PRE-LIE ALGEBRAS IN POSITIVE CHARACTERISTIC

IOANNIS DOKAS

Abstract. In prime characteristic we introduce the notion of restricted pre-Lie algebra. We prove in the pre-Lie context the analogue to Jacobson's theorem for restricted Lie algebras. In particular, we prove that any dendriform algebra over a field of positive characteristic is a restricted pre-Lie algebra. As a consequence we prove that Rota-Baxter algebras and quasitriangular algebras are restricted pre-Lie algebras. We construct a left adjoint functor to the functor $(-)_{preLie} : Dend \to preLie$. Finally, we define the notion of restricted enveloping dendriform algebra and we construct a left adjoint functor for the functor $(-)_{p-preLie} : Dend \to p - preLie$.

Introduction

In many cases, in the theory of Lie algebras over a field of prime characteristic, the notion of Lie algebra has to be replaced by the notion of restricted Lie algebra. The theory of restricted Lie algebras was developed by N. Jacobson [8]. The notion of restricted Lie algebra arises naturally in positive characteristic. Indeed, by a theorem of N. Jacobson the associated Lie algebra A_{Lie} of any associative algebra A over a field of characteristic $p \neq 0$ is endowed with the structure of restricted Lie algebra $(A_{Lie}, [-,-], F)$, where F denotes the Frobenius map.

The structure of pre-Lie algebra appears in many fields of mathematics under the names left symmetric algebras, right symmetric algebras or Vinberg algebras. Pre-Lie algebras are Lie-admissible algebras which generalize the notion of associative algebra. The notion of pre-Lie algebra has been introduced by Gerstenhaber in [7]. In connection with the theory of operads, if $\mathcal{P}$ is an operad the space $\bigoplus_n P(n)$ admits structure of pre-Lie algebra [11].

In this paper we study pre-Lie algebras in prime characteristic. In the pre-Lie context the role of associative algebra is now played by the structure of dendriform algebra introduced by J.-L. Loday in [10]. Dendriform algebras are algebras with two binary operations, which dichotomize the notion of associative algebra. Moreover, any dendriform algebra structure induces a pre-Lie algebra structure. Therefore, there is a functor $(-)_{preLie} : Dend \to preLie$ from the category of dendriform algebras to the category of pre-Lie algebras.

In section 2, in a similar way with the case of restricted Lie algebra, we introduce the notion of restricted pre-Lie algebra generalizing the definition given by A. Dzhumadil'daev in [6]. We define a restricted pre-Lie algebra as a pair $(P, \{-,-\}, (-)^{[p]})$ where P is a pre-Lie algebra together with a map $(-)^{[p]} : P \to P$ which verifies specific relations. From the definition the induced Lie algebra $(P_{Lie}, [-,-])$ of a restricted pre-Lie algebra $(P, \{-,-\}, (-)^{[p]})$ is actually a restricted Lie algebra

1991 *Mathematics Subject Classification.* 17B63, 17A32.
Key words and phrases. Restricted Lie algebra, dendriform algebra, pre-Lie algebra.

$(P_{Lie}, [-,-], (-)^{[p]})$. Therefore we obtain a functor from the category of restricted pre-Lie algebras to the category of restricted Lie algebras.

Next in Theorem 2.11 we prove the analogue to Jacobson's theorem in the pre-Lie context: Any dendriform algebra over a field of characteristic $p \neq 0$ is a restricted pre-Lie algebra. Thus we obtain a functor $(-)_{p-preLie} : Dend \to p-preLie$ from the category of dendriform algebras to the category of restricted pre-Lie algebras. In section 3, we prove by Propositions 3.1 and 3.2 that Rota-Baxter algebras and quasitriangular algebras over a field of positive characteristic are restricted pre-Lie algebras. Thus, we obtain a large amount of examples of restricted pre-Lie algebras. In section 4 we define for any pre-Lie algebra P the notion of enveloping dendriform algebra $U(P)$. By Proposition 4.2 we prove that $U(P)$ has the appropriate universal property. Thus we construct a left adjoint functor to the functor $(-)_{preLie} : Dend \to preLie$. In the final section 5 we define for $(P, \{-,-\}, (-)^{[p]})$ a restricted pre-Lie algebra the notion of restricted enveloping dendriform algebra $U_p(P)$ and we construct a left adjoint functor for the functor $(-)_{preLie} : Dend \to preLie$.

1. Restricted Lie algebras

In order to study structures with additional operations (see for example [4]), many ideas and inspiration comes from the category $p - Lie$ of restricted Lie algebras. We recall Jacobson's definition of restricted Lie algebra.

Definition 1.1. A *restricted* Lie algebra $(L, (-)^{[p]})$ over a field k of characteristic $p \neq 0$ is a Lie algebra L over k together with a map $(-)^{[p]} : L \to L$ called the p-map such that the following relations hold:

$$(\alpha x)^{[p]} = \alpha^p \, x^{[p]} \tag{1.1}$$

$$[x^{[p]}, y] = [\underbrace{x, [x, [\cdots [x}_{p}, y]]]] \tag{1.2}$$

$$(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x,y) \tag{1.3}$$

where $is_i(x,y)$ is the coefficient of λ^{i-1} in the formal $(p-1)$-fold product

$$[\underbrace{\lambda x + y, [\cdots [\lambda x + y}_{p-1}, x] \cdots]]$$

where $x, y \in L$ and $\alpha \in k$.

Remark 1.2. We note that if $(L, (-)^{[p]})$ is a restricted Lie algebra and H denotes the Lie sub-algebra of L generated by $x, y \in L$, then $s_i(x,y) \in H^p$ where H^i is defined by $H^1 = H$ and $H^i := [H^{i-1}, H]$.

The next theorem due to N. Jacobson [8] justifies the above definition.

Theorem 1.3. *Let $(A, \star)$ be an associative algebra over a field k of characteristic $p \neq 0$. Then $(A, [-,-], F)$ is a restricted Lie algebra where for all $x, y \in A$ we denote by $[x,y] := x \star y - y \star x$, and F is the Frobenius map $F : A \to A$ given by $F(x) := x^{\star p}$.*

2. Restricted pre-Lie algebras

We recall that M. Gerstenhaber has defined the notion of pre-Lie algebra in [7]. In particular, the Lie bracket involved in Gerstenhaber structure on Hochschild cohomology comes from a pre-Lie structure. Moreover, pre-Lie algebras appeared in E. Vinberg's work on convex homogeneous cones [14] and they play an important role in the differential geometry of flat manifolds [12]. Besides, the theory of pre-Lie algebras has received a great impulse, because of its applications to renormalization, as formalized by A. Connes and D. Kreimer [2].

Definition 2.1. A *left pre-Lie algebra* $(A, -, -)$ is a k-vector space equipped with a binary operation $\{-,-\} : A \to A$ such that the following relation holds:

$$\{x, \{y, z\}\} - \{\{x, y\}, z\} = \{y, \{x, z\}\} - \{\{y, x\}, z\}.$$

If we denote by $a(x, y, z)$ the associator then the above relation is written

$$a(x, y, z) = a(y, x, z).$$

For this reason in the literature left pre-Lie algebras are called also left symmetric algebras. As we mentioned the category of pre-Lie algebras is Lie admissible. In particular we have the following proposition.

Proposition 2.2. *If $(A, \{-,-\})$ is a pre-Lie algebra, then A is equipped with a structure of Lie algebra with a Lie bracket given by $[x, y] := \{x, y\} - \{y, x\}$.*

Proof. See Lemma 1.4.2 in [11]. □

Free pre-Lie algebras. Let X be a set, then an explicit description of the pre-Lie algebra preLie(X) generated by X is given in [3] in terms of rooted trees labeled by X.

In a similar way with the case of restricted Lie algebras we propose the following definition for restricted pre-Lie algebras.

Definition 2.3. *A restricted pre-Lie algebra* $(P, (-)^{[p]})$, is a pre-Lie algebra P over a field k of characteristic $p \neq 0$ together with a map $(-)^{[p]} : P \to P$ called the p-map such that:

$$(\alpha x)^{[p]} = \alpha^p \, x^{[p]} \tag{2.1}$$

$$\{x^{[p]}, y\} = \{\underbrace{x, \{x, \{\cdots \{x}_{p}, y\}\}\} \tag{2.2}$$

$$\{y, x^{[p]}\} = \{\underbrace{x, \{x, \{\cdots \{x}_{p}, y\}\}\} - [\underbrace{x, [x, [\cdots [x}_{p}, y]]]] \tag{2.3}$$

$$(x + y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x, y) \tag{2.4}$$

where $[x, y] := \{x, y\} - \{y, x\}$ is the induced bracket, the $s_i(x, y)$ are defined as in the Definition 1.1 and $x, y \in P$ and $\alpha \in k$.

If $(P, (-)^{[p]})$ and $(P', (-)^{[p]})$ are restricted pre-Lie algebras, then a pre-Lie morphism $f : P \to P'$ is called restricted if $f(x^{[p]}) = f(x)^{[p]}$. We denote by $p-\mathrm{preLie}$ the category of restricted pre-Lie algebras over k.

From the definition above we see that if $(P, \{-,-\}, (-)^{[p]})$ is a restricted pre-Lie algebra, then the induced Lie algebra is a restricted Lie algebra $(P, [-,-], (-)^{[p]})$.

Therefore there is a functor from the category of restricted pre-Lie algebras to the category of restricted Lie algebras and we obtain the following proposition.

Proposition 2.4. *The following diagram of categories of algebras is commutative:*

$$\begin{array}{ccc} p-\text{preLie} & \longrightarrow & p-\text{Lie} \\ \downarrow & & \downarrow \\ \text{preLie} & \longrightarrow & \text{Lie} \end{array}$$

Remark 2.5. The Definition 2.3 of restricted pre-Lie algebra generalizes the Definition 1.3 given by A. Dzhumadil'daev in [6]. Indeed, in [6] a restricted pre-Lie algebra is defined as a pre-Lie algebra $(P, \{-,-\})$ such that

$$\{x^{\{p\}}, y\} = \{\underbrace{x, \{x, \{\cdots \{x}_{p}, y\}\}\},$$

where

$$x^{\{p\}} := \{\underbrace{x, \{x, \{\cdots \{x, x}_{p}\}\}\}\}.$$

Moreover, by Corollary 2.4 in [6], it is proved that the induced Lie algebra is a restricted Lie algebra $(P, [-,-], (-)^{\{p\}})$. We easily see that the relations of the Definition 2.3 are verified, thus a restricted pre-Lie algebra P in terms of the Definition 1.3 in [6] is a restricted pre-Lie algebra $(P, \{-,-\}, (-)^{[p]})$ in terms of the Definition 2.3 where $x^{[p]} := x^{\{p\}}$.

We have seen that the structure of associative algebra in prime characteristic is the prototype in order to define the notion of restricted Lie algebra. The question which arises naturally is: what structure in the context of pre-Lie algebras will play the role of associative algebras? As we will see below this role is played by the structure of dendriform algebras introduced by J.-L. Loday in [10]. Let us recall the definition of dendriform algebra.

2.1. **Dendriform algebras.** In many cases in combinatorial constructions we obtain associative products which have the nice property to split into components. This phenomenon arises also in other contexts and the category of dendrifrom algebras play a crucial role.

Definition 2.6. A *dendriform algebra* $(D, \prec, \succ)$ is a k-vector space endowed with two binary operations $\prec : D \to D$ and $\succ : D \to D$ such that the following relations hold:

$$\begin{aligned} (x \prec y) \prec z &= x \prec (y \prec z + y \succ z), \\ (x \succ y) \prec z &= x \succ (y \prec z), \\ (x \prec y + x \succ y) \succ z &= x \succ (y \succ z). \end{aligned}$$

for all elements $x, y, z \in D$.

If we denote by $\star : D \to D$ the operation given by $x \star y := x \prec y + x \succ y$ then by Lemma 5.2 in [10] we have that $(D, \star)$ is an associative algebra. The above relations are concisely written as follows:

$$(x \prec y) \prec z = x \prec (y \star z), \tag{2.5}$$

$$(x \succ y) \prec z = x \succ (y \prec z), \tag{2.6}$$

$$(x \star y) \succ z = x \succ (y \succ z). \tag{2.7}$$

Shuffle algebra. J.-L. Loday proves in [10] that the shuffle algebra is a dendriform algebra. Indeed the shuffle product can be split in two products in a way that the axioms of dendriform algebra are satisfied.

Remark 2.7. A dendriform algebra D for which $x \succ y = y \prec x$ for all $x, y \in D$ is called Zinbiel algebra. The notion of Zinbiel algebra is related to the notion of commutative algebra with divided powers (see [5]).

Free Dendriform algebras. There exists an explicit description of free dendriform algebras in terms of planar binary trees. (cf. Theorem 5.8 in [10]).

The next proposition is well known and proves that there is a functor from the category of dendriform algebras to the category of pre-Lie algebras.

Proposition 2.8. *Let $(D, \prec, \succ)$ be a dendriform algebra. Then D is equipped with a left pre-Lie algebra operation $\{-,-\}$ given by $\{x, y\} := x \succ y - y \prec x$ for all $x, y \in D$.*

Therefore, there is a functor from the category of dendriform algebras to the category of pre-Lie algebras.

Remark 2.9. If $(D, \prec, \succ)$ is a dendriform algebra, then the Lie structures associated to $(D, \star)$ and $(D, \{-,-\}$ coincide.

As a consequence, considering the associated categories of algebras we have the following proposition.

Proposition 2.10. *The following diagram of categories of algebras is commutative:*

$$\begin{array}{ccc} \text{Dend} & \longrightarrow & \text{preLie} \\ \downarrow & & \downarrow \\ \text{Ass} & \longrightarrow & \text{Lie} \end{array}$$

Next we prove a theorem analogue to the Jacobson's Theorem 1.3 in the pre-Lie algebra context.

Theorem 2.11. *Let $(D, \prec, \succ)$ be a dendriform algebra over a field k of characteristic $p \neq 0$. Then $(D, \{-,-\}, (-)^{\star p})$ is a restricted pre-Lie algebra.*

Proof. First we prove the relation 2.2. Let $x, y \in D$. We denote by R_x the right $\prec$ action by x namely $R_x(y) := y \prec x$. Similarly we denote by L_x the left $\succ$ action by x, i.e $L_x(y) := x \succ y$. From relation 2.6 in the axioms of the definition of dendriform algebra we obtain that

$$\begin{aligned} R_x(L_x(y)) &= (x \succ y) \prec x \\ &= x \succ (y \prec x) \\ &= L_x(R_x(y)). \end{aligned}$$

Besides we have

$$\begin{aligned}\{x,y\} &= x \succ y - y \prec x \\ &= L_x(y) - R_x(y) \\ &= (L_x - R_x)(y).\end{aligned}$$

Since L_x and R_x commute we obtain that $(L_x - R_x)^p = \sum_{i=0}^{i=p} \binom{p}{i} (L_x)^i (R_x)^{p-i}$. Therefore in prime characteristic p we obtain that $(L_x - R_x)^p = (L_x)^p - (R_x)^p$. Moreover from relation 2.7 we have $(L_x)^p(y) = x^{\star p} \succ y$. Besides from relation 2.5 we get $(R_x)^p(y) = y \prec x^{\star p}$.

Finally we have

$$\begin{aligned}(L_x - R_x)^p &= \{\underbrace{x, \{x, \cdots \{x}_{p}, y\} \cdots\},\} \\ &= (L_x)^p - (R_x)^p \\ &= x^{\star p} \succ y - y \prec x^{\star p} \\ &= \{x^{\star p}, y\}.\end{aligned}$$

Now by Jacobson's Theorem 1.3, the associative operation endows D with a Frobenius and therefore $(D, [-,-], (-)^{\star p})$ is a restricted Lie algebra for the induced Lie bracket. Therefore the relations 2.1 and 2.4 follows obviously.

Finally, we have

$$[x^{\star p}, y] = [\underbrace{x[x[\cdots[x}_{p}, y]]]]$$

and the relation 2.3 is a consequence of $[x^{\star p}, y] = \{x^{\star p}, y\} - \{y, x^{\star p}\}$. □

Therefore from Theorem 2.12 we obtain, in prime characteristic, the analogue of Proposition 2.7 :

Proposition 2.12. *The following diagram of categories of algebras is commutative:*

$$\begin{array}{ccc} \text{Dend} & \longrightarrow & p-\text{preLie} \\ \downarrow & & \downarrow \\ \text{Ass} & \longrightarrow & p-\text{Lie} \end{array}$$

3. Rota Baxter algebras and infinitesimal bialgebras.

In this section we obtain examples of restricted Lie algebras. We note that from Remark 2.5 any example of restricted pre-Lie algebra in terms of the definition of A. Dzhumadil'daev in [6] is an example in terms of Definition 2.3. Moreover, as a consequence of Theorem 2.12 we obtain a large amount of examples of restricted pre-Lie algebras.

3.1. **Rota-Baxter algebras.** Gian-Carlo Rota introduced a special types of operators (see in [13]), in the category of associative algebras beyond the usual derivations. In particular, let $(A, \cdot)$ be an associative algebra then an operator $\beta : A \to A$ is called *Rota-Baxter* operator if

$$\beta(x) \cdot \beta(y) := \beta(\beta(x) \cdot y + x \cdot \beta(y)).$$

An associative algebra (A, β) equipped with a Rota-Baxter operator β is called a Rota-Baxter algebra. Rota-Baxter algebras appear in many fields in theoretical physics and mathematics.

Proposition 3.1. *A Rota Baxter algebra $(A, \cdot, \beta)$ over a field k of characteristic $p \neq 0$ is equipped with the structure of a restricted pre-Lie algebra.*

Proof. From Aguiar' s Proposition 4.5 in [1] any Rota Baxter algebra is equipped with the structure of dendriform algebra with operations given by

$$x \succ y := \beta(x) \cdot y$$

and

$$x \prec y := x \cdot \beta(y).$$

Therefore by Theorem 2.12 the Rota Baxter algebra (A, β) is equipped with the structure of restricted pre-Lie algebra $(A, \{-,-\}, (-)^{\cdot p})$ where the pre-Lie bracket is given by $\{x, y\} := \beta(x) \cdot y - y \cdot \beta(x)$. □

3.2. **Quasitriangular algebras.** Infinitesimal algebras were introduced by Joni and Rota in [9]. An infinitesimal bialgebra or a ϵ-bialgebra is a triple (A, μ, Δ) where (A, μ) is an associative algebra, (A, Δ) is a coassociative coalgebra, and Δ is a derivation. A special class of ϵ-bialgebras are called quasitriangular ϵ-bialgebras. M. Aguiar in [1] proves that there is functor from the category of quasitriangular ϵ-bialgebras to the category of dendriform algebras. Thus by Theorem 2.12 we obtain the following proposition:

Proposition 3.2. *Any quasitriangular ϵ-bialgebra over a field k of characteristic $p \neq 0$ is equipped with the structure of a restricted pre-Lie algebra.*

4. Enveloping dendriform algebra

In this section we define the notion of enveloping dendriform algebra of a pre-Lie algebra. Moreover, we construct a left adjoint functor to the functor:

$$(-)_{\text{preLie}} : Dend \to preLie.$$

Definition 4.1. Let P be a pre-Lie algebra. A universal enveloping dendriform algebra of P is a pair (U, i) where U is a dendriform algebra and $i : P \to U$ is a pre-Lie homomorphism and the following holds: for any dendriform algebra A and any pre-Lie homomorphism $f : P \to A_{preLie}$ there exists a unique dendriform homomorphism $\theta : U \to A$ such that $\theta i = f$.

Let P be a pre-Lie algebra we consider the free dendriform algebra $Dend(P)$ generated by the vector space P. Let $\mathcal{R}$ be the dendriform ideal of $Dend(P)$ generated by the elements:

$$\{\{x, y\} - (x \succ y - y \prec x)\}$$

where $x, y \in P$. We denote by $U(P) := Dend(P)/\mathcal{R}$. Let $i : P \to U(P)$ denote the restriction to P of the canonical homomorphism of $Dend(P)$ onto $U(P)$.

Proposition 4.2. *Let P a pre-Lie algebra the pair $(U(P), i)$ is a universal enveloping dendriform algebra for P.*

Proof. Let $f : P \to A_{preLie}$ be a pre-Lie homomorphism, where P is a pre-Lie algebra and A a dendriform algebra. Since $Dend(P)$ is the free dendriform algebra over P there is a dendriform homomorphism $\theta' : Dend(P) \to A$ which extends f. Moreover we have:

$$\begin{aligned}\theta'(\{x,y\}) - \theta'(x \succ y - y \prec x) &= f(\{x,y\}) - \theta'(x) \succ \theta'(y) + \theta'(y) \prec \theta'(x) \\ &= \{f(x), f(y)\} - f(x) \succ f(y) + f(y) \prec f(x) \\ &= 0\end{aligned}$$

Since the dendriform algebra $Dend(P)$ is generated by P, the homomorphism θ' induces a unique dendriform homomorphism $\theta : U_p(P) \to A$ such that $\theta i = f$. □

Corollary 4.3. *The functor U : $preLie \to Dend$ is left adjoint to the functor $(-)_{preLie} : Dend \to preLie$. Therefore we have:*

$$Hom_{preLie}(P, A_{preLie}) \simeq Hom_{Dend}(U(P), A).$$

5. Restricted enveloping dendriform algebra

Similarly with the notion of restricted enveloping algebra in the Lie context we define the notion of restricted enveloping dendriform algebra for a restricted pre-Lie algebra. Also, we construct a left adjoint functor to the functor:

$$(-)_{p-\text{preLie}} : Dend \to p - preLie.$$

We recall from Theorem 2.8 that there is a functor $-_{p-\text{preLie}}$: Dend $\to$ p-preLie from the category of dendriform algebras to the category of p-preLie algebras. Next we construct a left adjoint functor $U_p : p-\text{preLie} \to \text{Dend}$ to the functor $-_{p-\text{preLie}}$.

Definition 5.1. Let P be a restricted pre-Lie algebra. A universal restricted enveloping dendriform algebra of P is a pair (U_p, i) where U is a dendriform algebra and $i : P \to U_p$ is a restricted pre-Lie homomorphism and the following holds: for any dendriform algebra A and any restricted pre-Lie homomorphism $f : P \to A_{p-preLie}$ there exist a unique dendriform homomorphism $\theta : U_p \to A$ such that $\theta i = f$.

Let $(P, (-)^{[p]})$ be a restricted pre-Lie algebra we consider the free dendriform $Dend(P)$ algebra generated by the vector space P. Let $\mathcal{R}_p$ be the dendriform ideal of $Dend(P)$ generated by the elements:

$$\{\{x,y\} - (x \succ y - y \prec x), x^{[p]} - x^{\star p}\}$$

where $x, y \in P$. We denote by $U_p(P) := Dend(P)/\mathcal{R}_p$. Let $i : P \to U(P)$ denote the restriction to P of the canonical homomorphism of $Dend(P)$ onto $U_p(P)$.

Proposition 5.2. *Let $(P, (-)^p)$ be a restricted pre-Lie algebra. The pair $(U_p(P), i)$ is a universal enveloping dendriform algebra for P.*

Proof. Let $f : P \to A_{p-preLie}$ be a restricted pre-Lie homomorphism, where $(P, (-)^{[p]})$ is a pre-Lie algebra and A is a dendriform algebra. From the universal property of $U(P)$ there is a dendriform homomorphism $\theta' : U(P) \to A$ which extends f. Moreover we have:

$$\begin{aligned}\theta'(x^{[p]}) - \theta'(x^{\star p}) &= f(x^{[p]}) - \theta'(x)^{\star p} \\ &= f(x)^{\star p} - \theta'(x)^{\star p} \\ &= 0\end{aligned}$$

The homomorphism θ' induces a unique dendriform homomorphism $\theta : U_p(P) \to A$ such that $\theta i = f$. □

Corollary 5.3. *The functor $U_p : p - preLie \to Dend$ is left adjoint to the functor $(-)_{p-preLie}$. Therefore we have:*

$$Hom_{p-preLie}(P, D_{p-preLie}) \simeq Hom_{Dend}(U_p(P), D).$$

References

[1] M. Aguiar, *Infinitesimal Hopf algebras*, in: New Trends in Hopf Algebra Theory, La Falda, 1999, in: Contemp. Math., vol. 267, Amer. Math. Soc., Providence, RI, 2000, pp. 129.

[2] A. Connes, D. Kreimer, *Hopf algebras, renormalization and non commutative geometry*, Commun. Math. Phys. 199, No. 1, 203-242 (1998).

[3] F. Chapoton, M. Livernet, *Pre-Lie algebras and the rooted trees operad*, Internat. Math. Res. Notices 2001, no. 8, 395–408.

[4] I. Dokas, J.-L. Loday, *On restricted Leibniz algebras*, Communications in Algebra , 34 (2006), no.12, 4467-4478.

[5] I. Dokas, *Zinbiel algebras and Commutative algebras with divided powers,* Glasgow Mathematical Journal, (2010), Volume 52, Issue 02, pp 303-313.

[6] A. Dzhumadil'daev, *Jacobson formula for right-symmetric algebras in characteristic p*, Comm. Algebra 29 (2001), no. 9, 3759–3771.

[7] M. Gerstenhaber, *The cohomology structure of an associative ring*, Ann. of Math. (2) 78 (1963), 267288.

[8] N. Jacobson, *Lie algebras,* Dover Publications, Inc, 1979.

[9] S. A. Joni and G. C. Rota, *Coalgebras and Bialgebras in Combinatorics*, Stud. Appl. Math. 61 (1979), no. 2, 93-139. Reprinted in Gian-Carlo Rota on Combinatorics: Introductory papers and commentaries (Joseph P.S. Kung, Ed.), Birkhauser, Boston (1995).

[10] J.-L. Loday, Dialgebras, in *Dialgebras and related operads*, Springer Lecture Notes in Math. 1763 (2001), 7-66.

[11] J.-L. Loday and B. Vallette, *Algebraic Operads*, (to appear).

[12] Milnor, John *On fundamental groups of complete affinely flat manifolds*, Advances in Math. 25 (1977), no. 2, 178–187.

[13] Gian-Carlo Rota, *Baxter Operators*, an Introduction, in Gian-Carlo Rota on Combinatorics: Introductory papers and commentaries (Joseph P.S. Kung, Ed.), Birkhauser, Boston, 1995. 504-512.

[14] E. B. Vinberg, Convex homogeneous cones, Trans. Moscow Math. Soc. 12 (1963) 340403.

E-mail address: `dokas@ucy.ac.cy`