On upper-critical graphs

José Antonio Martín H.*

Faculty of Computer Science, Complutense University of Madrid, Spain

Abstract

This work introduces the concept of *upper-critical graphs*, in a complementary way of the conventional (lower)critical graphs: an element x of a graph G is called *critical* if $\chi(G - x) < \chi(G)$. It is said that G is a *critical graph* if every element (vertex or edge) of G is critical. Analogously, a graph G is called *upper-critical* if there is no edge that can be added to G such that G preserves its chromatic number, i.e. $\{ e \in \overline{E}(G) \mid \chi(G + e) = \chi(G) \} = \emptyset$. A characterization in terms of hereditary properties under some transformations, e.g. subgraphs and minors and in terms of construction and counting is given.

Key words: Graph-coloring; critical-graphs; upper-critical graphs; prime-numbers.

1. Preliminary definitions and basic terminology

Unless we state it otherwise, all graphs in this work are connected and simple (finite, and have no loops or parallel edges).

Partitioning the set of vertices $V(G)$ of a graph G into separate classes, in such a way that no two adjacent vertices are grouped into the same class, is called the vertex graph coloring problem. In order to distinguish such classes, a set of colors C is used, and the division into these *(color) classes* is given by a proper-coloring (we will use here just the single term coloring) $\varphi : V(G) \to C$, where $\varphi(u) \neq \varphi(v)$ for all uv belonging to the set of edges $E(G)$ of G. Given a graph vertex coloring problem over a graph G with a set of colors C, if C has cardinality k, then φ is a *k-coloring* of G. The *Chromatic number* of a graph $\chi(G)$ is the minimum number of colors necessary to color the vertices of a graph G in such a way that no two adjacent vertices get colored with the same color, thus, if $\chi(G) \leq k$ then one says that G is *k-colorable* (i.e. can be colored with k different colors) and if $\chi(G) = k$ then one says that G is *k-chromatic*.

The set off all adjacent vertices to a vertex $x \in V(G)$ is called its *neighborhood* and is denoted by $N_G(x)$. The *closed neighborhood* of a vertex x includes also the vertex x, i.e. $N_G(x) \cup x$.

The degree of a vertex x ($deg(x)$) is equal to the cardinality of its neighborhood $deg(x) = |N_G(x)|$. A *complete vertex* is any $x \in V(G)$ such that $N_G(x) \cup x = V(x)$ and a graph is called a *complete graph* if every vertex is a complete vertex.

An *independent set* (also called *stable set*) I is a set of vertices of G such that there are no edges between any two vertices in I.

An *edge contraction* denoted by G/xy or G/e is the process of replacing two adjacent vertices x, y of G, i.e $xy \in E(G)$, by a new vertex z such that $N_G(z) = N_G(x) \cup N_G(y)$.

A graph H is called a *minor* of the graph G ($H \prec G$) if H is isomorphic to a graph that can be obtained from a subgraph of G by zero or more edge deletions, edge contractions or vertex deletions on a subgraph of G. In particular, G is minor of itself.

A *vertex identification* denoted by $G/x, y$ is the process of replacing two non-adjacent vertices x, y of G, i.e $xy \notin E(G)$, by a new vertex z such that $N_G(z) = N_G(x) \cup N_G(y)$.

*address: Facultad de Informática, Universidad Complutense de Madrid, C. Prof. José García Santesmases, s/n., 28040, Madrid, (Spain). Tel.: +34 91.394.7656, Fax: +34 91.394.7510

Email address: jamartinh@fdi.ucm.es (José Antonio Martín H.)

A graph $H = G/S$ is called a *contraction* of the graph G if H is isomorphic to a graph that can be obtained by zero or more vertex identifications on G, where S is the set of vertex identifications to obtain H. In particular, G is a contraction of itself.

When S induces a contraction H of G such that $H = K_k$ then H is called a *collapse* of G. Thus, since a collapse $H = K_k$ is also a partition of all the vertices of G in k independent sets then $\chi(G)$ is the size of the minimum collapse of G and if $k = \chi(G)$ then $H = K_k$ induces a k-coloring of G.

Critical graphs where first studied by Dirac [1, 2, 3]. An element x of a graph G is called *critical* if $\chi(G-x) < \chi(G)$. If all the vertices of a graph G are critical we say that G is *vertex-critical* and if every element (vertex or edge) of G is critical we say that G is a *critical graph* and also if $\chi(G) = k$ we say that G is k-critical.

Examples of critical graphs in general are the *complete graphs* K_k of size $|V(G)| = k$, odd Cycles are the only 3-critical and odd-Wheels are just one case of 4-critical graphs. If x is a critical vertex of a graph G then x is a color class itself, that is, there is at least one k-coloring of G where x is the only vertex with the $\varphi(x)$ color.

Also k-critical graphs posses the next well known properties:

- G has only one component.
- G is finite
- Every vertex is adjacent to at least k-1 others.
- If G is (k-1)-regular, meaning every vertex is adjacent to exactly k-1 others, then G is either K_k or an odd cycle.
- $|V(G)| \neq k + 1$
- If G is different from K_k then $|V(G)| \geq k + 2$.

2. Upper-critical graphs

This work introduces the concept of *upper-critical graphs*, in a complementary way of the conventional (lower)critical graphs: In conventional (lower)critical graphs, an element x of a graph G is called *critical* if $\chi(G - x) < \chi(G)$. If all the vertices of a graph G are critical we say that G is *vertex-critical*.

If every element (vertex or edge) of G is critical we say that G is a *critical graph* and if $\chi(G) = k$ we say that G is k-critical.

Of analogous form:

Definition 1. *A graph G is called* upper-critical *if there is no edge (e) that can be added to G such that G preserves its chromatic number, i.e.*

$$\{e \in \overline{E}(G) \mid \chi(G + e) = \chi(G)\} = \emptyset$$

or alternatively,

$$\chi(G + e) > \chi(G) \ \ \forall e \notin E(G), \quad or \quad G = K_k$$

2.1. Hereditary properties of transformations, subgraphs and minors

Theorem 1. *If G is a k-chromatic upper-critical graph then every k-coloring of G induce the same partition of vertices of G, in k different color classes, i.e. G is uniquely colorable (uniquely k-colorable, to be more precise).*

Proof. Given a k-chromatic upper-critical graph G, i.e. $\chi(G+e) > \chi(G)$. Let us suppose that there is at least a vertex x that can be assigned to either a color class c_1 or a color class c_2 then it is possible to add a new edge $e = xy$ to G $(G+e)$ from vertex x towards an element of c_2. But, in this case, $G + e$ has at least one k-coloring and thus $\chi(G + e) = \chi(G)$ which is a contradiction. □

Theorem 2. *If G is an upper-critical graph and x a vertex of G then:*

$$N_G(x) = \overline{c}(x),$$

where $\overline{c}(x)$ *is the subset of vertices of G that are the complement (in G) of the color class of x.*

Proof. Suppose $N_G(x) \neq \overline{c}(x)$. Then there is a vertex y such that $y \notin c(x)$ and $y \notin N_G(x)$. Hence $\chi(G+e) = \chi(G)$ for a new edge $e = xy$ which is a contradiction. □

Theorem 3. *If G is a k-chromatic upper-critical graph then: G contains K_k as a subgraph.*

Proof. Since G has k color classes, let $x_1, x_2, x_3, ..., x_k$ be k vertices such that each x_i belongs to a different color class c_i. Now, the induced subgraph $x_1, x_2, x_3, ..., x_k$ is a complete graph K_k since, by theorem 2: $x_i x_j \in E(G)\ \forall i \neq j$:

$$\bigcup_{i=1}^{k} x_i = K_k; \quad x_i x_j \in E(G)\ \forall i \neq j$$

□

Theorem 4. *If G is an upper-critical graph and x, y are two vertices of G such that $xy \notin E(G)$ then:*

$$N_G(x) = N_G(y) : x = y, \quad \textit{i.e. vertex } y \textit{ is a "copy" of vertex } x$$

Proof. Let us suppose there is a vertex z in $N_G(y)$ not in $N_G(x)$ then:

1. $\chi(G + xz) > \chi(G)$ hence $c(x) = c(z)$.
2. $\chi(G + xy) > \chi(G)$ hence $c(x) = c(y)$.
3. Then $c(z) = c(y)$, which is a contradiction.

□

Theorem 5. *If G is an upper-critical graph and x a vertex of G then:*

$$G + y : y = x \quad \textit{is also upper-critical}$$

Proof. Let G be an upper-critical graph different from a complete graph. Let $G + y$ be the graph obtained by adding a copy (y) of vertex x of G to itself.
Since $c(y) = c(x)$ then $G + y$ is k-chromatic and since $\chi(G + y + e) > \chi(G) \forall e \notin E(G + y)$ (G is upper-critical) then $G + y$ is upper-critical. □

Theorem 6. *If G is an upper-critical graph and x a vertex of G then:*

$$G - x \quad \textit{is also upper-critical}$$

Proof. Let us suppose that G is a k-chromatic upper-critical graph, such that $G - x$ is not an upper-critical graph.
There are two cases for a vertex x:

1. $\chi(G - x) = k - 1$. Then $(G - x) + e$ has no (k-1)-coloring, otherwise there is a k-coloring of $G + e$ hence $G - x$ is also upper-critical.
2. $\chi(G - x) = k$. Then, since $G - x$ is not upper-critical, there is a k-coloring of $(G - x) + e$. But there is no k-coloring of $G + e$, since G is upper-critical. Hence $\chi(G - x) < \chi(G - x + e)$ and thus $\chi(G - x) < k$ which is a contradiction of the case $\chi(G - x) = k$.

□

Theorem 7. *If G is an upper-critical graph and x, y are two vertices of G such that $xy \notin E(G)$ then:*

$$G/x, y \quad \textit{is also upper-critical}$$

Proof. Since G is upper-critical and $xy \notin E(G)$ then, by theorem 4, vertex y is a copy of x and hence $G/x, y = G - x$. Now, since $G - x$ is upper-critical then $G/x, y$ is upper-critical. □

Theorem 8. *If G is an upper-critical graph and $xy \in E(G)$ then:*

$$G/xy \text{ is also upper-critical}$$

Proof. by induction on the number $n = V(G)$ of vertices of G. Base n=2: The K_2 graph is upper-critical by definition and $K_2/e = K_1$ and K_1 is upper-critical by definition.

Assume true for every upper-critical graph in n vertices.

Proof for $n+1$: Let G be an upper-critical graph on $n+1$ vertices such that G/e is not upper-critical.

If $G = K_k$ then G/e is also a complete graph hence G/e is upper-critical which is a contradiction.

If $G \neq K_k$ then there are two cases:

1. x is a complete vertex (this case includes all three cases: x, y or both). It is easy to see that $G/xy = G - x$, but $G - x$ is upper-critical, thus there is a contradiction.
2. Since $G \neq K_k$ and x is not a complete vertex: we can delete a copy (z) of x from G, obtaining a new graph $H = G - z$. Since G is upper-critical then H is upper-critical and thus $(G - z)/e$ is upper-critical by the inductive hypothesis. But now since z is a copy of x then:

   $$(G - z)/e + z = G/e$$

   is upper-critical by theorem 5 which is a contradiction.

□

Theorem 9. *If G is an upper-critical graph, x, y are two vertices of G and $e = xy \notin E(G)$ then:*

$$G + e \text{ is also upper-critical}$$

Proof. Let z be any arbitrary vertex adjacent to both x and y. There are two cases:

1. Every z is a complete vertex: Fix G to be k-chromatic. Then $G + e$ has K_{k+1} as a subgraph and $G + e + e_2$ will have K_{k+2} as a subgraph, for a new edge e_2, so $\chi(G + e + e_2) > \chi(G + e)$. Hence $G + e$ is upper-critical.
2. There is at least one z which is not a complete vertex: Let $\dot{z}$ be a copy of z. Then $G + \dot{z}$ is upper-critical and so $(G + \dot{z})/x\dot{z}$. Now:

   $$(G + \dot{z})/x\dot{z} = G + xy = G + e$$

Therefore $G + e$ is upper-critical. □

2.2. *Construction, characterization and counting*

An easy procedure to obtain an upper-critical graph G is to find a k-coloring of some k-chromatic graph H and add the edges $x_i x_j$ to H whenever the color class of vertex x_i is different from the color class of x_j, i.e. $x_i x_j \in E(G)\ \forall i, j : c(x_i) \neq c(x_j)$. Furthermore, we will see that there is a quite easy and direct way to obtain any arbitrary upper-critical graph.

Upper-critical graphs, contrary to (lower)critical-graphs, has a very easy general characterization and description: the n-vertex list notation $G = \{n_1, n_2, n_3, ..., n_k\}$ is the k-chromatic upper-critical graph where the positive integers $\{n_1, n_2, n_3, ..., n_k\}$ indicate de number of vertices in each color class (c_i) respectively, e.g. $K_3 = \{1, 1, 1\}$, $K_4^- = \{1, 1, 2\}$. A sample of the upper-critical graphs in n-vertex list notation, up to $|V| = 5$ and $k = 5$, is shown in table 1.

It is immediate to see that the n-vertex list is unique in the sense that two upper-critical graphs share the same n-vertex list *iif* they are isomorphic.

From this it follows that it is possible to specify directly an arbitrary upper-critical graph using the n-vertex list notation subject to just one constraint:

$$|V(G)| = \sum_{i=1}^{k} n_i \tag{1}$$

where G is a k-chromatic upper-critical graph and n_i is the number of vertices belonging to the kth color class.

Table 1: Upper-critical graphs on $|V(G)|$ vertices vs. Chromatic number $k = \chi(G)$

$|V| \setminus k$	1	2	3	4	5	...
1	K_1					
2		K_2				
3		$\{1,2\}$	K_3			
4		$\{1,3\}$, $\{2,2\}$	$\{1,1,2\}$	K_4		
5		$\{1,4\}$, $\{2,3\}$	$\{1,1,3\}$, $\{1,2,2\}$	$\{1,1,1,2\}$	K_5	
$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Also, it is possible to determine directly the number of upper-critical graphs given the number ($N = |V|$) of vertices and the chromatic number (k). The number $U_c(N,k)$ of k-chromatic upper-critical graphs of size N is:

$$U_c(N,k) = \left\lfloor \frac{N}{k} \right\rfloor, \tag{2}$$

since we are assigning N vertices to k color classes.

Other interesting fact of the distribution of upper-critical graphs arranged in vertices vs. chromatic number (as in table 1) is that there is no regular upper-critical graphs (different from K_k) when $|V|$ is prime, since each regular k-chromatic graph will appear at every factor of ($|V|$), e.g. for $|V| = 9$ we have $k = 3 : \{3,3,3\}$ and for $|V| = 15$ we have $k = 5 : \{3,3,3,3,3\}$ and $k = 3 : \{5,5,5\}$.

Furthermore, following the column $k = 2$ of table 1, it will be very interesting to prove (or disprove) the following conjecture:

Conjecture 1. *If $|V| > 2$ and $|V|$ is even then there is at least one bipartite upper-critical graph $G = \{n_1, n_2\}$ such that there is no regular upper-critical graph on n_1 vertices nor in n_2 vertices.*

References

[1] G. A. Dirac. A property of 4-chromatic graphs and some remarks on critical graphs. *J. London Math. Soc.*, 27:85–92, 1952.
[2] G. A. Dirac. Some theorems on abstract graphs. *Proc. London. Math. Soc.*, (3) 2:69–81, 1952.
[3] G. A. Dirac. The structure of k-chromatic graphs. *Fund. Math.*, 40:42–55, 1953.