THREE-MANIFOLDS AND KÄHLER GROUPS

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ABSTRACT. We give a simple proof of a result originally due to Dimca and Suciu [6]: a group that is both Kähler and the fundamental group of a closed three-manifold is finite. We also prove that a group that is both the fundamental group of a closed three-manifold and of a non-Kähler compact complex surface is $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}_2$.

1. INTRODUCTION

In the late 1980s the study of Kähler groups, that is, fundamental groups of closed Kähler manifolds, took off in spectacular fashion. While restrictions on such groups were previously known because of Hodge theory and because of rational homotopy theory, several deep new results were proved around 1988. I will only recall two of them here. These and many other results on Kähler groups are discussed in detail in [2].

Firstly, generalising partial results of Johnson and Rees [11], Gromov proved:

Theorem 1 (Gromov [7]). *A Kähler group does not split as a nontrivial free product.*

Secondly, building on work of Siu, Sampson and others, Carlson and Toledo proved:

Theorem 2 (Carlson–Toledo [5]). *No fundamental group of a closed real hyperbolic n-manifold with $n \geq 3$ is a Kähler group.*

When these results were proved, several people, including Donaldson and Goldman, noticed the contrast between Kähler groups on the one hand and three-manifold groups on the other: the latter are closed under free products, and, according to Thurston, most three-manifolds with freely indecomposable fundamental group are hyperbolic. Moreover, a case by case check of the Thurston geometries as explained in [23] shows the following: closed three-manifolds carrying one of the geometries $S^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, $\mathbb{R}^3$ or Sol^3 have virtually odd first Betti number, and so their fundamental groups cannot be Kähler. Moreover, closed three-manifolds carrying one of the geometries Nil^3 or $\mathrm{SL}_2(\mathbb{R})$ have positive first Betti numbers with trivial cup product from H^1 to H^2. Their fundamental groups cannot be Kähler by the Hard Lefschetz Theorem. Now, the only Thurston geometry that has not been excluded is S^3, where every fundamental group is finite. Since all finite groups are Kähler, it was natural to expect that the intersection of three-manifold groups with the Kähler groups should consist exactly of the finite groups appearing as fundamental groups of three-manifolds with geometry S^3. The obstacle to turning this expectation into a theorem, indeed a corollary of the above Theorems 1 and 2, came from three-manifolds with a non-trivial JSJ decomposition along incompressible tori. While one could imagine that those manifolds containing

at least some hyperbolic piece might yield to a generalisation of the harmonic map techniques of Carlson and Toledo [5][1], the case of graph manifolds seemed intractable.

Twenty years ago one thought about such questions modulo Thurston's geometrisation conjecture. Since this has now been proved by Perelman [21, 22, 13, 19], an unconditional result can finally be obtained. Indeed, Dimca and Suciu recently proved:

Theorem 3 (Dimca–Suciu [6])**.** *Assume that a group Γ is the fundamental group both of a closed Kähler manifold and of a closed three-manifold. Then Γ is finite, and, therefore, a finite subgroup of $O(4)$.*

Once one proves Γ to be finite, it follows from Perelman's work [21, 22, 13, 19] that Γ is a finite subgroup of $O(4)$ acting freely on S^3. Note that by a classical construction due to Serre, every finite group is the fundamental group of a smooth complex projective variety, hence a closed Kähler manifold. By the Lefschetz hyperplane theorem one may assume this variety to be a surface.

To me, a surprising aspect of the proof given by Dimca and Suciu is that it does not follow the above outline at all, and makes little use of the Thurston approach to three-manifolds. In fact, their proof does not use Theorems 1 and 2. Instead, they consider separately the cases of trivial and of nontrivial first Betti number. If the first Betti number of the fundamental group of a closed oriented three-manifold is positive, then they prove it is not Kähler using a lot of machinery of a very different sort: characteristic and resonance varieties, Catanese's approach to the Siu–Beauville theorem, a commutative algebra result of Buchsbaum–Eisenbud, ... Then, for the case of zero first Betti number, Dimca and Suciu appeal to results of Reznikov and Fujiwara pertaining to Kazhdan's property T. It is only at this point that their proof depends on geometrisation via Fujiwara's arguments.

The present paper arose from my attempt to understand the argument of Dimca and Suciu [6]. From their high-tech treatment of the positive Betti number case I extracted the following strategy for obtaining a contradiction: *If Γ has positive first Betti number and is both the fundamental group of a closed oriented three-manifold and of a closed Kähler manifold, then $H^1(\Gamma; \mathbb{R})$ comes from a complex curve. Therefore all cup products of classes in $H^1(\Gamma; \mathbb{R})$ also come from a curve, and this is incompatible with three-dimensional Poincaré duality.*

One can actually implement this strategy in several different ways to prove Theorem 3. Here I will give a distinctly low-tech implementation, leading to an elementary proof of the following:

Theorem 4. *If Γ is a group with $b_1(\Gamma) > 0$ whose real cohomology algebra $H^*(\Gamma; \mathbb{R})$ satisfies 3-dimensional oriented Poincaré duality, then Γ is not a Kähler group.*

To put this into perspective, recall that many Kähler groups are Poincaré duality groups (of even dimension), cf. [11, 25, 14]. Also recall that, for every $k \geq 3$, Toledo [25] constructed examples of Kähler groups of cohomological dimension $2k - 1$. Moreover, his examples are duality (though not Poincaré duality) groups.

Of course, to exclude a group from being a Kähler group, it is enough that some finite index subgroup satisfy the assumptions of Theorem 4.

Corollary 5. *Let M be a closed aspherical three-manifold. If M has a finite orientable covering that is not an $\mathbb{R}$-homology sphere, then $\pi_1(M)$ is not a Kähler group.*

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[1]A first step in this direction was soon taken by Hernández-Lamoneda, although his paper [10] was only published much later.

Theorem 4 is more general than the Corollary because the theorem makes no assumption in integral cohomology and does not assume that Γ is torsion-free. Even if one adds these assumptions, it is not known whether every such group is the fundamental group of an aspherical three-manifold. This issue is a version of the three-dimensional Borel conjecture, where we of course assume Γ to be finitely presentable; see Problem 3.77 on Kirby's problem list [12].

Corollary 5 proves most of Theorem 3, since it handles not only manifolds with a nontrivial JSJ decomposition, but also gives a uniform treatment of geometric cases that no longer need to be checked case by case, so we obtain quite a simple proof of Theorem 3 for groups with virtually positive first Betti number. Using Perelman's geometrisation theorem, the case of first Betti number zero can actually be reduced to Theorem 2. In Section 2 below we first prove Theorem 4, and then spell out the resulting straightforward proof of Theorem 3, avoiding the difficult arguments of Dimca–Suciu [6], and the appeals to the works of Reznikov and Fujiwara. Like the original proof of [6], the proof of Theorem 3 given here uses geometrisation only to handle the case of trivial (virtual) first Betti number.

Using the Kodaira classification of non-Kähler complex surfaces we shall also prove the following:

Theorem 6. *Assume that a group Γ is the fundamental group both of a closed complex surface S and of a closed three-manifold. Then either Γ is a finite subgroup of $O(4)$ and S is a Kähler surface, or Γ is $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}_2$ and S is a Hopf surface.*

This is interesting since in complex dimensions ≥ 3 every finitely presentable group is the fundamental group of a compact complex manifold, as proved by Taubes [24]. Thus, for fundamental group questions, complex surfaces are at the watershed between curves and the general case of complex three-folds, just like three-manifolds are at the watershed between real surfaces and the general case of four-manifolds, where all finitely presentable groups appear.

2. PROOFS

Proof of Theorem 4. Suppose for a contradiction that X is a closed Kähler manifold with fundamental group Γ, and let $\alpha_X \colon X \longrightarrow T^{b_1(\Gamma)}$ be its Albanese map. Since the target is aspherical, α_X factors up to homotopy into a composition

$$X \xrightarrow{c_X} B\Gamma \xrightarrow{a} B\mathbb{Z}^{b_1(\Gamma)} = T^{b_1(\Gamma)} \ ,$$

where c_X is the classifying map of the universal covering of X, and a is induced by the Abelianisation followed by dividing out the torsion in first homology. One concludes that $\alpha_X^* = c_X^* \circ a^*$ is trivial in real cohomology of degree > 3 because $B\Gamma$ has no such cohomology, and so the image of α_X cannot have complex dimension 2 or more. Thus the image of α_X is a complex curve.

Note that all the maps above induce isomorphisms in degree one cohomology. Moreover, by the Hard Lefschetz Theorem, α_X^* is nontrivial in degree 2 cohomology, and so the same is true for a^*. However, there is no class in $H^1(\Gamma; \mathbb{R})$ that has a nontrivial cup product with the image of a^* in $H^2(\Gamma; \mathbb{R})$, since this cup product comes from the image of α_X, which has real dimension $= 2$. This contradicts the assumption that Γ satisfies 3-dimensional Poincaré duality. $\qquad\square$

Proof of Theorem 3. We need to show that an infinite three-manifold group Γ cannot be Kähler. Since finite coverings of Kähler manifolds are Kähler, we only need to exclude some finite index subgroup of Γ, and so three-manifolds can be replaced by their finite coverings. In particular we may assume that all three-manifolds are orientable.

We may restrict our attention to three-manifolds that are prime in the sense of being indecomposable under connected sums, since a nontrivial free product is never a Kähler group by Theorem 1. Such a prime three-manifold is either $S^1 \times S^2$, or is aspherical, cf. [18]. Since a Kähler group cannot be infinite cyclic, we are reduced to the consideration of aspherical three-manifolds, so that, for all 3-manifolds with positive (virtual) first Betti number, Theorem 3 follows from Corollary 5.

To complete the proof of Theorem 3 it remains to deal with groups with vanishing first Betti number. Thus consider a closed oriented aspherical three-manifold M with infinite fundamental group Γ having $b_1(\Gamma) = 0$. If M contains an incompressible torus, then by a result of Luecke [17], see also [15], M has a finite covering with positive first Betti number, so that Corollary 5 applied to this covering shows that Γ is not Kähler. Thus we are left with the case of an aspherical M that contains no incompressible torus. Such manifolds are hyperbolic by the work of Perelman [21, 22, 13], and fundamental groups of hyperbolic three-manifolds are never Kähler by Theorem 2. □

Proof of Theorem 6. Suppose that Γ is the fundamental group of both a compact complex surface S and a closed three-manifold M. As before we may assume M to be orientable.

If S is Kähler, then Γ is finite by Theorem 3. Conversely, if Γ is finite, then the first Betti number of S vanishes, and so S is Kählerian, cf. [4].

If S is not Kählerian, then its first Betti number is odd, see again [4]. We now use the Enriques–Kodaira classification to conclude that either S is properly elliptic with $b_1(S) \geq 3$, or S is of class VII with $b_1(S) = 1$, cf. [3, 20]. In the first case Γ is freely indecomposable and is a Poincaré duality group of dimension 4 by results of Kodaira described in [2, Section 3 of Ch. 1]. In the second case, it is known only that $\pi_1(S)$ cannot split into $\Gamma_1 \star \Gamma_2$ with both Γ_i containing proper subgroups of finite index; see [2, Thm. 1.35]. However, since three-manifold groups are residually finite [9], this is enough to conclude that in our case, where $\pi_1(S) = \Gamma = \pi_1(M)$, Γ is indeed freely indecomposable.

Thus we may assume that M is prime. If it is aspherical, then Γ is a three-dimensional Poincaré duality group. This means that Γ is not the fundamental groups of properly elliptic surfaces with $b_1(S) \geq 3$ since they are four-dimensional Poincaré duality groups. If Γ is the fundamental group of a class VII surface, then we have $b_1(\Gamma) = 1$, and, by Poincaré duality on M, $b_2(\Gamma) = 1$. Under the classifying map of the universal covering of S, $H^2(\Gamma; \mathbb{R})$ injects into $H^2(S; \mathbb{R})$, where it becomes an isotropic subspace for the cup product for dimension reasons. (Its cup square comes from the three-dimensional M.) Thus the intersection form of S would have to be indefinite, which contradicts the known fact that the intersection forms of class VII surfaces are negative definite; see [2, Lemma 1.45].

Thus we are left to consider the case of an M that is prime but not aspherical. This means that M is $S^1 \times S^2$ if it is orientable; cf. [18]. However, for a nonorientable M we could also have the nontrivial S^2-bundle over S^1, also with fundamental group $\mathbb{Z}$, and $S^1 \times \mathbb{R}P^2$, with fundamental group $\mathbb{Z} \oplus \mathbb{Z}_2$; cf. [23]. Since both $\mathbb{Z}$ and $\mathbb{Z} \oplus \mathbb{Z}_2$ occur as fundamental groups of Hopf surfaces, and, by a result of Kodaira (cf. [20]), only of Hopf surfaces, this completes the proof of Theorem 6. □

3. DISCUSSION

3.1. **Avoiding the use of Theorem 1.** In the proof of Theorem 3 in Section 2, I found it most straightforward to reduce to the consideration of prime three-manifolds by using Gromov's result on free products, stated as Theorem 1 in the introduction. However, one can completely bypass the use of Theorem 1, as we now explain.

Lemma 7. *Assume that Γ_1 and Γ_2 each have a non-trivial finite quotient $f_i \colon \Gamma_i \longrightarrow Q_i$. Then their free product $\Gamma_1 \star \Gamma_2$ has a finite index subgroup with odd first Betti number.*

Proof. Consider the induced homomorphism $f \colon \Gamma_1 \star \Gamma_2 \longrightarrow Q_1 \times Q_2$. By the Kurosh subgroup theorem, its kernel is of the form $F_k \star \Gamma$, where F_k is a free group of rank $k = (|Q_1| - 1)(|Q_2| - 1)$, and Γ is a free product of copies of the kernels of the f_i. For a finite quotient $g \colon F_k \longrightarrow Q$ of order d we consider the kernel Δ of $\bar{g} \colon F_k \star \Gamma \longrightarrow Q$, where $\bar{g}$ restricts to F_k as g and is trivial on Γ. Then Δ is isomorphic to $F_l \star \Gamma \star \ldots \star \Gamma$ with d copies of Γ appearing, and $l = 1 + d(k - 1)$. Thus $\Delta \subset \Gamma_1 \star \Gamma_2$ is a finite index subgroup with

$$b_1(\Delta) = l + d \cdot b_1(\Gamma) = 1 + d \cdot (k - 1 + b_1(\Gamma)) \,.$$

Choosing d to be even, we have found the desired subgroup. $\qquad\square$

Since three-manifold groups are residually finite [9], we have the following:

Corollary 8. *If M is a non-prime three-manifold, then it has a finite covering with odd first Betti number.*

At the expense of appealing to residual finiteness, we can use this Corollary in place of Theorem 1 to exclude non-prime manifolds form consideration in the proof of Theorem 3. More generally, without restricting to three-manifold groups, Lemma 7 tells us that an arbitrary free product whose free factors admit finite quotients cannot be a Kähler group. This is exactly the special case of Theorem 1 originally proved by Johnson and Rees [11]. Indeed our proof of the Lemma is a simplification of the argument in [11].

Finally, there is another way to avoid appealing to Theorem 1, without using the residual finiteness of three-manifold groups. If we consider a non-prime oriented three-manifold M, then its prime summands are aspherical manifolds plus, perhaps, copies of $S^1 \times S^2$. Therefore, the classifying space $B\pi_1(M)$ is the one-point union of the aspherical summands plus, possibly, some circles. Now if $\pi_1(M)$ were a Kähler group, then one could use this classifying space in the argument from the proof of Theorem 4 to derive a contradiction with the Hard Lefschetz theorem.

3.2. **The necessity to discuss $\mathbb{R}$-homology spheres.** In the proof of Theorem 3 it was necessary to consider separately the case of groups with zero first Betti number. This step would be superfluous, if it were known that every closed three-manifold has a finite covering with positive first Betti number. If such a statement were available, then one would not need Theorem 2 for the proof of Theorem 3 given here.

Apparently the question of whether every closed three-manifold with infinite fundamental group has virtually positive first Betti number was raised long ago by Waldhausen, Thurston, and others; see Problems 3.2 and 3.50 in Kirby's problem list [12] and the references given there. Curiously, those references do not include [8, 17] and other papers quoted in [8], all of which contain a wealth of information about this problem. In any case, this problem seems to be still open.

3.3. **The second Betti number of infinite Kähler groups.** Carlson and Toledo have asked whether an infinite Kähler group has virtually positive second Betti number[2]. If this were known to be true, then, because of three-dimensional Poincaré duality, we would not have to consider $\mathbb{R}$-homology 3-spheres in the proof of Theorem 3. Moreover, we would not need to use geometrisation, and we would not need Theorem 2 either! We refer to the paper of Klingler [14] for a recent discussion of this question of Carlson and Toledo.

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[2]The original reference for their question is Section 18.16 in [16], where only a more specific version is formulated.

Unfortunately, a slight misstatement occurs in [2, Prop. 3.44 (i)], which implicitly asserts a positive answer to the question of Carlson and Toledo. The statement $b_2(\pi_1(X)) \geq 1$ there should be replaced by $b_2(X) \geq 1$ (which is trivial). The Proposition in question was proved by Amorós [1], whose paper does not contain the misstatement.

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