The local integration of Leibniz algebras

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Abstract

This article gives a local answer to the coquecigrue problem. Hereby we mean the problem, formulated by J-L. Loday in [Lod93], is that of finding a generalization of the Lie's third theorem for Leibniz algebra. That is, we search a manifold provided with an algebraic structure which generalizes the structure of a (local) Lie group, and such that the tangent space at a distinguished point is a Leibniz algebra structure. Moreover, when the Leibniz algebra is a Lie algebra, we want that the integrating manifold is a Lie group. In his article [Kin07], M.K. Kinyon solves the particular case of split Leibniz algebras. He shows, in particular, that the tangent space at the neutral element of a Lie rack is provided with a Leibniz algebra structure. Hence it seemed reasonable to think that Lie racks give a solution to the coquecigrue problem, but M.K. Kinyon also showed that a Lie algebra can be integrated into a Lie rack which is not a Lie group. Therefore, we have to specify inside the category of Lie racks, which objects are the coquecigrues. In this article we give a local solution to this problem. We show that every Leibniz algebra becomes integrated into a *local augmented Lie rack*. The proof is inspired by E. Cartan's proof of Lie's third theorem, and, viewing a Leibniz algebra as a central extension by some center, proceeds by integrating explicitly the corresponding Leibniz 2-cocycle into a rack 2-cocycle. This proof gives us a way to construct local augmented Lie racks which integrate Leibniz algebras, and this article ends with examples of the integration of non split Leibniz algebras in dimension 4 and 5.

Introduction

The main result of this article is a local answer to the *coquecigrue problem*. By coquecigrue problem, we mean the problem of integrating Leibniz algebras. This question was formulated by J.-L. Loday in [Lod93] and consists in finding a generalisation of the Lie's third theorem for Leibniz algebras. This theorem establishes that for every Lie algebra $\mathfrak{g}$, there exists a Lie group G such that its tangent space at 1 is provided with a structure of Lie algebra isomorphic to $\mathfrak{g}$. Leibniz algebras are generalisations of Lie algebras, they are their non-commutative analogues. Precisely, a *(left) Leibniz algebra* (over $\mathbb{R}$) is an $\mathbb{R}$-vector space $\mathfrak{g}$ provided with a bilinear map $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called the bracket and satisfying the *(left) Leibniz identity* for all x, y and z in $\mathfrak{g}$

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]]$$

Hence, a natural question is to know if, for every Leibniz algebra, there exists a manifold provided with an algebraic structure generalizing the group structure, and such that the tangent space in a distinguished point, called 1, can be provided with a Leibniz algebra structure isomorphic to the given Leibniz algebra. As we want this integration to be the generalization of the Lie algebra case, we also require that, when the Leibniz algebra is a Lie algebra, the integrating manifold is a Lie group.

One result about this question was given by M.K. Kinyon in [Kin07]. In his article, he solves the particular case of *split* Leibniz algebras, that is Leibniz algebras which are isomorphic to the *demisemidirect product* of a Lie algebra and a module over this Lie algebra. That is, Leibniz algebras which are isomorphic to $\mathfrak{g} \oplus \mathfrak{a}$ as vector space, where the bracket is given by $[(x,a),(y,b)] = ([x,y], x.a)$. In this case, he shows that the algebraic structure which answers the problem is the structure of a *digroup*. A digroup is a set with two binary operation $\vdash$ and $\dashv$, a neutral element 1 and some compatibility conditions. More precisely, Kinyon shows that a digroup structure induces a *pointed rack* structure (pointed in 1), and it is this algebraic structure which gives the tangent space at 1 a Leibniz algebra structure. Of course, not every Leibniz algebra is isomorphic to a demisemidirect product, so we have to find a more general structure to solve the problem. One should think that the right structure is that of a pointed rack, but M.K. Kinyon showed in [Kin07] that the second condition (Lie algebra becomes integrated into a Lie group) is not always fulfilled. Thus we have to specify the structure inside the category of pointed racks.

In this article we don't give a complete answer to the coquecigrue problem in the sense that we only construct a *local* algebraic structure and not a global one. Indeed, to define an algebraic structure on a tangent space at a given point on a manifold, we just need an algebraic structure in a neighborhood of this point. We will show in chapter 3 that a local answer to the problem is given by the pointed augmented local racks which are abelian extensions of a Lie group by an anti-symmetric module.

Our approach to the problem is similar to the one given by E. Cartan in [Car30]. The main idea comes from the fact that we know the Lie's first and second theorem on a class of Lie algebras. For example, every abelian Lie algebra or every Lie subalgebra of the Lie algebra $End(V)$ is integrable (using the Lie's first theorem). More precisely, let $\mathfrak{g}$ be a Lie algebra, $Z(\mathfrak{g})$ its center and $\mathfrak{g}_0$ the quotient of $\mathfrak{g}$ by $Z(\mathfrak{g})$. The Lie algebra $Z(\mathfrak{g})$ is abelian and $\mathfrak{g}_0$ is a Lie subalgebra of $End(\mathfrak{g})$, thus there exist Lie groups, respectively $Z(\mathfrak{g})$ and G_0, which integrate these Lie algebras. As a vector space, $\mathfrak{g}$ is isomorphic to the direct sum $\mathfrak{g}_0 \oplus Z(\mathfrak{g})$, thus the tangent space at $(1,0)$ of the manifold $G_0 \times Z(\mathfrak{g})$ is isomorphic to $\mathfrak{g}$. As a Lie algebra, $\mathfrak{g}$ is isomorphic to the central extension $\mathfrak{g}_0 \oplus_\omega Z(\mathfrak{g})$ where ω is a Lie 2-cocycle on $\mathfrak{g}_0$ with coefficients in $Z(\mathfrak{g})$. That is, the bracket on $\mathfrak{g}_0 \oplus_\omega Z(\mathfrak{g})$ is defined by

$$[(x,a),(y,b)] = \big([x,y], \omega(x,y)\big) \tag{1}$$

where ω is an anti-symmetric bilinear form on $\mathfrak{g}_0$ with value on $Z(\mathfrak{g})$ which satisfies the Lie algebra cocycle identity

$$\omega([x,y],z) - \omega(x,[y,z]) + \omega(y,[x,z]) = 0$$

Hence we have to find a group structure on $G_0 \times Z(\mathfrak{g})$ which gives this Lie algebra structure on the tangent space at $(1,0)$. It is clear that the bracket (1) is completely determined by the bracket on $\mathfrak{g}_0$ and the cocycle ω. Hence, the only thing we have to understand is ω. The Lie algebra $\mathfrak{g}$ is a central extension of $\mathfrak{g}_0$ by $Z(\mathfrak{g})$, thus we can hope that the Lie group which integrates $\mathfrak{g}$ should be a central extension of G_0 by $Z(\mathfrak{g})$. To follow this idea, we have to find a group 2-cocycle on G_0 with coefficients in $Z(\mathfrak{g})$. In this case, the group structure on $G_0 \times Z(\mathfrak{g})$ is given by

$$(g,a).(h,b) = \big(gh, a + b + f(g,h)\big) \tag{2}$$

where f is a map from $G \times G \to Z(\mathfrak{g})$ vanishing on $(1,g)$ and $(g,1)$ and satisfying the group cocycle identity

$$f(h,k) - f(gh,k) + f(g,hk) - f(g,h) = 0$$

With such a cocycle, the conjugation in the group is given by the formula

$$(g,a).(h,b).(g,a)^{-1} = \left(ghg^{-1}, a + f(g,h) - f(ghg^{-1}, g)\right) \tag{3}$$

and by imposing a smoothness condition on f in a neighborhood of 1, we can differentiate this formula twice, and obtain a bracket on $\mathfrak{g}_0 \oplus Z(\mathfrak{g})$ defined by

$$[(x,a),(y,b)] = \left([x,y], D^2 f(x,y)\right)$$

where $D^2 f(x,y) = d^2 f(1,1)((x,0),(0,y)) - d^2 f(1,1)((y,0),(0,x))$. Thus, if $D^2 f(x,y)$ equals $\omega(x,y)$, we recover the bracket (1). Hence, if we associate to ω a group cocyle f satisfying some smoothness conditions and such that $D^2 f = \omega$, then our integration problem is solved. This can be done in two steps. The first one consists in finding a local Lie group cocycle defined around 1. Precisely, we want a map f defined on a subset of $G_0 \times G_0$ containing $(1,1)$ with values in $Z(\mathfrak{g})$ which satisfies the local group cocycle identity (cf. [Est54] for a definition of local group). We can construct explicitly such a local group cocycle. This construction is the following one (cf. Lemma 5.2 in [Nee04]) :

Let V be an open convex 0-neighborhood in $\mathfrak{g}_0$ and $\phi : V \to G_0$ a chart of G_0 with $\phi(0) = 1$ and $d\phi(0) = id_{\mathfrak{g}_0}$. For all $(g,h) \in \phi(V) \times \phi(V)$ such that $gh \in \phi(V)$ let us define $f(g,h) \in Z(\mathfrak{g})$ by the formula

$$f(g,h) = \int_{\gamma_{g,h}} \omega^{inv}$$

where $\omega^{inv} \in \Omega^2(G_0, Z(\mathfrak{g}))$ is the invariant differential form on G_0 associated to ω and $\gamma_{g,h}$ is the smooth singular 2-chain defined by

$$\gamma_{g,h}(t,s) = \phi\left(t\left(\phi^{-1}\left(g\phi(s\phi^{-1}(h))\right)\right) + s\left(\phi^{-1}\left(g\phi((1-t)\phi^{-1}(h))\right)\right)\right)$$

The formula for f defines a smooth function such that $D^2 f(x,y) = \omega(x,y)$. We now only have to check whether f satisfies the local group cocycle identity. Let $(g,h,k) \in \phi(V)^3$ such that gh, hk and ghk are in $\phi(V)$. We have

$$f(h,k) - f(gh,k) + f(g,hk) - f(g,h) = \int_{\gamma_{h,k}} \omega^{inv} - \int_{\gamma_{gh,k}} \omega^{inv} + \int_{\gamma_{g,hk}} \omega^{inv} - \int_{\gamma_{g,h}} \omega^{inv}$$

$$= \int_{\partial\gamma_{g,h,k}} \omega^{inv}$$

where $\gamma_{g,h,k}$ is a smooth singular 3-chain in $\phi(V)$ such that $\partial\gamma_{g,h,k} = g\gamma_{h,k} - \gamma_{gh,k} + \gamma_{g,hk} - \gamma_{g,h}$ (such a chain exists because $\phi(V)$ is homeomorphic to the convex open subset V of $\mathfrak{g}_0$). Thus

$$f(h,k) - f(gh,k) + f(g,hk) - f(g,h) = \int_{\partial\gamma_{g,h,k}} \omega^{inv}$$

$$= \int_{\gamma_{g,h,k}} d_{dR}\omega^{inv}$$

$$= 0$$

3

because ω^{inv} is a closed 2-form. Hence, we have associated to ω a local group 2-cocycle, smooth in a neighborhood in 1, and such that $D^2 f(x, y) = \omega(x, y)$. Thus we can define a local Lie group structure on $G_0 \times Z(\mathfrak{g})$ by setting

$$(g, a)(h, b) = \big(gh, a + g.b + f(g, h)\big),$$

and the tangent space at $(1, 0)$ of this local Lie group is isomorphic to $\mathfrak{g}$. If we want a global structure, we have to extend this local cocycle to the whole group G_0. First P.A. Smith ([Smi50, Smi51]), then W.T. Van Est ([Est62]) have shown that it is precisely this enlargement which may meet an obstruction coming from both $\pi_2(G_0)$ and $\pi_1(G_0)$. In finite dimension, $\pi_2(G_0) = 0$, thus there is no obstruction to integrate Lie algebras. This equality is no longer true in infinite dimension, hence this obstruction prevents the integration of infinite dimension Lie algebras into global Lie groups (cf. [Nee02, Nee04]).

To integrate Leibniz algebras into pointed racks, we follow a similar approach. In this context, we use the fact that we know how to integrate any (finite dimensional) Lie subalgebra of $End(V)$ for V a vector space. In a similar way as the Lie algebra case, we associate to any Leibniz algebra an abelian extension of a Lie algebra $\mathfrak{g}_0$ by an anti-symmetric representation $Z_L(\mathfrak{g})$. As we have the theorem for Lie algebras, we can integrate $\mathfrak{g}_0$ and $Z_L(\mathfrak{g})$ into the Lie groups G_0 and $Z_L(\mathfrak{g})$, and, using the Lie's second theorem, $Z_L(\mathfrak{g})$ is a G_0-module. Then, the main difficulty becomes the integration of the Leibniz cocycle into a local Lie rack cocycle. In chapter 3 we explain how to solve this problem. We make a similar construction as in the Lie algebra case, but in this context, there are several difficulties which appear. One of them is that our cocycle is not anti-symmetric, so we can't consider the equivariant form associated to it and integrate this form. To solve this problem, we will use Proposition 1.1 which, in particular, establishes an isomorphism from the 2-nd cohomology group of a Leibniz algebra $\mathfrak{g}$ with coefficients in an anti-symmetric representation $\mathfrak{a}^a$ to the 1-st cohomology group of $\mathfrak{g}$ with coefficients in the symmetric representation $Hom(\mathfrak{g}, \mathfrak{a})^s$. In this way, we get a 1-form that we can now integrate. Another difficulty is to specify on which domain this 1-form should be integrated. In the Lie algebra case, we integrate over a 2-simplex and the cocycle identity is verified by integrating over a 3-simplex, whereas in our context we will replace the 2-simplex by the 2-cube and the 3-simplex by a 3-cube.

Let us describe the content of the article section-wise.

Section 1: Leibniz algebras

This whole section, except the Proposition 1.1, is based on [Lod93, LP93, Lod98]. We first give the basic definitions we need about Leibniz algebras. Unlike J.-L. Loday and T. Pirashvili, who work with right Leibniz algebras, we study left Leibniz algebras. Hence, we have to translate all the definitions needed into our context. As we have seen above, we translate our integration problem into a cohomological problem, thus we need a cohomology theory for Leibniz algebras and, a fortiori, a notion of representation. We take the definition of a representation of a Leibniz algebra given by J.-L. Loday and T. Pirashvili in [LP93]. We end this section by a fondamental result (Proposition 1.1). This proposition establishes an isomorphism of cochain complexes from $CL^n(\mathfrak{g}, \mathfrak{a}^a)$ to $CL^{n-1}(\mathfrak{g}, Hom(\mathfrak{g}, \mathfrak{a})^s)$. The important fact in this result is the transfer from an anti-symmetric representation to a symmetric one. This will be useful when we will have to associate a local Lie rack 2-cocycle to a Leibniz 2-cocycle.

4

Section 2: Lie racks

The notion of rack comes from topology, in particular, the theory of invariants of knots and links (cf. for example [FR]). It is M.K. Kinyon in [Kin07] who was the first to link racks to Leibniz algebras. The idea of linking these two structures comes from the case of Lie groups and Lie algebras and in particular from the construction of the bracket using the conjugation. Indeed, a way to define a bracket on the tangent space at 1 of a Lie group is to differentiate the conjugation morphism twice. Let G a Lie group, the conjugation is the group morphism $c : G \to Aut(G)$ defined by $c_g(h) = ghg^{-1}$. If we differentiate this expression with respect to the variable h at 1, we obtain a Lie group morphism $Ad : G \to Aut(\mathfrak{g})$. We can still derive this morphism at 1 to obtain a linear map $ad : \mathfrak{g} \to End(\mathfrak{g})$. Then, we are allowed to define a bracket $[-,-]$ on $\mathfrak{g}$ by setting $[x,y] = ad(x)(y)$. We can show that this bracket satisfies the left Leibniz identity, and that this identity is induced by the equality $c_g(c_h(k)) = c_{c_g(h)}(c_g(k))$. Thus, if we denote $c_g(h)$ by $g \triangleright h$, the only properties we use to define a Lie bracket on $\mathfrak{g}$ are

1. $g \triangleright : G \to G$ is a bijection for all $g \in G$.

2. $g \triangleright (h \triangleright k) = (g \triangleright h) \triangleright (g \triangleright k)$ for all $g, h, k \in G$

3. $g \triangleright 1 = 1$ and $1 \triangleright g = g$ for all $g \in G$.

Hence, we call *(left) rack*, a set provided with a binary operation $\triangleright$ satisfying the first and the second condition. A rack is called *pointed* if there exists an element 1 which satisfies the third condition. We begin this chapter by giving definitions and examples, for this we follow [FR]. They work with right racks, hence, as in the Leibniz algebra case, we translate the definitions to left racks. In particular, we give the most important example called *(pointed) augmented rack*. This example presents similarities with crossed modules of groups, and in this case, the rack structure is induced by a group action.

As in the group case, we want to construct a pointed rack associated to a Leibniz algebra using an abelian extension. Hence, we need a cohomology theory where the second cohomology group corresponds to the extension classes of a rack by a module. We take the definitions given by N. Andruskiewitsch and M. Graña in [AG03].

At the end of this section, we give the definitions of local rack cohomology and (local) Lie rack cohomology.

Section 3: Lie racks and Leibniz algebras

This section is the heart of this article. It gives the local solution for the coquecigrue problem. To our knowledge, all the results in this chapter are new, except Proposition 3.1 due to M.K. Kinyon ([Kin07]). First, we recall the link between (local) Lie racks and Leibniz algebras explained by M.K. Kinyon in [Kin07] (Proposition 3.1). Then, we study the passage from smooth $As(X)$-modules to Leibniz representations (Proposition 3.9) and (local) Lie rack cohomology to Leibniz cohomology. We define a morphism from the (local) Lie rack cohomology of a rack X with coefficients in a $As(X)$-module A^s (resp. A^a) to the Leibniz cohomology of the Leibniz algebra associated to X with coefficients in $\mathfrak{a}^s = T_0A$ (resp. $\mathfrak{a}^a$) (Proposition 3.10).

The end of this section (section 3.4 to 3.7) is on the integration of Leibniz algebras into local Lie racks. We use the same approach as E. Cartan for the Lie groups case. That is, for every Leibniz algebra, we consider the abelian extension by the left center and integrate it. This extension is

caracterized by a 2-cocycle, and we construct (Proposition 3.22) a local Lie rack 2-cocycle integrating it by an explicit construction similar to the one explained in the Lie group case. This construction is summarized in our main theorem (Theorem 3.28). We remark that the constructed 2-cocycle has more structure (Proposition 3.26). That is, the rack cocycle identity is induced by another one. This other identity permits us to provide our constructed local Lie rack with a structure of augmented local Lie rack (Proposition 3.31). We end this section with examples of the integration of non split Leibniz algebras in dimension 4 and 5.

1 Leibniz algebras

As it is written in the introduction, we work with left Leibniz algebras instead of right Leibniz algebras. The main reason comes from the fact that M.K. Kinyon works in this context in his article [Kin07]. Indeed, this article is our starting point of the integration problem for Leibniz algebras. Thus, we have chosen to work in this context.

A *(left) Leibniz algebra (over $\mathbb{R}$)* is a vector space $\mathfrak{g}$ (over $\mathbb{R}$) provided with a bracket $[-,-]$: $\mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$, which satisfies the *left Leibniz identity*

$$[x,[y,z]] = [[x,y],z] + [y,[x,z]].$$

Remark that an equivalent way to define a left Leibniz algebra is to say that, for all $x \in \mathfrak{g}$, $[x,-]$ is a derivation for the bracket $[-,-]$. The first example of a Leibniz algebra is a Lie algebra. Indeed, if the bracket is anti-symmetric, then the Leibniz identity is equivalent to the Jacobi identity. Hence, we have a functor $inc : Lie \to Leib$. This functor has a left adjoint $(-)_{Lie} : Leib \to Lie$ which is defined on the objects by $\mathfrak{g}_{Lie} = \mathfrak{g}/\mathfrak{g}_{ann}$, where $\mathfrak{g}_{ann}$ is the two-sided ideal of $\mathfrak{g}$ generated by the set $\{[x,x] \in \mathfrak{g} \,|\, x \in \mathfrak{g}\}$. We can remark that there are other ways to construct a Lie algebra from a Leibniz algebra. One is to quotient $\mathfrak{g}$ by the *left center* $Z_L(\mathfrak{g}) = \{x \in \mathfrak{g} \,|\, [x,-] = 0\}$, but this construction is not functorial.

To define a cohomology theory for Leibniz algebras, we need a notion of representation of such algebraic structure. As we work with left Leibniz algebra, we have to translate the definition given by J.L. Loday and T. Pirashvili in their article [LP93]. In our context, a *representation* over a Leibniz algebra $\mathfrak{g}$, becomes a vector space M provided with two linear maps $[-,-]_L : \mathfrak{g} \otimes M \to M$ and $[-,-]_R : M \otimes \mathfrak{g} \to \mathfrak{g}$, satisfying the following three axioms

$$[x,[y,m]_L]_L = [[x,y],m]_L + [y,[x,m]_L]_L \qquad (LLM)$$
$$[x,[m,y]_R]_L = [[x,m]_L,y]_R + [m,[x,y]]_R \qquad (LML)$$
$$[m,[x,y]]_R = [[m,x]_R,y]_R + [x,[m,y]_R]_L \qquad (MLL)$$

Recall that, for a Lie algebra $\mathfrak{g}$, a representation of $\mathfrak{g}$ is a vector space M provided with a linear map $[-,-] : \mathfrak{g} \otimes M \to M$ satisfying $[[x,y],m] = [x,[y,m]] - [y,[x,m]]$. A Lie algebra is a Leibniz algebra, hence we want that a Lie representation M of a Lie algebra $\mathfrak{g}$, is a Leibniz representation of $\mathfrak{g}$. We have two canonical choices for putting a Leibniz representation structure on M. One possibility is by setting $[-,-]_L = [-,-]$ and $[-,-]_R = -[-,-]$, and a second one is by setting $[-,-]_L = [-,-]$ and $[-,-]_R = 0$. These Leibniz representations are examples of particular Leibniz representations. The first one is an example of a *symmetric* representation, and the second one is an example of an *anti-symmetric* representation. A symmetric representation is a Leibniz representation where $[-,-]_L = -[-,-]_R$ and an anti-symmetric representation is a Leibniz representation

where $[-,-]_R = 0$. A Leibniz representation which is symmetric and anti-symmetric is called *trivial*.

Now, we are ready to define a cohomology theory for Leibniz algebras. The existence of a cohomology (and homology) theory for these algebras is one of the main motivation for studying them because, restricted to Lie algebras, this theory gives us new invariants (cf. [Lod93]). For $\mathfrak{g}$ a Leibniz algebra and M a representation of $\mathfrak{g}$, we define a cochain complex $\{CL^n(\mathfrak{g}, M), dL^n\}_{n \in \mathbb{N}}$ by setting

$$CL^n(\mathfrak{g}, M) = Hom(\mathfrak{g}^{\otimes n}, M)$$

and

$$
\begin{aligned}
dL^n\omega(x_0, \ldots, x_n) &= \sum_{i=0}^{n-1}(-1)^i[x_i, \omega(x_0, \ldots, \hat{x}_i, \ldots, x_n)]_L + (-1)^{n-1}[\omega(x_0, \ldots, x_{n-1}), x_n]_R \\
&+ \sum_{0 \leq i < j \leq n}(-1)^{i+1}\omega(x_0, \ldots, x_{j-1}, [x_i, x_j], x_{j+1}, \ldots, x_n)
\end{aligned}
$$

To prove that $dL^{n+1} \circ dL^n = 0$, we use *Cartan's formulas*. These formulas are described in [LP93] in the right Leibniz algebra context, but we can adapt them easily in our context.

Like for many algebraic structures, the second cohomology group of a Leibniz algebra $\mathfrak{g}$ with coefficients in a representation M is in bijection with the set of equivalence classes of abelian extensions of $\mathfrak{g}$ by M (cf. [LP93]). *An abelian extension* of a Leibniz algebra $\mathfrak{g}$ by M is a Leibniz algebra $\hat{\mathfrak{g}}$ such that, $M \overset{i}{\hookrightarrow} \hat{\mathfrak{g}} \overset{p}{\twoheadrightarrow} \mathfrak{g}$ is a short exact sequence of Leibniz algebra (where M is considered as an abelian Leibniz algebra) and the representation structure of M is compatible with the representation structure induced by this short exact sequence. That is, $[m, x]_R = i^{-1}[i(m), s(x)]$ and $[x, m]_L = i^{-1}[s(x), i(m)]$ where s is a section of p and the bracket is that of $\hat{\mathfrak{g}}$ (of course, we have to justify that this representation structure of $\mathfrak{g}$ on M induced by the short exact sequence doesn't depend on s, but we deduce it easily from the fact that the difference of two sections of p is in $i(M)$).

There are canonical abelian extensions associated to a Leibniz algebra. The one we will use to integrate Leibniz algebra is the *abelian extension by the left center*

$$Z_L(\mathfrak{g}) \overset{i}{\hookrightarrow} \mathfrak{g} \overset{p}{\twoheadrightarrow} \mathfrak{g}_0$$

where $\mathfrak{g}_0 := \mathfrak{g}/Z_L(\mathfrak{g})$. This is an extension of a Lie algebra by an anti-symmetric representation. In a sense, a symmetric representation is closer to a Lie representation that of an anti-symmetric representation. Hence, it is convenient to pass from a anti-symmetric representation to a symmetric representation. Let $\mathfrak{g}$ be a Lie algebra and M a Lie representation of $\mathfrak{g}$, then we define a Lie representation structure on $Hom(\mathfrak{g}, M)$ by setting

$$(x.\alpha)(y) := x.(\alpha(y)) - \alpha([x, y])$$

for all $x, y \in \mathfrak{g}$ and $\alpha \in Hom(\mathfrak{g}, M)$. The following proposition establishes an isomorphism from $HL^n(\mathfrak{g}, M^a)$ to $HL^{n-1}(\mathfrak{g}, Hom(\mathfrak{g}, M)^s)$, where M^a (resp. $Hom(\mathfrak{g}, M)^s$) means that M (resp. $Hom(\mathfrak{g}, M)$) is provided with a anti-symmetric (resp. symmetric) $\mathfrak{g}$-representation structure.

Proposition 1.1. *Let $\mathfrak{g}$ be a Lie algebra and M a Lie representation of $\mathfrak{g}$. We have an isomorphism of cochain complexes*

$$CL^n(\mathfrak{g}, M^a) \overset{\tau^n}{\to} CL^{n-1}(\mathfrak{g}, Hom(\mathfrak{g}, M)^s)$$

given by $\omega \mapsto \tau^n(\omega)$ where

$$\tau^n(\omega)(x_1, \ldots, x_{n-1})(x_n) = \omega(x_1, \ldots, x_n)$$

Proof : This morphism is clearly an isomorphism $\forall n \geq 0$. Moreover, we have

$$
\begin{aligned}
dL\tau^n(\omega)(x_0, \ldots, x_{n-1})(x_n) &= \sum_{i=0}^{n-2}(-1)^i[x_i, \tau^n(\omega)(x_0, \ldots, \hat{x}_i, \ldots, x_{n-1})](x_n) \\
&\quad + (-1)^{n-1}[x_{n-1}, \tau^n(\omega)(x_0, \ldots, x_{n-2})](x_n) \\
&\quad + \sum_{0 \leq i < j \leq n-1}(-1)^{i+1}\tau^n(\omega)(x_0, \ldots, x_{j-1}, [x_i, x_j], x_{j+1}, \ldots, x_{n-1})(x_n) \\
&= \sum_{i=0}^{n-1}(-1)^i([x_i, \omega(x_0, \ldots, \hat{x}_i, \ldots, x_{n-1}, x_n)] - \omega(x_0, \ldots, \hat{x}_i, \ldots, x_{n-1}, [x_i, x_n])) \\
&\quad + \sum_{0 \leq i < j \leq n-1}(-1)^{i+1}\omega(x_0, \ldots, x_{j-1}, [x_i, x_j], x_{j+1}, \ldots, x_{n-1}, x_n) \\
&= \sum_{i=0}^{n-1}(-1)^i[x_i, \omega(x_0, \ldots, \hat{x}_i, \ldots, x_{n-1}, x_n)] \\
&\quad + \sum_{0 \leq i < j \leq n}(-1)^{i+1}\omega(x_0, \ldots, x_{j-1}, [x_i, x_j], x_{j+1}, \ldots, x_{n-1}, x_n) \\
&= dL\omega(x_0, \ldots, x_{n-1}, x_n) \\
&= \tau^{n+1}(dL\omega)(x_0, \ldots, x_{n-1})(x_n)
\end{aligned}
$$

Hence $\{\tau^n\}_{n \geq 0}$ is a morphism of cochain complexes.

$\square$

2 Lie racks

2.1 Definitions and examples

Like in the Leibniz algebras case, we can define left racks and right racks. Because we have made the choice to work with left Leibniz algebras, we take the definition of left racks. A *(left) rack* is a set X provided with a product $\rhd : X \times X \to X$, which satisfies the *left rack identity*, that is for all $x, y, z \in X$:

$$x \rhd (y \rhd z) = (x \rhd y) \rhd (x \rhd z),$$

and such that $x \rhd \_ : X \to X$ is a bijection for all $x \in X$. A rack is said to be *pointed* if there exists an element $1 \in X$, called the neutral element, which satisfies $1 \rhd x = x$ and $x \rhd 1 = 1$ for all $x \in X$. A *rack morphism* is a map $f : X \to Y$ satisfying $f(x \rhd y) = f(x) \rhd f(y)$, and a *pointed rack morphism* is a rack morphism f such that $f(1) = 1$.

The first example of a rack is a group provided with the conjugation. Indeed, let G be a group, we define a rack product $\rhd$ on G by setting $g \rhd h = ghg^{-1}$ for all $g, h \in G$. Clearly, $g \rhd \_$ is a bijection with inverse $g^{-1} \rhd \_$ and, an easy computation shows that the rack identity is satisfied. Hence, we have a functor $Conj : Group \to Rack$. This functor has a left adjoint $As : Rack \to Group$ defined on the objects by $As(X) = F(X)/ < \{xyx^{-1}(x \rhd y^{-1}) \,|\, x, y \in X\} >$ where $F(X)$ is the free group generated by X, and $< \{xyx^{-1}(x \rhd y^{-1}) \,|\, x, y \in X\} >$ is the normal

subgroup generated by $\{xyx^{-1}(x \triangleright y^{-1}) \mid x, y \in X\}$. We can remark that $Conj(G)$ is a pointed rack. Indeed, we have $1 \triangleright g = g$ and $g \triangleright 1 = 1$ for all $g \in G$, where 1 is the neutral element for the group product. Hence, $Conj$ is a functor from $Group$ to $PointedRack$. This functor has a left adjoint $As_p : PointedRack \to Group$, defined on the objects by $As_p(X) = As(X)/ < \{[1]\} >$, where $< \{[1]\} >$ is the subgroup of $As(X)$ generated by the class $[1] \in As(X)$.

A second example, and maybe the most important, is the example of augmented racks. An *augmented rack* is the data of a group G, a G-set X, and a map $X \overset{p}{\to} G$ satisfying the *augmentation identity*, that is for all $g \in G$ and $x \in X$

$$p(g.x) = gp(x)g^{-1}.$$

Then, we define a rack structure on X by setting $x \triangleright y = p(x).y$. If there exists an element $1 \in X$ such that $p(1) = 1$ and $g.1 = 1$ for all $g \in G$, then the augmented rack $X \overset{p}{\to} G$ is said *pointed*, and the associated rack $(X, \triangleright)$ is pointed. We can remark that crossed modules and precrossed modules of groups are examples of augmented racks.

2.2   Pointed rack cohomology

To define a pointed rack cohomology theory, we need a good notion of pointed rack module. In this article, we take the definition given by N. Andruskiewitsch and M. Graña in [AG03]. Let X be a pointed rack, an *X-module* is an abelian group A, provided with two families of homomorphisms of the abelian group A, $(\phi_{x,y})_{x,y \in X}$ and $(\psi_{x,y})_{x,y \in X}$, satisfying the following axioms

(M_0) $\phi_{x,y}$ is an isomorphism.

(M_1) $\phi_{x,y \triangleright z} \circ \phi_{y,z} = \phi_{x \triangleright y, x \triangleright z} \circ \phi_{x,z}$

(M_2) $\phi_{x,y \triangleright z} \circ \psi_{y,z} = \psi_{x \triangleright y, x \triangleright z} \circ \phi_{x,y}$

(M_3) $\psi_{x,y \triangleright z} = \phi_{x \triangleright y, x \triangleright z} \circ \psi_{x,z} + \psi_{x \triangleright y, x \triangleright z} \circ \psi_{x,y}$

(M_4) $\phi_{1,y} = id_A$ $\forall y \in X$ and $\psi_{x,1} = 0$ $\forall x \in X$

Remark 2.1. There is a more general definition of (pointed) rack module given by N. Jackson in [Jac07], but we don't need this degree of generality. This definition of pointed rack module coincides with the definition of homogeneous pointed rack module given in [Jac07].

For example, there are two canonical X-module structures on an $As_p(X)$-module. Indeed, let A be a $As_p(X)$-module, that is an abelian group provided with a group morphism $\rho : As_p(X) \to Aut(A)$, the first X-module structure, called *symmetric*, that we can define on A is given for all $x, y \in X$ by

$$\phi_{x,y}(a) = \rho_x(a)$$
$$\psi_{x,y}(a) = a - \rho_{x \triangleright y}(a).$$

The second, called *anti-symmetric*, is given for all $x, y \in X$ by

$$\phi_{x,y}(a) = \rho_x(a)$$
$$\psi_{x,y}(a) = 0.$$

9

With this definition of module, N. Andruskiewitsch and M. Graña define a cohomology theory for pointed racks. For X a pointed rack and A a X-module, they define a cochain complex $\{CR^n(X,A), d_R^n\}_{n\in\mathbb{N}}$ by setting

$$CR^n(X,A) = \{f : X^n \to A \mid f(x_1,\ldots,1,\ldots,x_n) = 0\}$$

and

$$d_R^n f(x_1,\ldots,x_{n+1}) =$$
$$\sum_{i=1}^n (-1)^{i-1}\big(\phi_{x_1\triangleright\ldots\triangleright x_i,x_1\triangleright\ldots\triangleright\widehat{x_i}\triangleright\ldots\triangleright x_{n+1}}(f(x_1,\ldots,\widehat{x_i},\ldots,x_{n+1})) - f(x_1,\ldots,x_i\triangleright x_{i+1},\ldots,x_i\triangleright x_{n+1})\big)$$
$$+(-1)^n \psi_{x_1\triangleright\ldots\triangleright x_n,x_1\triangleright\ldots\triangleright x_{n-1}\triangleright x_{n+1}}(f(x_1,\ldots,x_n))$$

This complex is the same as the one defined in [Jac07], but in the left rack context. Adapting the proof given by N. Jackson in [Jac07], one easily sees that the second cohomology group $HR^2(X,A)$ is in bijection with the set of equivalence classes of abelian extensions of a pointed rack X by a X-module A. An *abelian extension* of a pointed rack X by a X-module A is a surjective pointed rack homomorphism $E \overset{p}{\twoheadrightarrow} X$ which satisfies the following axioms

(E_0) for all $x \in X$, there is a simply transitively right action of A on $p^{-1}(x)$.

(E_1) for all $u \in p^{-1}(x), v \in p^{-1}(y), a \in A$, we have $(u.a)\triangleright v = (u\triangleright v).\psi_{x,y}(a)$.

(E_2) for all $u \in p^{-1}(x), v \in p^{-1}(y), a \in A$, we have $u\triangleright(v.a) = (u\triangleright v).\phi_{x,y}(a)$.

and two extensions $E_1 \overset{p_1}{\twoheadrightarrow} X$, $E_2 \overset{p_2}{\twoheadrightarrow} X$ are called *equivalent*, if there exists a pointed rack isomorphism $E_1 \overset{\theta}{\to} E_2$ which satisfies the following axioms

1. $p_2 \circ \theta = p_1$.

2. for all $x \in X, u \in p^{-1}(x), a \in A$, we have $\theta(u.a) = \theta(u).a$.

2.3 Lie racks

To generalize Lie groups, we need a pointed rack provided with a differentiable structure compatible with the algebraic structure. This is the notion of Lie racks. A *Lie rack* is a smooth manifold X provided with a pointed rack structure such that the product $\triangleright$ is smooth, and such that for all $x \in X$ c_x is a diffeomorphism. We will see in section 3 that the tangent space at the neutral element of a Lie rack is provided with a Leibniz algebra structure.

Let X be a Lie rack, a X-module A is said *smooth* if A is a abelian Lie group, and if $\phi : X\times X\times A \to A$ and $\psi : X\times X\times A \to A$ are smooth. Then we can define a cohomology theory for Lie racks with values in a smooth module. For this we define a cochain complex $\{CR_s^n(X,A), d_R^n\}_{n\in\mathbb{N}}$ where $CR_s^n(X,A)$ is the set of functions $f : X^n \to A$ which are smooth in a neighborhood of $(1,\ldots,1) \in X^n$ and such that $f(x_1,\ldots,1,\ldots,x_n) = 0$ for all $x_1,\ldots,x_n \in X$. The formula for the differential d_R is the same as the one defined previously. We will see that a Lie rack cocycle (respectively a coboundary) derives itself in a Leibniz algebra cocycle (respectively coboundary).

2.4 Local racks

To define a Lie algebra structure on the tangent space at the neutral element of a Lie group, we can remark that we only use the local Lie group structure in the neighborhood of 1. We will see that this remark remains true for Lie racks and Leibniz algebras.

A *local rack* is a set X provided with a product $\rhd$ defined on a subset Ω of $X \times X$ with values in X, and such that the following axioms are satisfied:

1. If $(x, y), (x, z), (y, z), (x, y \rhd z)$ and $(x \rhd y, x \rhd z) \in \Omega$, then $x \rhd (y \rhd z) = (x \rhd y) \rhd (x \rhd z)$.

2. If $(x, y), (x, z) \in \Omega$ and $x \rhd y = x \rhd z$, then $y = z$.

A local rack is said to be *pointed* if there is a element $1 \in X$ such that $1 \rhd x$ and $x \rhd 1$ are defined for all $x \in X$ and respectively equals to x and 1. We called this element the *neutral element*. Then a *local Lie rack* is a pointed local rack $(X, \Omega, 1)$ where X is a smooth manifold, Ω is a open subset of X, and $\rhd : \Omega \to X$ is smooth. For example, every Lie rack open subset containing the neutral element is a local Lie rack. Given such a local Lie rack, we can define a associated cohomology theory.

Let X be a Lie rack, U a subset of X containing the neutral element 1 and A a smooth X-module. We define for all $n \in \mathbb{N}$, $CR_s^n(U, A)$ as the set of maps $f : U_{n-loc} \to A$, smooth in a neighborhood of the neutral element, and such that $f(x_1, \ldots, 1, \ldots, x_n) = 0$. If A is not anti-symmetric, then U_{n-loc} is the subset of elements $(x_1, \ldots, x_n)$ of $X \times U^{n-1}$ satisfying $x_{i_1} \rhd \ldots \rhd x_{i_j} \in U$, for all $i_1 < \cdots < i_j, 2 \le j \le n$. If A is anti-symmetric, U_{n-loc} is the subset of elements $(x_1, \ldots, x_n)$ of $X^{n-1} \times U$ satisfying $x_{i_1} \rhd \ldots \rhd x_{i_j} \rhd x_n \in U$, for all $i_1 < \cdots < i_j < n, 1 \le j \le n-1$. One easily checks that the formula for the differential d_R allows us to define a cochain complex $\{CR_s^n(U, A), d_R^n\}_{n \in \mathbb{N}}$. Then we define *U-local Lie rack cohomology of X with coefficients in A* as the cohomology of the cochain complex $\{CR_s^n(U, A), d_R^n\}_{n \in \mathbb{N}}$.

3 Lie racks and Leibniz algebras

3.1 From Lie racks to Leibniz algebras

In this section we construct a functor from the category of Lie racks to the category of Leibniz algebras. The following proposition is in [Kin07].

Proposition 3.1. *Let X be a Lie rack, then T_1X is a Leibniz algebra.*

Proof : Let X be a Lie rack, we denote by $\mathfrak{x}$ the tangent space of X at 1. The conjugation $\rhd$ induces for all $x \in X$ an automorphism of Lie racks $c_x : X \to X$ defined by $c_x(y) = x \rhd y$. We define for all $x \in X$

$$Ad_x = T_1 c_x \in GL(\mathfrak{x}).$$

By definition we have $c_{x \rhd y} = c_x \circ c_y \circ c_x^{-1}$ and $c_1 = id$. Hence it gives us a morphism of Lie racks

$$Ad : X \to GL(\mathfrak{x}).$$

We can differentiate Ad at 1, and we obtain a linear map

$$ad : \mathfrak{x} \to \mathfrak{gl}(\mathfrak{x}).$$

Then we define a bracket $[-, -]$ on $\mathfrak{r} = T_1 X$ by setting

$$[u, v] = ad(u)(v).$$

Now, we have to verify that this bracket satisfies the Leibniz identity, that is

$$[u, [v, w]] = [[u, v], w] + [v, [u, w]].$$

To show this identity, we use the rack identity

$$x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z).$$

Let $u, v, w \in \mathfrak{r}$ and γ_u (resp. γ_v and γ_w) be a smooth path in X, such that $\gamma_u(0) = 1$ and $\left.\frac{\partial}{\partial s}\right|_{s=0} \gamma_u(s) = u$ (resp. $\gamma_v(0) = \gamma_w(0) = 1$, $\left.\frac{\partial}{\partial s}\right|_{s=0} \gamma_v(s) = v$ and $\left.\frac{\partial}{\partial s}\right|_{s=0} \gamma_w(s) = w$). We have

$$\left.\frac{\partial}{\partial t}\right|_{t=0} (\gamma_u(r) \triangleright (\gamma_v(s) \triangleright \gamma_w(t))) = \left.\frac{\partial}{\partial t}\right|_{t=0} ((c_{\gamma_u(r)} \circ c_{\gamma_v(s)})(\gamma_w(t)))$$
$$= (Ad_{\gamma_u(r)} \circ Ad_{\gamma_v(s)})(w)$$

and

$$\left.\frac{\partial}{\partial t}\right|_{t=0} ((\gamma_u(r) \triangleright \gamma_v(s)) \triangleright (\gamma_u(r) \triangleright \gamma_w(t))) = \left.\frac{\partial}{\partial t}\right|_{t=0} ((c_{\gamma_u(r) \triangleright \gamma_v(s)} \circ c_{\gamma_u(r)})(\gamma_w(t)))$$
$$= (Ad_{\gamma_u(r) \triangleright \gamma_v(s)} \circ Ad_{\gamma_u(r)})(w)$$
$$= (Ad_{c_{\gamma_u(r)}(\gamma_v(s))} \circ Ad_{\gamma_u(r)})(w).$$

Moreover, if we differentiate with respect to the variable s

$$\left.\frac{\partial}{\partial s}\right|_{s=0} ((Ad_{\gamma_u(r)} \circ Ad_{\gamma_v(s)})(w)) = (Ad_{\gamma_u(r)} \circ ad(v))(w)$$

and

$$\left.\frac{\partial}{\partial s}\right|_{s=0} (Ad_{c_{\gamma_u(r)}(\gamma_v(s))} \circ Ad_{\gamma_u(r)})(w) = (ad(Ad_{\gamma_u(r)}(v)) \circ Ad_{\gamma_u(r)})(w).$$

And if we differentiate with respect to the variable r, we obtain

$$\left.\frac{\partial}{\partial r}\right|_{r=0} (Ad_{\gamma_u(r)} \circ ad(v))(w) = ad(u)(ad(v)(w))$$

and

$$\left.\frac{\partial}{\partial r}\right|_{r=0} (ad(Ad_{\gamma_u(r)}(v)) \circ Ad_{\gamma_u(r)})(w) = ad(ad(u)(v))(w) + ad(v)(ad(u)(w))$$

Hence, by identification, we have the Leibniz identity.

$\square$

Proposition 3.2. *Let $f : X \to Y$ be a morphism of Lie racks, then $T_1 f : T_1 X \to T_1 Y$ is a morphism of Leibniz algebras.*

12

Proof : The fact that f is a morphism of Lie rack implies that $f(1) = 1$ and $f(x \triangleright y) = f(x) \triangleright f(y)$ for all $x, y \in X$. Let $u, v \in \mathfrak{x}$ and γ_u (resp. γ_v) be a smooth path in X pointed at 1 such that $\frac{\partial}{\partial s}\Big|_{s=0} \gamma_u(s) = u$ (resp. $\frac{\partial}{\partial s}\Big|_{s=0} \gamma_v(s) = v$). We have

$$\frac{\partial}{\partial t}\Big|_{t=0} (f(\gamma_u(s) \triangleright \gamma_v(t))) = T_1 f(Ad_{\gamma_u(s)}(v))$$

and

$$\frac{\partial}{\partial t}\Big|_{t=0} (f(\gamma_u(s)) \triangleright f(\gamma_v(t))) = Ad_{f(\gamma_u(s))}(T_1 f(v)).$$

Now, if we differentiate with respect to the variable s, we have

$$\frac{\partial}{\partial s}\Big|_{s=0} (T_1 f(Ad_{\gamma_u(s)}(v))) = T_1 f([u, v])$$

and

$$\frac{\partial}{\partial s}\Big|_{s=0} (Ad_{f(\gamma_u(s))}(T_1 f(v))) = [T_1 f(u), T_1 f(v)].$$

This means that $T_1 f$ is a morphism of Leibniz algebras.

$\square$

Example 3.3 (Group)**.** Let G be a Lie group, then we get in this way the canonical Lie algebra structure on $T_1 G$.

Example 3.4 (Augmented rack)**.** Let $X \xrightarrow{p} G$ be an augmented Lie rack, then $T_1 X \xrightarrow{T_1 p} T_1 G$ is a Lie algebra in the category of linear maps (see [LP98]). This structure induces a Leibniz algebra structure on $T_1 X$ which is isomorphic to the one induces by the Lie rack structure on X.

We remark that a local smooth structure around 1 is sufficient to provide $T_1 X$ with a Leibniz algebra structure.

Proposition 3.5. *Let X be a local Lie rack, then $T_1 X$ is a Leibniz algebra.*

3.2   From $As_p(X)$-modules to Leibniz representations

Let X be a rack. An $As_p(X)$-module is an abelian group A provided with a morphism of groups $\phi : As_p(X) \to Aut(A)$. By adjointness, this is the same thing as a morphism of pointed racks $\phi : X \to Conj(Aut(A))$.

Definition 3.6. *Let X be a Lie rack, a **smooth $As(X)$-module** is an $As_p(X)$ module A such that*

1. A is an abelian Lie group.

2. $\phi : X \times A \to A$ is smooth.

Recall that, given a Leibniz algebra $\mathfrak{g}$, a $\mathfrak{g}$-representation $\mathfrak{a}$ is a vector space provided with two linear maps
$$[-,-]_L : \mathfrak{g} \otimes \mathfrak{a} \to \mathfrak{a}$$
and
$$[-,-]_R : \mathfrak{a} \otimes \mathfrak{g} \to \mathfrak{a},$$
satisfying the axioms $(LLM), (LML)$ and (MLL) given in section 1.

There are two particular classes of modules. The first, that we called *symmetric*, are the modules where $[-,-]_L = -[-,-]_R$. The second, that we called *anti-symmetric*, are the modules where $[-,-]_R = 0$. Given a Leibniz algebra $\mathfrak{g}$ and $\mathfrak{a}$ a vector space equipped with a morphism of Leibniz algebra $\phi : \mathfrak{g} \to End(\mathfrak{a})$, we can put two structures of $\mathfrak{g}$-representation on $\mathfrak{a}$. One is *symmetric* and defined by
$$[x,a]_L = \phi_x(a) \text{ and } [a,x]_R = -\phi_x(a), \quad \forall x \in \mathfrak{g}, a \in \mathfrak{a},$$
and the other is *anti-symmetric* and defined by

$$[x,a]_L = \phi_x(a) \text{ and } [a,x]_R = 0, \quad \forall x \in \mathfrak{g}, a \in \mathfrak{a}.$$

Moreover, given a rack X and A a (smooth) $As(X)$-module, we can put two structures of (smooth) X-module on A. One is called *symmetric* and defined by

$$\phi_{x,y}(a) = \phi_x(a) \text{ and } \psi_{x,y}(a) = a - \phi_{x \triangleright y}(a), \quad \forall x, y \in X, a \in A,$$

and the other is called *anti-symmetric* and defined by

$$\phi_{x,y}(a) = \phi_x(a) \text{ and } \psi_{x,y}(a) = 0, \quad \forall x, y \in X, a \in A.$$

These constructions are similar to each other because one is the infinitesimal version of the other. Indeed, let (A, ϕ, ψ) be a smooth symmetric X-module. We have by definition two smooth maps
$$\phi : X \times X \times A \to A \text{ and } \psi : X \times X \times A \to A$$
with $\phi_{1,1} = id, \psi_{1,1} = 0$. Thus the differentials of these maps at $(1,1)$ give us two maps
$$\epsilon : X \times X \to Aut(\mathfrak{a}); \epsilon(x,y) = T_1 \phi_{x,y}$$
and
$$\chi : X \times X \to End(\mathfrak{a}); \chi(x,y) = T_1 \psi_{x,y}.$$
These maps are smooth, so we can differentiate them at $(1,1)$ to obtain
$$T_{(1,1)}\epsilon : \mathfrak{x} \oplus \mathfrak{x} \to End(\mathfrak{a})$$
and
$$T_{(1,1)}\chi : \mathfrak{x} \oplus \mathfrak{x} \to End(\mathfrak{a}).$$
Then we put
$$[-,-]_L : \mathfrak{g} \otimes \mathfrak{a} \to \mathfrak{a}; [u,m]_L = T_{(1,1)}\epsilon(u,0)(m)$$
and
$$[-,-]_R : \mathfrak{a} \otimes \mathfrak{g} \to \mathfrak{a}; [m,u]_R = T_{(1,1)}\chi(0,u)(m).$$

14

Lemma 3.7. *The linear map $[-,-]_L$ satisfies the axiom (LLM).*

Proof : We want to show that for all $u, v \in \mathfrak{x}, m \in \mathfrak{a}$

$$[u, [v, m]_L]_L = [[u, v], m]_L + [v, [u, m]_L]_L.$$

By hypothesis, ϕ satisfies the following relation for all $x, y, z \in X$

$$\phi_{x, y \triangleright z} \circ \phi_{y,z} = \phi_{x \triangleright y, x \triangleright z} \circ \phi_{x,z},$$

and if we take $z = 1$, we obtain

$$\phi_{x,1} \circ \phi_{y,1} = \phi_{x \triangleright y, 1} \circ \phi_{x,1},$$

and so

$$T_1 \phi_{x,1} \circ T_1 \phi_{y,1} = T_1 \phi_{x \triangleright y, 1} \circ T_1 \phi_{x,1},$$

that is

$$\epsilon(x, 1) \circ \epsilon(y, 1) = \epsilon(x \triangleright y, 1) \circ \epsilon(x, 1).$$

Let $u, v \in \mathfrak{x}$ and γ_u (resp. γ_v) be a smooth path in X pointed at 1 such that $\left.\dfrac{\partial}{\partial s}\right|_{s=0} \gamma_u(s) = u$

(resp. $\left.\dfrac{\partial}{\partial s}\right|_{s=0} \gamma_v(s) = v$). We have

$$\left.\frac{\partial}{\partial t}\right|_{t=0} \left(\epsilon(\gamma_u(s), 1) \circ \epsilon(\gamma_v(t), 1) \right) = \epsilon(\gamma_u(s), 1) \circ \left.\frac{\partial}{\partial t}\right|_{t=0} \epsilon(\gamma_v(t), 1)$$
$$= \epsilon(\gamma_u(s), 1) \circ [v, -]_L$$

and

$$\left.\frac{\partial}{\partial s}\right|_{s=0} \left(\epsilon(\gamma_u(s), 1) \circ [v, 0]_L \right) = \left(\left.\frac{\partial}{\partial s}\right|_{s=0} \epsilon(\gamma_u(s), 1) \right) \circ [v, -]_L$$
$$= [u, -]_L \circ [v, -]_L$$
$$= [u, [v, -]_L]_L.$$

On the other hand, we have

$$\left.\frac{\partial}{\partial t}\right|_{t=0} \left(\epsilon(\gamma_u(s) \triangleright \gamma_v(t), 1) \circ \epsilon(\gamma_u(s), 1) \right) = [Ad_{\gamma_u(s)}(v), -]_L \circ \epsilon(\gamma_u(s), 1)$$

and

$$\left.\frac{\partial}{\partial s}\right|_{s=0} \left([Ad_{\gamma_u(s)}(v), -]_L \circ \epsilon(\gamma_u(s), 1) \right) = \left.\frac{\partial}{\partial s}\right|_{s=0} \left([Ad_{\gamma_u(s)}(v), -]_L \right) \circ \epsilon(1, 1)$$
$$+ [Ad_{\gamma_u(0)}(v), -]_L \circ \left.\frac{\partial}{\partial s}\right|_{s=0} \left(\epsilon(\gamma_u(s), 1) \right)$$
$$= [[u, v], -]_L + [v, -]_L \circ [u, -]_L$$
$$= [[u, v], -]_L + [v, [u, -]]_L.$$

Hence by identification

$$[u, [v, -]_L]_L = [[u, v], -]_L + [v, [u, -]]_L.$$

That is, $[-,-]_L$ satisfies (LLM).

$\square$

Lemma 3.8. *If (A, ϕ, ψ) is symmetric, then $[-,-]_L = -[-,-]_R$ and if (A, ϕ, ψ) is anti-symmetric, then $[-,-]_R = 0$.*

Proof : Suppose that (A, ϕ, ψ) is anti-symmetric, then $\psi = 0$, and it is clear that $[-,-]_R = 0$. Now, suppose that (A, ϕ, ψ) is symmetric, that is $\psi_{x,y}(a) = a - \phi_{x \rhd y, 1}(a)$. We have $\chi(x, y) = T_1 \psi_{x,y} = id - \epsilon(x \rhd y, 1)$. Let $u \in \mathfrak{x}$ and γ_u be a path in X pointed at 1 such that $\left. \dfrac{\partial}{\partial s} \right|_{s=0} \gamma_u(s) = u$. We have

$$
\begin{aligned}
[-, u]_R &= T_{(1,1)} \chi(0, u) \\
&= \left. \frac{\partial}{\partial s} \right|_{s=0} \chi(1, \gamma_u(s)) \\
&= \left. \frac{\partial}{\partial s} \right|_{s=0} (id - \epsilon(\gamma_u(s), 1)) \\
&= - \left. \frac{\partial}{\partial s} \right|_{s=0} \epsilon(\gamma_u(s), 1) \\
&= -[u, -]_L.
\end{aligned}
$$

$\square$

Finally, we have shown the following proposition:

Proposition 3.9. *Let X be a Lie rack, let $\mathfrak{x}$ be its Leibniz algebra, let A be an abelian Lie group and let $\mathfrak{a}$ be its Lie algebra. If (A, ϕ, ψ) is a smooth symmetric (resp. anti-symmetric) X-module, then $(\mathfrak{a}, [-,-]_L, [-,-]_R)$ is a symmetric (resp. anti-symmetric) $\mathfrak{x}$-module.*

3.3   From Lie rack cohomology to Leibniz cohomology

Proposition 3.10. *Let X be a Lie rack and let A be a smooth $As(X)$-module. We have morphisms of cochains complexes*

$$
CR_p^n(X, A^s)_s \xrightarrow{\delta^n} CL^n(\mathfrak{x}, \mathfrak{a}^s)
$$

and

$$
CR_p^n(X, A^a)_s \xrightarrow{\delta^n} CL^n(\mathfrak{x}, \mathfrak{a}^a),
$$

given by $\delta^n(f)(a_1, \dots, a_n) = d^n f(1, \dots, 1)((a_1, 0, \dots, 0), \dots, (0, \dots, 0, a_n))$ (where $d^n f$ is the n-th differential of f).

Proof : Let $f \in CR_p^n(X, A^s)$ and $(x_0, \dots, x_n) \in X^{n+1}$. We have

$$
d_R f(x_0, \dots, x_n) = \sum_{i=1}^{n+1} (-1)^{i-1} \phi_{x_1 \rhd \dots \rhd x_i} (f(x_1, \dots, \hat{x}_i, \dots, x_{n+1})) - f(x_1, \dots, x_i \rhd x_{i+1}, \dots, x_i \rhd x_{n+1})
$$

Let $(\gamma_0(t_0), \dots, \gamma_n(t_n))$ be a family of paths $\gamma_i :] - \epsilon_i, +\epsilon_i[\to V$ such that

$$
\gamma_i(0) = 1 \text{ and } \frac{\partial}{\partial s}_{s=0} \gamma_i(s) = x_i.
$$

16

We want to show that

$$\delta^{n+1}(d_R^n f) = d_L^n(\delta^n(f)).$$

We have by definition

$$\delta^{n+1}(d_R^n f)(a_0, \ldots, a_n) = \left. \frac{\partial^{n+1}}{\partial t_0 \ldots \partial t_n} \right|_{t_i=0} df(\gamma_0(t_0), \ldots, \gamma_n(t_n)).$$

Lemma 3.11. $\forall i \in \{1, \ldots, n\}$

$$\left. \frac{\partial^{n+1}}{\partial t_0 \ldots \partial t_n} \right|_{|t_i=0} \phi_{\gamma_0(t_0) \rhd \ldots \rhd \gamma_i(t_i)}(f(\gamma_0(t_0), \ldots, \gamma_{i-1}(t_{i-1}), \gamma_{i+1}(t_{i+1}), \ldots, \gamma_n(t_n))) = a_i.d_n(f)(a_0, \ldots, \widehat{a_i}, \ldots, a_n)$$

Proof :

Notation:

$$(t_0, \ldots, t_i) \mapsto \phi_{\gamma_0(t_0) \rhd \ldots \rhd \gamma_i(t_i)}$$

will be denoted by

$$(t_0, \ldots, t_i) \mapsto A(t_0, \ldots, t_i)$$

and

$$(t_0, \ldots, \widehat{t_i}, \ldots, t_n) \mapsto f(\gamma_0(t_0), \ldots, \gamma_{i-1}(t_{i-1}), \gamma_{i+1}(t_{i+1}), \ldots, \gamma_n(t_n))$$

will be denoted by

$$(t_0, \ldots, \widehat{t_i}, \ldots, t_n) \mapsto m(t_0, \ldots, \widehat{t_i}, \ldots, t_n)$$

We have

$$
\begin{aligned}
\left. \frac{\partial^{n+1}}{\partial t_0 \ldots \partial t_n} \right|_{t_k=0} A(t_0, \ldots, t_i)(m(t_0, \ldots, t_n)) &= \left. \frac{\partial^n}{\partial t_0 \ldots \partial t_{n-1}} \right|_{t_k=0} \left(\left. \frac{\partial}{\partial t_n} \right|_{t_n=0} A(t_0, \ldots, t_i)(m(t_0, \ldots, 0)) \right. \\
&+ \left. \frac{\partial^n}{\partial t_0 \ldots \partial t_{n-1}} \right|_{t_k=0} \left(A(t_0, \ldots, t_i)\left(\left. \frac{\partial}{\partial t_n} \right|_{t_n=0} m(t_0, \ldots, t_n) \right) \right) \\
&= \left. \frac{\partial^n}{\partial t_0 \ldots \partial t_{n-1}} \right|_{t_k=0} \left(A(t_0, \ldots, t_i)\left(\left. \frac{\partial}{\partial t_n} \right|_{t_n=0} m(t_0, \ldots, t_n) \right) \right)
\end{aligned}
$$

because by definition we have $f(x_0, \ldots, 1, \ldots, x_n) = 0$.
Thus by the same argument we have

$$\left. \frac{\partial^{n+1}}{\partial t_0 \ldots \partial t_n} \right|_{t_k=0} A(t_0, \ldots, t_i)(m(t_0, \ldots, t_n)) = \left. \frac{\partial}{\partial t_i} \right|_{t_i=0} A(0, \ldots, 0, t_i)\left(\left. \frac{\partial^n}{\partial t_0 \ldots \partial t_n} \right|_{t_k=0} m(t_0, \ldots, t_n) \right)$$

Hence

$$\left. \frac{\partial^{n+1}}{\partial t_0 \ldots \partial t_n} \right|_{t_i=0} A(t_0, \ldots, t_n)(m(t_0, \ldots, t_n)) = x_i.d_n(f)(x_0, \ldots, \hat{x}_i, \ldots, x_n)$$

$\square$

Lemma 3.12. $\forall i \in \{1, \ldots, n\}$

$$\left. \frac{\partial^{n+1}}{\partial t_0 \ldots \partial t_n} \right|_{t_i=0} f(\gamma_0(t_0), \ldots, \gamma_i(t_i) \rhd \gamma_{i+1}(t_{i+1}), \ldots, \gamma_i(t_i) \rhd \gamma_n(t_n)) = \sum_{k=i+1}^n \delta_n f(a_0, \ldots, [a_i, a_k], \ldots, a_n)$$

17

Proof : We have $\left.\dfrac{\partial^{n+1}}{\partial t_0 \ldots \partial t_n}\right|_{t_i=0} f(\gamma_0(t_0), \ldots, \gamma_i(t_i) \triangleright \gamma_{i+1}(t_{i+1}), \ldots, \gamma_i(t_i) \triangleright \gamma_n(t_n))$ which is equal to

$$\left.\frac{\partial}{\partial t_i}\right|_{t_i=0} d^n f(1, \ldots, 1)((a_0, 0, \ldots, 0), \ldots, (0, \ldots, Ad_{\gamma_i(t_i)}(a_{i+1}), \ldots, 0), \ldots, (0, \ldots, 0, Ad_{\gamma_i(t_i)}(a_n))$$

Moreover, this is equal to

$$\sum_{k=i+1}^{n} d^n f(1, \ldots, 1)((a_0, 0, \ldots, 0), \ldots, (0, \ldots, [a_i, a_k], \ldots, 0), \ldots, (0, \ldots, 0, a_n))$$

and this expression is equal to $\displaystyle\sum_{k=i+1}^{n} \delta^n f(a_0, \ldots, [a_i, a_k], \ldots, a_n)$.

$\square$

Hence

$$
\begin{aligned}
\delta^{n+1}(d_R^n f)(a_0, \ldots, a_n) &= \sum_{i=0}^{n}(-1)^i\left(a_i . \delta^n(f)(a_0, \ldots, \hat{a}_i, \ldots, a_n) - \sum_{k=i+1}^{n} \delta^n f(a_0, \ldots, [a_i, a_k], \ldots, a_n)\right) \\
&= \sum_{i=0}^{n}(-1)^i a_i . \delta^n(f)(a_0, \ldots, \hat{a}_i, \ldots, a_n) + \sum_{0 \le i < k \le n}(-1)^{i+1} \delta^n f(a_0, \ldots, [a_i, a_k], \ldots, a_n)
\end{aligned}
$$

that is

$$\delta^{n+1}(d_R^n f) = d_L^n(\delta^n(f))$$

This is exactly the same proof as for the case where A is anti-symmetric.

$\square$

We remark that we only need a local cocyle identity around 1. Thus we have

Proposition 3.13. *Let X be a Lie rack, let U be a 1-neighborhood in X and let A be a smooth $As(X)$-module. We have morphisms of cochain complexes*

$$CR_p^n(U, A^s) \xrightarrow{\delta^n} CL^n(\mathfrak{x}, \mathfrak{a}^s)$$

and

$$CR_p^n(U, A^a) \xrightarrow{\delta^n} CL^n(\mathfrak{x}, \mathfrak{a}^a),$$

given by $\delta^n(f)(a_0, \ldots, a_n) = d^n f(1, \ldots, 1)((a_1, 0, \ldots, 0), \ldots, (0, \ldots, 0, a_n))$.

3.4 From Leibniz cohomology to local Lie rack cohomology

In this section, we study two cases of Leibniz cocyles integration. This section will be used in the following section to integrate a Leibniz algebra into a local augmented Lie rack.

First, we study the integration of a 1-cocycle in $ZL^1(\mathfrak{g}, \mathfrak{a}^s)$ into a Lie rack 1-cocycle in $ZR_p^1(G, \mathfrak{a}^s)_s$, where G is a simply connected Lie group with Lie algebra $\mathfrak{g}$ and $\mathfrak{a}$ a representation of G.

Secondly, we use the result of the first part to study the integration of a 2-cocycle in $ZL^2(\mathfrak{g}, \mathfrak{a}^a)$ into a local Lie rack 2-cocycle in $ZR_p^2(U, \mathfrak{a}^a)_s$, where U is a 1-neighborhood in a simply connected Lie group G with Lie algebra $\mathfrak{g}$, and $\mathfrak{a}$ a representation of G. It is this second part that we will use to integrate Leibniz algebras.

3.4.1 From Leibniz 1-cocycles to Lie rack 1-cocycles

Let G be a simply connected Lie group and $\mathfrak{a}$ a representation of G. We want to define a morphism I^1 from $ZL^1(\mathfrak{g}, \mathfrak{a}^s)$ to $ZR_p^1(G, \mathfrak{a}^s)_s$ which sends $BL^1(\mathfrak{g}, \mathfrak{a}^s)$ into $BR_p^1(G, \mathfrak{a}^s)_s$. For this, we put

$$I^1(\omega)(g) = \int_{\gamma_g} \omega^{eq},$$

where $\omega \in ZL^1(\mathfrak{g}, \mathfrak{a}^s)$, $\gamma : G \times [0, 1] \to G$ is a smooth map such that γ_g is a path from 1 to g, γ_1 is the constant path equal to 1, and ω^{eq} is the closed left equivariant differential form in $\Omega^1(G, \mathfrak{a})$ defined by

$$\omega^{eq}(g)(m) = g.(\omega(T_g L_{g^{-1}}(m))).$$

By definition, it is clear that $I^1(\omega)(1) = 0$.

For the moment, $I^1(\omega)$ depends on γ, but because ω is a cocycle and G is simply connected, the dependence with respect to γ disappears.

Proposition 3.14. I^1 *does not depend on* γ.

Proof : Let $\gamma, \gamma' : G \times [0, 1] \to G$ such that $\gamma_g(0) = \gamma'_g(0) = 1$ and $\gamma_g(1) = \gamma'_g(1) = g$. We are going to show that

$$\int_{\gamma_g} \omega^{eq} = \int_{\gamma'_g} \omega^{eq}.$$

We have $\int_{\gamma_g} \omega^{eq} - \int_{\gamma'_g} \omega^{eq} = \int_{\gamma_g - \gamma'_g} \omega^{eq}$. As $H_1(G) = 0$ and $\partial(\gamma_g - \gamma'_g) = 0$, there exists $\sigma_g : [0, 1]^2 \to G$ such that $\gamma_g - \gamma'_g = \partial \sigma_g$. So

$$\int_{\gamma_g} \omega^{eq} - \int_{\gamma'_g} \omega^{eq} = \int_{\gamma_g - \gamma'_g} \omega^{eq}$$
$$= \int_{\partial \sigma_g} \omega^{eq}$$
$$= \int_{\sigma_g} d_{dR} \omega^{eq}$$
$$= 0.$$

Hence I^1 does not depend on γ.

$\square$

Proposition 3.15. I^1 *sends cocycles to cocycles and coboundaries to coboundaries.*

Proof : First, let $\omega \in ZL^1(\mathfrak{g}, \mathfrak{a}^s)$, we have

$$d_R I(\omega)(g, h) = g.I(\omega)(h) - I(\omega)(g \triangleright h) - (g \triangleright h).I(\omega)(g) + I(\omega)(g)$$

$$= g. \int_{\gamma_h} \omega^{eq} - \int_{\gamma_{g \triangleright h}} \omega^{eq} - (g \triangleright h). \int_{\gamma_g} \omega^{eq} + \int_{\gamma_g} \omega^{eq}$$

$$= \int_{\gamma_h} g.\omega^{eq} - \int_{\gamma_{g \triangleright h}} \omega^{eq} - \int_{\gamma_g} (g \triangleright h).\omega^{eq} + \int_{\gamma_g} \omega^{eq}$$

$$= \int_{g\gamma_h} \omega^{eq} - \int_{\gamma_{g \triangleright h}} \omega^{eq} - \int_{(g \triangleright h)\gamma_g} \omega^{eq} + \int_{\gamma_g} \omega^{eq}$$

$$= \int_{g\gamma_h - \gamma_{g \triangleright h} - (g \triangleright h)\gamma_g + \gamma_g} \omega^{eq}.$$

As $H^1(G) = 0$ and $\partial(g\gamma_h - \gamma_{g \triangleright h} - (g \triangleright h)\gamma_g + \gamma_g) = 0$, there exists $\gamma_{g,h} : [0,1]^2 \to G$ such that $\partial\gamma_{g,h} = g\gamma_h - \gamma_{g \triangleright h} - (g \triangleright h)\gamma_g + \gamma_g$. Hence, we have

$$d_R I(\omega)(g, h) = \int_{\partial\gamma_{g,h}} \omega^{eq}$$

$$= \int_{\gamma_{g,h}} d_{dR}\omega^{eq}$$

$$= 0.$$

Hence $ZL^1(\mathfrak{g}, \mathfrak{a}^s)$ is sent to $ZR_p^1(G, \mathfrak{a}^s)_s$.

Secondly, let $\omega \in BL^1(\mathfrak{g}, \mathfrak{a}^s)$. There exists $\beta \in \mathfrak{a}$ such that $\omega(m) = m.\beta$. We have

$$I(\omega)(g) = \int_{\gamma_g} \omega^{eq}$$

$$= \int_{\gamma_g} (d_L\beta)^{eq}$$

$$= \int_{\gamma_g} d_{dR}\beta^{eq}$$

$$= \beta^{eq}(g) - \beta^{eq}(1)$$

$$= g.\beta - \beta$$

$$= d_R\beta(g).$$

Hence $BL^1(\mathfrak{g}, \mathfrak{a}^s)$ is sent to $BR_p^1(G, \mathfrak{a}^s)_s$.

$\square$

Proposition 3.16. I^1 *is a left inverse for* δ^1.

Proof : Let $\omega \in ZL^1(\mathfrak{g}, \mathfrak{a}^s)$. Let $\varphi : U \to \mathfrak{g}$ be a local chart around 1 such that $\varphi(1) = 0$ and $d\varphi^{-1}(0) = id$.
We define for $x \in \mathfrak{g}$ the smooth map $\alpha_x :\ ] - \epsilon, +\epsilon[\to U$ by setting

$$\alpha_x(s) = \varphi^{-1}(sx),$$

20

and we define for all $s \in]-\epsilon, +\epsilon[$, the smooth map $\gamma_{\alpha_x(s)} : [0,1] \to U$ by setting

$$\gamma_{\alpha_x(s)}(t) = \varphi^{-1}(tsx).$$

We have

$$\begin{aligned}
\delta^1(I^1(\omega))(x) &= \left.\frac{\partial}{\partial s}\right|_{s=0} I^1(\omega)(\alpha_x(s)) \\
&= \left.\frac{\partial}{\partial s}\right|_{s=0} \int_{\gamma_{\alpha_x(s)}} \omega^{eq} \\
&= \left.\frac{\partial}{\partial s}\right|_{s=0} \int_{[0,1]} \gamma^*_{\alpha_x(s)}\omega^{eq} \\
&= \left.\frac{\partial}{\partial s}\right|_{s=0} \int_{[0,1]} \omega^{eq}(\gamma_{\alpha_x(s)}(t))(\left.\frac{\partial}{\partial t}\right|_{t=0}\gamma_{\alpha_x(s)}(t))dt.
\end{aligned}$$

Moreover, $\left.\dfrac{\partial}{\partial t}\right|_{t=0}\gamma_{\alpha_x(s)}(t) = \left.\dfrac{\partial}{\partial t}\right|_{t=0}\varphi^{-1}(stx) = sx$, thus

$$\begin{aligned}
\delta^1(I^1(\omega))(x) &= \left.\frac{\partial}{\partial s}\right|_{s=0} \int_{[0,1]} \omega^{eq}(\gamma_{\alpha_x(s)}(t))(sx)dt \\
&= \int_{[0,1]} \left.\frac{\partial}{\partial s}\right|_{s=0} \omega^{eq}(\gamma_{\alpha_x(s)}(t))(sx)dt \\
&= \int_{[0,1]} \left.\frac{\partial}{\partial s}\right|_{s=0} \omega^{eq}(\varphi^{-1}(tsx))(sx)dt \\
&= \int_{[0,1]} \left.\frac{\partial}{\partial s}\right|_{s=0} (\varphi^{-1})^*\omega^{eq}(tsx)(sx)dt \\
&= \int_{[0,1]} \left.\frac{\partial}{\partial s}\right|_{s=0} s(\varphi^{-1})^*\omega^{eq}(tsx)(x)dt \\
&= \int_{[0,1]} (\varphi^{-1})^*\omega^{eq}(0)(x)dt \\
&= \omega(x)\int_{[0,1]} dt \\
&= \omega(x).
\end{aligned}$$

Hence $\delta^1 \circ I^1 = id$.

$\square$

Remark 3.17. In fact, $I^1(\omega)$ is also a Lie group 1-cocyle. Indeed, the formula to define $I^1(\omega)$ is the same as the one defined by K.H. Neeb in Section 3 of [Nee04], and in this article he shows that $I^1(\omega)$ is a group cocycle. The following calculation shows that this group cocycle identity satisfied by $I^1(\omega)$ implies the rack cocycle identity satisfied by $I^1(\omega)$. Indeed, $I^1(\omega)$ is a group cocycle, thus we have

$$d_{Gp}I^1(\omega)(g,h) - d_{Gp}I^1(\omega)(g \triangleright h, g) = 0.$$

Moreover, we have

$$d_{Gp}I^1(\omega)(g,h) - d_{Gp}I^1(\omega)(g \triangleright h, g) = g.I^1(\omega)(h) - I^1(\omega)(gh) + I^1(\omega)(g) - g \triangleright h.I^1(\omega)(g)$$
$$+ I^1(\omega)(gh) - I^1(\omega)(g \triangleright h)$$
$$= d_R I^1(\omega)(g,h).$$

Thus $d_R I^1(\omega)(g,h) = 0$, and we see clearly that the rack cocycle identity is implied by the group cocycle identity. We will use this remark in Proposition 3.26.

3.4.2   From Leibniz 2-cocycles to Lie local rack 2-cocycles

Let G be a simply connected Lie group, let U be a 1-neigbourhood in G such that log is defined on U and let $\mathfrak{a}$ be a representation of G. In Proposition 3.13 we have defined for all $n \in \mathbb{N}$ the maps

$$HR_s^n(U, \mathfrak{a}^a) \xrightarrow{[\delta^n]} HL^n(\mathfrak{g}, \mathfrak{a}^a).$$

In the next section, we will see that a Leibniz algebra can be integrated into a local Lie rack since the morphism $[\delta^2]$ is surjective. More precisely, if we can construct a left inverse for $[\delta^2]$, then it gives us an explicit method to construct the local Lie rack which integrates the Leibniz algebra.

In this section, we define a morphism $[I^2]$ from $HL^2(\mathfrak{g}, \mathfrak{a}^a)$ to $HR_s^2(U, \mathfrak{a}^a)$, and we show that it is a left inverse for $[\delta^2]$. To construct the map $[I^2]$, we adapt an integration method of Lie algebra cocycles into Lie group cocycles by integration over simplex. This method is due to W.T. Van Est ([Est54]) and used by K.H. Neeb ([Nee02, Nee04]) for the infinite dimensional case.

Definition of I^2

We want to define a map from $ZL^2(\mathfrak{g}, \mathfrak{a}^a)$ to $ZR_p^2(U, \mathfrak{a}^a)_s$ such that $BL^2(\mathfrak{g}, \mathfrak{a}^a)$ is sent to $BR_p^2(U, \mathfrak{a}^a)_s$. In the previous section, we have integrated a Leibniz 1-cocycle on a Lie algebra $\mathfrak{g}$ with coefficients in a symmetric module $\mathfrak{a}^s$. In Proposition 1.1, we have shown that there is an isomorphism between $CL^2(\mathfrak{g}, \mathfrak{a}^a)$ and $CL^1(\mathfrak{g}, Hom(\mathfrak{g}, \mathfrak{a})^s)$, which sends $ZL^2(\mathfrak{g}, \mathfrak{a}^a)$ to $ZL^1(\mathfrak{g}, Hom(\mathfrak{g}, \mathfrak{a})^s)$ and $BL^2(\mathfrak{g}, \mathfrak{a}^a)$ to $BL^1(\mathfrak{g}, Hom(\mathfrak{g}, \mathfrak{a})^s)$. Hence, we can define a map

$$I : ZL^2(\mathfrak{g}, \mathfrak{a}^a) \to ZR_p^1(G, Hom(\mathfrak{g}, \mathfrak{a})^s)_s,$$

which sends $BL^2(\mathfrak{g}, \mathfrak{a}^a)$ into $BR_p^1(G, Hom(\mathfrak{g}, \mathfrak{a})^s)_s$. This is the composition

$$ZL^2(\mathfrak{g}, \mathfrak{a}^a) \xrightarrow{\tau^2} ZL^1(\mathfrak{g}, Hom(\mathfrak{g}, \mathfrak{a})^s) \xrightarrow{I^1} ZR_p^1(G, Hom(\mathfrak{g}, \mathfrak{a})^s)_s.$$

Now, we want to define a map from $ZR_p^1(G, Hom(\mathfrak{g}, \mathfrak{a})^s)_s$ to $ZR_p^2(U, \mathfrak{a}^a)$. Let $\beta \in CR_p^1(G, Hom(\mathfrak{g}, \mathfrak{a})^s)_s$, β has values in the representation $Hom(\mathfrak{g}, \mathfrak{a})$, so for all $g \in G$, we can consider the equivariant differential form $\beta(g)^{eq} \in \Omega^1(G, \mathfrak{a})$ defined by

$$\beta(g)^{eq}(h)(m) := h.(\beta(g)(T_h L_{h^{-1}}(m))).$$

Then we define an element in $CR_p^2(U, \mathfrak{a}^a)$ by setting

$$f(g,h) = \int_{\gamma_{g \triangleright h}} (\beta(g))^{eq},$$

where $\gamma : G \times [0,1] \to G$ is a smooth map such that for all $g \in G$, γ_g is a path from 1 to g in G and $\gamma_1 = 1$.

For the moment, an element of $ZR_p^1(G, Hom(\mathfrak{g}, \mathfrak{a})^s)_s$ is not necessarily sent to an element of $ZR_p^2(U, \mathfrak{a}^a)_s$. To reach our goal, we have to specify the map γ, and we define it by setting

$$\gamma_g(s) = exp(s \log(g)).$$

Then, we define $I^2 : ZL^2(\mathfrak{g}, \mathfrak{a}^a) \to CR_p^2(U, \mathfrak{a}^a)_s$ by setting for all $(g, h) \in U_{2-loc}$ (cf. notation in Section 2.4)

$$I^2(\omega)(g, h) = \int_{\gamma_{g \triangleright h}} (I(\omega)(g))^{eq}.$$

By definition, it is clear that $I^2(\omega)(g, 1) = I^2(\omega)(1, g) = 0$.

Properties of I^2

Proposition 3.18. I^2 *sends* $ZL^2(\mathfrak{g}, \mathfrak{a}^a)$ *into* $ZR_p^2(U, \mathfrak{a}^a)_s$.

To prove this proposition we need some lemmas.

Lemma 3.19. *For all* $(g, h) \in U_{2-loc}$, *we have* $\gamma_{g \triangleright h} = g \triangleright \gamma_h$.

Proof : Let $(g, h) \in U_{2-loc}$, by definition we have

$$\gamma_{g \triangleright h}(s) = exp(s(\log(g \triangleright h))).$$

By naturality of the exponential, and a fortiori of the logarithm, we have

$$\begin{aligned}
exp(s(log(g \triangleright h))) &= exp(sAd_g(\log(h))) \\
&= exp(Ad_g(s \log(h))) \\
&= c_g(exp(s \log(h))) \\
&= c_g(\gamma_h(s)).
\end{aligned}$$

Hence $\gamma_{g \triangleright h} = g \triangleright \gamma_h$.

$\square$

Lemma 3.20. *Let G be a Lie group, let $\mathfrak{a}$ be a representation of G and $\omega \in Hom(\mathfrak{g}, \mathfrak{a})$. We have for all $g \in G$*

$$g.(\omega^{eq}) = c_g^*((g.\omega)^{eq}).$$

Proof : Let $g, h \in G$ and $x \in T_h G$, we have:

$$\begin{aligned}
(c_g^*((g.\omega)^{eq}))(h)(x) &= (g.\omega)^{eq}(g \triangleright h)(d_h c_g(x)) \\
&= (g \triangleright h).(g.\omega)(d_{g \triangleright h} L_{g \triangleright h^{-1}}(d_h c_g(x))) \\
&= (g \triangleright h).(g.\omega)(d_h(c_g \circ L_{h^{-1}})(x)) \\
&= (g \triangleright h).(g.(\omega(Ad(g^{-1})(d_h(c_g \circ L_{h^{-1}})(x))))) \\
&= gh.(\omega(d_h(c_{g^{-1}} \circ c_g \circ L_{h^{-1}})(x))) \\
&= g.(h.(\omega(d_h L_{h^{-1}}(x)))) \\
&= g.(\omega^{eq}(h)(x)).
\end{aligned}$$

Hence $c_g^*((g.\omega)^{eq}) = g.(\omega)^{eq}$.

23

$\square$

Proof of proposition : Let $\omega \in ZL^2(\mathfrak{g}, \mathfrak{a}^a)$ and $(g, h, k) \in U_{3-loc}$. We have

$$d_R(I^2(\omega))(g, h, k) = g.I^2(\omega)(h, k) - I^2(\omega)(g \triangleright h, g \triangleright k) - (g \triangleright h).I^2(\omega)(g, k) + I^2(\omega)(g, h \triangleright k)$$

$$= g. \int_{\gamma_{h \triangleright k}} (I(\omega)(h))^{eq} - \int_{\gamma_{g \triangleright (h \triangleright k)}} (I(\omega)(g \triangleright h))^{eq} - (g \triangleright h). \int_{\gamma_{g \triangleright k}} (I(\omega)(g))^{eq}$$

$$+ \int_{\gamma_{g \triangleright (h \triangleright k)}} (I(\omega)(g))^{eq}$$

$$= \int_{\gamma_{h \triangleright k}} g.((I(\omega)(h))^{eq}) - \int_{\gamma_{g \triangleright (h \triangleright k)}} (I(\omega)(g \triangleright h))^{eq} - \int_{\gamma_{g \triangleright k}} (g \triangleright h).((I(\omega)(g))^{eq})$$

$$+ \int_{\gamma_{g \triangleright (h \triangleright k)}} (I(\omega)(g))^{eq}.$$

Because of Lemma 3.20, we have

$$d_R(I^2(\omega))(g, h, k) = \int_{\gamma_{h \triangleright k}} c_g^*((g.I(\omega)(h))^{eq}) - \int_{\gamma_{g \triangleright (h \triangleright k)}} I(\omega)(g \triangleright h)^{eq} - \int_{\gamma_{g \triangleright k}} c_{g \triangleright h}^*(((g \triangleright h).I(\omega)(g))^{eq})$$

$$+ \int_{\gamma_{g \triangleright (h \triangleright k)}} I(\omega)(g)^{eq}$$

$$= \int_{c_g \circ \gamma_{h \triangleright k}} (g.I(\omega)(h)^{eq} - \int_{\gamma_{g \triangleright (h \triangleright k)}} I(\omega)(g \triangleright h)^{eq} - \int_{c_{g \triangleright h} \circ \gamma_{g \triangleright k}} ((g \triangleright h).I(\omega)(g))^{eq}$$

$$+ \int_{\gamma_{g \triangleright (h \triangleright k)}} I(\omega)(g)^{eq},$$

and because of Lemma 3.19, we have

$$d_R(I^2(\omega))(g, h, k) = \int_{\gamma_{g \triangleright (h \triangleright k)}} (g.I(\omega)(h))^{eq} - \int_{\gamma_{g \triangleright (h \triangleright k)}} I(\omega)(g \triangleright h)^{eq} - \int_{\gamma_{g \triangleright (h \triangleright k)}} ((g \triangleright h).I(\omega)(g))^{eq}$$

$$+ \int_{\gamma_{g \triangleright (h \triangleright k)}} I(\omega)(g)^{eq}$$

$$= \int_{\gamma_{g \triangleright (h \triangleright k)}} (g.I(\omega)(h))^{eq} - I(\omega)(g \triangleright h)^{eq} - ((g \triangleright h).I(\omega)(g))^{eq} + I(\omega)(g)^{eq}$$

$$= \int_{\gamma_{g \triangleright (h \triangleright k)}} (g.I(\omega)(h) - I(\omega((g \triangleright h) - (g \triangleright h).I(\omega)(g) + I(\omega)(g))^{eq}$$

$$= \int_{\gamma_{g \triangleright (h \triangleright k)}} d_R(I(\omega))(g, h)$$

$$= 0.$$

Hence $ZL^2(\mathfrak{g}, \mathfrak{a}^a)$ is sent to $ZR_p^2(U, \mathfrak{a}^a)_s$.

$\square$

Proposition 3.21. I^2 *sends* $BL^2(\mathfrak{g}, \mathfrak{a}^a)$ *into* $BR_p^2(U, \mathfrak{a}^a)_s$.

Proof : Let $\omega \in BL^2(\mathfrak{g}, \mathfrak{a}^a)$, there exists an element $\beta \in CL^1(\mathfrak{g}, \mathfrak{a}^a)$ such that $\omega = d_L\beta$. We have

$$I(\omega)(g) = I^1(\tau^2(\omega))(g)$$

$$= \int_{\gamma_g} (\tau^2(\omega))^{eq}$$

$$= \int_{\gamma_g} (\tau^2(d_L\beta))^{eq}.$$

The fact that $\{\tau^n\}_{n\in\mathbb{N}}$ is a morphism of cochain complexes implies that

$$I(\omega)(g) = \int_{\gamma_g} (d(\tau^1(\beta)))^{eq}$$

$$= \int_{\gamma_g} d_{dR}((\tau^1(\beta))^{eq})$$

$$= \int_{\partial\gamma_g} (\tau^1(\beta))^{eq}$$

$$= g.\beta - \beta.$$

Hence for $(g, h) \in U_{2-loc}$ we have using Lemma 3.20

$$I_2(\omega)(g, h) = \int_{\gamma_{g\triangleright h}} (I(\omega)(g))^{eq}$$

$$= \int_{\gamma_{g\triangleright h}} ((g.\beta) - \beta)^{eq}$$

$$= \int_{\gamma_{g\triangleright h}} (g.\beta)^{eq} - \int_{\gamma_{g\triangleright h}} \beta^{eq}$$

$$= \int_{\gamma_{g\triangleright h}} (c^*_{g^{-1}}(g.(\beta^{eq}))) - \int_{\gamma_{g\triangleright h}} \beta^{eq}$$

$$= \int_{c_{g^{-1}}\circ\gamma_{g\triangleright h}} g.(\beta^{eq}) - \int_{\gamma_{g\triangleright h}} \beta^{eq}$$

$$= g.\int_{c_{g^{-1}}\circ\gamma_{g\triangleright h}} \beta^{eq} - \int_{\gamma_{g\triangleright h}} \beta^{eq}$$

$$= g.\int_{\gamma_h} \beta^{eq} - \int_{\gamma_{g\triangleright h}} \beta^{eq}$$

$$= d_R(I^1(\beta))(g, h).$$

Hence $BL^2(\mathfrak{g}, \mathfrak{a}^a)$ is sent to $BR_p^2(U, \mathfrak{a})_s$.

$\square$

Proposition 3.22. I^2 *is a left inverse for* δ^2.

Proof : Let $x, y \in \mathfrak{g}$, and I_x (resp I_y) be an interval in $\mathbb{R}$ such that $\epsilon_x(s) = exp(sx)$ (resp $\epsilon_y(s) = exp(sy)$) be defined for all $s \in I_x$ (resp for all $s \in I_y$). The map $\epsilon_x\triangleright\epsilon_y : I_x \times I_y \to G$ is

25

continuous, thus there exists W an open subset of $I_x \times I_y$ such that $(\epsilon_x \triangleright \epsilon_y)(W) \subseteq U$. Hence, there exists an interval $J \subseteq I_x \cap I_y$ such that $\epsilon_x(s) \triangleright \epsilon_y(t) \in U$ for all $(s,t) \in J \times J$.

We have to show

$$\delta^2 \circ I^2 = id.$$

Let $\omega \in ZL^2(\mathfrak{g}, \mathfrak{a}^a)$. By definition, we have

$$\delta^2(I^2(\omega))(x,y) = \left.\frac{\partial^2}{\partial s \partial t}\right|_{s,t=0} I^2(\omega)(\epsilon_x(s), \epsilon_y(s))$$

$$= \left.\frac{\partial^2}{\partial s \partial t}\right|_{s,t=0} \int_{\gamma_{\epsilon_x(s) \triangleright \epsilon_y(t)}} (I(\omega)(\epsilon_x(s)))^{eq}$$

$$= \left.\frac{\partial}{\partial s}\right|_{s=0} \left(\left.\frac{\partial}{\partial t}\right|_{t=0} \int_{\gamma_{\epsilon_y(t)}} c_{\epsilon_x(s)}^*(I(\omega)(\epsilon_x(s)))^{eq} \right).$$

First, we compute

$$\left.\frac{\partial}{\partial t}\right|_{t=0} \int_{\gamma_{\epsilon_y(t)}} c_{\epsilon_x(s)}^*(I(\omega)(\epsilon_x(s)))^{eq}$$

For the sake of clarity, we put $\alpha = c_{\epsilon_x(s)}^*(I(\omega)(\epsilon_x(s)))^{eq}$ and $\beta_t = \gamma_{\epsilon_y(t)}$. We have

$$\left.\frac{\partial}{\partial t}\right|_{t=0} \int_\beta \alpha = \left.\frac{\partial}{\partial t}\right|_{t=0} \int_{[0,1]} \beta^* \alpha$$

$$= \left.\frac{\partial}{\partial t}\right|_{t=0} \int_{[0,1]} f_t(r) dr$$

$$= \int_{[0,1]} \left.\frac{\partial}{\partial t}\right|_{t=0} f_t(r) dr,$$

where $f_t(r) = \alpha(\beta_t(r))(\beta_t'(r))$.
We have

$$\left.\frac{\partial}{\partial t}\right|_{t=0} f_t(r) = \left(\left.\frac{\partial}{\partial t}\right|_{t=0} \alpha(\beta_t(r)) \right) \beta_0'(r) + (\alpha(\beta_0(r))) \left(\left.\frac{\partial}{\partial t}\right|_{t=0} \beta_t'(r) \right).$$

Moreover, we have

$$\alpha(\beta_0(r)) = \alpha(1),$$
$$\beta_0'(r) = 0,$$

and

$$\left.\frac{\partial}{\partial t}\right|_{t=0} \beta_t'(r) = y.$$

So we have

$$\left.\frac{\partial}{\partial t}\right|_{t=0} \int_\beta \alpha = \int_{[0,1]} \alpha(1)(x) dr = \alpha(1)(y)$$

and

$$\delta^2(I^2(\omega))(x,y) = \left.\frac{\partial}{\partial s}\right|_{s=0} (c_{\epsilon_x(s)}^*(I(\omega)(\epsilon_x(s)))^{eq})(1)(y).$$

Furthermore we have

$$c^*_{\epsilon_x(s)}(I(\omega)(\epsilon_x(s))^{eq})(1)(y) = (I(\omega)(\epsilon_x(s)))^{eq}(c_{\epsilon_x(s)}(1))(Ad_{\epsilon_x(s)}(y))$$
$$= I(\omega)(\epsilon_x(s))(Ad_{\epsilon_x(s)}(y))$$
$$= (\int_{\gamma_{\epsilon_x(s)}} \tau^2(\omega)^{eq})(Ad_{\epsilon_x(s)}(y)).$$

If we put $\int_{\gamma_{\epsilon_x(s)}} \tau^2(\omega)^{eq} = \sigma(s)$ and $Ad_{\epsilon_x(s)}(y) = \lambda(s)$, we have

$$\frac{\partial}{\partial s}\Big|_{s=0} ((\int_{\gamma_{\epsilon_x(s)}} \tau^2(\omega)^{eq})(Ad_{\epsilon_x(s)}(y))) = \frac{\partial}{\partial s}\Big|_{s=0} \sigma(s)(\lambda(s))$$
$$= \sigma'(0)(\lambda(0)) + \sigma(0)(\lambda'(0)).$$

We have

$$\sigma(0) = 0,$$
$$\lambda(0) = y,$$

and

$$\sigma'(0) = \tau^2(\omega)(x).$$

Thus

$$\frac{\partial}{\partial s}\Big|_{s=0} \left((\int_{\gamma_{\epsilon_x(s)}} \tau^2(\omega)^{eq})(Ad_{\epsilon_x(s)}(y))\right) = \tau^2(\omega)(x)(y).$$

Hence $\delta^2(I^2(\omega))(x, y) = \omega(x, y)$.

$\square$

Remark 3.23. Suppose that we have a Leibniz 2-cocycle ω which is also a Lie 2-cocycle. In this case, we can integrate ω into a local Lie rack cocycle, but also into a local Lie group cocycle (cf. Introduction). Then it is natural to ask if the two constructions are related to each other.

Proposition 3.24. *Let G be a Lie group, let $\mathfrak{g}$ be its Lie algebra, let $\mathfrak{a}$ be a representation of G, $\omega \in \Lambda^2(\mathfrak{g}, \mathfrak{a})$ and γ_1, γ_2 smooth paths in G pointed in 1. Then*

$$\int_{\gamma_1} (\int_{\gamma_2} (\tau^2(\omega))^{eq})^{eq} = \int_{\gamma_1\gamma_2} \omega^{eq}$$

where $\gamma_1\gamma_2 : [0,1]^2 \to G; (s,t) \mapsto \gamma_1(t)\gamma_2(s)$.

Proof : On the one hand, we have

$$\int_{\gamma_1\gamma_2} \omega^{eq} = \int_{[0,1]^2} (\gamma_1\gamma_2)^*\omega^{eq}$$
$$= \int_{[0,1]^2} \omega^{eq}(\gamma_1\gamma_2)(\frac{\partial}{\partial s}\gamma_1(t)\gamma_2(s), \frac{\partial}{\partial t}\gamma_1(t)\gamma_2(s))dsdt,$$

27

and this expression is equal to

$$\int_{[0,1]^2} \gamma_1(t)\gamma_2(s).\omega\big(d_{\gamma_2(s)}L_{\gamma_2(s)^{-1}}(\frac{\partial}{\partial s}\gamma_2(s)), Ad_{\gamma_2(s)^{-1}}(d_{\gamma_1(t)}L_{\gamma_1(t)^{-1}}(\frac{\partial}{\partial t}\gamma_1(t)))\big)dsdt. \qquad (4)$$

On the other hand, we have

$$\int_{\gamma_1}\big(\int_{\gamma_2}(\tau^2(\omega))^{eq}\big)^{eq} = \int_{[0,1]}\gamma_1^*\big(\int_{\gamma_2}(\tau^2(\omega)^{eq})^{eq}\big)$$

$$= \int_{[0,1]}\big(\int_{\gamma_2}(\tau^2(\omega))^{eq}\big)^{eq}(\gamma_1(t))(\frac{\partial}{\partial t}\gamma_1(t))dt$$

$$= \int_{[0,1]}\gamma_1(t).\big(\int_{\gamma_2}(\tau^2(\omega))^{eq}\big)(d_{\gamma_1(t)}L_{\gamma_1(t)^{-1}}(\frac{\partial}{\partial t}\gamma_1(t)))dt$$

$$= \int_{[0,1]}\gamma_1(t).\big(\int_{[0,1]}(\tau^2(\omega))^{eq}(\gamma_2(s))(\frac{\partial}{\partial s}\gamma_2(s))\big)(d_{\gamma_1(t)}L_{\gamma_1(t)^{-1}}(\frac{\partial}{\partial t}\gamma_1(t)))dt.$$

This expression is equal to

$$\int_{[0,1]}\gamma_1(t).\big(\int_{[0,1]}\gamma_2(s).(\tau^2(\omega))(d_{\gamma_2(s)}L_{\gamma_2(s)^{-1}}(\frac{\partial}{\partial s}\gamma_2(s)))ds\big)(d_{\gamma_1(t)}L_{\gamma_1(t)^{-1}}(\frac{\partial}{\partial t}\gamma_1(t)))dt,$$

which is equal to

$$\int_{[0,1]}\gamma_1(t).\big(\int_{[0,1]}\gamma_2(s).\omega(d_{\gamma_2(s)}L_{\gamma_2(s)^{-1}}(\frac{\partial}{\partial s}\gamma_2(s)), Ad_{\gamma_2(s)^{-1}}(\cdot))ds\big)(d_{\gamma_1(t)}L_{\gamma_1(t)^{-1}}(\frac{\partial}{\partial t}\gamma_1(t)))dt.$$

Using the Fubini theorem, we show that this expression is equal to (4).

$\square$

If we apply this result to the case where $\gamma_1(s) = \gamma_{g \triangleright h}(s) = exp(s\log(g \triangleright h))$ and $\gamma_2(s) = \gamma_g(s) = exp(s\log(g))$ for $(g,h) \in U_{2-loc}$, then we obtain

Corollary 3.25. *If $\omega \in ZL^2(\mathfrak{g}, \mathfrak{a}^a) \cap Z^2(\mathfrak{g}, \mathfrak{a})$, then for all $g, h \in U_{2-loc}$ such that $gh \in U_{2-loc}$ we have*

$$I^2(\omega)(g,h) = \iota^2(\omega)(g,h) - \iota^2(\omega)(g \triangleright h, g), \qquad (5)$$

with

$$\iota^2(\omega)(g,h) = \int_{\gamma_{g,h}}\omega^{eq},$$

and where $\gamma_{g,h}$ is a smooth singular 2-chain in G such that $\partial\gamma_{g,h} = \gamma_g - \gamma_{gh} + g\gamma_h$.

We can remark that I^2 is more than a local Lie rack cocycle. Precisely, if ω is in $ZL^2(\mathfrak{g}, \mathfrak{a}^a)$ then the local rack cocycle identity satisfied by $I^2(\omega)$, comes from another identity satisfied by $I^2(\omega)$. Indeed, I^2 is defined using I, and to verify that I^2 sends Leibniz cocycles into local rack cocycles, we have used Proposition 3.15. This proposition establishes that I^1 sends Lie cocycles into rack cocycle. But, we have remarked (Remark 3.17) that the rack cocycle identity satisfied by $I^1(\omega)$, comes from the group cocycle identity. Hence, we can think that we forgot structure on $I^2(\omega)$. The following proposition points out the identity satisfied by $I^2(\omega)$ which induced the local rack identity.

28

Proposition 3.26. *If $\omega \in ZL^2(\mathfrak{g}, \mathfrak{a}^a)$, then $I^2(\omega)$ satisfies the identity*

$$g.I^2(\omega)(h, k) - I^2(\omega)(gh, k) + I^2(\omega)(g, h \triangleright k) = 0, \quad \forall (g, h, k) \in U_{3-loc}.$$

Moreover, this identity induces the local rack cocycle identity.

Proof : Let $\omega \in ZL^2(\mathfrak{g}, \mathfrak{a}^a)$ and $(g, h, k) \in U_{3-loc}$ we have:

$$g.I^2(\omega)(h, k) - I^2(\omega)(gh, k) + I^2(\omega)(g, h \triangleright k) = g. \int_{\gamma_{h \triangleright k}} (I(\omega)(h))^{eq} - \int_{\gamma_{(gh) \triangleright k}} (I(\omega)(gh))^{eq}$$

$$+ \int_{\gamma_{g \triangleright (h \triangleright k)}} (I(\omega)(g))^{eq}$$

$$= \int_{\gamma_{h \triangleright k}} g.((I(\omega)(h))^{eq}) - \int_{\gamma_{g \triangleright (h \triangleright k)}} (I(\omega)(gh))^{eq}$$

$$+ \int_{\gamma_{g \triangleright (h \triangleright k)}} (I(\omega)(g))^{eq}$$

$$= \int_{\gamma_{h \triangleright k}} c_g^*((g.I(\omega)(h))^{eq}) - \int_{\gamma_{g \triangleright (h \triangleright k)}} (I(\omega)(gh))^{eq}$$

$$+ \int_{\gamma_{g \triangleright (h \triangleright k)}} (I(\omega)(g))^{eq}$$

$$g.I^2(\omega)(h, k) - I^2(\omega)(gh, k) + I^2(\omega)(g, h \triangleright k) = \int_{c_g \circ \gamma_{h \triangleright k}} (g.I(\omega)(h))^{eq} - \int_{\gamma_{g \triangleright (h \triangleright k)}} (I(\omega)(gh))^{eq}$$

$$+ \int_{\gamma_{g \triangleright (h \triangleright k)}} (I(\omega)(g))^{eq}$$

$$= \int_{\gamma_{g \triangleright (h \triangleright k)}} (g.I(\omega)(h))^{eq} - \int_{\gamma_{g \triangleright (h \triangleright k)}} (I(\omega)(gh))^{eq}$$

$$+ \int_{\gamma_{g \triangleright (h \triangleright k)}} (I(\omega)(g))^{eq}$$

$$= \int_{\gamma_{g \triangleright (h \triangleright k)}} (g.I(\omega)(h))^{eq} - (I(\omega)(gh))^{eq} + I(\omega)(g)^{eq}$$

$$= \int_{\gamma_{g \triangleright (h \triangleright k)}} (g.I(\omega)(h) - I(\omega)(gh) + I(\omega)(g))^{eq}$$

$$= \int_{\gamma_{g \triangleright (h \triangleright k)}} d(I(\omega))(g, h)$$

$$= 0.$$

Hence $I^2(\omega)$ satisfies the wanted identity.

Moreover, let $(g, h, k) \in U_{3-loc}$, we have

$$
\begin{aligned}
d_R(I^2(\omega))(g, h, k) &= g.(I^2(\omega)(h, k)) - I^2(\omega)(g \triangleright h, g \triangleright k) - (g \triangleright h).I^2(\omega)(g, k) + I^2(\omega)(g, h \triangleright k) \\
&= g.(I^2(\omega)(h, k)) - I^2(\omega)(g \triangleright h, g \triangleright k) - (g \triangleright h).I^2(\omega)(g, k) + I^2(\omega)(g, h \triangleright k) \\
&\quad - I^2(\omega)(gh, k) + I^2(\omega)(gh, k) \\
&= \big(g.(I^2(\omega)(h, k)) - I^2(\omega)(gh, k) + I^2(\omega)(g, h \triangleright k) \big) \\
&\quad - \big((g \triangleright h).I^2(\omega)(g, k) - I^2(\omega)(gh, k) + I^2(\omega)(g \triangleright h, g \triangleright k) \big) \\
&= 0 - 0 \\
&= 0.
\end{aligned}
$$

Hence $I^2(\omega)$ is a local Lie rack cocycle.

$\square$

We will see in the next section that this identity makes it possible to integrate a Leibniz algebra into a local augmented Lie rack.

3.5 From Leibniz algebras to local Lie racks

In this section, we present the main theorem of this article. In Proposition 3.1 we have seen that the tangent space at 1 of a (local) Lie rack is provided with a Leibniz algebra structure. Conversely, we now show that every Leibniz algebra can be integrated into an augmented local Lie rack. Our construction is explicit, and by this construction, a Lie algebra is integrated into a Lie group. Conversely, we show that an augmented local Lie rack whose tangent space at 1 is a Lie algebra is necessarily a (local) Lie group. That is, there is a structure of Lie group on this augmented local Lie rack, and the conjugation on the augmented local Lie rack is the conjugation in the group.

The idea of the proof is simple and uses the knowledge of the Lie's first theorem and Lie's second theorem. Let $\mathfrak{g}$ be a Leibniz algebra. First, we decompose the vector space $\mathfrak{g}$ into a direct sum of Leibniz algebras $\mathfrak{g}_0$ and $\mathfrak{a}$ that we know how to integrate. As we know the theorem for Lie subalgebras of endomorphisms of a finite dimensional vector space V, the factors are integrable if $\mathfrak{g}$ is isomorphic (as a vector space) to an abelian extension of a Lie subalgebra $\mathfrak{g}_0$ of $End(V)$ by a $\mathfrak{g}_0$-representation $\mathfrak{a}$. Hence $\mathfrak{g}$ is isomorphic to $\mathfrak{a} \oplus_\omega \mathfrak{g}_0$, the Leibniz algebra $\mathfrak{a}$ is abelian so becomes integrated into $\mathfrak{a}$, and $\mathfrak{g}_0$ is a Lie subalgebra of $End(V)$ so becomes integrated into a simply connected Lie subgroup G_0 of $Aut(V)$. Now, we have to understand how to patch $\mathfrak{a}$ and G_0. That is, we have to understand how the gluing data ω becomes integrated into a gluing data f between $\mathfrak{a}$ and G_0. It is the local Lie rack cocycle $I^2(\omega)$, constructed in the preceding section, which answers this question. Hence, we showed that a Leibniz algebra $\mathfrak{g}$ becomes integrated into a local Lie rack of the form $\mathfrak{a} \times_f G_0$.

Let $\mathfrak{g}$ be a Leibniz algebra, there are several ways to see $\mathfrak{g}$ as an abelian extension of a Lie subalgebra $\mathfrak{g}_0$ of $End(V)$ by a $\mathfrak{g}_0$-representation $\mathfrak{a}$. Here, we take the abelian extension associated to the (left) center of $\mathfrak{g}$. By definition the left center is

$$
Z_L(\mathfrak{g}) = \{ x \in \mathfrak{g} \,|\, [x, y] = 0 \;\; \forall y \in \mathfrak{g} \}.
$$

The left center $Z_L(\mathfrak{g})$ is an ideal in $\mathfrak{g}$ and we can consider the quotient of $\mathfrak{g}$ by $Z_L(\mathfrak{g})$. By definition, $Z_L(\mathfrak{g})$ is the kernel of the *adjoint representation* $ad_L : \mathfrak{g} \to End(\mathfrak{g}), x \mapsto [x, -]$. Thus this quotient is isomorphic to a Lie subalgebra of $End(\mathfrak{g})$. We denote this quotient by $\mathfrak{g}_0$. Hence, to a Leibniz algebra $\mathfrak{g}$ there is a canonical abelian extension given by

$$Z_L(\mathfrak{g}) \overset{i}{\hookrightarrow} \mathfrak{g} \overset{p}{\twoheadrightarrow} \mathfrak{g}_0.$$

This extension gives a structure of $\mathfrak{g}_0$-representation to $Z_L(\mathfrak{g})$, and by definition of $Z_L(\mathfrak{g})$, this representation is anti-symmetric. The equivalence class of this extension is caracterised by a cohomology class in $HL^2(\mathfrak{g}_0, Z_L(\mathfrak{g}))$. Hence there is $\omega \in ZL^2(\mathfrak{g}_0, Z_L(\mathfrak{g}))$ such that the abelian extension $Z_L(\mathfrak{g}) \overset{i}{\hookrightarrow} \mathfrak{g} \overset{p}{\twoheadrightarrow} \mathfrak{g}_0$ is equivalent to

$$Z_L(\mathfrak{g}) \overset{i}{\hookrightarrow} \mathfrak{g}_0 \oplus_\omega Z_L(\mathfrak{g}) \overset{\pi}{\twoheadrightarrow} \mathfrak{g}_0.$$

Here $\mathfrak{g}_0$ is a Lie subalgebra of $End(\mathfrak{g})$, so becomes integrated into a simply connected Lie subgroup G_0 of $Aut(\mathfrak{g})$, and $Z_L(\mathfrak{g})$ is an abelian Lie algebra, so becomes integrated into itself. $Z_L(\mathfrak{g})$ is a $\mathfrak{g}_0$-representation (in the sense of Lie algebra) and G_0 is simply connected, thus by the Lie's second theorem, $Z_L(\mathfrak{g})$ is a smooth G_0-module (in the Lie group sense) and we can provide $Z_L(\mathfrak{g})$ with an anti-symmetric smooth G_0-module structure. The cocycle $\omega \in ZL^2(\mathfrak{g}, Z_L(\mathfrak{g}))$ becomes integrated into the local Lie rack cocycle $I^2(\omega) \in ZR_p^2(G_0, Z_L(\mathfrak{g}))_s$, and we can put on the cartesian product $G_0 \times Z_L(\mathfrak{g})$ a structure of local Lie rack by setting

$$(g, a) \triangleright (h, b) = (g \triangleright h, \phi_{g,h}(b) + \psi_{g,h}(a) + I^2(\omega)(g, h)),$$

where $\phi_{g,h}(b) = g.b$ and $\psi_{g,h}(a) = 0$. That is we have

$$(g, a) \triangleright (h, b) = (g \triangleright h, g.b + I^2(\omega)(g, h))).$$

It is clear by construction that this local Lie rack has its tangent space at 1 provided with a Leibniz algebra structure isomorphic to $\mathfrak{g}$. Finally, we showed the following theorem

Theorem 3.27. *Every Leibniz algebra $\mathfrak{g}$ can be integrated into a local Lie rack of the form*

$$G_0 \times_{I^2(\omega)} \mathfrak{a}^a,$$

with conjugation

$$(g,a) \rhd (h,b) = (g \rhd h, g.b + I^2(\omega)(g,h)), \tag{6}$$

and neutral element $(1,0)$, where G_0 is a Lie group, $\mathfrak{a}$ a G_0-module and $\omega \in ZL^2(\mathfrak{g}_0, \mathfrak{a}^a)$. Conversely, every local Lie rack of this form has its tangent space at 1 provides with a Leibniz algebra structure.

We ask more in our original problem. Indeed, we ask that, using the same procedure, a Lie algebra becomes integrated into a Lie group. That is, we have to show that when $\mathfrak{g}$ is a Lie algebra, then $G_0 \times Z_L(\mathfrak{g})$ is provided with a Lie group structure, and the conjugation on $G_0 \times_{I^2(\omega)} Z_L(\mathfrak{g})$ is induced by the rack product in $Conj(G_0 \times Z_L(\mathfrak{g}))$.

Let $\mathfrak{g}$ be a Lie algebra, the left center $Z_L(\mathfrak{g})$ is equal to the center $Z(\mathfrak{g})$. The abelian extension $Z_L(\mathfrak{g}) \overset{i}{\hookrightarrow} \mathfrak{g} \overset{p}{\twoheadrightarrow} \mathfrak{g}_0$ provides $Z_L(\mathfrak{g})$ with an anti-symmetric structure but also a symmetric structure, so a trivial structure. This extension becomes a central extension and the cocycle $\omega \in ZL^2(\mathfrak{g}_0, Z(\mathfrak{g}))$ is also in $Z^2(\mathfrak{g}_0, Z(\mathfrak{g}))$. On the hand, with ω we can construct a local Lie rack cocycle $I^2(\omega)$, and on the other hand, we can construct a Lie group cocycle $\iota^2(\omega)$. Hence, using the formula (5) relating $I^2(\omega)$ and $\iota^2(\omega)$, the conjugation in $G_0 \times_{I^2(\omega)} Z(\mathfrak{g})$ can be written

$$(g,a) \rhd (h,b) = (g \rhd h, I^2(\omega)(g,h)) = (g \rhd h, \iota^2(\omega)(g,h) - \iota^2(\omega)(g \rhd h, g)),$$

and a easy calculation shows that this is the formula for the conjugation in the group $G_0 \times_{\iota^2(\omega)} Z(\mathfrak{g})$, where the product is defined by

$$(g,a)(h,b) = (gh, \iota^2(g,h)).$$

Conversely, suppose that a local Lie rack of the form $G_0 \times_{I^2(\omega)} \mathfrak{a}^a$ has its tangent space at 1, $\mathfrak{g}_0 \oplus_\omega \mathfrak{a}^a$, provided with a Lie algebra structure. Necessarily, $\mathfrak{a}$ is a trivial $\mathfrak{g}_0$-representation and $\omega \in Z^2(\mathfrak{g}_0, \mathfrak{a})$. Hence, as before we have the formula (5) relating $I^2(\omega)$ and $\iota^2(\omega)$ and the conjugation defined by the formula (6) is induced by the conjugation coming from the group structure on $G_0 \times_{\iota^2(\omega)} \mathfrak{a}$. Finally, we have the following refinement of Theorem 3.27.

Theorem 3.28. *Every Leibniz algebra $\mathfrak{g}$ becomes integrated into a local Lie rack of the form*

$$G_0 \times_{I^2(\omega)} \mathfrak{a}^a,$$

with conjugation

$$(g,a) \rhd (h,b) = (g \rhd h, g.b + I^2(\omega)(g,h)), \tag{7}$$

and neutral element $(1,0)$, where G_0 is a Lie group, $\mathfrak{a}$ a representation of G_0 and $\omega \in ZL^2(\mathfrak{g}_0, \mathfrak{a}^a)$. Conversely, every local Lie rack of this form has its tangent space at 1 provided with a Leibniz algebra structure.

Moreover, in the special case where $\mathfrak{g}$ is a Lie algebra, the above construction provides $G_0 \times_{I^2(\omega)} \mathfrak{a}^a$ with a rack product coming from the conjugation in a Lie group. Conversely, if the tangent space at 1 of $G_0 \times_{I^2(\omega)} \mathfrak{a}^a$ is a Lie algebra, then $G_0 \times_{I^2(\omega)} \mathfrak{a}^a$ can be provided with a Lie group structure, and the conjugation induced by the Lie group structure is the one defined by (7).

3.6   From Leibniz algebras to local augmented Lie racks

Let $\mathfrak{g}_0$ be a Lie algebra, $\mathfrak{a}$ a $\mathfrak{g}$-representation and $\omega \in ZL^2(\mathfrak{g}_0, \mathfrak{a}^a)$. In Proposition 3.18, we showed that $I^2(\omega)$ is a local Lie rack cocycle. We showed also that it satisfies the identity

$$g.I^2(\omega)(h,k) - I^2(\omega)(gh,k) + I^2(\omega)(g, h \triangleright k) = 0 \tag{8}$$

for all $(g, h, k) \in U_{3-loc}$.
Hence the natural question is

What is the algebraic structure on $G_0 \times_{I^2(\omega)} \mathfrak{a}^a$ encoded by this identity?

We will see that the answer is the structure of a local augmented Lie rack.

Definition 3.29. *Let G be a group. A **local G-set** is a set X provides with a map ρ defined on a subset Ω of $G \times X$ with values in X such that the followings axioms are satisfied*

1. *If $(h, x), (gh, x), (g, \rho(h, x)) \in \Omega$, then $\rho(g, \rho(h, x)) = \rho(gh, x)$.*

2. *For all $x \in X$, we have $(1, x) \in \Omega$ and $\rho(1, x) = x$.*

*A **local topological** (resp.(**smooth**)) G-set is a topological set (resp. smooth manifold) X with a structure of a local G-set such that*

1. *Ω is an open subset of X.*

2. *$\rho : \Omega \to X$ is continuous (resp. smooth).*

*A **fixed point** is an element $x_0 \in X$ such that for all $g \in G, (g, x_0) \in \Omega$ and $\rho(g, x_0) = x_0$.*

In the following proposition, we show that the identity (8) provides $G_0 \times_{I^2(\omega)} \mathfrak{a}^a$ with a structure of a local G_0-set.

Proposition 3.30. *$G_0 \times_{I^2(\omega)} \mathfrak{a}^a$ is a local smooth G_0-set, and $(1, 0)$ is a fixed point.*

Proof : We define an open subset Ω and a smooth map ρ by

1. $\Omega = \{(g, (h, b)) \in G_0 \times (G_0 \times_{I^2(\omega)} \mathfrak{a}^a) | (g, h) \in U_{2-loc}\}$.

2. $\rho(g, (h, b)) = (g \triangleright h, g.b + I^2(\omega)(g, h))$.

Let $(h, (k, z)), (gh, (k, z)), (g, \rho(h, (k, z))) \in \Omega$. This is equivalent to the condition $(h, k), (gh, k), (g, h \triangleright k) \in U_{2-loc}$, that is $(g, h, k) \in U_{3-loc}$. We have

$$\rho(g, \rho(h, (k, z))) = \rho(g, (h \triangleright k, h.z + I^2(\omega)(h, k)))$$
$$= (g \triangleright (h \triangleright k), g.(h.z) + g.I^2(\omega)(h, k) + I^2(\omega)(g, h \triangleright k)).$$

Using the identities (2) and $(gh) \triangleright k = g \triangleright (h \triangleright k)$, we have

$$\rho(g, \rho(h, (k, z))) = ((gh) \triangleright k, (gh).z + I^2(\omega)(gh, k))$$
$$= \rho(gh, \rho(k, z)).$$

Thus $\rho(g, \rho(h, (k, z))) = \rho(gh, \rho(k, z))$.
Moreover, we have $\rho(1, (k, z)) = (1 \triangleright k, 1.z + I^2(\omega)(1, k)) = (k, z)$ and $\rho(g, (1, 0)) = (g \triangleright 1, g.0 + I^2(\omega)(g, 1)) = (1, 0)$. Hence $G_0 \times_{I^2(\omega)} \mathfrak{a}^a$ is a local smooth G_0-set and $(1, 0)$ is a fixed point for this local action.

$\square$

We remark that we can reconstruct the rack product in $G_0 \times_{I^2(\omega)} \mathfrak{a}^a$ from the formula of the G_0-action. Indeed, we have

$$(g, a) \triangleright (h, b) = g.(h, b) = p(g, a).(h, b),$$

where p is the projection on the first factor $G_0 \times_{I^2(\omega)} \mathfrak{a}^a \xrightarrow{p} G_0$. Furthermore, $p(1, 0) = 1$ and p is equivariant. Indeed, let $(g, (h, y)) \in \Omega$, we have $p(\rho(g.(h, y))) = p(g \triangleright h, g.y + I^2(\omega)(g, h)) = g \triangleright h = g.p(h, y)$. Hence we showed the following proposition

Proposition 3.31. $G_0 \times_{I^2(\omega)} \mathfrak{a}^a \xrightarrow{p} G_0$ *is a local augmented Lie rack.*

Hence we can rewrite our main theorem

Theorem 3.32. *Every Leibniz algebra* $\mathfrak{g}$ *becomes integrated into a local augmented Lie rack of the form*

$$G_0 \times_{I^2(\omega)} \mathfrak{a}^a \xrightarrow{p} G_0,$$

with local action

$$g.(h, b) = (g \triangleright h, g.b + I^2(\omega)(g, h)),$$

and neutral element $(1, 0)$*, where* G_0 *is a Lie group,* $\mathfrak{a}$ *a representation of* G_0 *and* $\omega \in ZL^2(\mathfrak{g}_0, \mathfrak{a}^a)$. *Conversely, every local augmented Lie rack of this form has its tangent space at 1 provided with a Leibniz algebra structure.*

Moreover, in the special case where $\mathfrak{g}$ *is a Lie algebra, the above construction provides* $G_0 \times_{I^2(\omega)} \mathfrak{a}^a$ *with a rack product coming from the conjugation in a Lie group. Conversely, if the tangent space at 1 of* $G_0 \times_{I^2(\omega)} \mathfrak{a}^a$ *is a Lie algebra, then* $G_0 \times_{I^2(\omega)} \mathfrak{a}^a$ *can be provided with a Lie group structure, and the conjugation induced by the Lie group structure is the one defined by* (7).

3.7 Examples of non split Leibniz algebra integrations

3.7.1 In dimension 4

Example 3.33. Let $\mathfrak{g} = \mathbb{R}^4$. We define a bilinear map on $\mathfrak{g}$ by

$$[e_1, e_1] = e_4$$
$$[e_1, e_2] = e_4$$
$$[e_2, e_1] = -e_4$$
$$[e_2, e_2] = e_4$$
$$[e_3, e_3] = e_4$$

We have

$$[(x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4)] = (0, 0, 0, x_1 y_1 + x_1 y_2 - x_2 y_1 + x_2 y_2 + x_3 y_3)$$

Proposition 3.34. $(\mathfrak{g}, [-, -])$ *is a Leibniz algebra.*

34

Proof : We have

$$[(x_1, x_2, x_3, x_4), [(y_1, y_2, y_3, y_4), (z_1, z_2, z_3, z_4)]] = (0, 0, 0, 0),$$

and

$$[(y_1, y_2, y_3, y_4), [(x_1, x_2, x_3, x_4), (z_1, z_2, z_3, z_4)]] = (0, 0, 0, 0),$$

and

$$[[(x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4)], (z_1, z_2, z_3, z_4)] = (0, 0, 0, 0).$$

Hence the bracket $[-, -]$ satisfies the Leibniz identity.

$\square$

To follow the method explained above, we have to determine the left center $Z_L(\mathfrak{g})$, the quotient $\mathfrak{g}_0$ of $\mathfrak{g}$ by $Z_L(\mathfrak{g})$ denoted $\mathfrak{g}_0$, the action of $\mathfrak{g}_0$ on $Z_L(\mathfrak{g})$ and the Leibniz 2-cocycle describing the abelian extension $Z_L(\mathfrak{g}) \hookrightarrow \mathfrak{g} \twoheadrightarrow \mathfrak{g}_0$.

Let $x \in Z_L(\mathfrak{g})$, for $y = (1, 0, 0, 0), y = (0, 1, 0, 0)$ or $y = (0, 0, 1, 0)$ in $\mathfrak{g}$, we have $[x, y] = 0$. This implies that $x_1 = x_2 = x_3 = 0$. Conversely, every element in $\mathfrak{g}$ with the first three coordinates equal to 0 is in $Z_L(\mathfrak{g})$. Hence $Z_L(\mathfrak{g}) = < e_4 >$ and $\mathfrak{g}_0 \simeq < e_1, e_2, e_3 >$. The bracket on $\mathfrak{g}_0$ is equal to zero, hence $\mathfrak{g}_0$ is an abelian Lie algebra. The action of $\mathfrak{g}_0$ on $ZL(\mathfrak{g})$ is given by

$$\rho_x(y) = [(x_1, x_2, x_3, 0), (0, 0, 0, y_4)] = (0, 0, 0, 0),$$

and the Leibniz 2-cocycle is given by

$$\omega(x, y) = [(x_1, x_2, x_3, 0), (y_1, y_2, y_3, 0)] = (0, 0, 0, x_1y_1 + x_1y_2 - x_2y_1 + x_2y_2 + x_3y_3).$$

Now, we have to determine the Lie group G_0 associated to $\mathfrak{g}_0$, the action of G_0 on $Z_L(\mathfrak{g})$ integrating $\rho : \mathfrak{g}_0 \to End(Z_L(\mathfrak{g}))$ (the action of $\mathfrak{g}_0$ on $Z_L(\mathfrak{g})$), and the Lie rack cocycle integrating ω.

The Lie algebra $\mathfrak{g}_0$ is abelian, thus a Lie group integrating $\mathfrak{g}_0$ is $G_0 = \mathfrak{g}_0$. Moreover, ρ is zero, hence the Lie group action of G_0 on $Z_L(\mathfrak{g})$ which integrates ρ is the trivial action. The facts that ρ is zero and $\mathfrak{g}_0$ abelian, imply that $d_L^1 : CL^1(\mathfrak{g}_0, Z_L(\mathfrak{g})) \to CL^2(\mathfrak{g}_0, Z_L(\mathfrak{g}))$ is zero. Hence, because $\mathfrak{g}_{ann} = Z_L(\mathfrak{g})$, $\mathfrak{g}$ is non split. That is, $\mathfrak{g}$ is not isomorphic to the direct sum of a Lie algebra $\mathfrak{h}$ and a Lie representation V over $\mathfrak{h}$ provided with the bracket $[(x, v), (y, w)] = ([x, y], x.w)$ (see the introduction). What remains to be done is the integration of the cocycle ω. A formula for f, a Lie rack cocycle integrating ω, is

$$f(a, b) = \int_{\gamma_b} (\int_{\gamma_a} \tau^2(\omega)^{eq})^{eq},$$

where $\gamma_a(s) = sa$ and $\gamma_b(t) = tb$. Let $a \in G_0$ and $x, y \in \mathfrak{g}_0$. We have

$$\int_{\gamma_a} \tau^2(\omega)^{eq} = \int_{[0,1]} \tau^2(\omega)^{eq}(\gamma_a(s))(\frac{\partial}{\partial s}\Big|_{s=0} \gamma_a(s))ds$$

$$= \int_{[0,1]} \Phi_{\gamma_a(s)}(\tau^2(\omega)(a))ds$$

$$= \int_{[0,1]} \tau^2(\omega)(a)ds$$

$$= \tau^2(\omega)(a).$$

35

Thus

$$f(a, b) = \int_{\gamma_b} (\tau^2(\omega)(a))^{eq}$$

$$= \int_{[0,1]} \gamma_b^*(\tau^2(\omega)(a))^{eq}$$

$$= \int_{[0,1]} \tau^2(\omega)(a))^{eq}(\gamma_b(t))(\frac{\partial}{\partial t}\Big|_{t=0} \gamma_b(t))dt$$

$$= \int_{[0,1]} \tau^2(\omega)(a))(b)dt$$

$$= \tau^2(\omega)(a)(b)$$

$$= \omega(a, b).$$

Hence, the conjugation in $G_0 \times_f Z_L(\mathfrak{g}) = \mathbb{R}^4$ is given by

$$(a_1, a_2, a_3, a_4) \triangleright (b_1, b_2, b_3, b_4) = (b_1, b_2, b_3, a_1 b_1 + a_2 b_2 + a_3 b_3 + a_1 b_2 - a_2 b_1 + b_4).$$

We have

$$\frac{\partial^2}{\partial s \partial t}\Big|_{s,t=0} (sa_1, sa_2, sa_3, sa_4) \triangleright (tb_1, tb_2, tb_3, tb_4) = \frac{\partial^2}{\partial s \partial t}\Big|_{s,t=0} (tb_1, tb_2, tb_3, st(a_1 b_1 + a_2 b_2 + a_3 b_3 + a_1 b_2 - a_2 b_1), tb_4)$$

$$= (0, 0, 0, a_1 b_1 + a_2 b_2 + a_3 b_3 + a_1 b_2 - a_2 b_1)$$

$$= [(a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4)].$$

Thus $(\mathbb{R}^4, \triangleright)$ integrates $(\mathbb{R}^4, [-, -])$.

Example 3.35. Let $\mathfrak{g} = \mathbb{R}^4$. We define a bilinear map on $\mathfrak{g}$ by

$$[e_1, e_1] = e_2$$
$$[e_1, e_2] = e_3$$
$$[e_1, e_3] = e_4$$

We have

$$[(x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4)] = (0, x_1 y_1, x_1 y_2, x_1 y_3)$$

Proposition 3.36. $(\mathfrak{g}, [-, -])$ *is a Leibniz algebra.*

Proof : We have

$$[(x_1, x_2, x_3, x_4), [(y_1, y_2, y_3, y_4), (z_1, z_2, z_3, z_4)]] = [(x_1, x_2, x_3, x_4), (0, y_1 z_1, y_1 z_2, y_1 z_3)]$$

$$= (0, 0, x_1 y_1 z_2, x_1 y_1 z_3),$$

and

$$[(y_1, y_2, y_3, y_4), [(x_1, x_2, x_3, x_4), (z_1, z_2, z_3, z_4)]] = (0, 0, x_1 y_1 z_2, x_1 y_1 z_3),$$

and

$$[[(x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4)], (z_1, z_2, z_3, z_4)] = 0.$$

Hence the bracket $[-, -]$ satisfies the Leibniz identity.

36

$\square$

To follow the method explained above, we have to determine the left center $Z_L(\mathfrak{g})$, the quotient $\mathfrak{g}_0$ of $\mathfrak{g}$ by $Z_L(\mathfrak{g})$ denoted $\mathfrak{g}_0$, the action of $\mathfrak{g}_0$ on $Z_L(\mathfrak{g})$ and the Leibniz 2-cocycle describing the abelian extension $Z_L(\mathfrak{g}) \hookrightarrow \mathfrak{g} \twoheadrightarrow \mathfrak{g}_0$.

Let $x \in Z_L(\mathfrak{g})$, for $y = (1,0,0,0), y = (0,1,0,0)$ or $y = (0,0,1,0)$ in $\mathfrak{g}$, we have $[x,y] = 0$. This implies that $x_1 = 0$. Conversely, every element in $\mathfrak{g}$ with the first coordinate equals to 0 is in $Z_L(\mathfrak{g})$. Hence $Z_L(\mathfrak{g}) = < e_2, e_3, e_4 >$ and $\mathfrak{g}_0 \simeq < e_1 >$. The bracket on $\mathfrak{g}_0$ is equal to zero, hence $\mathfrak{g}_0$ is an abelian Lie algebra. The action of $\mathfrak{g}_0$ on $ZL(\mathfrak{g})$ is given by

$$\rho_x(y) = [(x_1,0,0,0),(0,y_2,y_3,y_4)] = (0,0,x_1y_2,x_1y_3),$$

and the Leibniz 2-cocycle is given by

$$\omega(x,y) = [(x_1,0,0,0),(y_1,0,0,0)] = (0,x_1y_1,0,0).$$

Moreover, we have $[x,x] = (0,x_1^2,x_1x_2,x_1x_3)$, hence $\mathfrak{g}_{ann} = Z_L(\mathfrak{g})$. This Leibniz algebra is not split because for $\alpha \in Hom(\mathfrak{g}_0,\mathbb{R})$ and $x,y \in \mathfrak{g}_0$, we have $d_L\alpha(x,y) = \rho_x(\alpha(y)) = (0,0,x_1\alpha(y)_2,x_1\alpha(y)_3)$.

Now, we have to determine the Lie group G_0 associated to $\mathfrak{g}_0$, the action of G_0 on $Z_L(\mathfrak{g})$ integrating $\rho : \mathfrak{g}_0 \to End(Z_L(\mathfrak{g}))$ (the action of $\mathfrak{g}_0$ on $Z_L(\mathfrak{g})$), and the Lie rack cocycle integrating ω.

The Lie algebra $\mathfrak{g}_0$ is abelian, thus a Lie group integrating $\mathfrak{g}_0$ is $G_0 = \mathfrak{g}_0$. Moreover, a simple calculation shows that the Lie group action of G_0 on $Z_L(gg)$ defined by

$$\phi_x(y) = y + \rho_x(y),$$

integrates ρ. What remains to be done is the integration of the cocycle ω. A formula for f, a Lie rack cocycle integrating ω, is

$$f(a,b) = \int_{\gamma_b} (\int_{\gamma_a} \tau^2(\omega)^{eq})^{eq},$$

where $\gamma_a(s) = sa$ and $\gamma_b(t) = tb$. Let $a \in G_0$ and $x,y \in \mathfrak{g}_0$. We have

$$\int_{\gamma_a} \tau^2(\omega)^{eq} = \int_{[0,1]} \tau^2(\omega)^{eq}(\gamma_a(s))(\frac{\partial}{\partial s}\Big|_{s=0} \gamma_a(s))ds$$

$$= \int_{[0,1]} \Phi_{\gamma_a(s)}(\tau^2(\omega)(a))ds$$

$$= \int_{[0,1]} \phi_{\gamma_a(s)} \circ \tau^2(\omega)(a)ds$$

$$= \int_{[0,1]} \begin{pmatrix} 0 \\ a \\ sa^2 \\ 0 \end{pmatrix} ds$$

$$= \begin{pmatrix} 0 \\ a \\ 0 \\ 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 \\ 0 \\ a^2 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ a \\ \frac{1}{2}a^2 \\ 0 \end{pmatrix}.$$

Thus

$$f(a,b) = \int_{\gamma_b} \left(\begin{pmatrix} 0 \\ a \\ \frac{1}{2}a^2 \\ 0 \end{pmatrix} \right)^{eq}$$

$$= \int_{[0,1]} \gamma_b^* \left(\begin{pmatrix} 0 \\ a \\ \frac{1}{2}a^2 \\ 0 \end{pmatrix} \right)^{eq}.$$

$$f(a,b) = \int_{[0,1]} \begin{pmatrix} 0 \\ a \\ \frac{1}{2}a^2 \\ 0 \end{pmatrix}^{eq} (\gamma_b(t)) \left(\frac{\partial}{\partial t} \Big|_{t=0} \gamma_b(t) \right) dt$$

$$= \int_{[0,1]} \phi_{\gamma_b(t)} \left(\begin{pmatrix} 0 \\ a \\ \frac{1}{2}a^2 \\ 0 \end{pmatrix} (b) \right) dt$$

$$= \int_{[0,1]} abe_2 + (tab^2 + \frac{1}{2}a^2 b)e_3 + \frac{1}{2}ta^2 b^2 e_4 dt$$

$$= abe_2 + \frac{1}{2}(ab^2 + a^2 b)e_3 + \frac{1}{4}a^2 b^2 e_4.$$

Hence, the conjugation in $G_0 \times_f Z_L(\mathfrak{g}) = \mathbb{R}^4$ is given by

$$(a_1, a_2, a_3, a_4) \triangleright (b_1, b_2, b_3, b_4) = (b_1, a_1 b_1, a_1 b_2 + \frac{1}{2}(a_1 b_1^2 + a_1^2 b_1), a_1 b_3 + \frac{1}{4}a_1^2 b_1^2).$$

We have

$$\frac{\partial^2}{\partial s \partial t}_{|s,t=0} (sa_1, sa_2, sa_3, sa_4) \triangleright (tb_1, tb_2, tb_3, tb_4) = \frac{\partial^2}{\partial s \partial t}_{|s,t=0} (tb_1, sta_1 b_1, sta_1 b_2 + \frac{1}{2}(st^2 a_1 b_1^2 + s^2 ta_1^2 b_1),$$

$$sta_1 b_3 + s^2 t^2 \frac{1}{4}a_1^2 b_1^2)$$

$$= (0, a_1 b_1, a_1 b_2, a_1 b_3)$$

$$= [(a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4)].$$

Thus $(\mathbb{R}^4, \triangleright)$ integrates $(\mathbb{R}^4, [-, -])$.

3.7.2 In dimension 5

Example 3.37. Let $\mathfrak{g} = \mathbb{R}^5$. We define a bilinear map on $\mathfrak{g}$ by

$$[e_1, e_1] = [e_1, e_2] = e_3$$
$$[e_2, e_1] = [e_2, e_2] = [e_1, e_3] = e_4$$
$$[e_1, e_4] = [e_2, e_3] = e_5$$

We have

$$[(x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4)] = (0, 0, x_1(y_1 + y_2), x_2(y_1 + y_2) + x_1 y_3, x_1 y_4 + x_2 y_3).$$

Proposition 3.38. $(\mathfrak{g}, [-,-])$ *is a Leibniz algebra.*

Proof : We have

$$[(x_1, x_2, x_3, x_4), [(y_1, y_2, y_3, y_4), (z_1, z_2, z_3, z_4)]] = (0, 0, 0, x_1 y_1(z_1 + z_2), x_1(y_2(z_1 + z_2) + y_1 z_3) + x_2 y_1(z_1 + z_2)),$$

and

$$[(y_1, y_2, y_3, y_4), [(x_1, x_2, x_3, x_4), (z_1, z_2, z_3, z_4)]] = (0, 0, 0, x_1 y_1(z_1 + z_2), x_1(y_2(z_1 + z_2) + y_1 z_3) + x_2 y_1(z_1 + z_2)),$$

and

$$[[(x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4)], (z_1, z_2, z_3, z_4)] = (0, 0, 0, 0).$$

Hence the bracket $[-,-]$ satisfies the Leibniz identity.

$\square$

To follow the method explained above, we have to determine the left center $Z_L(\mathfrak{g})$, the quotient of $\mathfrak{g}$ by $Z_L(\mathfrak{g})$ denoted $\mathfrak{g}_0$, the action of $\mathfrak{g}_0$ on $Z_L(\mathfrak{g})$ and the Leibniz 2-cocycle describing the abelian extension $Z_L(\mathfrak{g}) \hookrightarrow \mathfrak{g} \twoheadrightarrow \mathfrak{g}_0$.

Let $x \in Z_L(\mathfrak{g})$, for $y = (0, 0, 1, 0, 0)$ in $\mathfrak{g}$, we have $[x, y] = 0$. This implies that $x_1 = x_2 = 0$. Conversely, every element in $\mathfrak{g}$ with the first two coordinates equal to 0 is in $Z_L(\mathfrak{g})$. Hence $Z_L(\mathfrak{g}) = <e_3, e_4, e_5>$ and $\mathfrak{g}_0 \simeq <e_1, e_2>$. The bracket on $\mathfrak{g}_0$ is equal to zero, hence $\mathfrak{g}_0$ is an abelian Lie algebra. The action of $\mathfrak{g}_0$ on $ZL(\mathfrak{g})$ is given by

$$\rho_x(y) = [(x_1, x_2, 0, 0, 0), (0, 0, y_3, y_4, y_5)] = (0, 0, 0, x_1 y_3, x_1 y_4 + x_2 y_3),$$

and the Leibniz 2-cocycle is given by

$$\omega(x, y) = [(x_1, x_2, 0, 0, 0), (y_1, y_2, 0, 0, 0)] = (0, 0, x_1(y_1 + y_2), x_2(y_1 + y_2), 0).$$

Moreover, we have $[x, x] = (0, 0, x_1(x_1 + x_2), x_2(x_1 + x_2) + x_1 x_3, x_1 x_4 + x_2 x_3)$, hence taking $x = (1, 0, 0, 0, 0), (0, 1, 0, 0, 0)$ and $(0, 1, 1, 0, 0)$, we see easily that $\mathfrak{g}_{ann} = Z_L(\mathfrak{g})$. This Leibniz algebra is not split because for $\alpha \in Hom(\mathfrak{g}, Z_L(\mathfrak{g}))$ and $x, y \in \mathfrak{g}_0$, we have $d_L\alpha(x, y) = \rho_x(\alpha(y)) = (0, 0, 0 x_1\alpha(y)_3, x_1\alpha(y)_4 + x_2\alpha(y)_3)$.

Now, we have to determine the Lie group G_0 associated to $\mathfrak{g}_0$, the action of G_0 on $Z_L(\mathfrak{g})$ integrating $\rho : \mathfrak{g}_0 \to End(Z_L(\mathfrak{g}))$ (the action of $\mathfrak{g}_0$ on $Z_L(\mathfrak{g})$), and the Lie rack cocycle integrating ω.

The Lie algebra $\mathfrak{g}_0$ is abelian, thus a Lie group integrating $\mathfrak{g}_0$ is $G_0 = \mathfrak{g}_0$. To integrate the action ρ, we use the exponential $exp : End(Z_L(\mathfrak{g})) \to Aut(Z_L(\mathfrak{g}))$. Indeed, for all $x \in \mathfrak{g}_0$, we have

$$\rho_x = \begin{pmatrix} 0 & 0 & 0 \\ x_1 & 0 & 0 \\ x_2 & x_1 & 0 \end{pmatrix}.$$

Hence, we define a Lie group morphism $\phi : G_0 \to Aut(Z_L(\mathfrak{g}))$ by setting

$$\phi_x = exp(\rho_x) = \begin{pmatrix} 1 & 0 & 0 \\ x_1 & 1 & 0 \\ x_2 + \frac{1}{2}x_1^2 & x_1 & 0 \end{pmatrix}.$$

It is easy to see that $d_1\phi = \rho$. What remains to be done is the integration of the cocycle ω. A formula for f, a Lie rack cocycle integrating ω, is

$$f(a,b) = \int_{\gamma_b} \left(\int_{\gamma_a} \tau^2(\omega)^{eq} \right)^{eq},$$

where $\gamma_a(s) = sa$ and $\gamma_b(t) = tb$. Let $a \in G_0$ and $x, y \in \mathfrak{g}_0$. We have

$$\int_{\gamma_a} \tau^2(\omega)^{eq} = \int_{[0,1]} \tau^2(\omega)^{eq}(\gamma_a(s)) \left(\frac{\partial}{\partial s}\Big|_{s=0} \gamma_a(s) \right) ds$$

$$= \int_{[0,1]} \Phi_{\gamma_a(s)}(\tau^2(\omega)(a)) ds$$

$$= \int_{[0,1]} \phi_{\gamma_a(s)} \circ \tau^2(\omega)(a) ds$$

$$= \int_{[0,1]} \begin{pmatrix} 1 & 0 & 0 \\ sa_1 & 1 & 0 \\ sa_2 + \frac{1}{2}(sa_1)^2 & sa_1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & a_1 \\ a_2 & a_2 \\ 0 & 0 \end{pmatrix} ds$$

$$= \int_{[0,1]} \begin{pmatrix} a_1 & a_1 \\ sa_1^2 + a_2 & sa_1^2 + a_2 \\ 2sa_1a_2 + \frac{1}{2}s^2a_1^3 & 2sa_1a_2 + \frac{1}{2}s^2a_1^3 \end{pmatrix} ds.$$

Thus

$$\int_{\gamma_a} \tau^2(\omega)^{eq} = \begin{pmatrix} a_1 & a_1 \\ \frac{1}{2}a_1^2 + a_2 & \frac{1}{2}a_1^2 + a_2 \\ a_1a_2 + \frac{1}{6}a_1^3 & a_1a_2 + \frac{1}{6}a_1^3 \end{pmatrix}.$$

Hence

$$f(a,b) = \int_{\gamma_b} \left(\int_{\gamma_a} \tau^2(\omega)^{eq} \right)^{eq}$$

$$= \int_{\gamma_b} \begin{pmatrix} a_1 & a_1 \\ \frac{1}{2}a_1^2 + a_2 & \frac{1}{2}a_1^2 + a_2 \\ a_1a_2 + \frac{1}{6}a_1^3 & a_1a_2 + \frac{1}{6}a_1^3 \end{pmatrix}^{eq}$$

$$= \int_{[0,1]} \gamma_b^* \begin{pmatrix} a_1 & a_1 \\ \frac{1}{2}a_1^2 + a_2 & \frac{1}{2}a_1^2 + a_2 \\ a_1a_2 + \frac{1}{6}a_1^3 & a_1a_2 + \frac{1}{6}a_1^3 \end{pmatrix}^{eq}$$

$$= \int_{[0,1]} \begin{pmatrix} a_1 & a_1 \\ \frac{1}{2}a_1^2 + a_2 & \frac{1}{2}a_1^2 + a_2 \\ a_1a_2 + \frac{1}{6}a_1^3 & a_1a_2 + \frac{1}{6}a_1^3 \end{pmatrix}^{eq} (\gamma_b(t))(\frac{\partial}{\partial t}\Big|_{t=0} \gamma_b(t))dt$$

$$= \int_{[0,1]} \phi_{\gamma_b(t)}(\begin{pmatrix} a_1 & a_1 \\ \frac{1}{2}a_1^2 + a_2 & \frac{1}{2}a_1^2 + a_2 \\ a_1a_2 + \frac{1}{6}a_1^3 & a_1a_2 + \frac{1}{6}a_1^3 \end{pmatrix} (b))dt.$$

We have

$$\phi_{\gamma_b(t)}(\begin{pmatrix} a_1 & a_1 \\ \frac{1}{2}a_1^2 + a_2 & \frac{1}{2}a_1^2 + a_2 \\ a_1a_2 + \frac{1}{6}a_1^3 & a_1a_2 + \frac{1}{6}a_1^3 \end{pmatrix} (b)) = \begin{pmatrix} 1 & 0 & 0 \\ tb_1 & 1 & 0 \\ tb_2 + \frac{1}{2}(tb_1)^2 & tb_1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & a_1 \\ \frac{1}{2}a_1^2 + a_2 & \frac{1}{2}a_1^2 + a_2 \\ a_1a_2 + \frac{1}{6}a_1^3 & a_1a_2 + \frac{1}{6}a_1^3 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$= \begin{pmatrix} a_1(b_1 + b_2) \\ (tb_1a_1 + a_2 + \frac{1}{2}a_1^2)(b_1 + b_2) \\ (a_1a_2 + \frac{1}{6}a_1^3 + \frac{1}{2}tb_1a_1^2 + tb_2a_1 + tb_1a_2 + \frac{1}{2}(tb_1)^2a_1)(b_1 + b_2) \end{pmatrix}.$$

Thus

$$f(a,b) = \begin{pmatrix} a_1(b_1 + b_2) \\ (\frac{1}{2}b_1a_1 + a_2 + \frac{1}{2}a_1^2)(b_1 + b_2) \\ (a_1a_2 + \frac{1}{6}a_1^3 + \frac{1}{4}b_1a_1^2 + \frac{1}{2}b_2a_1 + \frac{1}{2}b_1a_2 + \frac{1}{6}(b_1)^2a_1)(b_1 + b_2) \end{pmatrix}.$$

and the conjugation in $G_0 \times_f Z_L(\mathfrak{g}) = \mathbb{R}^5$ is given by

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} \triangleright \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 + a_1(b_1 + b_2) \\ a_1b_3 + b_4 + (\frac{1}{2}b_1a_1 + a_2 + \frac{1}{2}a_1^2)(b_1 + b_2) \\ (a_2 + \frac{1}{2}a_1^2)b_3 + a_1b_4 + b_5 + (a_1a_2 + \frac{1}{6}a_1^3 + \frac{1}{4}b_1a_1^2 + \frac{1}{2}b_2a_1 + \frac{1}{2}b_1a_2 + \frac{1}{6}(b_1)^2a_1)(b_1 + b_2) \end{pmatrix}.$$

With a simple computation we verify that

$$\frac{\partial^2}{\partial s \partial t}\Big|_{s,t=0} (sa_1, sa_2, sa_3, sa_4, sa_5) \triangleright (tb_1, tb_2, tb_3, tb_4, \ tb_5) = [(a_1, a_2, a_3, a_4, a_5), (b_1, b_2, b_3, b_4, \ b_5)].$$

Thus $(\mathbb{R}^5, \triangleright)$ integrates $(\mathbb{R}^5, [-,-])$.

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