elekes-Sharir paper as well.)
Proposition 2.3. No more than $N$ lines of $\mathfrak{L}$ meet at a single point. No more than $N$ lines of $\mathfrak{L}$ lie in a single plane. No more than $O(N)$ lines of $\mathfrak{L}$ lie in a single regulus (doubly ruled surface.)

Proof. For each $p \in P$, we consider the subset $\mathfrak{L}_{p} \subset \mathfrak{L}$ given by

$$
\mathfrak{L}_{p}=\left\{\rho\left(L_{p q} \cap G^{\prime}\right)\right\}_{q \in P} .
$$

To prove the first two assertions, note that from the definition of $L_{p q}$ no two of the lines may intersect and from equation [2.3 that the lines of $\mathfrak{L}_{p}$ all have different directions so that they must be pairwise skew. Therefore no two intersect at a point and no two lie in the same plane. Thus any set of lines meeting at a point or lying in a plane must come from different $\mathfrak{L}_{p}$ 's and this limits their number to $N$.

The situation for the third assertion is slightly more complicated. We define the set of lines $\mathfrak{L}_{p}^{\prime}=\left\{\rho\left(L_{p q} \cap G^{\prime}\right\}_{q \in \mathbb{R}^{2}}\right.$. (We have $\mathfrak{L}_{p} \subset \mathfrak{L}_{p}^{\prime}$.) If a regulus contains more than $O(1)$ lines of $\mathfrak{L}_{p}^{\prime}$, then all the lines in one ruling of the regulus are contained in $\mathfrak{L}_{p}^{\prime}$. If $p_{1} \neq p_{2}$, then $\mathfrak{L}_{p_{1}}^{\prime}$ and $\mathfrak{L}_{p_{2}}^{\prime}$ are disjoint. So if a regulus has more than $O(1)$ lines from one of its rulings in $\mathfrak{L}_{p_{1}}$, then none of the lines in that ruling lie in any other $\mathfrak{L}_{p_{2}}$. Thus for any regulus $R$ there may be at most two exceptional points $p$ for which $\mathfrak{L}_{p}$ contains up to $N$ lines lying in the regulus, but all other $\mathfrak{L}_{p}$ contribute at most $O(1)$, leaving a total of $O(N)$.

In light of the preceding, we may break down the proof of our main theorem into the following theorems of incidence geometry.
Theorem 2.4. Let $\mathfrak{L}$ be any set of $N^{2}$ lines in $\mathbf{R}^{3}$ for which no more than $N$ lie in a common plane and no more than $O(N)$ lie in a common regulus. Then the number of points of intersection of two lines in $\mathfrak{L}$ is $O\left(N^{3}\right)$.

Theorem 2.5. Let $\mathfrak{L}$ be any set of $N^{2}$ lines in $\mathbf{R}^{3}$ for which no more than $N$ lie in a common plane, and let $k$ be a number $3 \leq k \leq N$. Let $Q_{k}$ be the set of points where at least $k$ lines and not as many as $2 k$ lines meet. Then

$$
\left|Q_{k}\right|=O\left(\frac{N^{3}}{k^{2}}\right)
$$

Elekes and Sharir essentially conjectured these two theorems (Conjecture 1 in [ES]), and they essentially proved Theorem [2.5 in the case $k=3$.

The theorems 2.4 and 2.5 together imply theorem 1.1 by giving an upper bound on the number of distance quadruplets $\left(p_{1}, p_{2}, p_{3}, p_{4}\right)$. The reader should note that because of our choice of parametrization, Theorems 2.4 and 2.5 do not say anything about the distance quadruples coming from translations, but these are in fact the quadruplets where

$$
p_{3}-p_{1}=p_{4}-p_{2},
$$

so that they are actually additive quadruplets and clearly bounded by $N^{3}$ since fixing three points defines the fourth. Counting the quadruples coming from translation separately, we see that

$$
|Q(P)| \lesssim N^{3}+\sum_{j=1}^{\infty} 2^{2 j} Q_{2^{j}}
$$

In light of the first part of propostion [2.3, we need only consider $j$ with $2^{j} \leq N$ and we can use the theorems to bound the sum by

$$
|Q(P)| \lesssim N^{3}+\sum_{j=1}^{\log N} N^{3} \lesssim N^{3} \log N
$$

This, of course, implies the main theorem. Thus, in the remainder of the paper, we shall devote ourselves to proving the two theorems.

## 3 Flecnodes

Our goal in this section is to prove Theorem [2.4. We will do this by purely algebraic methods following essentially the proof strategy of [GK]. That is, we will show that an important subset of our lines lies in the zero set of a fairly low degree polynomial $p$. What requires a new idea is the next step. We need a polynomial $q$ derived from $p$ with similar degree on which the lines also vanish. With that information we will apply a variant of Bezout's lemma.

Lemma 3.1. Let $p(x, y, z)$ and $q(x, y, z)$ be polynomials on $\mathbf{R}^{3}$ of degrees $m$ and $n$ respectively. If there is a set of $m n+1$ distinct lines simultaneously contained in the zero set of $p$ and the zero set of $q$ then $p$ and $q$ have a common factor.

Thus we will conclude that $p$ and the derived polynomial $q$ must have a common factor and we will arrive at some geometrical conclusion from this based on the way that $q$ was derived. In the paper GK, the derived polynomials that we used were the gradient of $p$ and the algebraic version of the second fundamental form of the surface given by $p=0$. These were good choices because when three or more lines were incident at each point, we knew on geometric grounds that one or the other would vanish at each point, because the point would be either singular or flat. However, here we are faced with points at which only two lines intersect, and so we must make a more clever choice of the derived polynomial.
We begin with the definition of a flecnode. Given an algebraic surface in $\mathbf{R}^{3}$ given by the equation $p(x, y, z)=0$ where p is a polynomial of degree $d$ at least 3 , a flecnode is a point $(x, y, z)$ where a line agrees with the surface to order three. To find all such points, we might solve the system of equations:

$$
p(x, y, z)=0 ; \quad \nabla_{v} p(x, y, z)=0 ; \quad \nabla_{v}^{2} p(x, y, z)=0 ; \quad \nabla_{v}^{3} p(x, y, z)=0
$$

These are four equations for six unknowns, $(x, y, z)$ and the components for the direction $v$. However the last three equations are homogeneous in $v$ and may be viewed as three equations in five unknowns (and the whole system as 4 equations in 5 unknowns.) We may reduce the last three equations to a single equation in three unknowns $(x, y, z)$. We write the reduced equation as

$$
F l(p)(x, y, z)=0
$$

The polynomial $F l(p)$ is of degree $11 d-24$. It is called the flecnode polynomial of $p$ and vanishes at any flecnode of any level set of $p$. (See [Salm] Art. 588 pages 277-78.)

The term flecnode was apparently first coined by Cayley. The polynomial $F L(p)$ was discovered by the Rev. George Salmon, but its most important property to us was communicated to him by Cayley.
Proposition 3.2. The surface $p=0$ is ruled if and only if $F l(p)$ is everywhere vanishing on it.

One direction of the proposition is obvious. If the surface is ruled, there is a line contained in the surface at every point. If the line is contained in the surface, it certainly agrees to order 3. The reverse direction is more computational. It is indicated in a footnote to Salm Art. 588 page 278 . One sees that setting $\mathrm{FL}(\mathrm{p})=0$ is a way of rewriting a differential equation on $p$ which implies ruledness. Proposition 3.2 was used in a famous paper of Segre [Seg]. For a generalization to manifolds in higher dimensions see [Land].

An immediate corollary of the proposition is

Corollary 3.3. Let $p=0$ be a degree d hypersurface in $\mathbf{R}^{d}$. Suppose that the surface contains more than $11 d^{2}-24 d$ lines. Then $p$ has a ruled factor.

Proof. By lemma 3.1, since both $p$ and $F L(p)$ vanish on the same set of more than $11 d^{2}-24 d$ lines, they must have a common factor $q$. Thus $F L(q)=0$ at points of $q=0$ which are regular for $p$. Thus $F L(q)=0$ identically on $q=0$.

Now we would like to consider ruled surfaces of degree less than $N$. Thus our surfaces are the sets

$$
p(x, y, z)=0
$$

for a polynomial $p$ (which we may choose square free) of degree less than $N$. We may uniquely factorize the polynomial into irreducibles:

$$
p=p_{1} p_{2} \ldots p_{m} .
$$

We say that $p$ is plane-free and regulus-free if the zero sets of none of the factors is a plane or a regulus. Thus if $p$ is plane-free and regulus-free, the zero-set of each of the factors is an irreducible algebraic singly-ruled surface. We now state the main geometrical lemma for proving Theorem 2.4,
Lemma 3.4. Let $p$ be a polynomial of degree less than $N$ so that $p=0$ is ruled and so that $p$ is plane-free and regulus-free. Let $\mathfrak{L}_{1}$ be a set of lines contained in the surface $p=0$ with $\left|\mathfrak{L}_{1}\right| \lesssim N^{2}$. Let $Q_{1}$ be the set of points of intersection of lines in $\mathfrak{L}_{1}$. Then

$$
\left|Q_{1}\right| \lesssim N^{3} .
$$

Proof. We must first notice that not every line contained in the ruled surface is necessarily a generator of the ruled surface. However, for any $p_{j}$, an irreducible component of $p$ having degree $d_{j}$, it must be that a line which is not a generator intersects every generator of the surface. Choosing three generators $l_{1}, l_{2}, l_{3}$ which are pairwise skew (if such do not exist then the component is a plane, which by hypothesis is impossible), we see that every line which is not a generator must lie in a particular regulus which is the union of all lines intersecting $l_{1}, l_{2}, l_{3}$. Since we know that $p$ is regulus-free, we know that there are at most $2 d_{j}$ such lines contained in the zero set of $p_{j}$ by Lemma 3.1. Summing over $j$ we see that there are at most $2 N$ lines which are not generators in $\mathfrak{L}_{1}$. We call these non-generators $\mathfrak{L}_{2}$, and we let $\mathfrak{L}_{3}$ denote the remaining lines in $\mathfrak{L}_{1}$. There are at most $2 N^{3}$ intersections between a line in $\mathfrak{L}_{2}$ and a line in $\mathfrak{L}_{3}$ and there are at most $4 N^{2}$ intersections between lines in $\mathfrak{L}_{2}$.

Thus we need only consider intersections between lines in $\mathfrak{L}_{3}$. Fix a line $l$ in $\mathfrak{L}_{3}$. We will prove that it intersects $\leq N$ other lines in $\mathfrak{L}_{3}$. Hence the number of intersections between lines in $\mathfrak{L}_{3}$ is $\leq N^{3}$. Each line of $\mathfrak{L}_{3}$ is a generator of our ruled surface. The generators within one irreducible component of the ruled surface don't intersect each other. We only
have to estimate the number of intersections between $l$ and generators of other components. It may happen that $l$ lies in some other components. If $l$ lies in another component, then it must be a generator of that component (otherwise $l$ would be in $\mathfrak{L}_{2}$ ). If $l$ is a generator in another component then it doesn't intersect any other generators from that component. So we can ignore those components. So we only need to consider the intersections between $l$ and components that don't contain $l$. A component of degree $d_{j}$ intersects $l$ at most $d_{j}$ times. Each of those points lies in only one generator of the component. So there are $\leq d_{j}$ intersections between $l$ and generators of that component. So the total number of intersections between $l$ and other lines in $\mathfrak{L}_{3}$ is $\leq \sum d_{j} \leq N$.

Now we are ready to begin the proof of Theorem [2.4. We assume we have a set $\mathfrak{L}$ of at most $N^{2}$ lines for which no more than $N$ lie in a plane and no more than $N$ lie in a regulus. We suppose, by way of contradiction, that for $Q$, a positive real number sufficiently large, there are $Q N^{3}$ points of intersection of lines of $\mathfrak{L}$ and we assume that this is an optimal example, so that for no $M<N$ do we have a set of $M^{2}$ lines so that no more than $M$ lie in a plane and no more than $M$ lie in a regulus is it the case that there are more than $Q M^{3}$ intersections. ( $N$ need not be an integer.)

We now apply a degree reduction argument similar to the one in GK. We let $\mathfrak{L}^{\prime}$ be the subset of $\mathfrak{L}$ consisting of lines which intersect at least $\frac{Q N}{10}$ lines of $\mathfrak{L}$. Since the lines not in $\mathfrak{L}^{\prime}$ participate in at most $\frac{Q N^{3}}{10}$ intersections. Thus there are at least $\frac{9 Q N^{3}}{10}$ intersections between lines of $\mathfrak{L}^{\prime}$. Moreover $\mathfrak{L}^{\prime}$ has $\alpha N^{2}$ lines with $0<\alpha<1$.

Now we select a random subset $\mathfrak{L}^{\prime \prime}$ of the lines of $\mathfrak{L}^{\prime}$ choosing lines independently with probability $\frac{100}{Q}$. With positive probability, there will be no more than $\frac{200 \alpha N^{2}}{Q}$ lines in $\mathfrak{L}^{\prime \prime}$ and each line of $\mathfrak{L}^{\prime}$ will intersect $\frac{N}{20}$ lines of $\mathfrak{L}^{\prime \prime}$. Now pick $\frac{R \sqrt{\alpha} N}{\sqrt{Q}}$ points on each line of $\mathfrak{L}^{\prime \prime}$. ( $R$ is a constant which is sufficiently large but universal.) Call the set of all of the points $\mathfrak{S}$. There are $O\left(\frac{R \alpha^{\frac{3}{2}} N^{3}}{Q^{\frac{3}{2}}}\right)$ points in $\mathfrak{S}$, so we may find a polynomial $p$ of degree $O\left(\frac{R^{\frac{1}{3}} \alpha^{\frac{1}{2}} N}{Q^{\frac{1}{2}}}\right)$ which vanishes on every point of $\mathfrak{S}$. With $R$ sufficiently large, $p$ must vanish identically on every line of $\mathfrak{L}^{\prime \prime}$. Since each line of $\mathfrak{L}^{\prime}$ meets $\frac{N}{20}$ lines of $\mathfrak{L}^{\prime \prime}$, it must be that $p$ vanishes identically on each line of $\mathfrak{L}^{\prime}$. Thus ends the degree reduction argument and we will now study the relatively low degree polynomial $p$.
We may factor $p=p_{1} p_{2}$ where $p_{1}$ is the product of the ruled irreducible factors of $p$ and $p_{2}$ is the product of unruled irreducible factors of $p$. Each of $p_{1}$ and $p_{2}$ is of degree $O\left(\frac{\alpha^{\frac{1}{2}} N}{Q^{\frac{1}{2}}}\right)$. (We have suppressed the $R$ dependence since $R$ is universal.) We break up the set of lines of $\mathfrak{L}^{\prime}$ into the disjoint subsets $\mathfrak{L}_{1}$ consisting of those lines in the zero set of $p_{1}$ and $\mathfrak{L}_{2}$ consisting of all the other lines in $\mathfrak{L}^{\prime}$.

There are no more than $O\left(N^{3}\right)$ points of intersection between lines of $\mathfrak{L}_{1}$ and $\mathfrak{L}_{2}$ since each
line of $\mathfrak{L}_{2}$ contains no more than $O\left(\frac{\alpha^{\frac{1}{2}} N}{Q^{\frac{1}{2}}}\right)$ points where $p_{1}$ is zero. Thus we are left with two (not mutually-exclusive) cases which cover all possibilities. There are either $\frac{3 Q N^{3}}{10}$ points of intersection between lines of $\mathfrak{L}_{1}$ or there are $\frac{3 Q N^{3}}{10}$ points of intersection between lines of $\mathfrak{L}_{2}$. We will handle these separately.
Suppose there are $\frac{3 Q N^{3}}{10}$ intersections between lines of $\mathfrak{L}_{1}$. We factor $p_{1}=p_{3} p_{4}$ where $p_{3}$ is plane-free and regulus-free and $p_{4}$ is a product of planes and reguli. We break $\mathfrak{L}_{1}$ into disjoint sets $\mathfrak{L}_{3}$ and $\mathfrak{L}_{4}$, with $\mathfrak{L}_{3}$ consisting of lines in the zero set of $p_{3}$ and $\mathfrak{L}_{4}$ consisting of all other lines of $\mathfrak{L}_{1}$. As before there $O\left(N^{3}\right)$ points of intersection between lines of $\mathfrak{L}_{3}$ and $\mathfrak{L}_{4}$ since lines of $\mathfrak{L}_{4}$ are not in the zero set of $p_{3}$. Moreover there are at most $O\left(N^{3}\right)$ points of intersection between lines of $\mathfrak{L}_{4}$ because they lie in at most $N$ planes and reguli each containing at most $N$ lines. (We just see that each line has at most $O(N)$ intersections with planes and reguli it is not contained in and there are at most $O\left(N^{2}\right)$ points of intersection between lines internal to each plane and regulus.) However there cannot be more than $O\left(N^{3}\right)$ points of intersection between lines of $\mathfrak{L}_{3}$ by applying the key lemma 3.4. (Here we used that $p_{3}$ is plane-free and regulus-free.)

Thus we must be in the second case, where many of the points of intersection are between lines of $\mathfrak{L}_{2}$, all of which lie in the zero set of $p_{2}$ which is totally unruled. Recall that $p_{2}$ is of degree $O\left(\frac{\alpha^{\frac{1}{2}} N}{Q^{\frac{1}{2}}}\right)$. Thus by Corollary [3.3, its zero-set contains no more than $O\left(\frac{\alpha N^{2}}{Q}\right)$ lines. We would like to now invoke the fact that the example we started with was optimal and reach a contradiction. But we can't quite do that. Our set $\mathfrak{L}_{2}$ has $\beta N^{2}$ lines with $\beta=O\left(\frac{\alpha}{Q}\right)$ and we only know that there are no more than $N$ lines in any plane or regulus, whereas we need to know that there are no more than $\sqrt{\beta} N$ lines. If this is the case we are done. If not we construct a subset $\mathfrak{L}_{5}$ as follows. If there is a plane or regulus containing more than $\sqrt{\beta} N$ lines of $\mathfrak{L}_{2}$, we put those lines in $\mathfrak{L}_{5}$ and remove them from $\mathfrak{L}_{2}$. We repeat as needed labelling the remaining lines $\mathfrak{L}_{6}$. Since we removed $O(N)$ planes and reguli, there are $O\left(N^{3}\right)$ points of intersection between lines of $\mathfrak{L}_{5}$. Since no lines of $\mathfrak{L}_{6}$ belong to any plane or regulus of $\mathfrak{L}_{5}$ there are fewer than $O\left(N^{3}\right)$ points of intersection between lines of $\mathfrak{L}_{5}$ and $\mathfrak{L}_{6}$. Now we apply optimality of our original example to rule out more than $O\left(\frac{N^{3}}{Q^{\frac{1}{2}}}\right)$ points of intersection between lines of $\mathfrak{L}_{6}$. Thus we have reached a contradiction.

## 4 Cell decompositions

In this section, we prove an estimate for incidences of lines when not too many lines lie in a plane.
Theorem 4.1. Let $k \geq 3$. Let $\mathfrak{L}$ be a set of $L$ lines in $\mathbb{R}^{3}$ with at most $B$ lines in any plane. Let $\mathfrak{S}$ be the set of points in $\mathbb{R}^{3}$ intersecting at least $k$ lines of $\mathfrak{L}$. Then the following
inequality holds:

$$
|\mathfrak{S}| \leq C\left[L^{3 / 2} k^{-2}+L B k^{-3}+L k^{-1}\right] .
$$

Theorem 4.1 implies Theorem 2.5 by setting $L=N^{2}$ and $B=N$.
To get a sense of the right-hand side, we consider some examples.
Example 1. Choose $L / k$ points. Let $\mathfrak{L}$ consist of $k$ lines through each point. The set $\mathfrak{L}$ has a k-fold incidence at each of the $L / k$ points. (We can also arrange that no three lines lie in a plane.)

Example 2. Choose $L / B$ planes. Put $B$ lines in each of the planes. The $B$ lines in each plane can be arranged to create $B^{2} k^{-3} \mathrm{k}$-fold incidences. (See the examples in SzT].) This set of lines has a total of $L B k^{-3} \mathrm{k}$-fold incidences.
Example 3. Let $G_{0}$ denote the integer lattice $\{(a, b, 0)\}$ with $1 \leq a, b \leq L^{1 / 4}$. Let $G_{1}$ denote the integer lattice $\{(a, b, 1)\}$ with $0 \leq a, b \leq L^{1 / 4}$. Let $\mathfrak{L}$ denote all the lines from a point of $G_{0}$ to a point of $G_{1}$. (This set of lines appears when we take $P$ to be a square grid in $\mathbb{R}^{2}$ and consider the corresponding incidence problem in the Elekes-Sharir framework.) Any plane contains at most $L^{1 / 4}$ points of either grid and so at most $L^{1 / 2}$ lines of $\mathfrak{L}$.
Theorem 4.2. Let $\mathfrak{L}$ be the set of lines in Example 3. For any $k$ in the range $2 \leq k \leq$ $(1 / 4) L^{1 / 2}$, the number of $k$-fold incidences of $\mathfrak{L}$ is $\gtrsim(\log L)^{-1} L^{3 / 2} k^{-2}$.

Proof. Consider a point $x$ in $\mathbb{R}^{3}$ contained in the slab $0<x_{3}<1$. We define a map $F_{x}: \mathbb{R}^{2} \rightarrow \mathbb{R}^{2}$ by saying that $F_{x}(a, b)=(c, d)$ if the line from $(a, b, 0)$ through $x$ hits $(c, d, 1)$. We define $G$ to be the integral grid in the plane given by $\{(a, b)\}$ with $1 \leq a, b \leq L^{1 / 4}$. The number of lines from $\mathfrak{L}$ which pass through $x$ is exactly the cardinality of $F_{x}(G) \cap G$. Now any intersection of two lines from $\mathfrak{L}$ will have rational coordinates, so we can assume the coordinates of $x$ are rational. Let us say that the $x_{3}$ coordinate of $x$ is $p / q$, written in lowest terms. By a similar triangles argument, $F_{x}(G)$ is a square grid with spacing $\frac{q-p}{p}$. The maximum possible number of intersections between a grid with this spacing and a grid with unit spacing is $\sim L^{1 / 2} q^{-2}$. Let us say that the middle half of $G$, written $G_{\text {middle }} \subset G$, is the integral grid $\{(a, b)\}$ with $(1 / 4) L^{1 / 4} \leq a, b \leq(3 / 4) L^{3 / 4}$. If $F_{x}$ maps a vertex from $G_{\text {middle }}$ into $G$, then the number of intersections between $F_{x}(G)$ and $G$ is $\sim L^{1 / 2} q^{-2}$. If we fix $p$ and $q$, then the number of choices of $x=\left(x_{1}, x_{2}, p / q\right)$ so that $F_{x}\left(G_{\text {middle }}\right) \cap G$ is non-empty is $\sim L^{1 / 2} q^{2}$.
Now we fix $k \leq(1 / 4) L^{1 / 2}$. We pick $q$ in the range $L^{1 / 4} k^{-1 / 2} \leq q \leq 2 L^{1 / 4} k^{-1 / 2}$. For each $q$ we pick $p$ coprime to $q$. For each $p, q$, we pick $x$ as above. For each $x, F_{x}(G) \cap G$ has $\gtrsim L^{1 / 2} q^{-2} \sim k$ elements. Therefore, each $x$ lies in $\sim k$ lines of $\mathfrak{L}$. So the number of k-fold incidences of $\mathfrak{L}$ is at least

$$
\sim \sum_{q=L^{1 / 4} k^{-1 / 2}}^{2 L^{1 / 4} k^{-1 / 2}} \phi(q) L^{1 / 2} q^{2} \sim L k^{-1} \sum_{q=L^{1 / 4} k^{-1 / 2}}^{2 L^{1 / 4} k^{-1 / 2}} \phi(q)
$$

Here $\phi(q)$, Euler's totient function, is the number of integers $0<p<q$ which are coprime to $q$. When $q$ is prime $\phi(q)=q-1$. There are $\gtrsim(\log L)^{-1} L^{1 / 4} k^{-1 / 2}$ primes $q$ in our sum, and so the number of k-fold incidences is $\gtrsim(\log L)^{-1} L^{3 / 2} k^{-2}$.

Remark. It looks likely that $\sum_{q=Q}^{2 Q} \phi(q) \gtrsim Q^{2}$, which would imply that the number of k -fold incidences in Example 3 is $\gtrsim L^{3 / 2} k^{-2}$ with no log term.

Before starting the proof of Theorem 4.1, let us describe why it looks difficult. Elekes and Sharir essentially proved this theorem when $k=3$ using the algebraic method. Using the degree reduction argument from Section 3, one can prove that most of the points lie in a surface of degree $\lesssim L^{2} S^{-1} k^{-2}$. For $k=3$, one can then apply the algebraic method using this surface to prove the desired inequality. But when $k$ is large, this degree is not low enough to run the algebraic method successfully. To use the algebraic method, we need the points and lines to lie in a surface of degree $L^{2} S^{-1} k^{-3}$. (In the section below called the algebraic case, we explain how to prove the conclusion once the points lie in a surface of degree $\lesssim L^{2} S^{-1} k^{-3}$. By examining that section, one can check that when the degree is any higher than this, we get a worse conclusion.)

Theorem 4.1 also does not hold over finite fields. Let $F_{p}$ denote the field with $p$ elements. Let $\mathfrak{L}$ denote all of the lines in $F_{p}^{3}$, and let $\mathfrak{S}$ denote all of the points in $F_{p}^{3}$. Each point lies in $\sim p^{2}$ lines, so we can take $k=p^{2}$. There are at most $\sim p^{2}$ lines in a plane, so we can take $B=2 p^{2}$. The set of lines in $F_{p}^{3}$ has $\sim p^{4}$ elements. If the theorem held, we would expect $|\mathfrak{S}| \leq C p^{2}$. But $|\mathfrak{S}|=p^{3}$. The fact that the theorem is false over finite fields and the fact that the degree in the last paragraph is too high seem related: since the theorem is false in finite fields, it probably has no purely algebraic proof.

The situation we just described is similar to the Szemeredi-Trotter theorem, which we now recall.
Theorem 4.3. (Szemeredi-Trotter, [SzT] ) If $\mathfrak{L}$ is a set of $L$ lines in the plane and $\mathfrak{S}$ is a set of $S$ points in the plane, and each point lies in at least $k$ lines of $\mathfrak{L}$, then

$$
S \leq C\left[L^{2} k^{-3}+L k^{-1}\right]
$$

The Szemeredi-Trotter theorem is also false over finite fields: let $\mathfrak{L}$ be all of the lines in $F_{p}^{2}$, and let $\mathfrak{S}$ be all of the points in $F_{p}^{2}$. All of the proofs of the Szemeredi-Trotter theorem use in some way the topology of $\mathbb{R}^{2}$. For example, the proof in [CEGSW] uses a cell decomposition of $\mathbb{R}^{2}$. They consider some (carefully chosen) lines in $\mathbb{R}^{2}$, and the
complement of the lines is a union of connected components which are the cells. They then consider how many lines intersect each cell. Anyway, this argument involves the topology of $\mathbb{R}^{2}$ : a set of lines in $F_{p}^{2}$ does not divide their complement into cells.
This analogy suggests using topological methods to attack Theorem 4.1, imitating some of the proofs of Szemeredi-Trotter. However, the topological methods have not been able to prove sharp estimates in 3 -dimensional incidence problems. For example, one may consider the joints problem, which is closely related to the case $k=3$ of Theorem 4.1.
Theorem 4.4. (Joints theorem, Guth-Katz, [GK]) A set of $L$ lines in $\mathbb{R}^{3}$ determines $\lesssim L^{3 / 2}$ joints. A joint is a point where three non-coplanar lines of the set intersect.

The joints theorem was proven using the algebraic method. Before that, it was attacked using topological and cell-decomposition ideas related to the proofs of Szemeredi-Trotter. These ideas made interesting progress in papers like [SW] and [FS]. For example, in [FS], Feldman and Sharir proved that the number of joints is at most $O\left(L^{1.62}\right)$. However, no argument of this kind has gotten a sharp result for joints.

To summarize, Theorem4.1 combines the difficulties of the joints problem and the SzemerediTrotter theorem. The proof requires a combination of algebra and topology. A key tool will be the polynomial ham sandwich theorem, which we will use to build a cell decomposition of $\mathbb{R}^{3}$ where the walls of the cells are defined by a polynomial of degree $\lesssim L^{2} S^{-1} k^{-3}$. Our cell decomposition creates a dichotomy. If most of the points lie in the interiors of the cells, then we can prove our inequality using the cellular method as in CEGSW. If most of the points lie in the walls of the cells, then the points lie in an algebraic surface of low degree, and we can prove our inequality using the algebraic method as in [GK] or [EKS].
We now quote the polynomial ham sandwich theorem of Stone and Tukey. It was proved in [ST]. For more discussion see [G].

We say that an algebraic hypersurface $p\left(x_{1}, \ldots, x_{n}\right)=0$ bisects a set $U$ of finite but positive volume if the volume of $U \cap\{p<0\}$ and $U \cap\{p>0\}$ is the same.
Theorem 4.5. Let $U_{1}, \ldots, U_{M}$ be any finite volume subsets of $\mathbf{R}^{n}$ having positive volume with $M=\binom{n+d}{n}-1$. Then there is a real algebraic hypersurface of degree at most $d$ that bisects each $U_{i}$.

We would now like to adapt theorem 4.5 to finite sets of points. Given a set of points $S$, we say that a polynomial $p$ bisects it if at least half the points in $S$ are in $\{p \geq 0\}$ and at least half the points in $S$ are in $\{p \leq 0\}$. Note that some of the points can be on the zero set. It is only important that there is not an absolute majority on one side or the other. We now give a discrete version of the ham sandwich theorem.
Corollary 4.6. Let $S_{1}, \ldots, S_{M}$ be disjoint finite set of points in $\mathbf{R}^{n}$ with $M=\binom{n+d}{n}-1$. Then there is a real algebraic hypersurface of degree at most d that bisects each $U_{i}$.

Proof. Fix $\epsilon>0$ to be the minimum distance between any of the points in the sets $S_{j}$.

For any $\frac{\epsilon}{2}>\delta>0$ define $U_{j, \delta}$ to be the union of closed $\delta$ balls centered at points of $S_{j}$. We obtain a polynomial $p_{\delta}$ of degree $d$ which bisects the $U_{j, \delta}$ by Theorem 4.5. We find a sequence $\left\{\delta_{j}\right\}$ converging to 0 so that $p_{\delta_{j}}$ converge in the space of polynomials of degree $d$. We let $p$ be the limiting polynomial.

Let $V_{j}^{+}$be the set of points at distance $\leq \delta_{j}$ from the set where $p_{\delta_{j}} \geq-\delta_{j}$ and $V_{j}^{-}$be the set of points at distance $\leq \delta_{j}$ from the set where $p_{\delta_{j}} \leq \delta_{j}$. Clearly for each $j$ and $i$, we have that $V_{j}^{+}$and $V_{j}^{-}$contain at least half the points in each $S_{i}$.

Now, we can work in the set $K$ defined as the union of all the $\epsilon$-balls centered around points in any of the $S_{i}$ 's. Clearly $K$ is compact so $p_{\delta_{j}}$ is converging uniformly on $K$. Thus if we let $V^{+}$and $V^{-}$be the sets where $p \geq 0$ and $p \leq 0$ respectively, we have that

$$
V^{+}=\operatorname{closure}\left(\bigcap_{m=1}^{\infty} \bigcup_{j=m}^{\infty} V_{j}^{+}\right)
$$

and

$$
V^{-}=\operatorname{closure}\left(\bigcap_{m=1}^{\infty} \bigcup_{j=m}^{\infty} V_{j}^{-}\right)
$$

Thus $p$ bisects each $S_{i}$.

An important special case of Theorem 4.11 is the uniform case where each point has $\sim k$ lines through it and each line contains about the same number of points. We will first prove the theorem under some uniformity hypotheses. Later we will reduce the general theorem to this case by simple arguments.
Lemma 4.7. Let $k \geq 3$. Let $\mathfrak{L}$ be a set of $L$ lines in $\mathbb{R}^{3}$ with at most $B$ lines in any plane. Let $\mathfrak{S}$ be a set of $S$ points in $\mathbb{R}^{3}$ so that each point intersects between $k$ and $2 k$ lines of $\mathfrak{L}$.

Also, we assume that there are $\geq \frac{1}{100} L$ lines in $\mathfrak{L}$ which each contain $\geq \frac{1}{100} S k L^{-1}$ points of $\mathfrak{S}$.
Then $S \leq C\left[L^{3 / 2} k^{-2}+L B k^{-3}+L k^{-1}\right]$.
Proof. If $S \leq L k^{-1}$, then we are done. So from now on, we assume that $S>L k^{-1}$.
Next we apply the Szemeredi-Trotter theorem. As stated the Szemeredi-Trotter theorem applies to lines and points in $\mathbb{R}^{2}$. But if we take a generic projection from $\mathbb{R}^{3}$ to $\mathbb{R}^{2}$ we will get a set of lines with at least as many k-fold incidences. Therefore, the Szemeredi-Trotter theorem also applies to lines in $\mathbb{R}^{3}$ and we get $S \lesssim L^{2} k^{-3}+L k^{-1}$. Since we know $S>L k^{-1}$, we can conclude that $S \lesssim L^{2} k^{-3}$.

We define $d=A L^{2} S^{-1} k^{-3}$, where $A$ is a large constant that we will choose later. Because of the Szemeredi-Trotter bound, we can assume $d \geq 1$, and so we can assume that $d$ is an integer by slightly adjusting our choice of $A$. If $S$ is close to the Szemeredi-Trotter bound then $d$ is close to 1 .

Now we come to the main idea of the paper. We are going to divide $\mathbb{R}^{3}$ into $\sim d^{3}$ cells using an algebraic surface of degree $\leq d$ using the discrete ham sandwich theorem, Corollary 4.6. Either many points of $\mathfrak{S}$ will lie in the interiors of the cells, or else many points will lie in the algebraic surface. If many points lie in the interiors of the cells, we will prove our estimate using the cell method of incidence geometry as in [CEGSW]. If many points lie in the algebraic surface, we will prove our estimate using the algebraic method as in [GK] and [EKS.
First we produce our cell decomposition by repeatedly using Corollary 4.6. We will define a sequence of polynomials $p_{1}, \ldots, p_{J}$ as follows. We choose $p_{1}$ to bisect $\mathfrak{S}$. Some of the points of $\mathfrak{S}$ live in the zero set of $p_{1}$. The remainder can be subdivided in $\mathfrak{S}_{+}$and $\mathfrak{S}_{-}$ living respectively in sets where $p_{1}$ is strictly positive or strictly negative. Then we apply corollary 4.6 again to find $p_{2}$ which bisects $\mathfrak{S}_{+}$and $\mathfrak{S}_{-}$. The degree of $p_{j}$ is $\lesssim 2^{j / 3}$. Then we define $p$ to be the product of these polynomials. It has degree $\lesssim \sum_{j=1}^{J} 2^{j / 3} \lesssim 2^{J / 3}$. We choose $J$ so that $2^{J} \sim d^{3}$ but so the degree of $p$ is $\leq d$. The zero set of $p$, which we call $Z$, divides $\mathbb{R}^{3}$ into $\leq 2^{J} \lesssim d^{3}$ cells. (A cell is defined by giving the sign of $p_{1}, p_{2}, \ldots, p_{j}$. There are $2^{j}$ different cells. Each cell is an open set of $\mathbb{R}^{3}$. Some cells may be empty. The boundary of any cell is contained in $Z$.) Because we bisect the points at every step, each cell contains $\leq S 2^{-J} \lesssim S d^{-3}$ points of $\mathfrak{S}$.

What we do next depends on how many points lie in the interiors of cells and how many points lie in $Z$. If there are $\geq 10^{-8} S$ points of $\mathfrak{S}$ in the interiors of cells, we say we are in the cellular case. Otherwise, we say we are in the algebraic case.

## Cellular case

Since the interiors of the cells contain $\gtrsim S$ points, and since we have $\lesssim d^{3}$ cells each with $\lesssim S d^{-3}$ points, there must be $\gtrsim d^{3}$ cells with $\gtrsim S d^{-3}$ points in the interior of each. We call these cells "full cells".

If a full cell has $\leq k$ points in it, then we compute $S \lesssim L^{3 / 2} k^{-2}$. (We have $S d^{-3} \lesssim k$. Plugging in for $d, S L^{-6} S^{3} k^{9} \lesssim k$. Rearranging, $S^{4} \lesssim L^{6} k^{-8}$. )

On the other hand, if every full cell has $>k$ points in it, we will reach a contradiction because there are too many intersections between the lines $\mathfrak{L}$ and the surface $Z$.

Consider a full cell, and let $S_{\text {cell }}$ denote the number of points of $\mathfrak{S}$ in the interior of the cell and $L_{\text {cell }}$ the number of lines of $\mathfrak{L}$ which intersect the interior of the cell. Using Szemeredi-Trotter, we see that

$$
S_{\text {cell }} \lesssim L_{\text {cell }}^{2} k^{-3}+L_{\text {cell }} k^{-1} .
$$

Because $S_{\text {cell }} \geq k$, the first term on the right-hand side is dominant and we see that

$$
S d^{-3} \lesssim S_{\text {cell }} \lesssim L_{\text {cell }}^{2} k^{-3} .
$$

Rearranging, we get a lower bound for the number of lines in the cell:

$$
L_{\text {cell }} \gtrsim S^{1 / 2} d^{-3 / 2} k^{3 / 2}
$$

Now each of these lines intersects $Z$ in the boundary of the full cell. There are $\sim d^{3}$ full cells, so the total number of intersections between lines of $\mathfrak{L}$ and $Z$ is

$$
\gtrsim S^{1 / 2} d^{3 / 2} k^{3 / 2}
$$

On the other hand, $Z$ has degree $\lesssim d$. If $l$ is a line of $\mathfrak{L}$ which goes through the interior of some cell, then $l$ does not lie in $Z$, and so $l$ intersects $Z \lesssim d$ times. Therefore, the total number of intersections between lines of $\mathfrak{L}$ not lying in $Z$ and $Z$ is $\lesssim L d$. Comparing the last two estimates, we see that

$$
S^{1 / 2} d^{3 / 2} k^{3 / 2} \lesssim L d
$$

Rearranging and plugging in the definition of $d$, we see that $A \lesssim 1$. Now we choose $A$ sufficiently large compared to the constants in our argument and we get a contradiction.

## Algebraic case

The arguments in the algebraic case are very similar to those in [GK] and EKS. We will give a self-contained presentation, but the reader may consult these papers for longer explanations of certain points.

Now we may assume that $<10^{-8} S$ points of $\mathfrak{S}$ lie in the interiors of the cells. We let $\mathfrak{S}_{1}$ denote the subset of $\mathfrak{S}$ lying in $Z$. We know that $\left|\mathfrak{S} \backslash \mathfrak{S}_{1}\right|<10^{-8} S$.
Now we define $\mathfrak{L}_{1}$ to be the set of lines in $\mathfrak{L}$ which contain at least $10^{-8} S k L^{-1}$ points of $\mathfrak{S}_{1}$.
If this number $10^{-8} S k L^{-1}$ is $\lesssim d$, then we compute that $S \lesssim L^{3 / 2} k^{-2}$, and we are done. (Plugging in the definition of $d$, we would have, $S k L^{-1} \lesssim L^{2} S^{-1} k^{-3}$ and so $S^{2} \lesssim L^{3} k^{-4}$. ) So we can assume that each line of $L_{1}$ meets $>d$ points of $\mathfrak{S}_{1} \subset Z$. Therefore, every line in $\mathfrak{L}_{1}$ lies in $Z$.

We have slightly pruned our lines and points, but the pair $\left(\mathfrak{L}_{1}, \mathfrak{S}_{1}\right)$ still contains many high incidence points. Each line $l$ in $\mathfrak{L} \backslash \mathfrak{L}_{1}$ contains $<10^{-8} S k L^{-1}$ incidences with $\mathfrak{S}_{1}$. So the total number of incidences between $\mathfrak{L} \backslash \mathfrak{L}_{1}$ and $\mathfrak{S}_{1}$ is $\leq 10^{-8} S k$. Each point of $\mathfrak{S}_{1}$ has $\geq k$ incidences with lines in $\mathfrak{L}$. Define $\mathfrak{S}_{2}$ to be the subset of $\mathfrak{S}_{1}$ that intersects at least $(4 / 5) k$ lines of $\mathfrak{L}_{1}$. If $p$ is a point in $\mathfrak{S}_{1} \backslash \mathfrak{S}_{2}$, then $p$ must intersect at least $(1 / 5) k$ lines from $\mathfrak{L} \backslash \mathfrak{L}_{1}$. Therefore,

$$
\left|\mathfrak{S}_{1} \backslash \mathfrak{S}_{2}\right|(1 / 5) k \leq 10^{-8} S k
$$

In other words, $\left|\mathfrak{S}_{1} \backslash \mathfrak{S}_{2}\right| \leq(1 / 2) 10^{-7} S$, and so $\left|\mathfrak{S} \backslash \mathfrak{S}_{2}\right| \leq 10^{-7} S$.
Now each point of $\mathfrak{S}_{2} \subset Z$ is a special point of the variety $Z$. A point $x \in \mathfrak{S}_{2}$ has $\geq(4 / 5) k \geq 12 / 5>2$ lines of $\mathfrak{L}_{1}$ passing through it. (Here we use $k \geq 3$.) These lines lie in the variety $Z$. If the lines are not coplanar, then $x$ is a singular point of $Z$. (If $x$ were a regular point of $Z$, all the lines would have to lie in the tangent plane to $Z$ at $x$.) If $x$ is non-singular, then the lines of $\mathfrak{L}_{1}$ thru $x$ are coplanar. In this case, $x$ is a flat point of $Z$ : a non-singular point at which the second fundamental form of $Z$ is zero. If the second fundamental form were non-zero, then the non-trivial quadratic surface that agrees with $Z$ to order 2 at the point $x$ would have to contain $>2$ lines, and this is impossible. We write $\mathfrak{S}_{2}=\mathfrak{S}_{\text {sing }} \cup \mathfrak{S}_{\text {flat }}$.
Now we define $\mathfrak{L}_{2} \subset \mathfrak{L}$ to be the set of lines containing $\geq(1 / 200) S k L^{-1}$ points of $\mathfrak{S}_{2}$. Since $\mathfrak{S}_{2} \subset \mathfrak{S}_{1}$, a line in $\mathfrak{L}_{2}$ contains $\geq 10^{-8} S k L^{-1}$ points of $\mathfrak{S}_{1}$, and so $\mathfrak{L}_{2} \subset \mathfrak{L}_{1}$.

Our uniformity hypothesis will allow us to prove that $\mathfrak{L}_{2}$ is not too small. Recall that we assumed there are $\geq \frac{1}{100} L$ lines in $\mathfrak{L}$ which each contain $\geq \frac{1}{100} S k L^{-1}$ points of $\mathfrak{S}$. Not all of these lines need to contain $\geq \frac{1}{200} S k L^{-1}$ points of $\mathfrak{S}_{2}$, but most of them do. There are only $10^{-7} S$ points in $\mathfrak{S} \backslash \mathfrak{S}_{2}$. Each of these points lies in $\leq 2 k$ lines of $\mathfrak{L}$. So they contribute only $2 \cdot 10^{-7} S k$ incidences. So there are still at least $\frac{1}{200}$ lines in $\mathfrak{L}$ which contain at least $\frac{1}{200} S k L^{-1}$ points of $\mathfrak{S}_{2}$. In other words, $\mathfrak{L}_{2}$ contains at least $(1 / 200) L$ lines.
We define $\mathfrak{L}_{\text {sing }} \subset \mathfrak{L}_{2}$ to be the set of lines in $\mathfrak{L}_{2}$ containing at least (1/400)SkL $L^{-1}$ points of $\mathfrak{S}_{\text {sing }}$. Similarly, we define $\mathfrak{L}_{\text {flat }} \subset \mathfrak{L}_{2}$ to be the set of lines in $\mathfrak{L}_{2}$ containing at least $(1 / 400) S k L^{-1}$ points of $\mathfrak{S}_{\text {flat }}$. Each line in $\mathfrak{L}_{2}$ is either in $\mathfrak{L}_{\text {sing }}$ or in $\mathfrak{L}_{\text {flat }}$, maybe both. So either $\left|\mathfrak{L}_{\text {sing }}\right| \geq(1 / 400) L$ or $\left|\mathfrak{L}_{\text {flat }}\right| \geq(1 / 400) L$. We call these cases the singular subcase and the flat subcase. They are subcases of the algebraic case.

## The singular subcase

Recall that $Z$ is the vanishing set of a polynomial $p$ of degree $\leq d$. We can assume that $p$ is square-free. We recall that the singular set of $Z$ is defined to be the set of points $x$ where $p(x)=0$ and $\nabla p(x)=0$. Recall that $\nabla p(x)$ is the gradient vector $\left(\partial_{1} p(x), \partial_{2} p(x), \partial_{3} p(x)\right)$. Because $p$ is square-free, the singular part of $Z$ is a subvariety of dimension $\leq 1$. Its

1-dimensional part has degree $\leq 6 d^{2}$, and so the singular part of $Z$ contains $\leq 6 d^{2}$ lines. This estimate is a consequence of Bezout's theorem - see Lemma 3.1. See [GK] for a more detailed explanation.

Each line in $\mathfrak{L}_{\text {sing }}$ contains $\gtrsim S k L^{-1}$ singular points of $Z$. If the number of singular points on a line is $\leq d$, then we compute $S \lesssim L^{3 / 2} k^{-2}$. (We have $S k L^{-1} \lesssim d \lesssim L^{2} S^{-1} k^{-3}$. Hence $S^{2} \lesssim L^{3} k^{-4}$. ) So we may assume each line of $\mathfrak{L}_{\text {sing }}$ contains $>d$ singular points of $Z$. Now $p$ vanishes at each of these points, so it vanishes on the entire line. Also, each $\partial_{i} p$ is a polynomial of degree $<d$, and it vanishes at the singular points, so it vanishes on the entire line. So we conclude that every line in $\mathfrak{L}_{\text {sing }}$ is contained in the singular set of $Z$.

Since $\mathfrak{L}_{\text {sing }}$ contains $\gtrsim L$ lines, and the singular set of $Z$ contains $\lesssim d^{2}$ lines, we can conclude that $L \lesssim d^{2} \lesssim L^{4} S^{-2} k^{-6}$. Rearranging we get $S^{2} \lesssim L^{3} k^{-6}$ and so $S \lesssim L^{3 / 2} k^{-3}$.

## The flat subcase

Recall that a non-singular point $x \in Z$ is called flat if the second fundamental form of $Z$ vanishes at $x$. This definition is not very algebraic at first sight, but it can be turned into an algebraic condition.
Lemma 4.8. Let $x$ be a non-singular point of the variety $Z$ defined by $p(x)=0$. Then $x$ is flat if and only if the following three polynomials vanish at $x$ :

$$
\nabla_{e_{j} \times \nabla p} p \times \nabla p, j=1,2,3 .
$$

Here, $e_{j}$ are the coordinate vectors of $\mathbb{R}^{3}$, and $\times$ denotes the cross product of vectors. For more explanation, see Section 3 of [GK].
We define the variety $Z_{f l a t}$ as $\left\{x \mid p(x)=0, \nabla_{e_{j} \times \nabla p} p \times \nabla p(x)=0, j=1,2,3\right\}$. The variety $Z_{\text {flat }}$ is the closure of the set of flat non-singular points of $Z$.

Each of the polynomials in Lemma 4.8 has degree $\leq 3 d$. (Recall that our polynomial $p$ has degree $\leq d$.) Therefore, if a line contains more than $3 d$ flat points of $Z$ then the line lies in $Z_{\text {flat }}$.
Each line in $\mathfrak{L}_{\text {flat }}$ contains $\gtrsim S k L^{-1}$ flat points of Z . If one of the lines has $\leq 3 d$ flat points, then $S k L^{-1} \lesssim d$ and we conclude $S \lesssim L^{3 / 2} k^{-2}$ as above. So we may assume that every line of $\mathfrak{L}_{\text {flat }}$ lies in $Z_{\text {flat }}$.
Now the variety $Z$ has several irreducible components. Some may be planes and others are not planes. We write $Z=Z_{\text {plane }} \cup Z^{\prime}$ where $Z_{\text {plane }}$ is a union of planes and $Z^{\prime}$ is a union of irreducible surfaces which are not planes. Each line in $Z$ lies in one of the irreducible components of $Z$ - in particular it lies either in $Z_{\text {plane }}$ or in $Z^{\prime}$.

If an entire irreducible component is flat, then the component is a plane. So the variety $Z_{\text {flat }}$ contains $Z_{\text {plane }}$, but its intersection with $Z^{\prime}$ has dimension $\leq 1$. The 1-dimensional part of the variety $Z_{f l a t}$ has degree $\lesssim d^{2}$, and so it contains $\lesssim d^{2}$ lines. In particular $Z_{f l a t}$ contains $\lesssim d^{2}$ lines in $Z^{\prime}$.

If half the lines in $\mathfrak{L}_{\text {flat }}$ lie in $Z^{\prime}$, then $(1 / 800) L \lesssim d^{2} \lesssim L^{4} S^{-2} k^{-6}$, and we conclude $S \lesssim L^{3 / 2} k^{-3}$. So from now on, we may assume that half the lines in $\mathfrak{L}_{\text {flat }}$ lie in $Z_{\text {plane }}$.

Now $Z_{\text {plane }}$ is a union of $\leq d$ planes. We are now in the case that these planes contain $\geq(1 / 800) L$ lines. One of the planes must contain $\gtrsim L / d$ lines. By assumption, any plane contains at most $B$ lines of $\mathfrak{L}$. Therefore, $L / d \lesssim B$. Plugging in the value of $d$, we see that $L^{-1} S k^{3} \lesssim B$ and so $S \lesssim B L k^{-3}$.

We now prove a more general case of Lemma 4.7, where we no longer assume that many lines have roughly the average number of points.
Theorem 4.9. Let $k \geq 3$. Let $\mathfrak{L}$ be a set of $L$ lines in $\mathbb{R}^{3}$ with $\leq B$ lines in any plane. Let $\mathfrak{S}$ be a set of $S$ points so that each point meets between $k$ and $2 k$ lines of $\mathfrak{L}$.
Then $S \leq C\left[L^{3 / 2} k^{-2}+L B k^{-3}+L k^{-1}\right]$.
Proof. Let $\mathfrak{L}_{1}$ be the subset of lines in $\mathfrak{L}$ which contain $\geq(1 / 100) S k L^{-1}$ points of $\mathfrak{S}$. If $\left|\mathfrak{L}_{1}\right| \geq(1 / 100) L$, then we have all the hypotheses of Lemma 4.7, and we may conclude

$$
S \leq C_{0}\left[L^{3 / 2} k^{-2}+L B k^{-3}+L k^{-1}\right] .
$$

We are going to prove that $S$ obeys this same estimate, with the same constant, regardless of the size of $\mathfrak{L}_{1}$. The proof will go by induction on the number of lines.

From now on we assume that $\left|\mathfrak{L}_{1}\right| \leq(1 / 100) L$. We define $\mathfrak{S}^{\prime} \subset \mathfrak{S}$ to be the set of points with $\geq(9 / 10) k$ incidences with lines of $\mathfrak{L}_{1}$. The number of incidences between $\mathfrak{S}$ and $\mathfrak{L} \backslash \mathfrak{L}_{1}$ is $\leq(1 / 100) S k$. Therefore, the size of $\mathfrak{S}^{\prime}$ is at least $(9 / 10) S$.

A point of $\mathfrak{S}^{\prime}$ has at least $(9 / 10) k$ incidences with $\mathfrak{L}_{1}$ and at most $2 k$ incidences with $\mathfrak{L}_{1}$. This is a slightly larger range than we have considered before. In order to do induction, we need to reduce the range. We observe $\mathfrak{S}^{\prime}=\mathfrak{S}_{+}^{\prime} \cup \mathfrak{S}_{-}^{\prime}$, where $\mathfrak{S}_{+}^{\prime}$ consists of points with $\geq k$ incidences to $\mathfrak{L}_{1}$ and $\mathfrak{S}_{-}^{\prime}$ consists of points with $\leq k$ incidences with $\mathfrak{L}_{1}$. We define $\mathfrak{S}_{1}$ to be the larger of $\mathfrak{S}_{+}^{\prime}$ and $\mathfrak{S}_{-}^{\prime}$. It has $\geq(9 / 20) S$ points in it.
If we picked $\mathfrak{S}_{1}=\mathfrak{S}_{+}^{\prime}$ then we define $k_{1}=k$. If we picked $\mathfrak{S}_{1}=\mathfrak{S}_{-}^{\prime}$ then we define $k_{1}$ to be the smallest integer $\geq(9 / 10) k$. Each point in $\mathfrak{S}_{1}$ has at least $k_{1}$ and at most $2 k_{1}$ incidences with lines of $\mathfrak{L}_{1}$. Also, $k_{1}$ is an integer $\geq(9 / 10) k \geq 27 / 10$, so $k_{1} \geq 3$.

The set of lines $\mathfrak{L}_{1}$ and the set of points $\mathfrak{S}_{1}$ obey all the hypotheses of Theorem 4.9 (using $k_{1}$ in place of $k$ and using the same $B$ ). There are fewer lines in $\mathfrak{L}_{1}$ than in $\mathfrak{L}$. Doing induction on the number of lines, we may assume that our result holds for these sets. If we denote $\left|\mathfrak{L}_{1}\right|=L_{1}$ and $\left|\mathfrak{S}_{1}\right|=S_{1}$, we get

$$
S_{1} \leq C_{0}\left[L_{1}^{3 / 2} k_{1}^{-2}+B L_{1} k_{1}^{-3}+L_{1} k_{1}^{-1}\right] .
$$

Now $S \leq(20 / 9) S_{1}$. Also, $L_{1} \leq(1 / 100) L$. And $k_{1} \geq(9 / 10) k$.
Therefore,

$$
S \leq(20 / 9) S_{1} \leq\left[(20 / 9)(1 / 100)(10 / 9)^{3}\right] C_{0}\left[L^{3 / 2} k^{-2}+L B k^{-3}+L k^{-1}\right]
$$

The bracketed product of fractions is $<1$, and so $S$ obeys the desired bound.

Finally, we can prove Theorem 4.1. This is an easy argument given Theorem 4.9,
Proof. Let $k \geq 3$. Suppose that $\mathfrak{L}$ is a set of $L$ lines with $\leq B$ in any plane. Suppose that $\mathfrak{S}$ is a set of points, each intersecting at least $k$ lines of $\mathfrak{L}$.

We subdivide the points $\mathfrak{S}=\cup_{j=0}^{\infty} \mathfrak{S}_{j}$, where $\mathfrak{S}_{j}$ consists of the points incident to at least $2^{j} k$ lines and at most $2^{j+1} k$ lines. We define $k_{j}$ to be $2^{j} k$. Then Theorem 4.9 applies to $\left(\mathfrak{L}, \mathfrak{S}_{j}, k_{j}, B\right)$, and we conclude that

$$
\begin{aligned}
& \left|\mathfrak{S}_{j}\right| \leq C_{0}\left[L^{3 / 2} k_{j}^{-2}+L B k_{j}^{-3}+L k_{j}^{-1}\right] \\
& \leq 2^{-j} C_{0}\left[L^{3 / 2} k^{-2}+L B k^{-3}+L k^{-1}\right] .
\end{aligned}
$$

Now $S \leq \sum_{j}\left|\mathfrak{S}_{j}\right| \leq 2 C_{0}\left[L^{3 / 2} k^{-2}+L B k^{-3}+L k^{-1}\right]$.

## References

[CSzT] F. R. K. Chung, E Szemerédi, and W.T. Trotter The number of different distances determined by a set of points in the Euclidean plane Discrete Comput. Geom. (1992) 7 1-11
[CEGSW] K.L. Clarkson, H. Edelsbrunner, L. Guibas, M Sharir, and E. Welzl Combinatorial Complexity bounds for arrangements of curves and spheres Discrete Comput. Geom. (1990) 5 99-160
[D] Z. Dvir On the size of Kakeya sets in finite fields J. Amer. Math Soc. (2009) 22 10931097
[E] P. Erdös On sets of distances of $n$ points Amer. Math. Monthly (1946) 53 248-250
[EKS] Gy. Elekes, H. Kaplan, and M. Sharir On lines, joints, and incidences in three dimensions, preprint
[ES] Gy. Elekes and M. Sharir Incidences in three dimensions and distinct distances in the plane Proceedings 26th ACM Symposium on Computational Geometry (2010) 413-422
[FS] S. Feldman and M. Sharir, An improved bound for joints in arrangements of lines in space, Discrete Comput. Geom. 33, 307-320, (2005).
[GIS] J. Garibaldi, A. Iosevich, S. Senger The Erdös Distance problem AMS Student Mathematical Library Volume 56 (2011)
[G] L. Guth The endpoint case of the Bennett-Carbery-Tao Multilinear Kakeya Conjecture Acta Math., to appear
[GK] L. Guth and N. H. Katz, Algebraic methods in discrete analogues of the Kakeya problem, Adv. in Math., (2010) 225 2828-2839
[KT] N.H. Katz and G. Tardos A new entropy inequality for the Erdös distance problem Towards a new theory of geometric graphs, Contemp Math (2004) 342 119-126
[Land] J.M. Landsberg, Is a linear space contained in a submanifold - On the number of the derivatives needed to tell Journal für die reine und angewandte Mathematik, (1999) 50853-60
[M] L. Moser On the different distances determined by $n$ points Amer. Math. Monthly (1952) 59 85-91
[Salm] G. Salmon, A Treatise on the Analytic Geometry of Three Dimensions, Vol. 2, 5th edition Hodges, Figgis And Co. Ltd. (1915)
[Seg] B. Segre, The number of lines lying on a quartic surface, Quart J. Math (1943) 86 96
[SW] M. Sharir and E. Welzl, Point-Line incidences in space, Combinatorics, Probability, and Computing 13(2) 203-220 (2004)
[SoTo] J. Solymosi and C. Tóth Distinct Distances in the plane Discrete Comput. Geom. (2001) 25 629-634
[ST] Stone, A.H. and Tukey, J. W. Generalized sandwich theorems Duke Math. Jour. (1942) 9 356-359
[SzT] E. Szemerédi and W. T. Trotter Jr., Extremal Problems in Discrete Geometry Combinatorica (1983) 3 381-392
[T] G. Tardos On distinct sums and distinct distances Adv. Math. (2003) 180 275-289
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