A SHORT AND ELEMENTARY PROOF OF HANNER'S THEOREM

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Abstract. Hanner's theorem is a classical theorem in the theory of retracts and extensors in topological spaces, which states that a local ANE is an ANE. While Hanner's original proof of the theorem is quite simple for separable spaces, it is rather involved for the general case. We provide a proof which is not only short, but also elementary, relying only on well-known classical point-set topology.

1. Introduction

Denote by $\mathscr{M}$ the class of metrizable spaces. Hanner's theorem is a fundamental theorem in the theory of extensors and retracts, stating that a space which admits an open covering by ANEs for $\mathscr{M}$, is an ANE for $\mathscr{M}$.

Proving that a space with a countable covering by open ANEs is an ANE is not hard [2], but the original proof of the Hanner's general theorem is rather complicated [3]. We give a short and elementary proof, based on reducing the uncountable covering by ANEs to a countable covering by ANEs using a technique originating with J. Milnor [4]. Another short proof of the theorem has been given by J. Dydak [1] as part of his framework for the extension dimension theory.

2. Preliminaries

A metrizable space Y is said to be an ANE for $\mathscr{M}$ if, given any space $X \in \mathscr{M}$ and any continuous map $f\colon A \to Y$ where A is a closed subset of X, there exists a neighborhood U of A in X and a continuous extension $F\colon U \to Y$ of f.

Theorem 1. i) *Any open subset of a space which is an ANE for $\mathscr{M}$ is an ANE for $\mathscr{M}$.*

ii) *If $X = \bigcup_{i\in I} U_i$ where the U_i are disjoint open subsets of X which are ANEs for $\mathscr{M}$, then X is an ANE for $\mathscr{M}$.*

iii) *If $X = \bigcup_{n\in\mathbb{N}} U_n$ where the U_n are open subsets of X which are ANEs for $\mathscr{M}$, then X is an ANE for $\mathscr{M}$.*

Proof. Claim *i)* is trivial, and proofs of claims *ii)*, and *iii)* can be found in Hanner's article [2]. □

3. Hanner's general theorem

Our theorem is the following

Theorem 2. *If $X \in \mathscr{M}$ and $X = \bigcup_{i\in I} U_i$ where the U_i are open subsets of X which are ANEs for $\mathscr{M}$, then X is an ANE for $\mathscr{M}$.*

Proof. Find a partition of unity $\{\varphi_i\colon X \to [0,1]\}_{i\in I}$ which is subordinate to the covering $\{U_i\}_{i\in I}$ of X. For each finite subset $T \subset I$ we denote

$$W(T) = \{x \in X | \varphi_i(x) > \varphi_j(x) \ \forall\ i \in T,\ \forall\ j \in I \setminus T\}.$$

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This set is open because $W(T) = u_T^{-1}(0,1]$ for the continuous map

$$u_T\colon X \to [0,1], \quad u_T(x) = \max\{0, \min\{\varphi_i(x) - \varphi_j(x) | i \in T,\ j \in I \setminus T\}\}.$$

Furthermore, $W(T) \subset \varphi_i^{-1}(0,1] \subset U_i$ for each $i \in T$ since $x \in W(T)$ implies $\varphi_i(x) > \varphi_j(x) \geq 0$ for each $i \in T$ and $j \in I \setminus T$. It follows that $W(T)$ is an ANE for $\mathscr{M}$ by Theorem 1 part *i)*.

Note that if $\mathrm{Card}(T) = \mathrm{Card}(T')$ and $T \neq T'$, then $W(T) \cap W(T') = \emptyset$, since otherwise for some $x \in W(T) \cap W(T')$, $i \in T \setminus T'$ and $j \in T' \setminus T$ we have simultaneously $\varphi_i(x) < \varphi_j(x)$ and $\varphi_j(x) > \varphi_i(x)$, which is impossible.

Define

$$W_n = \bigcup \{W(T) | \mathrm{Card}(T) = n\}.$$

Then W_n is an ANE for $\mathscr{M}$ by Theorem 1 part *ii)*.

But now $X = \bigcup_{n \in \mathbb{N}} W_n$ is an ANE for $\mathscr{M}$ by Theorem 1, part *iii)*. □

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