A symmetry result on Reinhardt domains

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Abstract We show the following symmetry property of a bounded Reinhardt domain Ω in $\mathbb{C}^{n+1}$: let $M = \partial\Omega$ be the smooth boundary of Ω and let h be the Second Fundamental Form of M; if the coefficient $h(T,T)$ related to the characteristic direction T is constant then M is a sphere. In Appendix we state the result from an hamiltonian point of view.

1 Introduction

A Reinhardt domain Ω (with center at the origin) is by definition an open subset of $\mathbb{C}^{n+1}$ such that

$$\text{if} \quad (z_1, \dots, z_{n+1}) \in \Omega \quad \text{then} \quad (e^{i\theta_1} z_1, \dots, e^{i\theta_{n+1}} z_{n+1}) \in \Omega \tag{1}$$

for all the real numbers $\theta_1, \dots, \theta_{n+1}$. These domains naturally arise in the theory of several complex variables as the logarithmically convex Reinhardt

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domains are the domains of convergence of power series (see for instance [4], [7]). We will suppose from now on that the Reinhardt domain Ω has a smooth boundary (it would be enough C^2). The boundary $M := \partial\Omega$ is then a smooth real hypersurface in $\mathbb{C}^{n+1}$ and thus a CR-manifold of CR-codimension equal to one, with the standard CR structure induced by the holomorphic structure of $\mathbb{C}^{n+1}$. Thus for every $p \in M$ the tangent space T_pM splits in two subspaces: the $2n-$dimensional horizontal subspace H_pM, the largest subspace in T_pM invariant under the action of the standard complex structure J of $\mathbb{C}^{n+1}$ and the vertical one-dimensional subspace generated by the characteristic direction $T_p := J \cdot N_p$, where N_p is the unit normal at p. Moreover, if $\widetilde{g}$ is the standard metric on $\mathbb{C}^{n+1}$, then it holds

$$T_pM = H_pM \oplus \mathbb{R}T_p$$

and the sum is $\widetilde{g}$-orthogonal.

Let us consider the complexified horizontal space

$$H^{\mathbb{C}}M := \{Z = X - iJ \cdot X : X \in HM\}$$

The Levi Form l is then the sesquilinear and hermitian operator on $H^{\mathbb{C}}M$ defined in the following way: $\forall Z_1, Z_2 \in H^{\mathbb{C}}M$

$$l(Z_1, Z_2) = \widetilde{g}(\widetilde{\nabla}_{Z_1}\bar{Z}_2, N) \tag{2}$$

where $\widetilde{\nabla}$ is the Levi-Civita connection for $\widetilde{g}$. Moreover by a direct computation it holds

$$l(Z, Z) = \widetilde{g}(\widetilde{\nabla}_Z\bar{Z}, N) = \widetilde{g}([X, Y], T) \tag{3}$$

where $Y = J \cdot X$. We will say M be (strictly) pseudoconvex if l is (strictly) positive definite as quadratic form.

In analogy with classical curvatures defined in terms of elementary symmetric functions of the eigenvalues of the Second Fundamental Form, one defines the j-th Levi curvatures L^j in terms of elementary symmetric functions of the eigenvalues of the Levi Form

$$L^j = \frac{1}{\binom{n}{j}} \sum_{1 \leq i_1 < \cdots < i_j \leq n} \lambda_{i_1} \cdots \lambda_{i_j},$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of l. In particular when $j = n$ we have the Total-Levi Curvature and when $j = 1$ we have the Levi-Mean Curvature L.

Being hypersurfaces in $\mathbb{C}^{n+1}$ real hypersurfaces in $\mathbb{R}^{2n+2}$, one can also compare the Levi Form with the Second Fundamental Form h of M by using the identity [3]

$$l(Z, Z) = h(X, X) + h(J(X), J(X)), \quad \forall X \in HM$$

Thus, a direct calculation leads to the relation between the classical Mean Curvature H and the Levi-Mean Curvature L [12]:

$$H = \frac{1}{2n+1}(2nL + h(T, T)) \tag{4}$$

where $h(T, T) = \widetilde{g}(\widetilde{\nabla}_T T, N)$ is the coefficient of the Second Fundamental Form related to the characteristic direction T.

Definition 1.1. *We will call $h(T, T)$ the characteristic curvature of M.*

By (4) the characteristic curvature is a sort of complementary of the Levi-Mean Curvature in computing the Mean Curvature. Moreover, for every hypersurface in $\mathbb{C}^{n+1}$, $h(T, T)$ is invariant under a biholomorphic (rigid) transformation, as the Levi curvatures are.

Following the pioneering result due to Alexandrov [1] on the classical Mean

Curvature of Euclidean surface, the problem of characterizing compact hypersurfaces with positive constant Levi-Mean Curvature has recently received a great amount of attention. Klingenberg in [8] gave a first positive answer to this problem by showing that if the characteristic direction is a geodesic and the Levi Form is diagonal, then M is a sphere. Monti and Morbidelli in [13] proved a Darboux-type theorem for $n \geq 2$: the unique Levi umbilical hypersurfaces in $\mathbb{C}^{n+1}$ with all constant Levi curvatures are spheres or cylinders. Later on Montanari and the author proved two results of this type: in [11] they relaxed Klingerberg conditions and they proved that if the characteristic direction is a geodesic, then Alexandrov Theorem holds for hypersurfaces with positive constant Levi-Mean Curvature; in [10] they proved some integral formulas for compact hypersurfaces, of independent interest, and then they follow the Reilly approach [14], [15], [16] to prove Isoperimetric estimates and a Alexandrov type theorem, namely: let M be a closed smooth real hypersurface bounding a star-shaped domain in $\mathbb{C}^{n+1}$, if the j-Levi curvature is a positive constant K and the maximum of the Mean Curvature of M is bounded from above by K then M is a sphere. In a couple of recent papers Hounie and Lanconelli proved Alexandrov type theorems for Reinhardt domains in $\mathbb{C}^2$ first and for Reinhardt domain in $\mathbb{C}^{n+1}$, $n \geq 1$, with an additional rotational symmetry then. In [5] they showed the result for bounded Reinhardt domain of $\mathbb{C}^2$, i.e. for domains Ω such that if $(z_1, z_2) \in \Omega$ then $(e^{i\theta_1} z_1, e^{i\theta_2} z_2) \in \Omega$ for all real θ_1, θ_2. Under this hypothesis, in a neighborhood of a point, there is a defining function F only depending on the radii $r_1 = |z_1|$, $r_2 = |z_2|$, $F(r_1, r_2) = f(r_2^2) - r_1^2$ with f the solution of the ODE

$$sff'' = sf'^2 - k(f + sf'^2)^{3/2} - ff' \tag{5}$$

Alexandrov Theorem follows from uniqueness of the solution of (5). Their technique has then been used in [6] to prove an Alexandrov Theorem for bounded Reinhardt domains in $\mathbb{C}^{n+1}$ with an additional rotational symmetry in two complementary sets of variables, for every n.

Here we prove a similar result of symmetry for Reinhardt domains in $\mathbb{C}^{n+1}$ starting from the characteristic curvature rather than the Levi ones.

Theorem 1.2. *Let $M := \partial\Omega$ be the smooth boundary of a bounded Reinhardt domain Ω in $\mathbb{C}^{n+1}$. If the characteristic curvature $h(T,T)$ is constant then M is a sphere of radius equal to $1/h(T,T)$.*

Let $\{X_1, \dots, X_n, Y_1, \dots, Y_n\}$, with $Y_k = J \cdot X_k$, be an orthonormal basis of the horizontal space HM; keeping in mind the structure of the Second Fundamental Form

$$h = \begin{pmatrix} h(X_k, X_k) & h(X_k, Y_j) & h(X_k, T) \\ h(Y_j, X_k) & h(Y_j, Y_j) & h(Y_j, T) \\ h(T, X_k) & h(T, Y_k) & h(T, T) \end{pmatrix}$$

with k and j running in $1, \dots, n$, we are making assumption only on the one-dimensional characteristic subspace of the tangent space rather than on the $2n-$dimensional horizontal one HM: moreover when in addition one assumes one of the Levi curvatures be non zero (as in the Alexandrov type results) then HM spans the whole tangent space; in fact the vector fields $\{X_1, \dots, X_n, Y_1, \dots, Y_n\}$ satisfy the Hörmander rank condition.

When there exists a defining function $f : \mathbb{C}^{n+1} \to \mathbb{R}$

$$\Omega = \{z \in \mathbb{C}^{n+1} : f(z) < 0\}, \quad M = \partial\Omega = \{z \in \mathbb{C}^{n+1} : f(z) = 0\}$$

such that $f(z) = g(r)$ depends only on the radii $r = (r_1, \dots, r_{n+1})$, where

$$r_k = z_k \bar{z}_k, \qquad k = 1, \dots, n+1$$

then we can find an explicit formula to compute the characteristic curvature $h(T,T)$. In fact by using the following identities

$$f_k = \bar{z}_k g_k, \qquad f_{\bar{k}} = z_k g_k, \qquad f_{\bar{j}k} = \delta_{jk} g_k + z_j \bar{z}_k g_{jk}$$

$$|\partial f|^2 = \sum_k r_k g_k^2$$

the unit normal N is

$$N = -\frac{1}{|\partial f|} \sum_k (z_k g_k \partial_{z_k} + \bar{z}_k g_k \partial_{\bar{z}_k})$$

and the characteristic direction T reads as

$$T = J \cdot N = -\frac{i}{|\partial f|} \sum_k (z_k g_k \partial_{z_k} - \bar{z}_k g_k \partial_{\bar{z}_k})$$

Then by a direct computation we have that

$$h(T,T) = \widetilde{g}(\widetilde{\nabla}_T T, N) = \sum_k^{n+1} \frac{r_k g_k^3}{|\partial f|^3} \tag{6}$$

Example 1.3 (characteristic curvature of the sphere)**.** *Let*

$$g(r_1, \ldots, r_{n+1}) = r_1 + \ldots + r_{n+1} - R^2$$

be the defining function of the sphere of radius equal to R in $\mathbb{C}^{n+1}$. By the formula (6) we have that the characteristic curvature of the sphere is

$$h(T,T) = \frac{1}{R}$$

Example 1.4 (characteristic curvature of ellipsoidal type domains)**.** *Let*

$$g(r_1, \ldots, r_{n+1}) = \frac{r_1}{a_1^2} + \ldots + \frac{r_{n+1}}{a_{n+1}^2} - 1$$

be the defining function of an ellipsoid in $\mathbb{C}^{n+1}$ with $(a_1, \ldots, a_{n+1})$ positive constants. By the formula (6) we have that at a point $p = (r_1, \ldots, r_{n+1}) \in M$ its characteristic curvature is

$$h_p(T,T) = \frac{\sum_k^{n+1} \frac{r_k}{a_k^6}}{\left(\sum_k^{n+1} \frac{r_k}{a_k^4}\right)^{3/2}}$$

In the next section we will prove the Theorem 1.2, then in the Appendix we will show an Hamiltonian point of view of the result.

2 Proof of Theorem 1.2

Let us identify $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \simeq \mathbb{C}^{n+1}$ so that $z = (x, y)$. First we prove a property of independent interest.

Lemma 2.1. *Let Ω be a Reinhardt domain in $\mathbb{C}^{n+1}$ and let*

$$p = (z_1, \ldots, z_{n+1}) = (x_1, \ldots, x_{n+1}, y_1, \ldots, y_{n+1})$$

the "position vector" of a point on $M := \partial\Omega$. If T_p is the characteristic direction at $p \in M$ then it holds identically

$$\tilde{g}(p, T_p) \equiv 0 \tag{7}$$

Proof. If M is any smooth hypersurface bounding a domain Ω in $\mathbb{C}^{n+1}$ with defining function $f : \mathbb{C}^{n+1} \to \mathbb{R}$ such that

$$\Omega = \{z \in \mathbb{C}^{n+1} : f(z) < 0\}, \quad M = \partial\Omega = \{z \in \mathbb{C}^{n+1} : f(z) = 0\}$$

then the unit normal N is:

$$N = -\frac{1}{|\partial f|} \sum_{k=1}^{n+1} (f_{\bar{k}} \partial_{z_k} + f_k \partial_{\bar{z}_k})$$

where $f_k = \dfrac{\partial f}{\partial z_k}$, with $k = 1, \ldots, n+1$. Thus the characteristic direction T is:

$$T = J \cdot N = -\frac{i}{|\partial f|} \sum_{k=1}^{n+1} (f_{\bar{k}} \partial_{z_k} - f_k \partial_{\bar{z}_k})$$

By identifying $f(z) = f(x, y)$, from the real point of view we have:

$$N = -\frac{1}{|\nabla f|} \sum_{k=1}^{n+1} (f_{x_k} \partial_{x_k} + f_{y_k} \partial_{y_k})$$

$$T = \frac{1}{|\nabla f|} \sum_{k=1}^{n+1} (f_{y_k} \partial_{x_k} - f_{x_k} \partial_{y_k})$$

Now, if Ω is a Reinhardt domain (with center at the origin) in $\mathbb{C}^{n+1}$ then we can find (at least locally) a defining function $f(z) = g(r)$ depending only on the radii $r = (r_1, \ldots, r_{n+1})$ where

$$r_k = z_k \bar{z}_k = x_k^2 + y_k^2, \qquad k = 1, \ldots, n+1$$

So if $g_k = \dfrac{\partial g}{\partial r_k}$ we obtain

$$f_{x_k} = 2 x_k g_k, \qquad f_{y_k} = 2 y_k g_k$$

with $k = 1, \ldots, n+1$. In vectorial notation then we have

$$T = \frac{1}{|\nabla f|} (f_{y_1}, \ldots, f_{y_{n+1}}, -f_{x_1}, \ldots, -f_{x_{n+1}}) =$$

$$= \frac{2}{|\nabla f|} (y_1 g_1, \ldots, y_{n+1} g_{n+1}, -x_1 g_1, \ldots, -x_{n+1} g_{n+1})$$

and thus it holds identically

$$\tilde{g}(p, T_p) = \frac{2}{|\nabla f(p)|} \sum_{k=1}^{n+1} \Big(x_k y_k g_k(p) - y_k x_k g_k(p) \Big) \equiv 0$$

for every $p \in M$ $\square$

In other words, the vector position p has generally a normal component and a tangential component; in turn, the tangential component has an horizontal component and a characteristic component: for Reinhardt domains the characteristic component of the vector position p identically vanishes.
Now we can prove the main result.

Proof. (of Theorem 1.2) Let us consider the function:

$$\varphi : M \to \mathbb{R}, \qquad \varphi(p) = \frac{|p|^2}{2} = \frac{\tilde{g}(p,p)}{2}$$

that represents one half the squared distance of the manifold from the origin. If $V \in TM$ is a tangent vector field to M then the derivative of φ along V is

$$V(\varphi(p)) = \frac{1}{2} V(\tilde{g}(p,p)) = \tilde{g}(p, V_p)$$

and by Lemma 2.1 we have

$$T(\varphi) = \tilde{g}(p, T) \equiv 0$$

Thus, if $\widehat{p}$ is a critical value of φ, then

$$X_k(\varphi)_{|\widehat{p}} = Y_k(\varphi)_{|\widehat{p}} = 0$$

Moreover, φ evaluated at a critical value is

$$\varphi(\widehat{p}) = \frac{|\widehat{p}|^2}{2} \tag{8}$$

and the position vector of any critical value $\widehat{p}$ is parallel to the (inner) unit normal direction N at $\widehat{p}$

$$\widehat{p} = \tilde{g}(\widehat{p}, N_{\widehat{p}}) N_{\widehat{p}} = -|\widehat{p}| N_{\widehat{p}}$$

Differentiating again φ along the characteristic direction T we obtain

$$0 \equiv T^2(\varphi) = T(\tilde{g}(p,T)) = \tilde{g}(T,T) + \tilde{g}(p, \widetilde{\nabla}_T T) = 1 + \tilde{g}(p, \widetilde{\nabla}_T T)$$

and if $\widehat{p}$ is a critical value for φ then we get

$$1 - |\widehat{p}|\tilde{g}(N_{\widehat{p}}, \widetilde{\nabla}_T T) = 1 - |\widehat{p}|h_{\widehat{p}}(T,T) = 0 \tag{9}$$

where $h_{\widehat{p}}(T,T)$ is the characteristic curvature of M at $\widehat{p}$.

Since M is a smooth compact hypersurface, then φ admits maximum and minimum which are critical values for φ. If $h(T,T)$ is constant then by (9) we have

$$|\widehat{p}| = \frac{1}{h_{\widehat{p}}(T,T)} = \frac{1}{h(T,T)} = const.$$

Then by (8) φ is constant on M and it holds

$$(2\varphi(p))^{1/2} = |p| = \frac{1}{h(T,T)} = const.$$

for every $p \in M$, and it means that M is a sphere of radius $\dfrac{1}{h(T,T)}$ $\square$

The boundedness hypothesis is crucial as the next example shows.

Example 2.2 (characteristic curvature of a cylinder type domain)**.** *Let*

$$g(r_1, r_2) = r_1 - R^2$$

be the defining function of a cylinder type domain in $\mathbb{C}^2$*. By the formula (6) we have that the its characteristic curvature is constant:*

$$h(T,T) = \frac{1}{R}$$

3 Appendix

Here we want to look at the Reinhardt domains from an hamiltonian point of view. First we recall that for every hypersurface M in $\mathbb{C}^{n+1}$, with f as defining function, the characteristic direction of M is exactly the (normalized) hamiltonian vector field for the hamiltonian function f. In fact let us consider a dynamic system with hamiltonian function (smooth enough) depending on position and momentum variables

$$H : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}, \qquad z = (q, p) \mapsto H(q, p)$$

and define the Action functional

$$A(z) = \int_{t_0}^{t_1} \Big(\langle p, \dot{q} \rangle - H(q, p) \Big) dt, \qquad z : [t_0, t_1] \to \mathbb{R}^{2n+2}$$

The first variation of A on a suitable space of curves leads to the following system of differential equations (Hamilton)

$$\begin{cases} \dot{q}_k = \dfrac{\partial H}{\partial p_k}(q, p) \\ \\ \dot{p}_k = -\dfrac{\partial H}{\partial q_k}(q, p) \end{cases} \qquad k = 1, \ldots, n+1 \tag{10}$$

Now, a Least Action Principle states that trajectories of motion (in the generalized phase space $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$) are solutions of (10). The isoenergetic surface of H of energy E is the following hypersurface in $\mathbb{R}^{2n+2}$: $M = \{z \in \mathbb{R}^{2n+2} : H(z) = E\}$. The conservation of energy principle ensures that if z is a critical point for A, then $z(t) \in M, \forall t \in [t_0, t_1]$. The hamiltonian vector field for H is the tangent vector field to M

$$X_z^H := \Big(\frac{\partial H}{\partial p}(q, p), -\frac{\partial H}{\partial q}(q, p) \Big) = J \cdot \nabla H(q, p)$$

where

$$J = \begin{pmatrix} 0 & I_{n+1} \\ -I_{n+1} & 0 \end{pmatrix}$$

is the canonical symplectic matrix in $\mathbb{R}^{2n+2}$ and in our case it coincides with the standard complex structure in $\mathbb{C}^{n+1}$.

The Hamilton system (10) rewrites as

$$\dot{z} = X_z^H$$

Now, if one identifies

$$\mathbb{C}^{n+1} \approx \mathbb{R}^{2n+2}, \qquad z = (z_1, \ldots, z_{n+1}), \qquad z_k = x + iy \simeq (x_k, y_k)$$

then the hypersurface M defined by

$$M = \{z \in \mathbb{C}^{n+1} : f(z) = 0\}, \qquad f : \mathbb{C}^{n+1} \to \mathbb{R}$$

is exactly the isoenergetic surfaces of $H = f + E$. Thus the hamiltonian vector field on M is

$$X_z^H = J \cdot \nabla H(z) = J \cdot \nabla f(z) = J \cdot N = T$$

where $N = \nabla f$ is the normal direction to M and T is the (not normalized) characteristic direction. Moreover the integral curves of X^H (the orbits in the phase space) coincide with that ones of T, eventually reparametrized. In this situation the characteristic curvature $h(T, T)$ is the normal curvature of the hamiltonian trajectories on the isoenergetic surface in the generalized phase space $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$.

Now, we recall that if Ω is a Reinhardt domain (with center at the origin) in $\mathbb{C}^{n+1}$ then we can find (at least locally) a defining function $f(z) = g(r)$ depending only on the radii $r = (r_1, \ldots, r_{n+1})$ where

$$r_k = z_k \bar{z}_k = x_k^2 + y_k^2, \qquad k = 1, \ldots, n+1$$

This means that the hamiltonian function depends only on the quantities $r_k = q_k^2 + p_k^2$ that represent the actions in the pair of variables action-angle. Thus the angle variables are cyclic and then the actions r_k (and all the functions depending on them) are conserved quantities along the trajectories of motion. In fact we have that the characteristic direction T is:

$$T = -\frac{i}{|\partial f|} \sum_k (z_k g_k \partial_{z_k} - \bar{z}_k g_k \partial_{\bar{z}_k})$$

then it holds

$$T(r_k) = 0, \qquad k = 1, \ldots, n+1$$

Moreover the system (10) reads as

$$\dot{z}_k = -i f_{\bar{k}} = -i z_k g_k \tag{11}$$

and since $g_k(t) = g_k(0)$, then the curve

$$z(t) = z_k(0) e^{-i g_k(0) t}$$

is an explicit solution of (11) with initial condition $z_k(0)$.

In particular, we have that the following curves

$$z(t) = z_k(0) e^{-i \frac{g_k(0)}{|\partial f(0)|} t}$$

are integral curves of the characteristic direction T.

We explicitly note that the trajectories of the characteristic direction belong to a $(n+1)$-dimensional torus $\mathbb{T}^{n+1}$ (eventually degenerate) identified by

$$\mathbb{T}^{n+1} = \mathbb{S}^1 \times \ldots \times \mathbb{S}^1 = \{z \in \Omega : |z_1| = c_1 \geq 0, \ldots, |z_{n+1}| = c_{n+1} \geq 0\} \tag{12}$$

and this is a particular case of the wellknown Liouville-Arnold Theorem [2].

In other words we have a symplectic toric action group on $\mathbb{C}^{n+1}$ with a fixed point at the origin.

Let us now consider the following explicit formula to compute the j-th Levi curvature of M in term of a defining function f (see [9]):

$$L^j = -\frac{1}{\binom{n}{j}} \frac{1}{|\partial f|^{j+2}} \sum_{1 \leq i_1 < \cdots < i_{j+1} \leq n+1} \Delta_{(i_1, \cdots, i_{j+1})}(f) \tag{13}$$

for all $j = 1, \dots, n$, where

$$\Delta_{(i_1,\cdots,i_{j+1})}(f) = \det \begin{pmatrix} 0 & f_{\bar{i}_1} & \cdots & f_{\bar{i}_{j+1}} \\ f_{i_1} & f_{i_1,\bar{i}_1} & \cdots & f_{i_1,\bar{i}_{j+1}} \\ \vdots & \vdots & \ddots & \vdots \\ f_{i_{j+1}} & f_{i_{j+1},\bar{i}_1} & \cdots & f_{i_{j+1},\bar{i}_{j+1}} \end{pmatrix} \tag{14}$$

If $f(z) = g(r)$ depends only on the radii $r = (r_1, \dots, r_{n+1})$ then by a direct computation we have that $\Delta_{(i_1,\cdots,i_{j+1})}(g)$ depends only on $(r_{i_1}, \cdots, r_{i_{j+1}})$. Thus all the j-th Levi curvatures are conserved quantities on every fixed $(n+1)$-dimensional torus $\mathbb{T}^{n+1}$: in particular they are constant along the trajectories of the characteristic direction T.

Moreover by the formula (6) also the characteristic curvature $h(T, T)$ is constant on every fixed $(n+1)$-dimensional torus. We explicitly recall that $h(T, T)$ (and all the conserved quantities as well) is constant along the trajectories of the characteristic direction T but the value of the constant changes accordingly to the initial condition of the equation (11).

Then our main result Theorem (1.2) states that if the value of the constant $h(T, T)$ is the same on all the trajectories of the characteristic direction T then M is a sphere.

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