

Orientable closed 3-manifolds with surface-complexity one

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Abstract

After a short summary of known results on surface-complexity of closed 3-manifolds, we will classify all closed orientable 3-manifolds with surface-complexity one.

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3-manifold, complexity, immersed surface, cubulation.

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Introduction

An approach to the study of closed 3-manifolds consists in filtering them. The aim is to find a function from the set of closed 3-manifolds to the set of natural (or real positive) numbers, so that the number associated to a closed 3-manifold is a measure of how much the manifold is complicated. For closed surfaces, this can be achieved by means of genus. For closed 3-manifolds, the problem has been studied very much and many possible functions has been found. For example, the Heegaard genus, the Gromov norm, the Matveev complexity have been considered.

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lens	other elliptic	flat
$L_{6,1}$	$(S^2, (2, 1), (2, 1), (2, -1))$	T^3
$L_{8,3}$	$(S^2, (2, 1), (2, 1), (3, -2))$	$(S^2, (2, 1), (4, 1), (4, -3))$
$L_{12,5}$		$(S^2, (3, 1), (3, 1), (3, -2))$
$L_{14,3}$		$(S^2, (2, 1), (2, 1), (2, 1), (2, -3))$ $(RP^2, (2, 1), (2, -1))$

Table 1: Irreducible orientable closed 3-manifolds with surface-complexity one.

All these functions fulfil many properties. For instance, they are additive under connected sum. However, some of them have drawbacks. The Heegaard genus and the Gromov norm are not finite-to-one, while the Matveev complexity is. Hence, in order to carry out a classification process, the third one is more suitable than the first two. The Matveev complexity is also a natural measure of how much the manifold is complicated, because if a closed 3-manifold is \mathbb{P}^2 -irreducible and different from the sphere S^3 , the projective space $\mathbb{R}P^3$ and the lens space $L_{3,1}$, then its Matveev complexity is the minimal number of tetrahedra in a triangulation of the manifold (the Matveev complexity of S^3 , $\mathbb{R}P^3$ and $L_{3,1}$ is zero). Such functions could also be tools to give proofs by induction.

The author [2] defined another function (called *surface-complexity*), from the set of closed 3-manifolds to the set of natural numbers, by means of triple points of particular immersions of closed surfaces. In this paper, we will give a short summary of known results on surface-complexity of closed 3-manifolds and we will classify all closed orientable 3-manifolds of surface-complexity one. Those with surface-complexity zero has been classified in [2], and the irreducible ones are S^3 , $\mathbb{R}P^3$ and the lens space $L_{4,1}$. Among those with surface-complexity one there are 11 irreducible ones, which are listed in Table 1. (For Seifert manifolds we have used the orbit invariants.) The list up to complexity two has been obtained independently by Kazakov [6].

Vigara [13] used triple points of particular transverse immersions of connected closed surfaces to define the triple point spectrum of a 3-manifold. The definition of the surface-complexity is similar to Vigara's one, but it has the advantage of being more flexible. This flexibility has allowed to

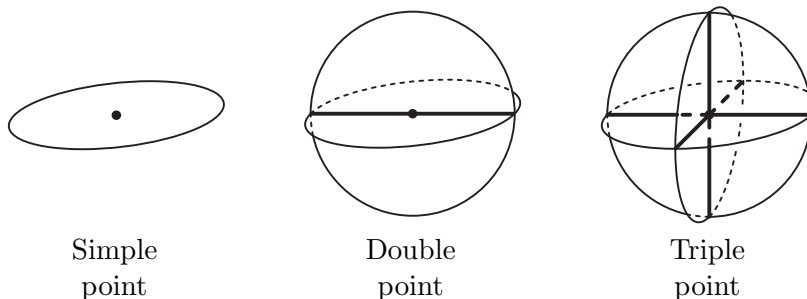


Figure 1: Neighbourhoods of points (marked by thick dots) of a Dehn surface.

prove many properties fulfilled by the surface-complexity, such as finiteness, naturalness and subadditivity.

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1 Definitions

Throughout this paper, all 3-manifolds are assumed to be connected and closed. By M , we will always denote such a (connected and closed) 3-manifold. Using the *Hauptvermutung*, we will freely intermingle the differentiable, piecewise linear and topological viewpoints.

Dehn surfaces A subset Σ of M is said to be a *Dehn surface of M* [10] if there exists an abstract (possibly non-connected) closed surface S and a transverse immersion $f: S \rightarrow M$ such that $\Sigma = f(S)$. By transversality, in Σ there are only the three types of points shown in Fig. 1 (called *simple*, *double* and *triple*, respectively). The set of triple points is denoted by $T(\Sigma)$; non-simple points are called *singular* and their set is denoted by $S(\Sigma)$. In all figures, triple points are always marked by thick dots and the singular set is also drawn thick.

(Quasi-)filling Dehn surfaces and surface-complexity A Dehn surface Σ of M will be called *quasi-filling* if $M \setminus \Sigma$ is made up of balls. Moreover, Σ is called *filling* [9] if its singularities induce a cell-decomposition of M ; more precisely,

- $T(\Sigma) \neq \emptyset$,
- $S(\Sigma) \setminus T(\Sigma)$ is made up of intervals (called *edges*),
- $\Sigma \setminus S(\Sigma)$ is made up of discs (called *regions*),
- $M \setminus \Sigma$ is made up of balls (i.e. Σ is quasi-filling).

Since M is connected, the quasi-filling Dehn surface Σ is connected. Moreover, M minus some (suitably chosen) balls is a regular neighbourhood of Σ and hence collapses to Σ . It is by now well-known that a filling Dehn surface determines M up to homeomorphism and that every M has filling Dehn surfaces (see, for instance, Montesinos-Amilibia [9] and Vigara [12], see also [3]).

Surface-complexity The *surface-complexity* $sc(M)$ of M is equal to c if M possesses a quasi-filling Dehn surface with c triple points and has no quasi-filling Dehn surface with less than c triple points [2].

2 Properties

In this section we will describe some results on surface-complexity and minimal quasi-filling Dehn surfaces. Details and proofs, unless explicitly stated, can be found in [2].

Minimality and finiteness A quasi-filling Dehn surface Σ of M is called *minimal* if it has a minimal number of triple points among all quasi-filling Dehn surfaces of M , i.e. $|T(\Sigma)| = sc(M)$.

Let us give some examples of quasi-filling Dehn surfaces without triple points, which are clearly minimal. Only two surfaces are quasi-filling Dehn surfaces of a 3-manifold: the sphere S^2 and the projective plane \mathbb{RP}^2 , which are quasi-filling Dehn surfaces of the sphere S^3 and the projective space \mathbb{RP}^3 , respectively. Two projective planes intersecting along a loop non-trivial in both of them form a quasi-filling Dehn surface of \mathbb{RP}^3 , which will be called *double projective plane*. A sphere intersecting a torus (resp. a Klein bottle) along a loop is a quasi-filling Dehn surface of $S^2 \times S^1$ (resp. $S^2 \rtimes S^1$). The *quadruple hat* (i.e. a disc whose boundary is glued four times along a circle) is a quasi-filling Dehn surface of the lens-space $L_{4,1}$. Therefore, we have that S^3 , \mathbb{RP}^3 , $S^2 \times S^1$, $S^2 \rtimes S^1$ and $L_{4,1}$ have surface-complexity zero.

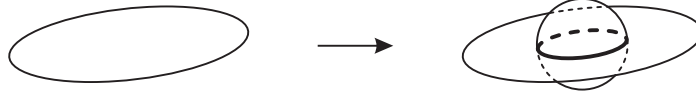


Figure 2: Bubble-move.

Minimal quasi-filling Dehn surfaces without triple points are clearly not filling, but if we take into account only \mathbb{P}^2 -irreducible 3-manifolds (except for three ones), we have minimal filling Dehn surfaces. More precisely, suppose M is \mathbb{P}^2 -irreducible, then we have two cases:

- if $sc(M) = 0$, then M is S^3 , $\mathbb{R}\mathbb{P}^3$ or $L_{4,1}$;
- if $sc(M) > 0$, then M has a minimal filling Dehn surface.

Note that the only 3-manifolds that are prime but not irreducible are $S^2 \times S^1$ and $S^2 \rtimes S^1$, hence the theorem implies that every prime 3-manifold, except the five manifolds described above, has a minimal filling Dehn surface.

Since there is a finite number of filling Dehn surfaces having a fixed number of triple points, we have that for any integer c there exists only a finite number of \mathbb{P}^2 -irreducible 3-manifolds having surface-complexity c .

Minimal quasi-filling Dehn surfaces Not all the minimal quasi-filling Dehn surfaces of a \mathbb{P}^2 -irreducible 3-manifold are indeed filling. However, they can be all constructed starting from filling ones (except for S^3 , $\mathbb{R}\mathbb{P}^3$ and $L_{4,1}$, for which non-filling ones must be used) and applying a simple move. The move acts on a quasi-filling Dehn surface near a simple point as shown in Fig. 2 and is called a *bubble-move*. If a quasi-filling Dehn surface Σ is obtained from a quasi-filling Dehn surface $\bar{\Sigma}$ by repeatedly applying bubble-moves, we will say that Σ is *derived from* $\bar{\Sigma}$.

The result above on minimal filling Dehn surfaces in the \mathbb{P}^2 -irreducible case can be improved by means of a slightly subtler analysis. Suppose M is \mathbb{P}^2 -irreducible. If Σ is a minimal quasi-filling Dehn surface of M , we have the following cases:

- If $sc(M) = 0$, one of the following holds:
 - M is S^3 and Σ is derived from S^2 ,
 - M is $\mathbb{R}\mathbb{P}^3$ and Σ is derived from $\mathbb{R}\mathbb{P}^2$ or from the double projective plane,

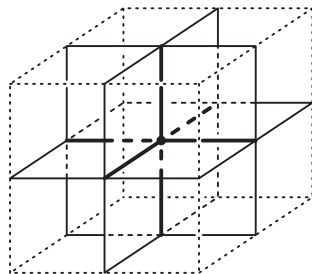


Figure 3: Local behaviour of duality.

- M is $L_{4,1}$ and Σ is derived from the four-hat.
- If $sc(M) > 0$, then Σ is derived from a minimal filling Dehn surface of M .

Cubulations and naturalness A *cubulation* of M is a cell-decomposition of M such that

- each 2-cell (called a *face*) is glued along 4 edges,
- each 3-cell (called a *cube*) is glued along 6 faces arranged like the boundary of a cube.

Note that self-adjacencies and multiple adjacencies are allowed.

The following construction is well-known (see [1, 5, 4], for instance). A filling Dehn surface Σ of M can be constructed from a cubulation \mathcal{C} of M by considering for each cube of \mathcal{C} the three squares shown in Fig. 3 and by gluing them together (up to isotopy, we can suppose that the squares fit together through the faces). Conversely, a cubulation \mathcal{C} of M can be constructed from a filling Dehn surface Σ of M by considering an abstract cube for each triple point of Σ and by gluing the cubes together along the faces (the identification of each pair of faces is chosen by following the four germs of regions adjacent to the respective edge of Σ). The cubulation and the filling Dehn surface constructed in such a way are said to be *dual* to each other.

The construction above allowed to prove that if M is \mathbb{P}^2 -irreducible and is different from S^3 , $\mathbb{R}\mathbb{P}^3$ and $L_{4,1}$, then its surface-complexity is equal to the minimal number of cubes in a cubulation of M .

Subadditivity An important feature of a complexity function is to behave well with respect to the cut-and-paste operations. The surface-complexity is subadditive under connected sum. Namely, the surface-complexity of the connected sum of 3-manifolds is less than or equal to the sum of their surface-complexities. We do not know whether it is indeed additive.

Estimations In general, calculating the surface-complexity $sc(M)$ of M is very difficult, but estimating it is relatively easy. More precisely, it is quite easy to give upper bounds for it. If one constructs a quasi-filling Dehn surface Σ of M , the number of triple points of Σ is an upper bound for the surface-complexity of M . Afterwards, the (usually difficult) problem of proving the sharpness of this bound arises.

There are explicit constructions of quasi-filling Dehn surfaces of M starting from triangulations, Heegaard splittings and Dehn surgery presentations of M . They allow to prove some estimations.

Triangulations Suppose that M has a triangulation with n tetrahedra. Then, the inequality $sc(M) \leq 4n$ holds.

Heegaard splittings Suppose that $H_1 \cup H_2$ is a Heegaard splitting of M such that the meridians of the handlebody H_1 intersect those of H_2 transversely in n points. Then, the inequality $sc(M) \leq 4n$ holds. This estimation can be improved for any connected sum of \mathbb{P}^2 -irreducible 3-manifolds M_k , such that no M_k is $L_{3,1}$. Indeed, suppose that $H_1 \cup H_2$ is a Heegaard splitting of such a 3-manifold M , that the meridians of the handlebody H_1 intersect those of H_2 transversely in n points, and that the closure of one of the components into which the meridians of H_1 and H_2 divide $\partial H_1 = \partial H_2$ contains m of these points. Then, the inequality $sc(M) \leq 4n - 4m$ holds.

Dehn surgery Suppose that M is obtained by Dehn surgery along a framed link L in S^3 (hence M is orientable). Moreover, suppose that L has a projection such that the framing is the blackboard one, such that there are n crossing points, and such that there are m components containing no overpass. Then, the inequality $sc(M) \leq 8n + 4m$ holds. If the framing is not the blackboard one, we have $sc(M) \leq 8n + 4m + 4 \sum_i |fr_i - w_i|$, where fr_i and w_i are, respectively, the framing and the writhe of the i -th component of the link.

Matveev complexity The surface-complexity is related to the Matveev complexity, at least in the \mathbb{P}^2 -irreducible case. The latter is defined using simple spines. A polyhedron P is *simple* if the link of each point of P can be embedded in the 1-skeleton of the tetrahedron. The points of P whose link is the whole 1-skeleton of the tetrahedron are called *vertices*. A sub-polyhedron P of M is a *spine* of M if $M \setminus P$ is a ball. The *Matveev complexity* $c(M)$ of M is the minimal number of vertices of a simple spine of M . The interested reader is referred to Matveev [7] for a complete discussion on Matveev complexity.

Suppose that M is \mathbb{P}^2 -irreducible and different from $L_{3,1}$ and $L_{4,1}$. Then, the inequalities $sc(M) \leq 4c(M)$ and $c(M) \leq 6sc(M)$ hold. The latter inequality has been proved in the weaker form $c(M) \leq 8sc(M)$ in [2] and then improved by Tarkaev [11]. For the two missing manifolds, we have $c(L_{3,1}) = 0$, $sc(L_{3,1}) = 2$, $c(L_{4,1}) = 1$ and $sc(L_{4,1}) = 0$.

3 Orientable 3-manifolds with surface-complexity one

In this section we will describe how we have obtained the list of the orientable 3-manifolds with surface-complexity one. The list of the irreducible orientable 3-manifolds with surface-complexity one is shown in Table 1. Note that six are elliptic and five are flat. Note also that the only missing flat 3-manifold is $(S^2, (2, 1), (3, 1), (6, -5))$. Finally, note that there are no \mathbb{P}^2 -irreducible orientable 3-manifolds having surface-complexity one of geometric-type $\mathbb{H}^2 \times \mathbb{R}$, $\widetilde{\text{SL}}_2\mathbb{R}$, Sol, hyperbolic or non-geometric.

We will now explain the steps to obtain this list; we will not go into detail. By means of duality, we can list cubulations with one cube up to homeomorphism. There are three inequivalent ways to pair the six faces of a cube. Each way involves three gluings of faces and each gluing can be done in four ways. Therefore, we need to analyse $3 \times 4^3 = 192$ cubulations. Among these we have ruled out those whose underlying topological space is not a closed 3-manifold, and we have removed duplicates (up to isomorphism of the cubic structure); in order to carry out this step, we have used a simple computer program. We have found 29 non-isomorphic cubulations of orientable 3-manifolds.

We have then examined each of these 29 cubulations to identify the underlying manifolds. Giving a name to many cubulations has been quite

easy. Sometimes the summands of a connected sum and the Seifert structure are clear from the cubic structure. For the other cases we have recognised the manifold by computing some topological invariants (homology, homotopy, ϵ -invariant [8, 7], Matveev complexity) and by searching in the list of Matveev [7].

In order to obtain the list of all orientable 3-manifolds with surface-complexity one, we apply the results on minimality and subadditivity described in the previous section. Since the only 3-manifolds that are prime but not irreducible are $S^2 \times S^1$ and $S^2 \times S^1$, each orientable 3-manifold with surface-complexity one is the connected sum of one irreducible orientable 3-manifold with surface-complexity one (listed in Table 1) and some (possibly none) among \mathbb{RP}^3 , $L_{4,1}$, $S^2 \times S^1$ and $S^2 \times S^1$.

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